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On the degree of the Gauss mapping of a submanifold of an Abelian variety

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Let X be an n-dimensional projective submanifold of an a-dimensional Abelian variety A. Since the holomorphic cotangent bundle, T_A^*, of A is trivial the surjection:

$$T_A^*|_X \to T_X^* \to 0$$

induces a classifying map:

$$\Gamma : X \to \mathrm{Gr}(n, a)$$

where $\mathrm{Gr}(n, a)$ denotes the Grassmannian of n dimensional quotients of $\mathbb{C}^a$. This mapping Γ is called the Gauss mapping. In this paper we bound the degree of Γ under the assumption that the normal bundle, N_X, of X in A is ample in the sense of Grothendieck [H2]. This condition is satisfied by a result of Hartshorne [H1] if $\dim X = 1$ and X generates A as a group or if A is simple.

(1.3.2) THEOREM. *Let X and A be as above then*

$$\deg \Gamma \leq \frac{|e(X)|}{\operatorname{cod} X}$$

where $e(X)$ is the topological Euler characteristic of X.

This theorem is stated and proved for immersed manifolds.

Examples (1.4.1), (1.4.2), (1.4.3) show the theorem is sharp and that it is false without the ampleness hypothesis.

The proof is based on a simple consequence, Theorem (0.1), of the result [G+L] of Gaffney and Lazarsfeld on ramification loci of branched coverings.

We follow the now standard practice of not distinguishing between vector bundles and their locally free sheaves of germs of holomorphic sections.

We would like to thank the referee for suggesting the proof of Theorem (0.1)

along the lines of an old argument of Remmert and Van de Ven [R+VdV]; our original argument which was slightly more involved used the theorem of Gaffney and Lazarsfeld [G+L] combined with symmetric products.

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§0. A result on branched covers of $\mathbb{P}_{\mathbb{C}}$

The following consequence of the result of Gaffney and Lazarsfeld [G+L] on ramification loci is the key step in the proof of our theorem [cf. also [L], Remark 2.3].

(0.1) THEOREM. *Let $f: W \to \mathbb{P}^n$ be a holomorphic finite to one surjection from an irreducible normal variety W onto $\mathbb{P}^n$. Assume that the degree of f is $k \leq n-1$. There is no surjective holomorphic map from W onto a positive dimensional variety of dimension $\leq n-k$.*

Proof. Assume that there was such a surjective map $g: W \to Y$ where $\dim Y \leq n-k$. Then $\dim f(g^{-1}(y))$ for any $y \in Y$ is at least k dimensional. Let Z_1 and Z_2 be the inverse images under g of a divisor and a point not on it respectively. The dimension of Z_1 is $n-1$ and the dimension of Z_2 is at least k. The set R_{k-1} of points of W where all sheets come together is of dimension at least $n-k+1$ by [G+L, Theorem 1]. Therefore

$$f(Z_1) \cap f(Z_2) \cap f(R_{k-1}) \neq \phi$$

Since f is one to one on R_{k-1}, we have the absurdity that Z_1 meets Z_2. □

The referee has pointed out that the assumption of normality of W is unnecessary.

§1. The degree of the Gauss mapping

(1.0) Let $\phi: X \to A$ be a holomorphic immersion of an n-dimensional projective manifold X into an $\dot{a}$-dimensional Abelian variety A. We have the natural map:

$$\phi^* T_A^* \to T_X^* \to 0 \tag{1.0.1}$$

Since T^*_A is trivial this defines the classifying map, called the Gauss map

$$\Gamma: X \to \mathrm{Gr}(n, a)$$

where $\mathrm{Gr}(n, a)$ is the Grassmannian of n dimensional quotients of $\mathbb{C}^a$.

(1.1) THEOREM. deg Γ *is a factor of all Chern numbers of* X.

Proof. Since Γ is the classifying map for (1.0.1) we conclude that $\Gamma^*Q \approx T^*_X$ where Q is the universal quotient bundle on $\mathrm{Gr}(n, a)$. Therefore any Chern number of T^*_X is deg Γ times the corresponding Chern number of $Q|_{\Gamma(X)}$. □

(1.2) The cokernel of (1.0.1) is denoted N^*_ϕ and called the conormal bundle of ϕ; if ϕ is an embedding it is the usual conormal bundle of X in A. The dual N_ϕ of N^*_ϕ is the normal bundle of ϕ.

By $\mathbb{P}(N_\phi)$ we mean $(N^*_\phi - X)/C^*$. There is a tautological line bundle ξ on $\mathbb{P}(N_\phi)$ such that direct image, $\pi_*(\xi)$, is isomorphic to N_ϕ where $\pi: \mathbb{P}(N_\phi) \to X$ is the projection induced from the projection of N_ϕ onto X.

We will be interested in maps, ϕ, such that N_ϕ is ample. By definition [H2] this means that there is an embedding $\psi: \mathbb{P}(N_\phi) \to \mathbb{P}_{\mathbb{C}}$ and some $k > 0$ such that $\psi^* \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1) \approx \xi^k$. A basic theorem of Hartshorne [H1] gives a condition for ampleness of normal bundles of submanifolds of Abelian varieties. It still holds with no changes of proof for immersions.

(1.2.1) THEOREM (Hartshorne [H1]). *Let* $\phi: X \to A$ *be a holomorphic immersion as in* (1.0). N_ϕ *is ample if either*:

(a) *A is a simple Abelian variety, i.e. A has no proper Abelian submanifold, or*

(b) dim $X = 1$ *and* $\phi(X)$ *generates* A *as a group.*

(1.3) Associated to the image of the a-dimensional vector space $\Gamma(T_A)$ into $\Gamma(N_\phi)$ under:

$$0 \to T_X \to \phi^* T_A \to N_\phi \to 0 \tag{*}$$

we have a holomorphic mapping

$$f: \mathbb{P}(N_\phi) \to \mathbb{P}^{a-1}$$

Here we identify sections of N_ϕ with sections of ξ to get our map.

It is not hard to see [cf. H+M] that we can identify $\mathbb{P}^{a-1}$ with $(\Gamma(T^*_A) - 0)/\mathbb{C}^*$ in such a way that

$$\pi_{f^{-1}(y)}: f^{-1}(y) \to X$$

maps $f^{-1}(y)$ biholomorphically onto:

$$\{x \in X \mid y(x) = 0\}$$

It follows from the definition of Γ that given a point $x \in X$ and a holomorphic one form η on X, induced by restriction from A

$$\eta(x) = 0 \text{ if and only if } \eta \text{ is zero on } (\Gamma^{-1}(\Gamma(x)) = 0. \tag{**}$$

Assume from here on that N_ϕ is ample. This is equivalent to the map f above being finite to one. From (**) we trivially see that Γ is finite to one. We refer the reader to the recent, pretty result of Z. Ran [R] for a proof of the finite to oneness of Γ whenever X is not fibred by tori.

Let Z denote the normalization of $\Gamma(X)$ and let $\Gamma': X \to Z$ denote the map induced by Γ. Let

$$\Phi: \mathbb{P}(N_\phi) \to \mathbb{P}^{a-1} \times Z$$

be the map given by $(f, \pi \circ \Gamma')$

The fibre degree of Γ and Γ' is the same. Denote it by $\deg \Gamma$. The fibre degree of f is $|e(X)|$, the absolute value of the topological Euler characteristic $e(X)$ of X. This follows from the usual identification of $c_n(X)[X] = (-1)^n c_n(T_X^*)[X]$ with $e(X)$ and the fact that f is finite to one.

By (**) and Γ being finite to one we see that $\mathcal{A} = \Phi(\mathbb{P}(N_\phi))$ maps finite to one onto $\mathbb{P}^{a-1}$ under the map $\tilde{f}$ induced by the projection of $\mathbb{P}^{a-1} \times Z$ onto $\mathbb{P}^{a-1}$. Let $f': \mathcal{A}' \to \mathbb{P}^{a-1}$ denote the map $\tilde{f}$ induced from the normalization $\mathcal{A}'$ of $\mathcal{A}$ onto $\mathbb{P}^{a-1}$. Its degree by the last paragraph is

$$\frac{|e(X)|}{\deg \Gamma}$$

Let g denote the map from $\mathcal{A}'$ onto Z induced by the projection $\mathbb{P}^{a-1} \times Z$ onto Z. Since Γ is finite to one we see that $\dim Z = \dim X = n$ and therefore that the fibres of g have dimension $a - n - 1$.

The following is now an immediate consequence of (0.1) with $W = \mathcal{A}'$.

(1.3.1) THEOREM. *Let $\phi: X \to A$ be a holomorphic immersion of a connected* *projective manifold X into an Abelian variety A. Assume that the normal bundle N_ϕ is ample, e.g. assume that A is simple or that X is a curve and $\phi(X)$ generates A. Then the degree of the Gauss mapping associated to $\phi : X \to A$ is bounded by:*

$$\frac{|e(X)|}{\operatorname{cod}\phi(X)}.$$

(1.4) Let us give some examples showing that the above is sharp.

(1.4.1) Let C be a smooth curve of genus $g>1$. Let $\phi: C \to \mathrm{Jac}(C)$ be the Albanese embedding of C into its Jacobian. Since $\phi(C)$ generates $\mathrm{Jac}(C)$, N_C is ample. Our theorem predicts that the degree of the Gauss mapping is ≤ 2. The Gauss mapping is easily checked to be the canonical mapping of C to $\mathbb{P}^{g-1}$ given by $\Gamma(K_C)$. This is an embedding unless C is hyperelliptic in which case it is 2 to one. For small codimension the result is also sharp. In codimension 2 it predicts degree $\le g-1$; in [N+S] will be found curves C of various genera immersed in complex 3-tori and having Gauss map of degree precisely $g-1$.

(1.4.2) Let X be a smooth ample divisor on a connected Abelian variety, A. Since $N_A \approx [A]$, our theorem applies and predicts that the degree is at most $|e(X)|$ to one. It is exactly this since $\mathbb{P}(N_X) \approx X$ and the map $f: \mathbb{P}(N_X) \to \mathbb{P}^{a-1}$ is then the Gauss mapping.

(1.4.3) Let $X = \prod_{i=1}^r X_i$ where X_i is a smooth connected ample divisor on a connected Abelian variety A_i for each $i = 1, \dots, r$. Then X is a submanifold $\prod_{i=1}^r A_i$ under the diagonal embedding. The Gauss mapping of X is easily seen to have degree $d_1 \cdot \dots \cdot d_r$ where d_i is the degree of the Gauss mapping of X_i in A_i for $i = 1, \dots, r$. By (1.4.2) we see this degree is $|e(X)|$. Therefore some condition such as ampleness is needed to bound the degree of the Gauss mapping.

One weak but curious consequence of the same sort of reasoning is the following.

(1.5) COROLLARY. *Let E be ample on a projective manifold X. Assume that E is spanned by global sections. Then the inverse of the total Chern class of E evaluated on X is $\ge rkE$.*

Proof. Let $f: \mathbb{P}(E) \to \mathbb{P}^{\dim X + rkE - 1}$ be the map from $\mathbb{P}(E)$ to projective space by a minimal spanning set of sections of E. The ampleness of E implies f is a finite to one surjection. As in the proof of our theorem, the inverse of the total Chern class of E evaluated on X is the degree of f. Use Theorem (0.1). $\square$

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