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Some spectral results for the Laplacian on line bundles over S^n

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Introduction

It is well known that the spectrum of the Laplace–Beltrami operator Δ on the standard n-sphere (S^n, g_0) consists of the eigenvalues

$$\lambda_k = k(k+n-1) = \left(k+\frac{n-1}{2}\right)^2 - \frac{(n-1)^2}{4}, \qquad k=0,1,2,\ldots$$

with the multiplicity $N_k = (n+2k-1)(n+k-2)(n+k-3)\cdots(n+1)n/k!$. A Weinstein [16], V. Guillemin [9], [10], Y. Colin de Verdière [3], [4], etc. studied the spectrum of a perturbed operator of the form $P = \Delta + q$, q being a C^∞ "potential" function, and obtained the following result among others.

THEOREM. *The spectrum of P (denoted by* Spec (P)*) consists (except for finitely many values) of clusters of eigenvalues in the interval*

$$[\lambda_k + \min \bar{V}(\gamma) - \varepsilon, \lambda_k + \max \bar{V}(\gamma) + \varepsilon],$$

for any $\varepsilon > 0$, where $\bar{V}: G \to \mathbf{R}$ is the function on the space G of closed geodesics of (S^n, g_0) defined by

$$\bar{V}(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} V(\gamma(t))\, dt.$$

Moreover, if a and b are regular values of $\bar{V}$, then

$$\frac{\#\{\nu \in \operatorname{Spec}(P);\ \lambda_k + a \leqq \nu \leqq \lambda_k + b\}}{N_k} \sim \frac{\mu\{\gamma \in G;\ a \leqq \bar{V}(\gamma) \leqq b\}}{\mu(G)} \quad (k \to \infty),$$

where μ is the measure on G.

In this paper we consider a perturbed operator of the form $\Delta+Q$, where Q is a first order differential operator. In particular, we are interested in the effect of the first order terms on the split of eigenvalues with multiplicity.

We note that the first order terms are related with a linear connection on a complex line bundle. Let E be a C^∞ Hermitian line bundle over a compact C^∞ Riemannian manifold (M, g), and $\tilde{d}$ be a linear connection compatible with the Hermitian structure (see [13], [17]). Let e be a local cross-section on E such that $|e|=1$. Then $\tilde{d}e=\omega e$ holds with a purely imaginary 1-form $\omega=i\alpha$ (α : real) on an open set of M. On a complex line bundle $(M, g; E, \tilde{d})$ with linear connection, there can be naturally defined a non-negative, second order, self-adjoint, elliptic differential operator L called the *Bochner–Laplacian* (or *Laplacian*, for short), which is locally expressed as

$$L=-\sum_{j,k=1}^{n} g^{jk}\nabla_j\nabla_k-2i\sum_{j=1}^{n} a^j\nabla_j+\sum_{j=1}^{n}(a_ja^j-i\nabla_ja^j),$$

where ∇ is the Levi–Civita connection defined by g and $\alpha=\sum a_j\,dx^j$, $a^j=\sum g^{jk}a_k$ (see [13]).

We will study the spectrum of the operator L on a complex line bundle E over (S^n, g_0), which operates on cross-sections of E.

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1. Results

As to $(S^n, g_0; E, \tilde{d})$, each linear connection $\tilde{d}$ is uniquely (up to gauge equivalence) determined by its curvature form $\Omega=d\omega$ on S^n, and the C^∞ bundle structure is given by $[\Omega/2\pi i]\in H^2(S^n, \mathbf{Z})$ (cf. Kostant [12]). The complex line bundle over S^n is always a trivial one, E_0, when $n\neq 2$, and the set of equivalence classes of line bundles over S^2 is $\{E_m\}$, each E_m being corresponding to $(1/2\pi i)\int_{S^2}\Omega=m\in\mathbf{Z}$.

On each line bundle E_m there is a unique harmonic connection $\tilde{d}_m$ whose curvature form is a harmonic 2-form on S^n (Hodge's theorem). Particularly, $(E_0, \tilde{d}_0)$ is the trivial bundle with the flat connection. Let L_m be the Laplacian defined from the harmonic connection $\tilde{d}_m$. (Notice that $L_0=\Delta$.) Then the spectrum of L_m (when $n=2$) consists of the eigenvalues

$$\lambda_k^{(m)}=\left(k+\frac{|m|+1}{2}\right)^2-\frac{m^2+1}{4},\qquad k=0,1,2,\ldots$$

with the multiplicity $N_k^{(m)}=|m|+2k+1$ (see [7], [13]). We set $\lambda_k^{(0)}=\lambda_k$, $N_k^{(0)}=N_k$ including the case $n\neq 2$.

Let $\tilde{d}$ be a linear connection on E_m, and L the Laplacian defined by $\tilde{d}$. Let $\mu_j^{(k)}$, $j=1,2,\ldots,N_k^{(m)}$, denote the eigenvalues of L which is split from $\lambda_k^{(m)}$ $(k=0,1,2,\ldots)$. Our first result is the following.

THEOREM 1. (1) $\max_j |\mu_j^{(k)}-\lambda_k^{(m)}|=O(k)$ $(k\to\infty)$ *holds.*
(2) $\max|\mu_j^{(k)}-\lambda_k^{(m)}|=O(1)$ *holds if and only if*

$$Q_{\tilde{d}}(\gamma)=Q_{\tilde{d}_m}(\gamma)\ (=(-1)^m) \tag{1.1}$$

holds for every closed geodesic γ *of* (S^n, g_0). *Here* $Q_{\tilde{d}}(\gamma)$ (*resp.* $Q_{\tilde{d}_m}(\gamma)$) *denotes the holonomy of the linear connection* $\tilde{d}$ (*resp.* $\tilde{d}_m$) *along* γ.

Remark. The condition (1.1) means that the parallel lift $\tilde{\gamma}$ of the closed geodesic γ is a closed curve in E_m. If m is odd (resp. even), $\tilde{\gamma}$ doubly (resp. simply) covers γ.

Our next result makes clear how the eigenvalues in the k-th cluster distribute as $k\to\infty$ when the condition (1.1) is satisfied for every closed geodesic.

Let S^*S^n be the unit cosphere bundle over S^n, on which there can be defined the induced volume form dvol from the symplectic volume form $dx^1\wedge\cdots\wedge dx^n\wedge d\xi_1\wedge\cdots\wedge d\xi_n$ on T^*S^n (cf. [9, §4]). Let ϕ_t be the Hamiltonian flow on $T^*S^n\backslash 0$ defined by the function $(x,\xi)\mapsto|\xi|$.

Under local coordinates of S^n and a local unitary frame of E_m, let $\omega=i\alpha=i\sum a_j\,dx^j$ and $\omega_m=i\alpha_m=i\sum \hat{a}_j\,dx^j$ be the connection form of $\tilde{d}$ and $\tilde{d}_m$, respectively. Then $\omega-\omega_m=i\beta=i\sum b_j\,dx^j$ (b_j : real) is a 1-form globally defined on S^n.

Remark. If ω and ω' are cohomologous, i.e. $\omega-\omega'$ is exact, then they define the gauge equivalent connections, and accordingly the spectra of their Laplacians are the same each other (see [13, §4]).

(I) *Case of the trivial bundle* E_0. We define a function $J^{(0)}$ on S^*S^n by

$$\begin{aligned} J^{(0)}=&\frac{1}{2\pi}\int_0^{2\pi}\Big[\phi_t^*\Big(\sum_j b_jb^j\Big)-\phi_t^*\Big(\sum_j b^j\xi_j/|\xi|\Big)^2\Big]\,dt\\ &-\frac{1}{2\pi}\int_0^{2\pi}dt\int_0^t\Big\{\phi_t^*\Big(\sum_j b^j\xi_j/|\xi|\Big),\,\phi_s^*\Big(\sum_j b^j\xi_j/|\xi|\Big)\Big\}\,ds. \end{aligned}\tag{1.2}$$

where $\{\,,\}$ is the Poisson bracket. Then we have the following.

THEOREM 2-0. *Assume that the condition* (1.1) *is satisfied* (*with* $m=0$) *for* *every closed geodesic γ of (S^n, g_0). If a and b are regular values of $J^{(0)}$, then*

$$\begin{aligned}&\#\{j;\ \mu_j^{(k)}\in[\lambda_k^{(0)}+a,\ \lambda_k^{(0)}+b],\ 1\leqq j\leqq N_k^{(0)}\}\\&\quad=(2\pi)^{-n}\operatorname{vol}\{(x,\xi)\in S^*S^n;\ a\leqq J^{(0)}(x,\xi)\leqq b\}k^{n-1}+O(k^{n-2}).\end{aligned}$$

(II) *Two dimensional case.* Let (θ, φ) be the polar coordinates of S^2 in $\mathbf{R}^3$. We consider various quantities on an open set $\mathring{S}^2=S^2\backslash\{\theta=0, \pi\}$. The restriction of E_m to $\mathring{S}^2$ is a trivial bundle, the Riemannian metric and the connection form of $\tilde{d}_m$ being given by

$$ds^2=d\theta^2+\sin^2\theta\, d\varphi^2,$$

$$\omega_m=i(\hat{a}_1\, d\theta+\hat{a}_2\, d\varphi)=\frac{i}{2}m(1-\cos\theta)\, d\varphi.$$

Let $(\theta, \varphi, \xi_1, \xi_2)$ be the canonical coordinates of $T^*\mathring{S}^2$, and let $\beta=\alpha-\alpha_m=b_1\, d\theta+b_2\, d\varphi$ on $\mathring{S}^2$. Now we define a function $J^{(m)}$ on $S_0^*S^2=S^*\mathring{S}^2\backslash\{\xi_2=0\}$ by

$$\begin{aligned}J^{(m)}=&\frac{1}{2\pi}\int_0^{2\pi}\Big[\phi_t^*\Big(\sum_j b_jb^j\Big)-\phi_t^*\Big(\sum_j b^j\xi_j/|\xi|\Big)^2\Big]\,dt\\&-\frac{1}{2\pi}\int_0^{2\pi}dt\int_0^t\Big\{\phi_t^*\Big(\sum_j b^j\xi_j/|\xi|\Big),\ \phi_s^*\Big(\sum_j b^j\xi_j/|\xi|\Big)\Big\}\,ds\\&+\frac{1}{\pi}\int_0^{2\pi}\phi_t^*\Big[b_2\hat{a}^2-(\hat{a}^2\xi_2/|\xi|)\Big(\sum_j b^j\xi_j/|\xi|\Big)\Big]\,dt\\&-\frac{1}{\pi}\int_0^{2\pi}dt\int_0^t\Big\{\phi_t^*(\hat{a}^2\xi_2/|\xi|),\ \phi_s^*\Big(\sum_j b^j\xi_j/|\xi|\Big)\Big\}\,ds,\end{aligned}\tag{1.3}$$

where

$$\hat{a}^2=\frac{m}{2(1+\cos\theta)}.$$

Then we have the following.

THEOREM 2-m. *Assume that the condition* (1.1) *is satisfied for every closed geodesic γ of (S^2, g_0). If a and b are regular values of $J^{(m)}$, then*

$$\begin{aligned}&\#\{j;\ \mu_j^{(k)}\in[\lambda_k^{(m)}+a,\ \lambda_k^{(m)}+b],\ 1\leqq j\leqq N_k^{(m)}\}\\&\quad=(2\pi)^{-2}\operatorname{vol}\{(x,\xi)\in S_0^*S^2;\ a\leqq J^{(m)}(x,\xi)\leqq b\}k+O(1).\end{aligned}$$

2. Preliminaries

In this section we first show that

$$\max_j |\mu_j^{(k)} - \lambda_k^{(m)}| = O(k)$$

holds as $k \to \infty$. Let $L(s)$ $(0 \leqq s \leqq 1)$ be the Laplacian defined by the linear connection $\tilde{d}(s)$ with the connection form $\omega(s) = \omega_m + s(\omega - \omega_m) = i\alpha_m + is\beta$, and let $\mu_j^{(k)}(s)$ be its eigenvalues with $\mu_j^{(k)}(0) = \lambda_k^{(m)}$. Since the coefficients of $L(s)$ vary analytically with respect to s, $\mu_j^{(k)}(s)$ depends also analytically on s (cf. [1, Lemma 3.15]). Moreover we have the following.

PROPOSITION 2.1. *For each $s \in [0, 1]$ there exists a positive constant C_s (not depending on j and k) such that*

$$\left|\frac{d\mu_j^{(k)}}{ds}(s)\right| \leqq C_s(1 + \mu_j^{(k)}(s))^{1/2}. \tag{2.1}$$

Proof. Let $\{\psi_j^{(k)}(s)\}$ be the system of orthonormal eigensections of $L(s)$ associated with $\mu_j^{(k)}(s)$, that is

$$L(s)\psi_j^{(k)}(s) = \mu_j^{(k)}(s)\psi_j^{(k)}(s).$$

Here we choose such $\psi_j^{(k)}(s)$'s that depend analytically on s. Differentiate this equation with respect to s, and we get

$$\frac{d\mu_j^{(k)}}{ds}(s) = (L'(s)\psi_j^{(k)}(s), \psi_j^{(k)}(s)), \tag{2.2}$$

where $L'(s) = dL(s)/ds$ and $(\ ,\ )$ is the natural inner product in the space, $C^\infty(E_m)$, of cross-sections of E_m. Therefore we have

$$\left|\frac{d\mu_j^{(k)}}{ds}(s)\right| \leqq \|L'(s)\psi_j^{(k)}(s)\|_{L^2} \|\psi_j^{(k)}(s)\|_{L^2} = \|L'(s)\psi_j^{(k)}(s)\|_{L^2}.$$

The first order differential operator $L'(s)$ defines a continuous map $H^l(E_m)$ into $H^{l-1}(E_m)$, $H^l(E_m)$ being a Sobolev space with the norm $\|u\|_l = \|(1 + L(s))^{l/2}u\|_{L^2}$. So we have

$$\begin{aligned}\|L'(s)\psi_j^{(k)}(s)\|_{L^2} &\leqq C_s \|\psi_j^{(k)}(s)\|_1 = C_s \|(1 + L(s))^{1/2}\psi_j^{(k)}(s)\|_{L^2} \\ &= C_s(1 + \mu_j^{(k)}(s))^{1/2}. \quad \square\end{aligned}$$

Obviously C_s depends continuously on s. Therefore, if we set $C=\max_{0\leqq s\leqq 1} C_s$, the inequality (2.1) implies that

$$|\mu_j^{(k)}-\lambda_k^{(m)}|\leqq Ck+o(k).$$

Thus Theorem 1, (1) is proved.

The following proposition is useful for the proof of Theorem 1, (2).

PROPOSITION 2.2. *If there is a pseudo-differential operator F on E_m of order* 0 *such that the operator*

$$G(s)=L'(s)+[L(s), F]$$

is of order 0, *then there exists a constant K_s such that*

$$\left|\frac{d\mu_j^{(k)}}{ds}(s)\right|\leqq K_s$$

for every j and k.

Proof. Using (2.2), we have

$$\begin{aligned}\frac{d}{ds}\mu_j^{(k)}(s)&=(G(s)\psi_j^{(k)}(s)-L(s)F\psi_j^{(k)}(s)+FL(s)\psi_j^{(k)}(s), \psi_j^{(k)}(s))\\&=(G(s)\psi_j^{(k)}(s), \psi_j^{(k)}(s)).\end{aligned}$$

Since $G(s)$ is a pseudo-differential operator of order 0, hence a bounded operator, we obtain the proposition. □

3. Return operator

We set for $m=0, \pm1, \pm2, \ldots$

$$A=\left(L+\frac{(n-1)^2+m^2}{4}\right)^{1/2},$$

$$A_m=\left(L_m+\frac{(n-1)^2+m^2}{4}\right)^{1/2},$$

where $m=0$ is the only case when $n\neq 2$. Then A and A_m are first order, self-adjoint, positive, elliptic pseudo-differential operators acting on $C^\infty(E_m)$. The spectrum of A_m is

$$\bar{\lambda}_k^{(m)} = k + \frac{|m| + (n-1)}{2}, \qquad k = 0, 1, 2, \ldots .$$

Let $U(t) = \exp(itA)$ and let $U_m(t) = \exp(itA_m)$. Then $U(t)$ and $U_m(t)$ are Fourier integral operators of order 0 for each $t \in \mathbf{R}$ (cf. [5]), and $U_m(2\pi) = (-1)^{|m|+n-1} I$ holds. We consider the so-called *return operator*

$$R = (-1)^{|m|+n-1} U(2\pi) = I + W,$$

which gives much information about the split of the spectrum of L from $\lambda_k^{(m)}$ or that of A from $\bar{\lambda}_k^{(m)}$. Let

$$R(t) = U(t) U_m(t)^{-1} = I + W(t). \tag{3.1}$$

Then $R = R(2\pi)$, and $R(t)$ is a pseudo-differential operator of order 0 because the Fourier integral operators $U(t)$ and $U_m(t)$ are associated with the same canonical transformation ϕ_t which is the Hamiltonian flow defined by the same principal symbol $|\xi|$ of A and A_m, so the geodesic flow. We remark that $W(t)$ is of order 0 (not -1) contrary to the case for the operator $P = \Delta + q$ (q: function) considered in [9], [10].

We compute the symbol of $W(t)$. Under fixed local coordinates of S^n and local cross-section of E_m, consider the Laplacian expressed as

$$L = -\sum g_0^{jk} \nabla_j \nabla_k - 2i \sum a^j \nabla_j + \sum (a_j a^j - i \nabla_j a^j),$$
$$L_m = -\sum g_0^{jk} \nabla_j \nabla_k - 2i \sum \hat{a}^j \nabla_j + \sum (\hat{a}_j \hat{a}^j - i \nabla_j \hat{a}^j),$$

with $\alpha = \sum a_j\, dx^j$, $\alpha_m = \sum \hat{a}_j\, dx^j$, and $\alpha - \alpha_m = \beta = \sum b_j\, dx^j$. We first list up the principal symbols of various operators:

$$\sigma(A) = \sigma(A_m) = |\xi|, \qquad \sigma(Q) = 2 \sum_j b^j \xi_j, \tag{3.2}$$

where $Q = L - L_m = -2i \sum b^j \nabla_j + \sum (b_j b^j + 2 b_j \hat{a}^j - i \nabla_j b^j)$, and

$$\sigma(A - A_m) = \frac{1}{|\xi|} \sum_j b^j \xi_j. \tag{3.3}$$

The last one is obtained by the equation $(A + A_m)(A - A_m) = Q + [A_m, A - A_m]$.

Remark. The principal symbol of a pseudo-differential operator on the vector bundle E over M is regarded as a map $\sigma: T^*M\backslash 0 \to \mathrm{End}(E)$ such that $\sigma(x, \xi) \in \mathrm{End}(E_x)$, where $\mathrm{End}(E)$ is the bundle of endomorphisms of E, which is isomorphic with $M \times \mathbf{C}$ in case of the line bundle, so the principal symbol is identified with a complex valued function on $T^*M\backslash 0$.

Now differentiate (3.1) with respect to t, and we get

$$\frac{1}{i}\dot{W}(t) = [A_m, W(t)] + (A - A_m)(I + W(t)). \tag{3.4}$$

Therefore using (3.2) and (3.3), we get the differential equation for the principal symbol $w(x, \xi; t)$ of $W(t)$:

$$\frac{1}{i}\dot{w}(x, \xi; t) = \frac{1}{i}(H_\sigma w)(x, \xi; t) + \frac{1}{|\xi|}\Big(\sum_j b^j \xi_j\Big)(w(x, \xi; t) + 1),$$

where H_σ denotes the Hamiltonian vector field with respect to $\sigma(A_m) = |\xi|$. Integrating this equation with $w(x, \xi; 0) = 0$, we obtain

$$w(x, \xi; t) = \exp\left[\frac{i}{|\xi|}\int_0^t \Big(\sum_j b^j(x(t-s))\xi_j(t-s)\Big)\, ds\right] - 1, \tag{3.5}$$

where $(x(s), \xi(s))$ is the integral curve of H_σ in $T^*S^n\backslash 0$ with initial point $(x(0), \xi(0)) = (x, \xi)$. From this formula we have

$$\sigma(W)(x, \xi) = w(x, \xi; 2\pi) = \exp\Big(i\int_{\gamma(x,\xi)} \beta\Big) - 1, \tag{3.6}$$

$\gamma(x, \xi)$ being the closed geodesic of (S^n, g_0) through x with $\dot{\gamma} = (\xi/|\xi|)^\#$ at x. ($\#: T^*S^n \to TS^n$ is the bundle isomorphism defined by g_0.)

The holonomy $Q_{\tilde{d}}(c)$ of the connection $\tilde{d}$ along a closed curve c in S^n is given by

$$Q_{\tilde{d}}(c) = \exp\Big(-\int_{\Sigma_c} \Omega\Big),$$

where Ω is the curvature form of $\tilde{d}$ and Σ_c is a surface in S^n with $\partial\Sigma_c = c$ (cf. [12, p. 108]). For the harmonic connection $\tilde{d}_m$,

$$Q_{\tilde{d}_m}(\gamma) = \exp\Big(-\int_{\Sigma_\gamma} \Omega_m\Big) = (-1)^m \tag{3.7}$$

holds for every closed geodesic γ of (S^n, g_0). From (3.6) we get

$$\sigma(W) = \exp\left(i\int_{\Sigma_\gamma} d\beta\right) - 1 = \exp\left(\int_{\Sigma_\gamma} (\Omega - \Omega_m)\right) - 1$$
$$= Q_{\bar{a}}(\gamma)^{-1} Q_{\bar{a}_m}(\gamma) - 1.$$

Thus we have

PROPOSITION 3.1. *The principal symbol of W is given by*

$$\sigma(W)(x, \xi) = Q_{\bar{a}}(\gamma(x, \xi))^{-1} Q_{\bar{a}_m}(\gamma(x, \xi)) - 1$$
$$= (-1)^m Q_{\bar{a}}(\gamma(x, \xi))^{-1} - 1. \tag{3.8}$$

4. Proof of Theorem 1

We first recall the following lemma concerning the norm of pseudo-differential operators of order 0 (cf. Nirenberg [15]).

LEMMA 4.1. *Let P be a pseudo-differential operator of order* 0 *on a Hermitian vector bundle E over a compact manifold and $p(x, \xi)$ be its principal symbol. Then,*

$$\inf_C \|P + C\| = \overline{\lim_{\xi \to \infty}} \sup_x |p(x, \xi)|,$$

where $\|\cdot\|$ denotes the operator norm as a map of $L^2(E)$ into $L^2(E)$, and infimum is taken over all compact operators C on $L^2(E)$; $|p(x, \xi)|$ represents the norm of $p(x, \xi)$ in End(E). *Particularly for the case where $p(x, \xi)$ is homogeneous in ξ, P is compact if and only if $p(x, \xi) \equiv 0$.*

Proof of Theorem 1, (2). Set $\bar{\mu}_j^{(k)} = (\mu_j^{(k)} + [(n-1)^2 + m^2]/4)^{1/2}$. Then $\{\bar{\mu}_j^{(k)}\}$ is the spectrum of A, and the condition $\max |\mu_j^{(k)} - \lambda_k^{(m)}| = O(1)$ $(k \to \infty)$ is equivalent to $\max |\bar{\mu}_j^{(k)} - \bar{\lambda}_k^{(m)}| = O(k^{-1})$. First, assume that $\max |\mu_j^{(k)} - \lambda_k^{(m)}| = O(1)$. Then, the eigenvalues of W

$$(-1)^{|m|+n-1} \exp(2\pi i \bar{\mu}_j^{(k)}) - 1 = \exp[2\pi i(\bar{\mu}_j^{(k)} - \bar{\lambda}_k^{(m)})] - 1$$

tend to 0 as $k \to \infty$, that is, W is a compact operator. Hence, by Lemma 4.1 we have $\sigma(W) \equiv 0$, so the condition (1.1) by virtue of Proposition 3.1.

In order to prove the converse we need the following lemmas.

LEMMA 4.2. *The condition* (1.1) *is replaced by*

$$\int_\gamma \beta = 0. \tag{4.1}$$

Proof. By virtue of (3.6) the condition (1.1) implies that $\hat{\beta}(\gamma) = \int_\gamma \beta$ takes an integer for every $\gamma \in \mathrm{Geod}(S^n)$ (the manifold of closed geodesics of (S^n, g_0)). Since $\hat{\beta}$ is a continuous function on $\mathrm{Geod}(S^n)$ which is connected, $\hat{\beta}$ is constant, and moreover is equal to zero by the fact $\hat{\beta}(\gamma^{-1}) = -\hat{\beta}(\gamma)$, where γ^{-1} denotes the closed geodesic with the inverse direction. □

Next, consider the first order differential operator $L'(s)$ discussed in §2, which is locally expressed as

$$L'(s) = -2i \sum_j b^j \nabla_j + \sum_j (2b_j \hat{a}^j - i \nabla_j b^j + 2sb_j b^j).$$

LEMMA 4.3. *There is a pseudo-differential operator F on E_m of order* 0 *such that $G(s) = L'(s) + [L(s), F]$ is of order* 0, *if and only if $\int_\gamma \beta = 0$ holds for every closed geodesic γ of (S^n, g_0).*

Proof. Suppose there is a operator F of order 0 with $\sigma(F) = f$ such that $G(s)$ is of order 0. Then the principal symbol of $G(s)$ is equal to zero, so

$$2\sum_j b^j \xi_j + \frac{1}{i}\left\{|\xi|^2, f\right\} = 0,$$

i.e.

$$\frac{1}{|\xi|}\sum_j b^j \xi_j + \frac{1}{i} H_\sigma f = 0. \tag{4.2}$$

Let $(x(t), \xi(t))$ be an integral curve of H_σ in $T^*S^n \backslash 0$ such that $\gamma(t) = x(t)$. Then from (4.2) we have

$$\begin{aligned}\int_\gamma \beta &= \frac{1}{|\xi|}\int_0^{2\pi}\Big(\sum_j b^j(x(t))\xi_j(t)\Big)\, dt = -\frac{1}{i}\int_0^{2\pi} (H_\sigma f)(x(t), \xi(t))\, dt \\ &= -\frac{1}{i}\int_0^{2\pi} \frac{d}{dt} f(x(t), \xi(t))\, dt = 0.\end{aligned}$$

Conversely, when $\int_\gamma \beta = 0$ for every closed geodesic γ, we define a C^∞ function f on $T^*S^n\backslash 0$ by

$$f(x,\xi) = \frac{i}{2\pi\,|\xi|}\int_0^{2\pi}\left[\int_0^t \left(\sum_j b^j(x(s))\xi_j(s)\right) ds\right] dt,$$

$(x(t), \xi(t))$ being the integral curve of H_σ with $(x(0), \xi(0)) = (x, \xi)$. Then f is homogeneous of degree 0 in ξ, and

$$\begin{aligned}(H_\sigma f)(x,\xi) &= \frac{i}{2\pi\,|\xi|}\int_0^{2\pi}\left[\int_0^t \frac{d}{ds}\left(\sum_j b^j(x(s))\xi_j(s)\right) ds\right] dt\\ &= \frac{i}{2\pi\,|\xi|}\int_0^{2\pi}\left[\left(\sum_j b^j(x(t))\xi_j(t)\right) - \left(\sum_j b^j(x)\xi_j\right)\right] dt\\ &= -i\sum_j b^j(x)\xi_j/|\xi|.\end{aligned}$$

Thus f satisfies (4.2). Let F be a pseudo-differential operator of order 0 whose principal symbol is f. Then $G(s)$ is of order 0. $\square$

Now, assume that the condition (1.1) is satisfied for every closed geodesic γ of (S^n, g_0). Then, by combining Lemmas 4.2 and 4.3 and Proposition 2.2, we get

$$|\mu_j^{(k)} - \lambda_k^{(m)}| \leqq K_{m,\tilde{a}}$$

for a constant $K_{m,\tilde{a}}$ not depending on j and k. Thus the proof of Theorem 1 is completed.

We conclude this section by a discussion about the condition (4.1).

PROPOSITION 4.4. *A* 1-*form* β *on* (S^n, g_0) *satisfies* $\int_\gamma \beta = 0$ *for every closed geodesic* γ, *if and only if*

$$\beta = df + \beta',$$

where f *is a* C^∞ *function and* β' *is an odd* 1-*form, that is,* $\tau^*\beta' = -\beta'$ *for the antipodal map* τ *of* S^n.

The proof will be given in Appendix.

As is easily shown the harmonic curvature form Ω_m is an odd 2-form. Hence, noting that $\tau^*\, d\beta = d\tau^*\beta$, we have the following theorem in terms of curvature forms.

THEOREM 4.5. *Let Ω be a closed 2-form on (S^n, g_0) such that $(1/2\pi i)\int_S \Omega = m \in \mathbf{Z}$ for every closed surface S in S^n, and let $(E, \tilde{d})$ be the line bundle with the linear connection whose curvature form is Ω. When and only when Ω is odd, the spectrum of the Laplacian L defined by $\tilde{d}$ consists of clusters of eigenvalues $\mu_j^{(k)}$, $j = 1, \ldots, N_k^{(m)}$, with* $\max |\mu_j^{(k)} - \lambda_k^{(m)}| = O(1)$ $(k \to \infty)$.

5. Proof of Theorem 2

We consider the case where the spectrum of L consists of clusters, that is the case where the operator W studied in §3 is of order -1. We apply the result due to A. Weinstein [16] and the improvement by Y. Colin de Verdiere [4] to our case.

Set

$$V = \frac{1}{2\pi i}(AW - W^*A),$$

W^* being the adjoint operator of W. Then we have

LEMMA 5.1. (1) *V is a self-adjoint pseudo-differential operator of order* 0 *with the principal symbol*

$$\sigma(V) = \frac{|\xi|}{2\pi i}(\sigma(W) - \overline{\sigma(W)}), \tag{5.1}$$

where $\sigma(W)$ denotes the symbol of order -1 of W.

(2) $[A, V] = 0$.

(3) *Let $\{\kappa_j^{(k)}\}$ $(j = 1, \ldots, N_k^{(m)}, k = 0, 1, 2, \ldots)$ be the spectrum of V. Then,*

$$\kappa_j^{(k)} = (\mu_j^{(k)} - \lambda_k^{(m)}) + O(k^{-1}) \quad (k \to \infty). \tag{5.2}$$

Proof. The proofs of (1) and (2) are obvious. (3) The eigenvalues of V are equal to

$$(1/2\pi i)\bar{\mu}_j^{(k)}[(-1)^{|m|+n-1}\{\exp(2\pi i \bar{\mu}_j^{(k)}) - \exp(-2\pi i \bar{\mu}_j^{(k)})\}],$$

which are asymptotically $(\mu_j^{(k)} - \lambda_k^{(m)}) + O(k^{-1})$ when $|\mu_j^{(k)} - \lambda_k^{(m)}| = O(1)$. □

The following lemma is due to Y. Colin de Verdière.

LEMMA 5.2. *For the operator A there are pseudo-differential operators A' of order* 1 *and A'' of order* -1 *such that*

(i) $A = A' + A''$,

(ii) $\operatorname{Spec}(A') \subset \left\{k + \dfrac{|m|+n-1}{2}\,;\, k \in \mathbf{Z}\right\}$, *and*

(iii) $[A', V] = 0$.

Proof. See Y. Colin de Verdière [3, Theorem 1.1].

Now consider the commuting pseudo-differential operators of order $1 : P_1 = A'$ and $P_2 = A'V$. Then by virtue of the result of Colin de Verdiere [4, Theorem 0.8] the asymptotic distribution of $\{\kappa_j^{(k)}\}$ as $k \to \infty$ is known by the function $J^{(m)} = \sigma(V)\,|_{S^*S^n}$. That is,

$$\begin{aligned} &\#\{\kappa_j^{(k)} \in [a, b];\ 1 \leqq j \leqq N_k^{(m)}\} \\ &\quad = (2\pi)^{-n} \operatorname{vol}\{J^{(m)-1}([a, b])\} k^{n-1} + O(k^{n-2}), \end{aligned}$$

a and b being regular values of $J^{(m)}$ (cf. [4, §4]). Hence, noticing Lemma 5.1, (3), we obtain Theorem 2.

We give the explicit formula of the function $J^{(m)}$. Noting the formula (5.1), we compute the symbol of order -1 of W. Let $v(x, \xi; t)$ be the subprincipal symbol of $W(t)$ discussed in §3. Consider the symbols of order -1 in eq. (3.4), and we get

$$\begin{aligned} \frac{1}{i}\,\dot{v}(t) = \frac{1}{i} H_\sigma v(t) + \sigma(A - A_m) v(t) + \frac{1}{i}\{\sigma_{\mathrm{sub}}(A_m), w(t)\} \\ + (\sigma_{\mathrm{sub}}(A - A_m))(1 + w(t)) + \frac{1}{2i}\{\sigma(A - A_m), w(t)\}. \end{aligned} \tag{5.3}$$

We need subprincipal symbols of some operators, which are computed as follows in local coordinates of S^n and a local frame of E_m:

$$\sigma_{\mathrm{sub}}(A_m) = \frac{1}{|\xi|} \sum_j \hat{a}^j \xi_j, \tag{5.4}$$

$$\sigma_{\mathrm{sub}}(Q) = \sum_j (b_j b^j + 2 b_j \hat{a}^j), \tag{5.5}$$

$$\sigma_{\text{sub}}(A-A_m)=-\frac{1}{|\xi|^3}\Big(\sum_j \hat{a}^i\xi_j\Big)\Big(\sum_j b^i\xi_j\Big)-\frac{1}{2|\xi|^3}\Big(\sum_j b^i\xi_j\Big)^2$$
$$+\frac{1}{2|\xi|}\sum_j(b_jb^i+2b_j\hat{a}^i). \tag{5.6}$$

See [7, Appendix] about various formulas for subprincipal symbols. In particular, (5.6) is derived by the identity:

$$2A_m(A-A_m)=Q+[A_m, A-A_m]-(A-A_m)^2.$$

The quantities (5.4) ~ (5.6) do not depend on the choice of local coordinates of S^n if we regard the operators as acting on $C^\infty(|\Lambda^n|^{1/2}\otimes E_m)$, $|\Lambda^n|^{1/2}$ being the bundle of half-densities on S^n (cf. [6]). They however depend on the choice of local frames of the bundle E_m. In fact, under a change of frame $e\mapsto e'=fe$, f being a non-vanishing function (called the transition function), the corresponding connection forms ω and ω' are related as $\omega'=\omega+f^{-1}\,df$. Therefore, if we admit only such frame transformations that satisfy $df=0$, the above quantities are invariantly defined as C^∞ functions on $T^*S^n\backslash 0$.

Integrating eq. (5.3) with $v(x, \xi; 0)=0$, we obtain

$$v(x,\xi;t)=\exp\left[\frac{i}{|\xi|}\int_0^t\Big(\sum_j b^j(x(t-s))\xi_j(t-s)\Big)\,ds\right]$$
$$\times\left\{\int_0^t\exp\left[-\frac{i}{|\xi|}\int_0^\tau\Big(\sum_j b^j(x(t-s))\xi_j(t-s)\Big)\,ds\right]C(x(t-\tau),\xi(t-\tau);\tau)\,d\tau\right\},$$

with

$$C(x,\xi;\tau)=i(\sigma_{\text{sub}}(A-A_m))(w(\tau)+1)+\tfrac{1}{2}\{\sigma(A-A_m)+2\sigma_{\text{sub}}(A_m), w(\tau)\}, \tag{5.7}$$

where $x(s)$ and $\xi(s)$ are the same as in the formula (3.5), and we assume that $x(s)$ $(0\leqq s\leqq t)$ is covered by a system of local frames $\{(U_j, e_j, f_{jk})\}$ of E_m such that $df_{jk}=0$.

LEMMA 5.3. *Let $(M, g; E, \tilde{d})$ be a complex line bundle with linear connection over a Riemannian manifold (M, g), and let c be a smooth, simply closed curve in M. Then there is a tubular neighborhood U of c such that $E\mid U$, the restriction of E to U, is a trivial bundle.*

The proof is obvious.

Noting this lemma and the condition $\int_{\gamma(x,\xi)} \beta = 0$ for every $(x, \xi) \in T^*S^n \backslash 0$, we have

$$\sigma(W) = v(x, \xi; 2\pi)$$
$$= \int_0^{2\pi} \exp\left[-\frac{i}{|\xi|}\int_0^{\tau}\left(\sum_j b^j(x(2\pi - s))\xi_j(2\pi - s)\right) ds\right] C(x(2\pi-\tau), \xi(2\pi-\tau), \tau)\, d\tau.$$

Plug (3.5) into (5.7), and we have

$$C(x(2\pi-\tau), \xi(2\pi-\tau), \tau)$$
$$= \exp\left[\frac{i}{|\xi|}\int_0^{\tau}\left(\sum_j b^j(x(2\pi-s))\xi_j(2\pi-s)\right) ds\right]$$
$$\times\left[i[\sigma_{\text{sub}}(A - A_m)](x(2\pi-\tau), \xi(2\pi-\tau)) + \frac{i}{2}\left\{\sigma(A-A_m) + 2\sigma_{\text{sub}}(A_m), \frac{1}{|\xi|}\int_0^{\tau} \phi^*_{\tau-s}\left(\sum_j b^j\xi_j\right) ds\right\}(x(2\pi-\tau), \xi(2\pi-\tau))\right],$$

and accordingly

$$\sigma(W) = i\int_0^{2\pi} \phi^*_{\tau}(\sigma_{\text{sub}}(A - A_m))\, d\tau - \frac{i}{2}\int_0^{2\pi} d\tau \int_0^{\tau}\left\{\phi^*_{\tau}(\sigma(A-A_m) + 2\sigma_{\text{sub}}(A_m)), \phi^*_s\left(\sum_j b^j\xi_j/|\xi|\right)\right\} ds, \tag{5.8}$$

by changing the variable $2\pi - \tau$ to τ.

(I) *The case of trivial bundle* E_0. In this case there is a global frame, and the harmonic connection form may be $\alpha_0 = \sum \hat{a}_j\, dx^j \equiv 0$, hence, plugging (5.4) and (5.6) into (5.8), we have the formula (1.2) for $J^{(0)}$ from (5.1).

(II) *Two dimensional case.* We explicitly express the symbols (5.4) and (5.6) on an open dense subset $S_0^*S^2$ of S^*S^2 discussed in §1, and plug them into (5.8). As a consequence we get the formula (1.3) for $J^{(m)}$ (restricted to $S_0^*S^2$).

6. Further outlook

1. In general the spectrum $\{\mu_j^{(k)}\}$ of L satisfies

$$\max |\mu_j^{(k)} - \lambda_k^{(m)}| = O(k) \quad (k \to \infty)$$

(Theorem 1, (1)). We are interested in the asymptotic behavior of $\mu_j^{(k)}$ is contained in the interval $[\lambda_k^{(m)}, \lambda_{k+1}^{(m)}]$ as $k\to\infty$. This is discussed in the paper [14], and it is shown that the limit distribution of $\mu_j^{(k)}$ in $[\lambda_k^{(m)}, \lambda_{k+1}^{(m)}]$ is known by the holonomy function $Q_{\tilde{d}}:\mathrm{Geod}(S^n)\to S^1$.

2. For the Schrödinger operator $P=\Delta+q$ on (S^n, g_0) it was proved by V. Guillemin [9] and H. Widom [18] that if $\mathrm{Spec}(P)=\mathrm{Spec}(\Delta)$, then $q\equiv 0$. In our case the following question is set up: *If* $\mathrm{Spec}(L)=\mathrm{Spec}(L_m)$, *is the connection* $\tilde{d}$ *harmonic*? This is an open question. It is affirmatively answered if we can show the claim that $J^{(m)}\equiv 0$ holds if and only if $\alpha-\alpha_m=\beta=\sum b_j\,dx^j$ is a closed 1-form.

3. It is natural to develop our discussions more generally for vector bundles over (M, g) all of whose geodesics are simply closed. Let E be a Hermitian vector bundle over M.

ASSUMPTION. *On E there exists a linear connection $\tilde{d}_*$ compatible with the Hermitian structure such that the spectrum of the Laplacian L_* defined by $\tilde{d}_*$ consists of*

$$\{\lambda_k^*=(k+\alpha)^2+\beta;\ k=0,1,2,\ldots\},$$

α and β being constants.

Under this assumption we can prove the following theorem similar to Theorem 1.

THEOREM. (1) $\max_j|\mu_j^{(k)}-\lambda_k^*|=O(k)$ $(k\to\infty)$ *holds.*
(2) $\max|\mu_j^{(k)}-\lambda_k^*|=O(1)$ *holds if and only if*

$$\int_\gamma(\omega-\omega_*)=0 \tag{6.1}$$

holds for every closed geodesic γ of (M, g). Here ω (resp. ω_*) is the connection matrix of $\tilde{d}$ (resp. $\tilde{d}_*$). Particularly for the case of line bundles, the condition (6.1) *is replaced by*

$$Q_{\tilde{d}}(\gamma)=Q_{\tilde{d}_*}(\gamma).$$

For the line bundles over a rank one symmetric space, we conjecture that
(i) Assumption is satisfied by the harmonic connection, and
(ii) except for (S^n, g_0), the spectrum of L consists of clusters only when $\tilde{d}$ is the harmonic connection.

The following problems arise for general vector bundles.

(1) *Does the given vector bundle E over* (M, g) *satisfy Assumption?*

(2) *Is Assumption satisfied by a harmonic* (or *Yang–Mills*) *connection* (*cf.* [2]) *on E* (*if exists*)?

Thus we are interested in spectra and holonomies of harmonic connections on vector bundles.

4. Assumption in 3 is probably not satisfied for line bundles over the Zoll manifold (not isometric with (S^n, g_0)). In such case we need different arguments in order to consider the condition that the spectrum consists of clusters. This will be developed in a subsequent article.

Appendix

We will prove Proposition 4.4. on the same lines as the case for functions (see [8, Appendix A]).

Let $C^\infty(\Lambda^p S^n)$ be the space of C^∞ real p-forms on (S^n, g_0). Then by Hodge's decomposition theorem we have

$$C^\infty(\Lambda^1 S^n) = dC^\infty(S^n) \oplus \delta C^\infty(\Lambda^2 S^n),$$

where d is the exterior differential and δ is its adjoint with respect to g_0. $SO(n+1)$ naturally acts on $C^\infty(\Lambda^1 S^n)$ and the decomposition is a direct sum of $SO(n+1)$-modules. Let $L^2_d(\Lambda^1 S^n)$ and $L^2_\delta(\Lambda^1 S^n)$ be the real Hilbert spaces generated by $dC^\infty(S^n)$ and $\delta C^\infty(\Lambda^2 S^n)$, respectively, and we have

$$L^2(\Lambda^1 S^n) = L^2_d(\Lambda^1 S^n) \oplus L^2_\delta(\Lambda^1 S^n)$$

(a direct sum as Hilbert spaces). Following Ikeda and Taniguchi [11] we will decompose $L^2(\Lambda^1 S^n)$ into irreducible $\mathrm{SO}(n+1)$-sub-modules. Consider real 1-forms on $\mathbf{R}^{n+1} = \{(x^0, \ldots, x^n)\}$. Let d_0 and δ_0 be the differential and codifferential on $C^\infty(\Lambda^1 \mathbf{R}^{n+1})$, respectively, and $\bar{\Delta} = d_0\delta_0 + \delta_0 d_0$. Let P^1_k denote the set of $\alpha \in C^\infty(\Lambda^1 \mathbf{R}^{n+1})$ of the form

$$\alpha = \sum_{j=0}^{n} \alpha_j \, dx^j,$$

with α_j to be homogeneous polynomials of (x^i) of degree k, and set

$$H^1_k = \{\alpha \in P^1_k;\ \bar{\Delta}\alpha = \delta_0 \alpha = 0\}.$$

Moreover, let

$$H_k^{1,d} = \{\alpha \in H_k^1;\ d_0\alpha = 0\},$$

$$H_k^{1,\delta} = \left\{\alpha \in H_k^1;\ \left(\sum_j x^j \frac{\partial}{\partial x^j}\right) \lrcorner \alpha = 0\right\} \quad (\lrcorner : \text{interior product}),$$

$$\mathcal{H}_k^{1,d} = i^*(H_k^{1,d}), \qquad \mathcal{H}_k^{1,\delta} = i^*(H_k^{1,\delta}),$$

where $i : S^n \to \mathbf{R}^{n+1}$ is the inclusion map. Then,

PROPOSITION A.1 (Ikeda–Taniguchi). *$SO(n+1)$ acts irreducibly on $\mathcal{H}_k^{1,d}$ and $\mathcal{H}_k^{1,\delta}$, and if $k \neq j$ the representations on $\mathcal{H}_k^{1,d}$ and $\mathcal{H}_j^{1,d}$ ($\mathcal{H}_k^{1,\delta}$ and $\mathcal{H}_j^{1,\delta}$) are inequivalent. Moreover,*

$$L^2(\Lambda^1 S^n) = \left(\sum_{k=0}^{\infty} \oplus \mathcal{H}_k^{1,d}\right) \oplus \left(\sum_{k=1}^{\infty} \oplus \mathcal{H}_k^{1,\delta}\right),$$

with

$$\sum_k \oplus \mathcal{H}_k^{1,d} \supset dC^\infty(S^n), \qquad \sum_k \oplus \mathcal{H}_k^{1,\delta} \supset \delta C^\infty(\Lambda^2 S^n).$$

Now consider the map $R : C^\infty(\Lambda^1 S^n) \to C^\infty(\mathrm{Geod}\,(S^n))$ given by $R\alpha(\gamma) = \hat{\alpha}(\gamma) = \int_\gamma \alpha$, which extends to a continuous map of $L^2(\Lambda^1 S^n)$ into $L^2(\mathrm{Geod}\,(S^n))$. Obviously, $R \mid \mathcal{H}_k^{1,d} = 0$. For $\alpha \in \mathcal{H}_k^{1,\delta}$, α is an odd (resp. even) 1-form if and only if k is even (resp. odd), so we have $R \mid \mathcal{H}_k^{1,\delta} = 0$ if k is even. $SO(n+1)$ naturally acts on $L^2(\mathrm{Geod}\,(S^n))$ and let

$$L^2(\mathrm{Geod}\,(S^n)) = \sum_\rho \oplus V_\rho$$

be the $SO(n+1)$-irreducible decomposition. Obviously R is a $SO(n+1)$-homomorphism, so (i) $\mathrm{Ker}\,(R \mid \mathcal{H}_k^{1,\delta}) = 0$ and $\overline{\mathrm{Im}\,(R \mid \mathcal{H}_k^{1,\delta})} = V_\rho$ for some ρ, or (ii) $\mathrm{Ker}\,(R \mid \mathcal{H}_k^{1,\delta}) = \mathcal{H}_k^{1,\delta}$. Therefore, in order to complete the proof of Proposition 4.4 we have only to show that for each odd k there is a 1-form α in $\mathcal{H}_k^{1,\delta}$ such that $R\alpha \neq 0$. For $k = 2l+1$ ($l = 0, 1, 2, \ldots$) set

$$\alpha = [a_{2l+1}(x^0)^{2l+1} + a_{2l}(x^0)^{2l}(x^1) + \cdots + a_1(x^0)(x^1)^{2l} + a_0(x^1)^{2l+1}]\, dx^2.$$

Then $\delta_0\alpha = 0$ always holds, and the equation $\bar{\Delta}\alpha = 0$ leads to

$$a_m(m-1) + a_{m-2}(2l-m+3)(2l-m+2) = 0, \qquad m = 2, \ldots, 2l+1.$$

Take such $a_0 \neq 0$, $a_1, \ldots, a_{2l+1}$ that satisfy this relation. Then $i^*\alpha$ belongs to $\mathcal{H}_k^1$, and $i^*\alpha = \alpha_d + \alpha_\delta$ according to the decomposition $\mathcal{H}_k^1 = \mathcal{H}_k^{1,d} \oplus \mathcal{H}_k^{1,\delta}$. Let $\gamma(t)$ $(0 \leqq t \leqq 2\pi)$ be a closed geodesic given by

$$\gamma(t) = (0, \cos t, \sin t, 0, \ldots, 0) \in S^n \subset \mathbf{R}^{n+1}.$$

Then by straightforward calculations we have

$$\begin{aligned} R\alpha_\delta(\gamma) = R(i^*\alpha)(\gamma) &= a_0 \int_0^{2\pi} (\cos t)^{2l+2}\, dt \\ &= 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2l+1)}{2 \cdot 4 \cdot 6 \cdots (2l+2)} a_0 \neq 0. \end{aligned}$$

This completes the proof of Proposition 4.4.

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