# Tight immersions of highly connected manifolds.

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# Tight immersions of highly connected manifolds[1]

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## 1. Introduction

One expects that most compact manifolds do not admit any tight immersions into a Euclidean space. We will support this in the case of highly connected manifolds. More precisely, we will give restrictions on the topology of $(k-1)$-connected $2k$-dimensional manifolds that admit tight immersions into Euclidean spaces. We will also determine the possible codimensions of such immersions.

Known examples of highly connected manifolds that admit tight immersions are $(S^k \times S^k) \# \cdots \# (S^k \times S^k)$, the projective planes $P_2\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ and all surfaces with the exception of the Klein bottle and the projective plane with one handle for which no tight immersions have yet been found; see section 2. It is not unlikely that these manifolds and their connected sums with copies of $P_2\mathbb{F}$ and $-P_2\mathbb{F}$ are the only examples.

Notice that the $k$-th Stiefel-Whitney class $w_k(M^{2k})$ vanishes if $M^{2k} = (S^k \times S^k) \# \cdots \# (S^k \times S^k)$. We have $w_k(M^{2k}) \neq 0$ if $M^{2k} = P_2\mathbb{F} \# N^{2k}$, where $k = \dim_{\mathbb{R}} \mathbb{F}$ and $N^{2k}$ is $(k-1)$-connected. In general one can show that $w_k(M^{2k}) = 0$ for a $(k-1)$-connected $2k$-dimensional manifold with $k \neq 1, 2, 4$ or $8$; i.e. if $2k$ is not the dimension of a projective plane; see section 4. A geometric interpretation of the condition "$w_k(M^{2k}) = 0$" for highly connected manifolds is that no homology class has self-intersection number 1 mod 2.

The following theorem together with results of Wall [Wa] show that there are many examples of highly connected manifolds which do not admit a tight immersion.

THEOREM A. *Let $M^{2k}$ be a $(k-1)$-connected compact manifold with $w_k(M^{2k}) = 0$ which admits a tight immersion into a Euclidean space and assume that $k > 2$. Then $M^{2k}$ has the same cohomology ring as $(S^k \times S^k) \# \cdots \# (S^k \times S^k)$ over the integers.*

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[1] Work partially done at IMPA in Rio de Janeiro and supported by the exchange program of GMD (Federal Republic of Germany) and CNPq (Brazil). It was completed at the University of Bonn.

We can prove much more if the dimension of the manifold is $4l$.

THEOREM B. *Let $M^{4l}$ be a $(2l-1)$-connected compact manifold with $w_{2l}(M^{4l})=0$ which admits a tight immersion into a Euclidean space and assume that $l>1$. Then $M^{4l}$ is diffeomorphic to the connected sum*

$$(S^{2l}\times S^{2l}) \# \cdots \# (S^{2l}\times S^{2l}) \# \Sigma$$

*where $\Sigma$ is a sphere with some differentiable structure. (The same conclusion is true for $k=3$ and 7 where $k$ is as in Theorem A.)*

We believe that $\Sigma$ in Theorem B can be proved to have the standard differentiable structure and hence be deleted from the connected sum. This would complete the classification of $4l$-dimensional ($l>1$) highly connected manifolds with $w_{2l}(M^{4l})=0$ which admit tight immersions. The proof of Theorem B is based on the methods of the paper [KW] by Kulkarni and Wood.

One expects that a $P_2\mathbb{F}$ can be decomposed off a highly connected manifold $M^{2k}$ with $w_k(M^{2k})\neq 0$ which admits a tight immersion into a Euclidean space. It can be proved that at least the cohomology ring is no obstruction; see Theorem D below.

We had to exclude four-dimensional manifolds in the above theorems. The following theorem is sufficient to prove that there are infinitely many four-dimensional manifolds which do not admit tight immersions.

THEOREM C. *Let $f: M^4 \to E^{4+l}$ be a substantial tight immersion of a simply connected compact manifold. Then, after a suitable choice of orientation:*

(i) *$l=2$ implies that $M^4$ can be decomposed diffeomorphically as $M^4=(S^2\times S^2)\# N^4$ and the Betti number $\beta_2(M^4;\mathbb{Z})$ is even. In particular, if the Stiefel-Whitney class $w_2(M)\neq 0$, then $\beta_2(M;\mathbb{Z})\geq 4$.*

(ii) *$l\geq 3$ implies that $M^4$ can be decomposed diffeomorphically as $M^4=P_2\mathbb{C}\# N^4$.*

The proof of Theorem C is much more difficult than that of Theorem B and relies in a more essential way on tightness.

As an application of Theorem C we will prove.

COROLLARY. *There are infinitely many simply connected compact four-dimensional manifolds, among them the Kummer surface, which do not admit any tight immersions into a Euclidean space.*

The Kummer surface is defined as

$$K = \{[z_0, \ldots, z_3] \in P_3\mathbb{C} \mid z_0^4 + \cdots + z_3^4 = 0\}.$$

It is the simplest algebraic surface which cannot be obtained as a connected sum of copies of $S^2 \times S^2$, $P_2\mathbb{C}$ and $-P_2\mathbb{C}$. It is a conjecture in differential topology that all simply connected compact four-manifolds can be written as a connected sum of copies of $S^4$, $S^2 \times S^2$, $P_2\mathbb{C}$, $-P_2\mathbb{C}$ and $K$.

The non-existence of a tight immersion of the Kummer surface follows immediately from Theorem C and a recent result of Donaldson [Do 2] which says that the Kummer surface cannot be decomposed diffeomorphically.

The other four-dimensional manifolds not admitting tight immersions are not explicitly given in the proof of the Corollary. They arise from algebraic surfaces of even degree in $P_3\mathbb{C}$ by splitting off copies of $S^2 \times S^2$ if the algebraic surface itself allows a tight immersion. It is only for the Kummer surface that we use Donaldson's results. The rest of the proof of the Corollary only uses the theory developed in this paper and some facts about characteristic numbers of algebraic surfaces.

We believe that the decomposition result in Theorem C can be generalized to higher dimensions. The next theorem, which refines a result of Kuiper, shows that the cohomology ring does not give an obstruction to such a generalization. We will also determine all possible codimensions of tight immersions of $2k$-dimensional $(k-1)$-connected manifolds, although one case remains open if one distinguishes the cases of vanishing and non-vanishing $k$-th Stiefel-Whitney class. Theorem D will be used in the proofs of Theorems A, B and C.

THEOREM D. *Let $f: M^{2k} \to E^{2k+l}$ be a substantial tight immersion of a compact $(k-1)$-connected manifold.*

(i) *Assume that the $k$-th Stiefel-Whitney class $w_k(M^{2k})$ vanishes. Then the codimension $l$ is* 1 *or* 2.

(ii) *Assume that the $k$-th Stiefel-Whitney class $w_k(M^{2k})$ does not vanish. Then $k = 1, 2, 4$ or 8.*

*If $k = 1$ or 2, then the codimension $l$ is $k$, $k+1$ or $k+2$.*

*If $k = 4$ or 8, then the codimension $l$ is $k+1$ or $k+2$.*

*Furthermore, the intersection form decomposes as $(\pm 1) \oplus \beta$ over the integers if $k > 1$. (We exclude the case $k = 1$ since the intersection form of a non-orientable surface is not defined over the integers.)*

*Remarks.* (a) This theorem refines a result of Kuiper [Ku 2], p. 231. In particular, (i) is equivalent to a part of his result. In (ii) the estimate $l \leq k+2$ is

due to Kuiper. Notice that it follows from the assumption $w_k(M^{2k}) \neq 0$ without using tightness that $k = 1, 2, 4$ or $8$ and that the codimension $l$ cannot be smaller than $k$. In part (ii) arguments involving tightness are therefore only needed to exclude the case $l = k$ and to decompose the intersection form.

(b) Using Theorems A and C, we can add the following conclusions to part (i): The Betti number $\beta_k(M^{2k}; \mathbb{Z})$ is even and, if $k > 2$, the intersection form is equivalent to

$$\begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \ (+ \text{ if } k \text{ is even}, \ - \text{ if } k \text{ is odd})$$

over the integers. For $k = 2$ we use that the intersection form is indefinite. By the classification of indefinite inner product spaces of type II in [MH] (here type II is equivalent to $w_2(M^4) = 0$) we therefore see that the intersection form for $k = 2$ is equivalent to

$$mE_8 \oplus n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

over the integers. We believe that $m$ can be proved to be 0. We have $m = -2$ and $n = 3$ for the Kummer surface which does not admit a tight immersion as we saw in the Corollary to Theorem C.

If one compares Theorem D with the examples in section 2, then one sees that all allowed codimensions actually occur in the examples with the exception of the following case which we formulate as a problem.

PROBLEM. *Does there exist a tight immersion $f: M^4 \to E^6$ of a simply connected compact manifold with non-vanishing second Stiefel-Whitney class? (There are such examples with vanishing Stiefel-Whitney class; e.g. the product embedding $S^2 \times S^2 \subset E^3 \times E^3 = E^6$).*

Theorem C gives as an obstruction that $\beta_2(M^4; \mathbb{Z}) \geq 4$. M. Hirsch gave in [Hi], p. 271, a necessary and sufficient condition for a compact four-dimensional manifold to admit a smooth immersion into $E^6$. This condition can be expressed as follows: $w_2(M^4)$ is the reduction modulo 2 of a class $\alpha \in H^2(M^4; \mathbb{Z})$ such that $\langle \alpha^2, [M] \rangle = -3\sigma(M)$, where $\sigma(M)$ is the signature of $M^4$. An immediate corollary is that $P_2\mathbb{C}$ cannot be immersed into $E^6$. It also follows at once that $P_2\mathbb{C} \# (-P_2\mathbb{C})$ can be immersed into $E^6$, but not tightly by Theorem C. A candidate for a simply connected compact four-dimensional manifold with

non-vanishing second Stiefel-Whitney class which could be tightly immersed into $E^6$ is $P_2\mathbb{C} \# P_2\mathbb{C} \# (-P_2\mathbb{C}) \# (-P_2\mathbb{C})$.

The paper is organized so that examples are given in section 2, definitions and technical results in section 3 and the proofs of the Theorems in section 4.

The paper is fairly self-contained. The book [CR 3] and the survey articles [Ku 2] and [Ku 3] are good introductions to the subject.

## 2. Examples

In this section we discuss systematically the $2k$-dimensional $(k-1)$-connected manifolds which are known to admit tight immersions and which codimensions of such immersions are known to occur.

(i) Kuiper has shown that all compact surfaces can be tightly immersed except maybe the Klein bottle and the projective plane with one handle for which no such immersions are yet known; see the papers [Ku 1, 2, 3]. The oriented surfaces (i.e. $w_1(M^2)=0$) can be tightly immersed with substantial codimensions 1 and 2 and there are tight immersions of non-orientable surfaces (i.e. $w_1(M^2) \neq 0$) with substantial codimensions 1, 2 and 3; see Theorem D.

(ii) Here we discuss examples of tight immersions of highly connected manifolds with $w_k(M^{2k})=0$. The only such manifolds known to admit tight immersions are the connected sums $(S^k \times S^k) \# \cdots \# (S^k \times S^k)$ (see Theorems B and A) and the occurring substantial codimensions are 1 and 2 (see Theorem D).

A tube of constant radius in $E^{2k+1}$ around $S^k \subset E^{k+1} \subset E^{2k+1}$ is a tight hypersurface homeomorphic to $S^k \times S^k$. Hebda [He] has shown that one can also realize the connected sum of arbitrarily many copies of $S^k \times S^k$ as a tight hypersurface by taking a tube in $E^{2k+1}$ around a ball in $E^{k+1}$ which has sufficiently many spherical holes. This example is not $C^\infty$ but it can be smoothed.

The product embedding of two convex hypersurfaces in $E^{k+1}$ is tight, homeomorphic to $S^k \times S^k$ and has substantial codimension two. To obtain a tight embedding of the connected sum of arbitrarily many copies of $S^k \times S^k$ with codimension two we imitate the above construction of Hebda and the examples of tight surfaces in $E^4$ in [Ku 2], p. 213. We choose two convex hypersurfaces in $E^{k+1}$ such that one of them has an open planar set $A$ and the other has a set $B$ which is congruent to $I \times S_\varepsilon^{k-1}$, where $I$ is an open interval and $S_\varepsilon^{k-1}$ is a $(k-1)$-sphere of radius $\varepsilon$. The product $A \times B$ lies in a $(2k+1)$-dimensional subspace and has a subset congruent to $I^{k+1} \times S_\varepsilon^{k-1}$. Now we can "make holes" in $A \times B \simeq I^{k+1} \times S^{k-1}$ as in Hebda's example above to get a tight embedding of $(S^k \times S^k) \# \cdots \# (S^k \times S^k)$ with substantial codimension two.

(iii) Examples of tight immersions of highly connected manifolds with $w_k(M^{2k}) \neq 0$ are the standard embeddings of the projective planes $P_2\mathbb{R}$, $P_2\mathbb{C}$, $P_2\mathbb{H}$ and $P_2\mathbb{O}$; see [Ta] and [Ku 3]. Their substantial codimension is $k+2$. They lie in spheres and can therefore be stereographically projected into a $E^{3k+1}$ where they are also tight; see Theorem D.

There are therefore two main questions: Do there exist tight immersions of connected sums of copies of $P_2\mathbb{F}$ and $-P_2\mathbb{F}$? and the question already mentioned in the introduction about tight immersions of $M^4$ into $E^6$.

## 3. Convex cycles

This section consists of basic definitions and some technical results which will be used in the proofs of the Theorems in the introduction. Particularly important is Lemma (3.6) on the existence of convex cycles.

3.1. DEFINITION. An immersion of a compact manifold into a Euclidean space is said to be *tight* if there is a field such that all singular cycles with coefficients in that field which bound in the manifold also bound in the intersection of the manifold with almost every halfspace containing the cycle.

This definition is the latest in a series of equivalent definitions of tightness which we will review for the sake of motivation. Finally we will give a new definition in terms of Čech cohomology which will be useful for technical reasons.

Chern and Lashof [CL] proved that the total absolute curvature of an immersion of a compact manifold into a Euclidean space is greater or equal to the sum of the Betti numbers for any field. Immersions satisfying equality for some field are now called tight, but they had other names in the beginning. Inspection of the proof of the Chern–Lashof theorem immediately leads to the following equivalent definition: An immersion is tight iff every height function which is a Morse function has the minimal number of critical points required by the Morse inequalities for some field. (Notice that almost every height function is a Morse function.) The proof of the Morse inequalities shows that this is equivalent to the following: An immersion $f: M \to E$ of a compact manifold into a Euclidean space is tight iff the induced homomorphism of singular homology groups with respect to some field $\mathbb{F}$

$$H_*(f^{-1}(S); \mathbb{F}) \to H_*(M; \mathbb{F})$$

is injective for every $*$ and almost every halfspace $S$; or equivalently iff there is a

field $\mathbb{F}$ such that

$$H^*(M;\mathbb{F}) \to H^*(f^{-1}(S);\mathbb{F})$$

is surjective for every $*$ and almost every halfspace $S$. It is possible to go to limits and replace "almost every halfspace" by "every halfspace" if the singular theory is replaced by Čech theory. We will not use Čech homology in this paper since it is not a standard theory. Čech cohomology on the other hand is well-known and can be found in the standard textbooks on topology. It is also used in the Duality Theorem which we will apply in the proof of (3.6). The symbol $\check{H}^*$ will refer to Čech cohomology and $H_*$ and $H^*$ to the singular theory.

It is important to define tightness for topological spaces for use with Kuiper's top sets which are the central technical tool of the theory. Their definition is given below.

3.2. DEFINITION. A continuous map $f:X \to E$ of a compact connected topological space into a Euclidean space is called *tight* if there is a field $\mathbb{F}$ such that the induced homomorphisms in Čech cohomology

$$\check{H}^*(M;\mathbb{F}) \to \check{H}^*(f^{-1}(S);\mathbb{F})$$

are surjective for every $*$ and every halfspace $S$. We will also say that $f$ is *tight with respect to* $\mathbb{F}$ or $\mathbb{F}$*-tight*. We will not refer explicitly to the field in the notation when the meaning is obvious from the context.

The convex hull of a subset $X$ in $E$ will be denoted by $\mathcal{H}(X)$. The boundary of the convex hull $\partial\mathcal{H}(X)$ will be called the *convex envelope of* $X$.

A *top set* is the preimage of the maximal value of a height function, i.e. a function of the type $\xi \cdot f : X \to \mathbb{R}$ where $\xi$ is a unit vector in $E$. A *top*$^2$ *set* is a top set of a top set. A *top*set* is inductively defined as the top set of a top$^{*-1}$ set. A *top*map* is $f$ restricted to a top*set. It is proved in [Ku 3], p. 102, that top*maps of tight maps are tight.

It is not true in general that top maps are injective. A nice counterexample for closed surfaces in $E^3$ due to Banchoff can be found in [CR 2]. A somewhat less natural such example can be obtained as follows: Take the product immersion $f \times c : M^2 \times S^1 \to E^5$ where $f$ is a tight surface in $E^3$ with self-intersections and $c$ a convex curve in $E^2$. Then for a fixed $t_0 \in S^1$ the top map $f \times c(t_0)$ is not injective. The following lemma about injectivity of top maps will be important in the proof of (3.6).

3.3. LEMMA. *A top*map of an* $\mathbb{F}$*-tight immersion is injective and its image is convex if the corresponding top*set is a Čech cohomology point with respect to* $\mathbb{F}$.

*Remark.* Kuiper proved in [Ku 3], p. 120, that the image under a tight map of a set which is a Čech homology point is convex if tightness is defined in terms of Čech homology. Lastufka proved in [La], p. 382, that a top*map is injective if its image is convex. We will use this result of Lastufka in the proof.

*Proof.* We prove the lemma by induction. Assuming $(I_l)$ we will prove $(I_{l+1})$ where the induction hypothesis $(I_l)$ is defined as

$(I_l)$ top*maps are injective and their image is convex if the corresponding top*sets are Čech cohomology points and their images lie in affine subspaces of dimension $\leq l$.

$(I_0)$ is of course trivial and $(I_1)$ follows easily. We therefore assume that $(I_l)$ is proved up to $l$ for $l \geq 1$.

Let $X$ be a top*set which is a cohomology point and such that $\dim \langle f(X) \rangle = l+1$. We know by $(I_l)$ that $f$ is injective on any top*set of $X$ and hence that $f$ is injective on the preimage of the convex envelope $\partial\mathscr{H}(f(X))$ of $f(X)$. It also follows from $(I_l)$ that $\partial\mathscr{H}(f(X)) \subset f(X)$. Denote the inverse image of the convex envelope $\partial\mathscr{H}(f(X))$ by $Y$.

We first prove that $f(X)$ is convex. Assume that $f(X)$ is not convex. Then there is a point $x$ in the interior of $\mathscr{H}(f(X))$ which does not lie in $f(X)$. The point $x$ can be used to define a retraction $r$ of $X$ on $Y$ as follows: A point $p \in X$ is mapped by $r$ onto the preimage of the point of $\partial\mathscr{H}(f(X))$ which lies on the ray from $x$ through $f(p)$. The map $r$ is well defined since $f$ is injective on $Y$ and it is obviously a retraction. The set $Y$ is homeomorphic to $S^l$ and hence it cannot be a retract of a space which is a cohomology point. Thus $f(X)$ must be convex.

Now it follows from the theorem of Lastufka that we quoted in the remark before the proof that $f$ is injective. ■

The next two lemmas are preliminary for the more important lemma (3.6).

3.4 LEMMA. *Assume that $f: M \to E$ is a tight immersion with respect to the field $\mathbb{F}$. If the height function $\xi \cdot f$ has a non-degenerate critical point $p$ of index $k$, then $\check{H}^k(f^{-1}(S); \mathbb{F}) \neq 0$, where $S$ is the halfspace $\{x \in E \mid \xi \cdot (x - f(p)) \leq 0\}$.*

*Proof.* We first remark that the definition of tightness immediately implies that the cohomology sequence of the pair $(f^{-1}(S_1), f^{-1}(S_2))$, where $S_1 \supset S_2$ are halfspaces, splits into short exact sequences

$$0 \to \check{H}^*(f^{-1}(S_1), f^{-1}(S_2)) \to \check{H}^*(f^{-1}(S_1)) \to \check{H}^*(f^{-1}(S_2)) \to 0.$$

This proves the lemma since it follows from Morse theory that there is an $\varepsilon > 0$ such that $\check{H}^k(f^{-1}(S), f^{-1}(S_{-\varepsilon})) \neq 0$, where $S_{-\varepsilon} = \{x \in E \mid \xi \cdot (x - f(p)) \leq -\varepsilon\}$. ■

Let $f: M \to E$ be an immersion. We denote the second fundamental form of $f$ by $\alpha$ and its normal bundle by $NM$. It is easy to show that $p \in M$ is a critical point of the height function $\xi \cdot f$ iff $\xi \in N_pM$. The Hessian of the critical point is $\xi \cdot \alpha$. A *convex point* of $f$ is a point whose image lies in the convex envelope $\partial\mathcal{H}(f(M))$ of $f(M)$. A *non-degenerate convex point* is a convex point which is a non-degenerate minimum of some height function. If $p \in M$ is a non-degenerate convex point, then $\alpha(X, X) \neq 0$ for every non-zero $X \in T_pM$ and the set $\{\alpha(X, X) \mid X \in T_pM\}$ is contained in a halfspace of $N_pM$. We denote the convex hull of $\{\alpha(X, X) \mid X \in T_pM\}$ by $K_p$. It is easy to see that $K_p$ spans $N_pM$ at a non-degenerate convex point if $f$ is tight.

3.5 LEMMA. *Let $p$ be a non-degenerate convex point of the tight immersion $f: M \to E$. Let $h \subset N_pM$ be a hyperplane of support of $K_p$ and let $\xi \in N_pM$ be orthogonal to $h$ with $\xi \cdot \alpha(X, X) \geq 0$ for every $X \in T_pM$.*

(i) *Then $\mathcal{E}_h = \{X \in T_pM \mid \alpha(X, X) \in h\}$ is a linear subspace and $\alpha(\mathcal{E}_h, T_pM) \subset h$.*

(ii) *Let $\xi(t)$ be a curve in $N_pM$ such that $\xi(0) = \xi$ and $\xi(t) \cdot \alpha(X, X) < 0$ for every non-zero $X \in \mathcal{E}_h$ and $t \neq 0$. Then there is an $\varepsilon > 0$ such that the height function $\xi(t) \cdot f$ has a non-degenerate critical point in $p$ of index equal to the dimension of $\mathcal{E}_h$ for every $0 < t \leq \varepsilon$.*

*Proof.* (i) The quadratic form $\xi \cdot \alpha(X, X)$ is positive semi-definite. Hence its nullspace is $\mathcal{E}_h$ which is therefore a linear subspace. We have $\alpha(\mathcal{E}_h, T_pM)$ since $\mathcal{E}_h$ is the nullspace of $\xi \cdot \alpha(X, Y)$.

(ii) Suppose there is a sequence $t_i \to 0$ such that the height functions $\xi_i \cdot f$ are degenerate where $\xi_i = \xi(t_i)$. The index of $\xi_i \cdot f$ is greater than or equal to $\dim \mathcal{E}_h$ for every $i$ since its Hessian is negative definite on $\mathcal{E}_h$ by assumption. Therefore there is for every $i$ a subspace $\mathcal{E}_i$ of dimension $d > \dim \mathcal{E}_h$ such that $\xi_i \cdot \alpha(X, X) \leq 0$ for every $X \in \mathcal{E}_i$. Let $(X_{ij}, 0 < j \leq d)$ be an orthonormal basis of $\mathcal{E}_i$. There is a subsequence $(i')$ of $(i)$ such that $(X_{i'j})$ converges to a unit vector $X_j$ for every $j$, $0 < j \leq d$. $(X_j)$ is an orthonormal basis of a space $\mathcal{E}$ of dimension $d$ such that $\xi \cdot \alpha(X, X) \leq 0$ for every $X \in \mathcal{E}$. This implies that $\alpha(X, X) \in h$ for every $X \in \mathcal{E}$ since $\xi \cdot \alpha(X, X) \geq 0$ for every $X \in T_pM$ and $\xi \cdot h = 0$. Hence $\mathcal{E} \subset \mathcal{E}_h$ which is a contradiction since $\dim \mathcal{E} = d > \dim \mathcal{E}_h$. ■

The following lemma is of central importance for the proofs of our theorems. It is intended to replace arguments involving integrable distributions in more special situations, see [Ch], [CR1], and [Th1]. Notice that neither do we assume that the dimension of the manifold $M$ is $2k$ nor do we make assumptions on the codimension. The notation used is introduced before and in lemma (3.5).

3.6 LEMMA (Existence of convex cycles). *Let $f: M \to E$ be a tight immersion of a $(k-1)$-connected compact manifold. Let $p \in M$ be a non-degenerate convex point. Assume that $H$ is a hyperplane in $E$ containing $T_pM$ such that $h = H \cap N_pM$ is a hyperplane of support of $K_p$ that only meets $K_p$ in a ray. Assume that* $\dim \mathscr{E}_h = k$. *Denote the convex envelope of $f(M) \cap H$ by $Q$. Then*

(i) *$H$ supports $f(M)$*

(ii) *$Q \subset f(M)$ and $f$ is injective on $f^{-1}(Q)$*

(iii) *$Q$ spans a $(k+1)$-dimensional affine subspace of $E$*

(iv) *the fundamental cycle of $f^{-1}(Q)$ is non-trivial in $M$, i.e. the homomorphism $H_k(f^{-1}(Q)) \to H_k(M)$ in singular homology is injective with respect to the same coefficient field as the tightness of $f$*

(v) *there is a neighborhood $U$ of $p$ such that $f(U) \cap Q = f(U) \cap H$ and $U \cap f^{-1}(Q)$ is a differentiable submanifold with tangent space $\mathscr{E}_h$ at $p$.*

*Remark.* Notice that the $(k+1)$-dimensional affine subspace of $E$ spanned by $Q$ does not in general contain the ray $h \cap K_p$. The cycle $f^{-1}(Q)$ will be called a (*non-degenerate*) *convex cycle.* We will sometimes write $Q$ instead of $f^{-1}(Q)$ although we mean a subset of $M$. We will also use the term convex cycle for $Q$ as a subset of $E$.

We first give an application of the lemma.

3.7 COROLLARY. *Let $f: M^{2k} \to E^{2k+l}$, $l \geq 2$, be a substantial tight immersion of a $(k-1)$-connected compact manifold. Let $p$ be a non-degenerate convex point. Then there pass at least two different (but possibly homologous) convex cycles $Q_1$ and $Q_2$ through $p$ that only have the point $p$ in common. Both $Q_1$ and $Q_2$ are differentiable around $p$ and they intersect transversally.*

*Remark.* More precisely, we have a family of different convex cycles which can be parameterized by $S^{l-2}$, but we do not know whether the family is continuous. For further details on this, see the proof of Theorem D in section 4.

*Proof.* Here we only prove the case $l = 2$. The general case will be proved in the proof of Theorem D. The image of $T_pM$ under $\alpha$, the set $\{\alpha(X, X) \mid X \in T_pM\}$, is a sector bounded by two rays on lines $h_1$ and $h_2$. The sector does not degenerate to a line, i.e. $h_1 \neq h_2$, since $f$ is substantial and $p$ a non-degenerate convex point. It is an immediate consequence of (3.4) and (3.5) that $\dim \mathscr{E}_{h_1} = \dim \mathscr{E}_{h_2} = k$. Thus we have two convex cycles $Q_1$ and $Q_2$ by (3.6) mapped into $H_1$ and $H_2$ respectively, where $H_i$ is the affine span of $h_i$ and $T_pM$. The tangent space $T_pM$ only meets $M$ in $p$ by tightness. Thus it follows that $Q_1$ and $Q_2$ only can meet in $p$. They meet there transversally since their tangent planes $\mathscr{E}_{h_1}$ and $\mathscr{E}_{h_2}$ meet there transversally, see (3.5). ■

*Proof of* (3.6). The proof will be divided into several steps to make the exposition clearer. The steps do not correspond to the different parts of the lemma.

(i) Let $\xi \in N_pM$ be a non-zero vector orthogonal to $h$ and such that $\xi \cdot \alpha(X, X) \geq 0$ for every $X \in T_pM$. This is possible since the set $\{\alpha(X, X) \mid X \in T_pM\}$ lies in a halfspace bounded by $h$. Then it follows from lemma (3.5) that there is a sequence $(\xi_i)$ of vectors in $N_p$ which converges to $\xi$ and has the properties that $p$ is a non-degenerate critical point of $\xi_i \cdot f$ of index $k$ and $\xi_i \cdot \alpha(X, X) < 0$ for every non-zero $X \in \mathscr{E}_h$. Furthermore the sequence can be chosen in such a way that $\xi$ and $(\xi_i)$ lie in a two-dimensional subspace and $\xi_{i+1}$ lies between $\xi$ and $\xi_i$ for every $i$. We denote the closed halfspace in $E$ bounded by the hyperplane orthogonal to $\xi_i$ and not containing $\xi_i$ by $S_i$. We now show that

$$f^{-1}(S_1) \supset f^{-1}(S_2) \supset \cdots \supset f^{-1}(S_i) \supset \cdots \supset f^{-1}(H).$$

It follows from the choice of the sequence $(\xi_i)$ as lying in a two-dimensional subspace and converging monotonically there to $\xi$ that if the sequence $(f^{-1}(S_i))$ would not be monotonically descending then there would be an $i_0 > 0$ and a $q \in M$ such that $f(q)$ were contained in the interior of $S_i$ for every $i > i_0$ and $f(q)$ and $\xi$ would lie on different sides of $H$. Hence the height function $\xi \cdot f$ would be negative in $q$. By turning $\xi$ slightly in the two-dimensional space spanned by $(\xi_i)$ and $\xi$ we would obtain a vector $\xi'$ such that $\xi' \cdot \alpha(X, X) > 0$ for non-zero $X \in T_pM$ and $\xi' \cdot f(q) < 0$. The point $p$ would be a non-degenerate relative minimum of $\xi' \cdot f$ with value 0. Hence there would be a height function $\xi'' \cdot f$, $\xi''$ close to $\xi$, which is a Morse function with at least two relative minima. This contradicts the tightness of $f$. The sequence $(f^{-1}(S_i))$ is therefore monotonically descending and $f^{-1}(S_i) \supset f^{-1}(H)$ for every $i$. It follows similarly that $f^{-1}(H) = \bigcap f^{-1}(S_i)$.

This argument also shows that $f(M)$ must lie on one side of $H$ which proves part (i) of the theorem. In other words, $f^{-1}(H)$ is a top set.

The cohomology group $\check{H}^k(f^{-1}(S_i))$ is non-trivial by lemma (3.4) since $\xi_i \cdot f$ has a non-degenerate critical point of index $k$ in $p$. The homomorphisms $\check{H}^k(M) \to \check{H}^k(f^{-1}(S_i)) \to \check{H}^k(f^{-1}(S_j))$, $i < j$, are surjective by tightness. The fact that $\check{H}^k(M)$ is a finite dimensional vector space implies that $\check{H}^k(f^{-1}(S_i)) \to \check{H}^k(f^{-1}(S_j))$ is an isomorphism for every $j > i > i_0$ for some $i_0$. Hence $\check{H}^k(f^{-1}(H)) = \lim_{\to} \check{H}^k(f^{-1}(S_i)) \neq 0$.

(ii) In this step we prove that there is an affine subspace $G$ in $H$ such that $\check{H}^*(f^{-1}(G)) \neq 0$ for some $* > 0$, and the convex envelope $Q'$ of $f(M) \cap G$ contains $f(p)$ and there is a neighborhood $V$ of $f(p)$ in $Q'$ contained in $f(M)$. Moreover, $f^{-1}(G)$ is a top*set of $M$.

We let $Q$ denote the convex envelope of $f(M) \cap H$ as in the statement of the theorem. Let $G_0$ denote the affine hull of $Q$. We first show that $f(p) \in Q$. Let $\eta \in N_pM$ be such that $\eta \cdot \alpha(X, X) > 0$ for every $X \in T_pM$. Then $\eta \cdot f$ has a non-degenerate relative minimum in $p$ which is an absolute minimum by tightness. Hence the hyperplane through $f(p)$ orthogonal to $\eta$ only meets $f(M)$ in $f(p)$ and $f(p)$ is therefore in the convex envelope of $f(M) \cap H$, i.e. $f(p) \in Q$. Assume there is in every neighborhood $V$ of $f(p)$ in $Q$ a point $x \in V - f(M)$. Let $(x_i)$ be a sequence of such points which converges to $f(p)$. Let $P_i \subset G_0$ be a hyperplane of support of $Q$ at $x_i$. We can assume that the sequence of hyperplanes $(P_i)$ converges to a hyperplane $P_0$ which is a hyperplane of support in $f(p)$. The plane $P_i$ meets $f(M)$ in some point for every $i$ since $P_i$ supports the convex hull of $f(M) \cap H$. The map $f \mid f^{-1}(P_i)$ is a top map of the top set $f^{-1}(P_i)$ of $f^{-1}(H)$ and hence tight. It follows that $\check{H}^*(f^{-1}(P_i)) \neq 0$ for some $* > 0$ since $P_i \cap f(M) = f(f^{-1}(P_i))$ would otherwise be convex by (3.3) and consequently $x_i \in f(M)$, which is a contradiction. The homomorphisms $\check{H}^*(f^{-1}(H)) \to \check{H}^*(f^{-1}(P_i))$ are surjective by tightness. Let $(U_i)$ be a monotonically descending sequence of closed neighborhoods of $f^{-1}(P_0)$ in $f^{-1}(H)$ such that $f^{-1}(P_0) = \bigcap U_i$, obtained as the preimage of halfspaces with boundaries parallel to $P_0$. By tightness of $f \mid f^{-1}(H)$ we have that $\check{H}^*(f^{-1}(H)) \to \check{H}^*(U_j) \to \check{H}^*(U_i)$, $i > j$, are surjective homomorphisms. It follows at once that for every $i$ there is a $k$ such that $f^{-1}(P_j) \subset U_i$ for $j > k$. Consequently $\check{H}^*(U_i) \neq 0$ for every $i$ and since they are all finite dimensional vector spaces, the surjective homomorphisms $\check{H}^*(U_j) \to \check{H}^*(U_i)$ are isomorphisms for every $i > j >$ some $i_0$. Hence $\check{H}^*(f^{-1}(P_0)) = \lim_{\to} \check{H}^*(U_i) \neq 0$ by the continuity of Čech cohomology. Notice that the dimension of the linear span of $f(M) \cap P_0$ is smaller than the dimension of the linear span of $f(M) \cap H = f(M) \cap G_0$. Let $G_1$ be the linear span of $f(M) \cap P_0$. If the convex envelope $Q_1$ of $f(M) \cap G_1$ does not have a neighborhood $V$ of $f(p)$ contained in $f(M)$, then we can repeat the argument above and find a hyperplane $P_1 \subset G_1$ supporting $Q_1$ at $f(p)$ and satisfying $\check{H}^*(f^{-1}(P_1)) \neq 0$ for some $* > 0$. Inductively we can continue this until we finally come to a plane $G$ such that the convex envelope $Q'$ of $f(M) \cap G$ has a neighborhood $V$ of $f(p)$ contained in $f(M)$ and $\check{H}^*(f^{-1}(G) \neq 0$ for some $* > 0$. We have by construction that $f^{-1}(G)$ is a top set.

(iii) In this step we prove that $\dim G \leq k + 1$. We also show that $Q = Q' \subset G$ if $\dim G = k + 1$. Finally the claim in part (v) will be proved under the assumption $Q = Q'$.

Assume that $\dim G = k' + 1 > k + 1$. We first prove that $\dim (G \cap T_pM) = k'$. We can represent $Q'$ locally around $f(p)$ as the graph of a convex function. The convex function is differentiable on a dense set $\mathscr{D}$ where its gradient is also continuous. The hyperplane of support is unique in a point $q$ above $\mathscr{D}$ and identical with its tangent plane $T_qQ' \subset T_qM$. We choose a sequence of points $(q_i)$

above $\mathscr{D}$ that converges to $p$ and a convergent subsequence of $(T_{q_i}Q')$ with limit $P$. The plane $P$ supports $Q'$ at $p$ and is contained in $T_pM$ since it is a limit of planes in $TM$. It follows that $P = G \cap T_pM$ since $G$ is not contained in $T_pM$. Let $G^\perp$ be the orthogonal complement of $G$ in $T_pM + G$. The space $G^\perp$ is $(n-k')$-dimensional since $\dim(T_pM + G) = n+1$, where $n = \dim M$. Let $(e_1, \ldots, e_{r-k'})$ be a basis of $G^\perp$. The map $F = (F_1, \ldots, F_{n-k'})\colon M \to \mathbb{R}^{n-k'}$ defined by $F_i = e_i \cdot f$ is of maximal rank in $p$ since $G^\perp \cap N_pM = 0$. The set $F^{-1}(0)$ is therefore a $k'$-dimensional manifold around $p$ which $f$ maps locally homeomorphically onto a neighborhood $V$ of $f(p)$ in $Q'$. The Hessian of the height function $\xi_i \cdot f$ is negative definite on the tangent space $f_*^{-1}P$ of $F^{-1}(0)$ at $p$ since $f^{-1}(S_i)$ contains a neighborhood of $F^{-1}(0)$ around $p$. But this is a contradiction since the index of $\xi_i \cdot f$ is $k < \dim P = k'$.

The above argument also shows that if $\dim G = k+1$, then a neighborhood $V$ of $f(p)$ in $Q'$ is a $k$-dimensional differentiable manifold and there is a neighborhood $U$ of $p$ in $M$ such that $f(U) \cap Q' = f(U) \cap G$ and $U \cap f^{-1}(Q') = f^{-1}(V)$. The tangent space of $f^{-1}(Q')$ is $\mathscr{E}_h$ since this is the only $k$-dimensional subspace of $T_pM$ on which the index form of every $\xi_i \cdot f$ is negative definite.

It also follows from $\dim G = k+1$ that $Q = Q'$. To prove this it suffices to show that $f(M) \cap H \subset G$. Assume that $f(M) \cap H$ is not contained in $G$. Then every neighborhood of $p$ in $M$ contains an element $q$ such that $f(q) \in H - G$, since otherwise there would be a hyperplane in $H$ that cuts $f(M) \cap H$ into more than two pieces which is impossible by tightness. There is a neighborhood $V$ of $p$ such that $F(q)$ is a regular value for every $q \in V$, where $F = (F_1, \ldots, F_{n-q})$ is defined as above. There is a curve $q(t)$, $0 \le t \le 1$, in $V$ such that $q(0) = p$, $q(1) \in H - G$ and such that $F^{-1}(F(q(t)))$ can for every $t$ be parameterized locally around $q(t)$ by a map $U(t)\colon D^{k'} \to M$ ($D^{k'}$ is the unit $k'$-disk) which is continuous in $t$ and such that $U(t)(\partial D^{k'}) \subset \{\xi_i \cdot f < 0\}$. The map $U(0)$ is a non-nullhomologous relative singular cycle of the pair $(\{\xi_i \cdot f \le 0\}, \{\xi_i \cdot f < 0\})$. On the other hand $U(0)$ is homologous to $U(1)$ with image in $\{\xi_i \cdot f < 0\}$ and hence nullhomologous which is a contradiction. Thus $f(M) \cap H \subset G$.

(iv) We first prove that $\dim G \ge k+1$. We know that $\check{H}^*(f^{-1}(G)) \neq 0$ for some $* > 0$ and that $\check{H}^*(M) \to \check{H}^*(f^{-1}(G))$ is surjective for every $*$. We will show that $\check{H}^*(f^{-1}(G)) = 0$ for $* \ge \dim G$ which by the assumptions on the topology of $M$ implies that $\dim G \ge k+1$. To this end we construct on open manifold $\mathscr{M}$ of the same dimension as $G$ and embed $f^{-1}(G)$ into $\mathscr{M}$. It then follows from the general Duality Theorem that $\check{H}^*(f^{-1}(G)) = 0$ if $* \ge \dim G$.

We define the manifold $\mathscr{M}$ and the embedding of $f^{-1}(G)$ into $\mathscr{M}$ as follows: For every point $q$ in $f^{-1}(G)$ we choose a neighborhood $U_q$ of $q$ on which $f$ is injective. We choose these neighborhoods so small that $f$ is injective on $U_{q_1} \cup U_{q_2}$ if $U_{q_1} \cap U_{q_2} \neq \varnothing$. We choose an open neighborhood $V_q$ of $f(q)$ such that the

connected component of $f^{-1}(V_q)$ which contains $p$ lies in $U_p$. This is possible since $f$ is locally injective. Now let $\mathcal{M}$ be the set of equivalence classes in the *disjoint* union of the family $\{V_q \mid q \in f^{-1}(G)\}$ defined such that $x \in V_{q_1}$ and $y \in V_{q_2}$ are equivalent iff $x = y$ and $U_{q_1} \cap U_{q_2} \neq \emptyset$. $\mathcal{M}$ has a canonical structure as an open manifold of the same dimension as $G$ and $f$ induces an embedding of $f^{-1}(G)$ into $\mathcal{M}$. We have thus proved that $\dim G \geq k+1$ and hence that $\dim G = k+1$ by (iii).

We use similar arguments to prove that $Q \subset f(M)$ and that $f \mid f^{-1}(Q)$ is injective. If a point $y \in Q$ is not in $f(M)$, then we choose a hyperplane $R$ in $G$ supporting $Q$ at $y$. The top*map $f \mid f^{-1}(R)$ is tight and consequently $\check{H}^*(f^{-1}(R)) \neq 0$ for some $* > 0$ since $f(M) \cap R$ would otherwise be convex by (3.3) and hence contain $y$. But on the other hand by arguments as above we see that this $*$ must be smaller than $\dim R < \dim G = k+1$ which implies that $\check{H}^*(M) \to \check{H}^*(f^{-1}(R))$ cannot be surjective which is a contradiction. Thus we have proved that $Q \subset f(M)$. Notice that we have proved that the top sets of $Q$ are cohomology points. This implies by (3.3) that $f \mid f^{-1}(Q)$ is injective.

It is only left to prove that $Q$ is not nullhomologous in $M$ or equivalently that $\check{H}^k(M) \to \check{H}^k(Q)$ is surjective. By tightness, this is the same as to prove that $\check{H}^k(f^{-1}(G)) \to \check{H}^k(Q)$ is surjective. This follows since $Q$ is a retract of $f^{-1}(G)$. The retraction can be defined exactly as the retraction in the proof of (3.3) using that we have already proved that $f$ is injective on $f^{-1}(Q)$ and that there is a point close to $p$ in the convex body bounded by $Q$ which does not lie in the image of $f$ by the already proved part (v) of the lemma. ■

## 4. Proofs of the theorems

We begin with the proof of Theorem D.

*Proof of Theorem D.* We will use in the proof that $w_k(M^{2k}) \neq 0$ iff the self-intersection number mod 2 of some $k$-dimensional $\mathbb{Z}_2$-homology class is one. This one sees as follows:

The Poincaré dual cohomology class $\alpha$ in $H^k(M; \mathbb{Z}_2)$ of a homology class in $H_k(M; \mathbb{Z}_2)$ with self-intersection number one has the property that $\alpha \cup \alpha$ is the fundamental cohomology class of $M$. We review that the Wu class $v_k$ is defined implicitly by the condition $\langle \beta \cup \beta, \bar{\mu} \rangle = \langle \beta \cup v_k, \bar{\mu} \rangle$ for every $\beta \in H^k(M, \mathbb{Z}_2)$ where $\bar{\mu}$ is the fundamental class of $M$, see [MS], p. 132. The Wu class $v_k$ of $M$ is clearly non-zero and hence by Wu's formula for the $k$-th Stiefel-Whitney class,

see [MS], p. 132,

$$w_k(M) = Sq^0(v_k) + Sq^k(1) = v_k$$

we have that $w_k(M) \neq 0$. Reversing the arguments proves the converse.

(i) We prove (i) by showing that there is a convex cycle with self-intersection number one if the codimension is greater than two. Let $p \in M$ be a non-degenerate convex point and let $K_p$ be the cone in $N_p$ spanned by $\{\alpha(X, X) \mid X \in T_pM\}$ as in (3.6). Let $h_0$ be a hyperplane in $N_p$ that supports $K_p$. We want to show that $h_0$ meets $K_p$ in a ray. Let $\exp(K_p)$ be the set of such hyperplanes. Then $K_p$ is the closure of the convex hull of the rays $h \cap K_p$ for $h \in \exp(K_p)$; in symbols $K_p = \overline{\mathscr{H}(\bigcup_{h \in \exp(K_p)} h \cap K_p)}$, see [Le], p. 44. Thus there is for every hyperplane $h_0$ a sequence $(h_i \cap K_p)$ of rays, $h_i \in \exp(K_p)$, that converges to a ray $R$ in $h_0 \cap K_p$. It follows from (3.4) and (3.5) that $\mathscr{E}_h$ is $k$-dimensional for every $h \in \exp(K_p)$. Thus it follows that $\dim\{X \mid \alpha(X, X) \in R\} \geq k$. If $R \neq h_0 \cap K$, then we would have that $\dim \mathscr{E}_{h_0} > k$ which is impossible by (3.4) and (3.5). It follows that every hyperplane of support $h$ of $K_p$ satisfies the hypotheses of (3.6) and these hyperplanes correspond continuously and one-to-one to the elements of a sphere $S^{l-2}$. Thus for every such $h$ we have a convex cycle $Q_h$ and any two only meet in $p$ and there they meet transversally which shows that they are representatives of non-trivial homology classes in $H_k(M, \mathbb{Z}_2)$. (We have not claimed that $Q_h$ depends continuously on $h$, but this seems likely.) There are only finitely many such classes, but an infinity of convex cycles, which shows that at least two of them must be homologous. This proves part (i).

(ii) If $w_k(M^{2k}) \neq 0$, then the $k$-th Stiefel-Whitney class of the normal bundle cannot vanish and the codimension $l$ of $f$ is at least $k$. On the other hand $l \leq k + 2$ by a theorem of Kuiper [Ku 2], p. 231. We will prove further below that $l \neq k$ if $k > 2$.

Next we prove that $k = 1, 2, 4$ or $8$ if $w_k(M^{2k}) \neq 0$. (Instead of using Lemma (3.6) about the existence of convex cycles we could argue without using tightness as in the proof of Theorem 3 in [Mi].) We will assume that $k \geq 3$ and prove that it is either 4 or 8. If $k \geq 3$, then we have proved that the codimension $l \geq k \geq 3$ and we can use the arguments in (i) to find a convex cycle $Q$ with self-intersection number one. If $\alpha \in H^k(M^{2k}, \mathbb{Z}_2)$ is the Poincaré dual of $Q$, then we have that $\alpha \cup \alpha$ is the fundamental cohomology class of $M^{2k}$. Hence we see that $\langle v_k, [Q]\rangle = \langle v_k, \alpha \cap \bar{\mu}\rangle = \langle v_k \cup \alpha, \bar{\mu}\rangle = \langle \alpha \cup \alpha, \bar{\mu}\rangle = 1$ and since $w_k(M) = v_k$ it follows that $w_k(M) \mid Q$ does not vanish. Thus we have proved that $TM \mid Q$ has non-trivial $k$-th Stiefel-Whitney class. Theorem 1 in [Mi] says that if an $O_m$-bundle over a $k$-sphere has non-vanishing $k$-th Stiefel-Whitney class, then $k = 1, 2, 4$ or $8$.

Now assume that $k > 2$ and $k = l$. Then we have just proved that there is a convex cycle $Q$ such that $TM \mid Q$ has a non-trivial $k$-th Stiefel-Whitney class. Any neighborhood $U$ of $Q$ therefore satisfies $w_k(U) \neq 0$. Let $H$ be a supporting hyperplane of $f(M)$ that contains $Q$. Then $H$ contains all tangent planes $T_pM$ at points $p \in Q$. Thus there is a neighborhood $U$ of $Q$ such that $\pi \circ f : U \to H$ is an immersion, where $\pi$ is the orthogonal projection. This is a contradiction since $w_k(U) \neq 0$ and the codimension of $\pi \circ f$ is $k - 1$.

Finally we prove that the intersection form decomposes as $(\pm 1) \oplus \beta$ if $k > 1$. If $k = 2$, then this follows from Donaldson's theorem [Do 1] and the classification of indefinite inner product spaces [MH]. If $k > 2$, then $l > 2$ and there is a convex cycle with self-intersection number $\pm 1$. Hence the intersection form decomposes. ■

*Proof of Theorem A.* The cohomology ring of $M^{2k}$ is of course completely determined if we know its intersection form. If $k$ is odd, then the intersection form is skew (symplectic). By the Corollary on p. 7 in [MH] there is a basis for $H_k(M^{2k}; \mathbb{Z})$ such that the intersection form takes the form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Thus $M^{2k}$ has the same cohomology ring as $(S^k \times S^k) \# \cdots \# (S^k \times S^k)$ for $k$ odd.

If $k$ is even, then the intersection form is symmetric. We first prove that the Pontrjagin classes of $M^{2k}$ vanish. Let $f : M^{2k} \to E^{2k+l}$ be a substantial tight immersion of $M^{2k}$. Then it follows from Theorem D that $l = 1$ or $2$. If $l = 1$, then the Pontrjagin classes of the normal bundle of course vanish. If $l = 2$, then we can consider the normal bundle to be a complex line bundle. The Chern classes of this line bundle vanish since $k > 2$, i.e. $H^2(M^{2k}; \mathbb{Z}) = 0$. Thus the Pontrjagin classes of the normal bundle vanish also in this case since they are a product in the Chern classes; see [MS], p. 177. Now it follows that the Pontrjagin classes of $M^{2k}$ all vanish since the cohomology of $M^{2k}$ does not have torsion; see [MS], p. 175. This implies by the Signature Theorem that the signature of $M^{2k}$ is 0. The intersection form of $M^{2k}$ is of type II since $w_k(M^{2k}) = 0$ and it is indefinite since its signature is 0. By the classification of indefinite symmetric bilinear forms we thus have that the intersection form of $M^{2k}$ is equivalent to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is the intersection form of $(S^k \times S^k) \# \cdots \# (S^k \times S^k)$ for $k$ even. ■

*Proof of Theorem B.* We use the methods of Kulkarni and Wood in [KW] to prove the theorem. We only sketch the proof and refer to section 12 of [KW] for more details and references to the literature.

We set $k = 2l$.

Since $k > 2$ we can by results of Whitney and Haefliger and Theorem A find embeddings of spheres into $M^{2k}$ which represent a basis of $H_k(M^{2k}; \mathbb{Z})$ and have the property that each sphere only meets one another sphere and this one it only meets once and transversally. Assume that the normal bundles of all these spheres are trivial. Let $S_1$ and $S_2$ be two such spheres with a common point. Then $S_1 \cup S_2$ has a neighborhood which is diffeomorphic to a neighborhood of the one point union $S^k \vee S^k$ in $S^k \times S^k$ whose complement is a $2k$-dimensional ball. Hence one can decompose $M^{2k}$ as claimed in the theorem.

Thus it is left to prove that the normal bundles are trivial. Let $f: M^{2k} \to E^{2k+l}$ be a substantial tight immersion. By Theorem D, $l \leq 2$. The normal bundle of $f$ restricted to the above embeddings of spheres are all trivial since $k > 2$. Thus one sees that the normal bundles in $M^{2k}$ of the spheres are stably trivial. The Euler characteristic of the normal bundles is 0 since the self-intersection numbers of the spheres are 0. For $k$ even (and $k = 1, 3, 7$) stably trivial $k$-plane bundles over $S^k$ are trivial if their Euler characteristic is 0. This finishes the proof of the theorem. ■

*Proof of Theorem C.* (i) We assume that the codimension $l = 2$. Let $p$ be a non-degenerate convex point and let $Q_1$ and $Q_2$ be the convex cycles through $p$ which exist by (3.7). One sees easily that $p$ can be chosen so that the convex cycles through neighboring points are homologous to $Q_1$ and $Q_2$ respectively. (If it turns out that the convex cycles depend continuously on the supporting hyperplane, then this is of course always the case; see also a remark in the proof of Theorem D.) The self-intersection numbers of $Q_1$ and $Q_2$ are 0 or 1 since two different convex cycles which meet intersect in a convex top set contained in the tangent plane of any of the common points. The projection of a sufficiently small neighborhood $U_1$ of $Q_1$ into the supporting hyperplane containing $Q_1$ is a codimension 1 immersion and hence it follows that $w_2(U_1)$ vanishes. Similarly one sees that $w_2(U_2)$ vanishes for a sufficiently small neighborhood $U_2$ of $Q_2$. By (3.7) the cycles $Q_1$ and $Q_2$ are smooth submanifolds around $p$. They can be smoothed everywhere to give smooth submanifolds of $M$ which are diffeomorphic to $S^k$ and arbitrarily close to $Q_1$ and $Q_2$. (This can e.g. be done as follows: One first notices that no supporting hyperplane in $\langle Q_1 \rangle$ of $Q_1$ meets the normal space of $f$. Hence $Q_1$ can first be approximated in the Euclidean space $\langle Q_1 \rangle$ by a $C^\infty$ convex hypersurface and then regularly projected into $f(U_1) \simeq U_1$. The same can be done with $Q_2$.) We denote these submanifolds also by $Q_1$ and $Q_2$. The second

Stiefel-Whitney classes of the normal bundles of $Q_1$ and $Q_2$ in $M$ vanish since $w_2(U_1)$, $w_2(U_2)$, $w_2(Q_1)$ and $w_2(Q_2)$ vanish. The Euler numbers of the normal bundles which coincide with the self-intersection numbers of $Q_1$ and $Q_2$ are therefore even. By the above we therefore see that the Euler numbers of the bundles are 0. By the classification of 2-plane (circle) bundles over $S^2$ in [St] we see that $Q_1$ and $Q_2$ have trivial normal bundles in $M$. By arguments as in the proof of Theorem B we can therefore split off $S^2 \times S^2$.

It follows from the result of Hirsch quoted in the introduction that a four-manifold which admits a (not necessarily tight) immersion into $E^6$ has even second Betti number.

(ii) We assume now that $l > 2$. With the methods in part (i) of this proof and in part (i) of the proof of Theorem D we find an embedding $\varphi$ of $S^2$ into $M$ whose normal bundle has Euler characteristic one (or minus one, but then we change the orientation). By the classification of 2-plane (circle) bundles over $S^2$ in [St] we see that the boundary of a tubular neighborhood of $\varphi(S^2)$, considered as a circle bundle over $S^2$, is equivalent to the Hopf fibration. One can now remove the tubular neighborhood and glue a four-cell instead to obtain a manifold $N$ such that $M = P_2\mathbb{C} \# N$. ■

*Proof of the Corollary.* Let $M$ be an algebraic surface of even degree $d \geq 4$ in $P_3\mathbb{C}$ without singularities. Then the second Stiefel-Whitney class $w_2(M)$ vanishes. ($w_2(M)$ is the mod 2 reduction of the first Chern class $c_1(M) = (4-d)g$, where $g$ is the Kähler class induced from $P_3\mathbb{C}$.) If $M$ admits a tight immersion into a Euclidean space, then the codimension $l$ is 1 or 2 by Theorem D. Codimension $l = 1$ can be excluded for $d \geq 4$ since the Pontrjagin class $p_1(M) \neq 0$. Thus $l = 2$ and we can use Theorem C to decompose $M$ as $(S^2 \times S^2) \# N_1$. The intersection form of $M$ is

$$mE_8 \oplus n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad m > 0,$$

since $w_2(M) = 0$ and the signature of $M$ does not vanish for $d \geq 4$. Hence the intersection form of $N_1$ is

$$mE_8 \oplus (n-1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Either $N_1$ cannot be tightly immersed or it decomposes as $(S^2 \times S^2) \# N_2$ where

the intersection form of $N_2$ is

$$mE_8 \oplus (n-2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can continue this until we arrive at a manifold $N_i$ which does not admit a tight immersion. This cannot take more than $n$ steps since a manifold $N_n$ with intersection form $mE_8$ cannot split as $(S^2 \times S^2) \# N_{n+1}$. (Of course we will stop after at least $n-1$ steps by Donaldson's Theorem, but we do not need this difficult result.) The manifold $N_i$ which does not admit a tight immersion has the same signature as the algebraic surface $M$ of degree $d$ that we began with, i.e. $\sigma(N_i) = (1/3)d(4-d^2)$. Two different even degrees $\geq 4$ thus give two different examples of manifolds that cannot be tightly immersed.

Now assume that $M$ is the Kummer surface, i.e. $d = 4$. Then it follows from Donaldson's result in [Do 2] that $M$ cannot be decomposed as $(S^2 \times S^2) \# N^4$. Hence there cannot exist a tight immersion of the Kummer surface. ■

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