

# **Growth of the coefficient auf quasiconformality and the boundary correspondence of automorphisms of a ball.**

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## Growth of the coefficient of quasiconformality and the boundary correspondence of automorphisms of a ball

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*Abstract.* A homeomorphism  $f: B^n \rightarrow B^n$  of the unit ball in  $R^n$  ( $n \geq 2$ ) whose coefficient of quasiconformality in the ball of radius  $r < 1$  has asymptotic rate of growth  $K(r) = \sup_{|x| \leq r} k(x, f) = O(\log(1/(1-r)))$  can be continued to a homeomorphism  $\bar{f}: \bar{B}^n \rightarrow \bar{B}^n$  of the closed ball  $\bar{B}^n$ . For  $n = 2$  this implies that the Caratheodory theory of prime ends for conformal mappings also holds for the class of homeomorphisms  $f: B^2 \rightarrow D$  with  $K(r) = O(\log(1/(1-r)))$ .

The following theorem was recently given by Zorić [10]:

If  $f: B^2 \rightarrow B^2$  is an automorphism of the unit disc  $B^2$  such that

$$\int^1 \frac{dr}{(1-r)K(r)} = \infty, \quad \int^1 K(r) dr < \infty,$$

where  $K(r)$  is the coefficient of quasiconformality of  $f$  in the disc  $B^2(r)$ , then  $f$  can be extended to a *continuous* mapping  $\bar{f}: \bar{B}^2 \rightarrow \bar{B}^2$  of the closed disc  $\bar{B}^2$  into itself.

Zorić [10] also made the conjecture that the above theorem holds for  $n \geq 3$  with  $K^{n-1}(r)$  instead of  $K(r)$ .

In this paper we prove that every homeomorphism  $f: B^n \rightarrow B^n$  of the unit ball  $B^n$  ( $n \geq 2$ ) such that  $K(r) = O(\log(1/(1-r)))$ , i.e.  $K(r)$  increases as the logarithm, can be continued to a *homeomorphism*  $\bar{f}: \bar{B}^n \rightarrow \bar{B}^n$  of the closed ball  $\bar{B}^n$ . We also give some consequences of this statement.

Turn to the precise formulations.

Let  $D$  and  $D'$  be regions in euclidean space  $R^n$  and  $f: D \rightarrow D'$  a homeomorphism. The number

$$k(x, f) = \limsup_{t \rightarrow 0} \frac{\max_{|y-x|=t} |f(y) - f(x)|}{\min_{|y-x|=t} |f(y) - f(x)|}$$

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will be called the coefficient of quasiconformality of  $f$  at  $x \in D$ . If  $D$  is the unit ball  $B^n$  let

$$K_f(r) = K(r) = \sup_{|x| \leq r} k(x, f).$$

In connection with the sequel recall that the coefficient of quasiconformality of a homeomorphism is a Borel measurable function (cf. [8]).

The rest of the notation and terminology that we use here is generally the same as in [8].

**LEMMA 1.** *Let  $f: B^n \rightarrow B^n$  be a homeomorphism with  $\int^1 K^{n-1}(r) dr < \infty$ . Then  $g = f^{-1}$  has a continuous extension  $\bar{g}: \bar{B}^n \rightarrow \bar{B}^n$  of the closed ball  $\bar{B}^n$  into itself.*

*Proof.* Since  $k(x, f)$  is bounded in every ball  $B^n(r)$  of radius  $0 < r < 1$ , it follows that  $k(y, g)$  is locally bounded and  $f$  is in the Sobolev space  $W_{n,\text{loc}}^1(B^n)$ , i.e.  $ACL^n$  in the sense of [8], (cf. [8], 32.3). So, for coordinate functions  $g^i$ ,  $1 \leq i \leq n$ , of  $g$  we have (cf. [5], [6]):

$$\begin{aligned} \int_{B^n} |\nabla g^i|^n dy &\leq \int_{B^n} k^{n-1}(y, g) J(y, g) dy \leq \int_{B^n} k^{n-1}(x, f) dx \\ &\leq \int_{S^{n-1}} d\omega_{n-1} \int_0^1 r^{n-1} K^{n-1}(r) dr \leq \omega_{n-1} \int_0^1 K^{n-1}(r) dr < \infty. \end{aligned}$$

By the standard argument (for example in the same way as in proof of theorem 10.1 in [3]), one concludes the proof of the lemma.

**LEMMA 2 (fundamental lemma).** *Let  $F$  be a compact subset of the unit ball  $B^n$ ,  $b \in S^{n-1} = \partial B^n$  and  $\Gamma$  the family of all curves  $\gamma$  in  $B^n$  such that  $\gamma$  has a common point with  $F$  and contains  $b$  in its closure. Let  $f: B^n \rightarrow D$  be a homeomorphism such that*

$$\int^1 \frac{dr}{(1-r)K(r)} = \infty, \tag{a}$$

and for some  $m > 1$

$$\int_{1-t^m}^1 K^{n-1}(r) dr = 0(t) \quad \text{when } t \rightarrow 0, (t > 0). \tag{b}$$

Then  $M(\Gamma') = 0$ , where  $\Gamma' = f(\Gamma)$ .

*Proof.* Let  $(r_k)$  be an increasing sequence in  $[0, 1)$  such that  $r_k \rightarrow 1$  when  $k \rightarrow \infty$  and  $F \subset B^n(r_0)$ . Let  $\Gamma_k$  be the family whose elements are subcurves of elements of  $\Gamma$  that connect through the spherical ring  $R_k = \{x \in R^n : 1 - r_k < |x - b| < 1 - r_{k-1}\}$  its boundary spheres  $S^{n-1}(b, 1 - r_k)$  and  $S^{n-1}(b, 1 - r_{k-1})$ . The condition (a) (as well as (b)) implies  $K(r) < \infty$  for  $0 \leq r < 1$  and by theorem 32.3 in [8] a homeomorphism  $f$  is in the class  $W_{n,\text{loc}}^1(B^n)$ . Consequently, families  $\Gamma'_k = f(\Gamma_k)$  are separate and  $\Gamma' > \Gamma'_k$  (cf. [8]). Therefore [2]

$$\frac{1}{M^{1-n}}(\Gamma') \geq \sum_{k=0}^{\infty} \frac{1}{M^{1-n}}(\Gamma'_k). \quad (2)$$

Standard arguments yield (cf. [4], Lemma 1)

$$M(\Gamma'_k) \leq \int_{R_k \cap B^n} \rho^n(x) k^{n-1}(x, f) dx,$$

for every  $\rho$  admissible for  $\Gamma_k$ . If for  $\rho$  we choose the extremal function of the ring  $R_k$  then we obtain

$$M(\Gamma'_k) \leq \frac{1}{\left(\log \frac{1 - r_{k-1}}{1 - r_k}\right)^n} \int_{R_k \cap B^n} \frac{k^{n-1}(x, f)}{|x - b|^n} dx. \quad (3)$$

Let  $(t, \omega) \xrightarrow{P} x$ ,  $\omega \in S^{n-1}(b, 1)$  be the spherical coordinate system with origin in  $b$ . Let  $\tau_m$  be the hypersurface defined by  $x \in \tau_m$  if and only if  $|x| = 1 - t^m$ , where  $m > 1$  is such that the condition (b) is satisfied. Denote by  $A_t$  the central projection from  $b$  of the set  $S^{n-1}(b, t) \cap \bar{B}^n$  onto the unit sphere  $S^{n-1}(b, 1)$ , by  $A'_t \subset A_t$  the projection of that part of the set  $S^{n-1}(b, t) \cap B^n$  which lies inside of the surface  $\tau_m$  and by  $A''_t$  the difference  $A_t - A'_t$ . Then, taking into account that  $k^{n-1}(P(t, \omega)) \leq K^{n-1}(|P(t, \omega)|)$ , we get

$$\begin{aligned} \int_{R_k \cap B^n} \frac{k^{n-1}(x, f)}{|x - b|^n} dx &\leq \int_{1-r_k}^{1-r_{k-1}} \frac{dt}{t} \int_{A_t \subset S^{n-1}(b, 1)} k^{n-1}(P(t, \omega)) dS^{n-1} \\ &\leq \int_{1-r_k}^{1-r_{k-1}} \frac{dt}{t} \int_{A'_t} K^{n-1}(|P(t, \omega)|) dS^{n-1}. \end{aligned} \quad (4)$$

Further, for  $1 - r_k \leq t \leq 1 - r_{k-1}$

$$\int_{A'_t} K^{n-1}(|P(t, \omega)|) dS^{n-1} \leq \omega_{n-1} K^{n-1}(1 - t_k^m), \quad (t_k = 1 - r_k), \quad (5)$$

and for  $0 < t \leq 1 - r_0$

$$\int_{A'_t} K^{n-1}(|P(t, \omega)|) dS^{n-1} = \int_{S^{n-2}} dS^{n-2} \int_{\theta_\tau}^{\theta_s} K^{n-1}(r(t, \theta)) d\theta,$$

where  $\theta$  is the angle between the vectors  $x - b$  and  $-b$ ,  $r(t, \theta) = |x|$  and  $\theta_\tau$  and  $\theta_s$  correspond to these points of  $S^{n-1}(b, t)$  that lie on  $\tau_m$  and  $S^{n-1}$  respectively. It is easy to see that for  $t \neq 0$

$$d\theta = \frac{1 - 2t \cos \theta + t^2}{rt \sin \theta} dr.$$

Consequently, there exist  $0 < t' < 1$  and a constant  $c > 0$  such that

$$d\theta \leq c \frac{dr}{t} \quad \text{for } 0 < t \leq t', \quad \theta_\tau < \theta < \theta_s (\theta_s < \pi).$$

So we have for  $0 < t \leq t'$

$$\int_{A'_t} K^{n-1}(|P(t, \omega)|) dS^{n-1} \leq \frac{c \omega_{n-2}}{t} \int_{r(t, \theta_\tau)}^1 K^{n-1}(r) dr, \quad (6)$$

with  $r(t, \theta_\tau) = 1 - t^m$ . According to (b) there exist  $0 < t'' < 1$  and  $c_1 > 0$  such that

$$\int_{1-t''^m}^1 K^{n-1}(r) dr \leq c_1 t \quad \text{for } 0 < t \leq t''. \quad (7)$$

Let  $t_0 = \min \{t', t''\}$ . Then from (6) and (7) it follows that

$$\int_{A''_t} K^{n-1}(|P(t, \omega)|) dS^{n-1} \leq \tilde{c} \omega_{n-2}, \quad (8)$$

for  $0 < t \leq t_0$  and some  $\tilde{c} > 0$ . From (3), (4), (5) and (8) it follows that there exist  $C > 0$  and  $0 \leq R_0 < 1$  such that

$$M(\Gamma'_k) \leq C \frac{K^{n-1}(1 - (1 - r_k)^m)}{\left( \log \frac{1 - r_{k-1}}{1 - r_k} \right)^{n-1}} \quad (9)$$

whenever  $r_{k-1} \geq R_0$ . From (2) and (9) one gets

$$M^{1/1-n}(\Gamma') \geq C^{1/1-n} \sum_{r_{k-1} \geq R_0} \frac{\ln(1-r_{k-1}) - \ln(1-r_k)}{K(1-(1-r_k)^m)}$$

for every increasing sequence  $(r_k)$ ,  $r_k \rightarrow 1$ . It follows that

$$M^{1/1-n}(\Gamma') \geq M \int_R^1 \frac{dr}{(1-r)K(1-(1-r)^m)},$$

for some  $M > 0$  and  $R \geq 0$ . Changing variable by  $1-(1-r)^m = u$  we finally have

$$M^{1/1-n}(\Gamma') \geq \frac{M}{m} \int^1 \frac{dr}{(1-r)K(r)}. \quad (10)$$

If  $M(\Gamma') > 0$  it follows from (10) that the integral in (a) converges. This yields a contradiction and the proof of the lemma is complete.

**LEMMA 3.** *Let  $f: B^n \rightarrow D$  be a homeomorphism such that  $\int^1 K^{n-1}(r) dr < \infty$ . Then  $D$  is a proper subset of  $R^n$ .*

*Proof.* Suppose on the contrary that  $D = R^n$ . Let  $p, q$  be two different points of the unit sphere  $S^{n-1} = \partial B^n$ , let  $s$  be a fixed element of  $(0, 1)$  and  $\Gamma$  the family of curves which through  $B^n$  join the segments  $[sp, p)$  and  $[sq, q)$ . Let  $a$  be the distance between the points  $sp$  and  $sq$ . Then the function  $x \mapsto \rho(x) = 1/a$  is admissible for  $\Gamma$ . Let  $\Gamma' = f(\Gamma)$ . Then we have

$$\begin{aligned} M(\Gamma') &\leq \int_{B^n} \rho^n(x) k^{n-1}(x, f) dx \leq \frac{1}{a^n} \int_{B^n} k^{n-1}(x, f) dx \\ &\leq \frac{\omega_{n-1}}{a^n} \int_0^1 K^{n-1}(r) dr < \infty. \end{aligned}$$

On the other side, since the modulus of curve family is a conformal invariant, we can suppose that  $\Gamma'$  is the family of curves which join two arcs that begin in the same point of  $R^n$ . This implies  $M(\Gamma') \geq c \log(b/t)$  for a fixed  $b$  and each  $0 < t \leq b$  (cf. [8], 10.12). This is a contradiction.

**THEOREM 1.** *If  $f: B^n \rightarrow B^n$  is a homeomorphism with  $K(r) = O(\log(1/(1-r)))$  than  $f$  can be extended to a homeomorphism  $\bar{f}: \bar{B}^n \rightarrow \bar{B}^n$  of the closed ball  $\bar{B}^n$ .*

*Proof.* Let  $b \in S^{n-1} = \partial B^n$ . Let  $C(f, b)$  be the cluster set of  $f$  at  $b$ . Let  $F, \Gamma, \Gamma'$  be as in Lemma 2,  $F$  being connected and having more than one point. Since the family  $\Delta(f(F), C(f, b), B^n)$  of all curves that join  $f(F)$  and  $C(f, b)$  through  $B^n$  is a subfamily of  $\Gamma'$ , because of the monotonicity of the modulus and Lemma 2, we obtain  $M(\Delta(f(F), C(f, b), B^n)) = 0$ . (If  $K(r) = O(\log(1/(1-r)))$  then the conditions (a) and (b) of Lemma 2 are satisfied). Since  $C(f, b)$  is connected this means that  $C(f, b)$  has exactly one point. It follows that  $f$  has a continuous extension  $\bar{f}: \bar{B}^n \rightarrow \bar{B}^n$ . On the base of Lemma 1 we conclude that  $f$  is a homeomorphism.

*Remarks.* 1) Theorem 1 was in fact proved under the hypothesis (a) and (b) of Lemma 2. But the condition (b) is slightly stronger than the condition  $\int^1 K^{n-1}(r) dr < \infty$  in [10]. 2) If  $K(r)$  increases faster than  $\log(1/(1-r))$  then Theorem 1 does not hold. According to [10] for every nondecreasing function  $h$  such that

$$\int^1 \frac{dr}{(1-r)h(r)} < \infty \quad \text{or} \quad \int^1 \frac{dr}{(1-r)h(r)} = \infty \quad \text{and} \quad \int^1 h^{n-1}(r) dr = \infty$$

there exists a diffeomorphism  $f: B^n \rightarrow B^n$  with  $K(r) \leq h(r)$  having no continuous extension from  $\bar{B}^n$  into itself. 3) Theorem 1 also holds (under the conditions (a) and (b)) if we replace the ball  $B^n$  in the range by a region  $D$  which has property  $P_2$  on the boundary (cf. [8], 17.5 and 17.15).

It was pointed out in [10] that the question about boundary behavior of different classes of homeomorphisms in the plane is reduced, from the metrical-point of view, to the study of boundary behavior of automorphisms of a disc  $B^2$ . (This is a consequence of the Riemann mapping theorem and the Caratheodory theory of prime ends).

**THEOREM 2.** *For the class of locally quasiconformal mappings  $f: B^2 \rightarrow D$  which satisfy the condition  $K(r) = O(\log(1/(1-r)))$  the Caratheodory theory of prime ends holds.*

*Proof.* It is enough to show that a region  $D$  is conformally equivalent to the unit disc  $B^2$ . But that is a consequence of Lemma 3.

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