A criterion for a variety to be cone.

Comment. Math. Helvetici 62 (1987) 417–422 0010-2571/87/030417-06$01.50 + 0.20/0

A criterion for a variety to be a cone

MAURO BELTRAMETTI and ANDREW JOHN SOMMESE

Let $X \subseteq \mathbb{P}_{\mathbb{C}}$ be a normal projective Cohen–Macaulay variety and let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. In this paper we give a criterion for X to be a cone. As a consequence we obtain the following theorem which settles affirmatively a conjecture of Conte and Murre [C–M] about 3-dimensional Gorenstein varieties with very ample anti-canonical bundle.

THEOREM. *Let X and L be as above and assume that $L^k = K_X^{-1}$ for some $k > 0$ where K_X is the dualizing sheaf of X. If the locus,* Irr (X), *of irrational singularities is not empty, then*

$$\dim \operatorname{Irr}(X) \geqq k - 1$$

with equality if and only if Irr (X) *is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$ and X is a cone with* Irr (X) *as vertex.*

In §0 we summarize background material and in §1 we obtain the above theorem as a consequence of a technical criterion for a variety to be a cone.

We would like to thank the Consiglio Nazionale delle Ricerche for making our collaboration possible. The second author would also like to thank the National Science Foundation for their support (NSF Grant DSM 8420315).

§0. Background material

We work over the complex numbers $\mathbb{C}$. By *variety* we mean an irreducible and reduced quasi-projective scheme X of dimension n. We denote its structure sheaf by $\mathcal{O}_X$. For any coherent sheaf $\mathcal{F}$ on X, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(X, \mathcal{F})$. If X is normal the *dualizing sheaf* K_X is defined to be $j_* K_{\operatorname{Reg}(X)}$ where $j: \operatorname{Reg}(X) \to X$ is the inclusion of the smooth points of X and $K_{\operatorname{Reg}(X)}$ is the canonical sheaf of holomorphic n-forms.

Let $p: \bar{X} \to X$ be a resolution of singularities of X. The Leray sheaves $p_{(i)}\mathcal{O}_{\bar{X}}$ are independent of the resolution, and we shall denote them by $\mathcal{S}_i(X)$. X is normal if and only if $p_{(0)}\mathcal{O}_{\bar{X}} = \mathcal{O}_X$. We denote by Irr (X) the irrational locus of X which is the union of the supports of the sheaves $\mathcal{S}_i(X)$ for $i > 0$.

Let $\mathcal{L}$ be a line bundle on a normal variety X. $\mathcal{L}$ is said to be *numerically effective* (*nef* for short) if $\mathcal{L} \cdot C \geqq 0$ for all effective curves C on X, and $\mathcal{L}$ is said to be *big* if $c_1(\mathcal{L})^n > 0$ where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$. If $\mathcal{L}$ is a nef and big line bundle on a normal projective variety X, then a convenient form (see [S], (0.2.1)) of the Kawamata–Viehweg vanishing theorem is

$$h^i(K_X \otimes \mathcal{L}) = 0 \quad \text{for} \quad i > \max\{0, \dim \operatorname{Irr}(X)\}. \tag{0.1}$$

If further X is Cohen–Macaulay and $\dim \operatorname{Irr}(X) = 0$, then ([$S$], (0.2.2))

$$h^0(K_X \otimes \mathcal{L}) = 0 \quad \text{implies} \quad \operatorname{Irr}(X) \quad \text{is empty.} \tag{0.2}$$

(0.3) *Basic Construction.* Let L be a line bundle on a possibly non-compact, irreducible, normal variety X. Assume that a finite dimensional vector space V of sections of L spans L off of a finite set F of X. There is a desingularization $p: \bar{X} \to X$ with a line bundle $\mathcal{L}$ on $\bar{X}$ such that:

a) $p_*\mathcal{L} \cong L \otimes \mathcal{J}_F$ where $\mathcal{J}_F$ is the image in $\mathcal{O}_X$ of $V \otimes L^{-1}$ under the natural map;

b) there is a vector space $\mathcal{V}$ of sections of $\mathcal{L}$ that spans $\mathcal{L}$ and such that the isomorphism in a) gives an isomorphism of $\mathcal{V}$ onto V.

Let us give a sketch of the construction. First blow up the ideal sheaf $\mathcal{J}_F = \operatorname{Image}(V \otimes L^{-1} \to \mathcal{O}_X)$ to obtain a modification $p': X' \to X$, a line bundle $\mathcal{L}'$ on X', and a space of sections $\mathcal{V}'$ of $\mathcal{L}'$ with properties a) and b) for p', X', $\mathcal{L}'$, $\mathcal{V}'$ instead of p, X, $\mathcal{L}$, $\mathcal{V}$. Now let $p: \bar{X} \to X$ be a desingularization obtained by composing a desingularization $q: \bar{X} \to X'$ with p'. Let $\mathcal{L} = q^*\mathcal{L}'$ and $V = q^*V'$.

(0.4) THEOREM. *Let L be a line bundle on a possibly non-compact, irreducible, normal variety X of dimension at least* 3. *Assume that a finite dimensional space V of global sections of L spans L off of a finite set $F \subset X$. Let $|V|$ denote the space of zero sets of elements of V. For a general $A \in |V|$:*

a) $\operatorname{Sing}(A - F) = A \cap \operatorname{Sing}(X - F)$ *and no component of* $\operatorname{Sing}(X - F)$ *is contained in $A - F$;*

b) *for each $i > 0$, the support of $\mathcal{S}_i(A - F)$ equals A intersected with the support of $\mathcal{S}_i(X - F)$, and no possibly embedded component of the support of $\mathcal{S}_i(X - F)$ is contained in $A - F$.*

In particular $\dim \operatorname{Irr}(X - F) \geqq \dim \operatorname{Irr}(A - F)$ *and* $\dim \operatorname{Sing}(X - F) \geqq \dim \operatorname{Sing}(A - F)$ *where the dimension of the empty set is taken to be $-\infty$. If X is Cohen–Macaulay then a general $A \in |V|$ is normal.*

Proof. Theorem (0.4.1) of [*S*] shows a) and b). To see normality let note that by the above the singular set of a general $A \in |V|$ has codimension 2 at least. Since A is Cohen–Macaulay this implies that A is normal. ■

We need the following result. For completeness we include a proof and refer also to [B], (2.6.1) and [B–S], (7.5).

(0.5) LEMMA. *Let X be a normal Cohen–Macaulay projective variety. If* Irr (X) *is finite then $\mathcal{S}_i(X)$ is* 0 *for* $1 \leqq i \leqq \dim X - 2$.

Proof. Since X is Cohen–Macaulay and projective we can choose X such that $H^i(L^{-1}) = 0$ for $i < \dim X$. By Theorem (0.4), choose a general element $A \in |L|$ such that A is normal, Irr (A) is empty and $\bar{A} = p^{-1}(A)$ is smooth for some desingularization $p : \bar{X} \to X$. Write $\bar{L} = p^*L$. We have

$$\begin{array}{ccccccccc}
\cdots \to & H^i(\bar{L}^{-1}) & \to & H^i(\mathcal{O}_{\bar{X}}) & \to & H^i(\mathcal{O}_{\bar{A}}) & \to \cdots \\
& \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*_{\bar{A}}} & \\
\cdots \to & H^i(L^{-1}) & \to & H^i(\mathcal{O}_X) & \to & H^i(\mathcal{O}_A) & \to \cdots
\end{array}$$

By the Kawamata–Viehweg vanishing theorem (0.1), $H^i(\bar{L}^{-1}) = 0$ for $i < \dim X$. Since A has only rational singularities and $\bar{A}$ is a desingularization of A, $p^*_{\bar{A}}$ is an isomorphism. Therefore from the above diagram we conclude that

$$H^i(\mathcal{O}_X) \xrightarrow{p^*} H^i(\mathcal{O}_{\bar{X}})$$

is an isomorphism for $i < \dim X - 1$ and an injection for $i = \dim X - 1$. A simple inspection of the Leray spectral sequence for p, using the assumption that the supports of $\mathcal{S}_i(X)$ are finite for $i \geqq 1$, shows that $H^0(\mathcal{S}_i(X)) = 0$ for $1 \leqq i \leqq \dim X - 2$. Since the supports of the $\mathcal{S}_i(X)$ are finite this implies that the supports of the $\mathcal{S}_i(X)$ are empty for $1 \leqq i \leqq \dim X - 2$.

(0.5.1) QUESTION. Is it true that if X is Cohen–Macaulay then for $i > 0$ the support of $\mathcal{S}_i(X)$, if not empty, is pure $(\dim X - i - 1)$-dimensional?

(0.6) Let $X \subset \mathbb{P}_{\mathbb{C}}$ be a projective variety and let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. Let V be the subspace of $\Gamma(X, L)$ consisting of sections that vanish at a point $x \in X$. Let $p : \bar{X} \to X$ be a desingularization of X with a $\mathcal{V}$ and $\mathcal{L}$ as in the basic construction (0.3). *Then X is a cone with vertex x if and only if $\mathcal{L}$ is not big.* In particular if X is not a cone on x then $H^i(\mathcal{L}^{-1}) = 0$ for $i < \dim X$.

For further background material we refer to [S], §0.

§1. On conditions for a variety to be a cone

(1.1) THEOREM. *Let $X \subset \mathbb{P}_{\mathbb{C}}$ be a normal Cohen–Macaulay projective variety of dimension $n \geqq 3$. Let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. Let* Irr (X) *be finite and non empty and let $N = h^0(\mathcal{S}_{n-1}(X))$. Let $x \in$* Irr (X) *and let $M = h^0(\mathcal{S}_{n-2}(A))$ for a general element A of $|L|$ that contains x. Then X is a cone on x if either*:

a) $h^0(K_X \otimes L) < M + N$, *or*

b) $h^{n-1}(\mathcal{O}_A) - h^{n-1}(\mathcal{O}_X) < M + N$ *and* $h^n(\mathcal{O}_X) = 0$.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0$$

By the Kawamata–Viehweg vanishing theorem (0.1) we have

$$h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_A) - h^{n-1}(\mathcal{O}_X) = h^n(-A) = h^0(K_X \otimes L) \tag{1.1.1}$$

Thus condition b) implies condition a). Therefore it suffices to prove that X is a cone on x if condition a) holds. Let V be the subspace of $\Gamma(X, L)$ consisting of sections that vanish at x and let $\mathcal{V}$, $\bar{X}$, $\mathcal{L}$ and $p: \bar{X} \to X$ be as in the basic construction (0.3). Since Irr (X) is finite and X is Cohen–Macaulay, $p_{(j)}(\mathcal{O}_{\bar{X}})$ is 0 for $1 \leqq j \leqq n-2$ by Lemma (0.5). Thus the Leray spectral sequence gives

$$\left.\begin{aligned} &h^n(\mathcal{O}_{\bar{X}}) = h^n(\mathcal{O}_X) - a \\ &h^{n-1}(\mathcal{O}_{\bar{X}}) = h^{n-1}(\mathcal{O}_X) + b \\ &h^j(\mathcal{O}_X) = h^j(\mathcal{O}_{\bar{X}}) \quad \text{for} \quad j \leqq n-2 \end{aligned}\right\} \tag{1.1.2}$$

where $a \geqq 0$, $b \geqq 0$ and $a + b = N$.

Let A be a general element of the linear space $|V|$ of Cartier divisors associated to V. It can be assumed that A is normal and $\bar{A}$, the proper transform in $\bar{X}$ of A, belongs to $|\mathcal{V}|$ and is smooth. Assuming that X is not a cone we have $h^i(\mathcal{L}^{-1}) = 0$ for $i < n$ in view of (0.6). Thus by (1.1.2), $h^i(\mathcal{O}_A) = h^i(\mathcal{O}_{\bar{A}})$ for $i \leqq n-2$. Therefore by the Leray spectral sequence for $p_{\bar{A}}$ and the fact that Irr (A) is finite we conclude that

$$h^{n-1}(\mathcal{O}_{\bar{A}}) = h^{n-1}(\mathcal{O}_A) - M \tag{1.1.3}$$

Therefore by (1.1.1), (1.1.2), (1.1.3) we have

$$\begin{aligned} h^0(K_X \otimes L) &= h^n(\mathcal{O}_{\bar{X}}) + a + h^{n-1}(\mathcal{O}_{\bar{A}}) + M - h^{n-1}(\mathcal{O}_{\bar{X}}) + b \\ &= h^n(\mathcal{O}_{\bar{X}}) + [h^{n-1}(\mathcal{O}_{\bar{A}}) - h^{n-1}(\mathcal{O}_{\bar{X}})] + M + N \end{aligned}$$

Note that the assumption $h^0(K_X \otimes L) < M + N$ implies that the middle term must be negative. But this is absurd since $h^{n-1}(\mathscr{L}^{-1}) = 0$.

(1.2) COROLLARY. *Let X and L be as in the above theorem. Let* $\operatorname{Irr}(X)$ *be finite and non empty and assume that $h^0(K_X \otimes L) \leqq 1$. Then* $\operatorname{Irr}(X)$ *consists of one point x and X is a cone from that point.*

Proof. Since $\operatorname{Irr}(X)$ is finite it follows from Lemma (0.5) that $h^0(\mathscr{S}_{n-1}(X)) > 0$. A general $A \in |L|$ passing through x has $\operatorname{Irr}(A)$ finite by Theorem (0.4). By Elkik's theorem [E], $x \in \operatorname{Irr}(A)$ since otherwise $x \notin \operatorname{Irr}(X)$. Thus $h^0(\mathscr{S}_{n-2}(A)) > 0$ and

$$h^0(K_X \otimes L) \leqq 1 < h^0(\mathscr{S}_{n-2}(A)) + h^0(\mathscr{S}_{n-1}(X))$$

This implies the result by the above Theorem (1.1).

(1.3) THEOREM. *Let $X \subset \mathbb{P}_{\mathbb{C}}$ be a normal Gorenstein projective variety of dimension $n \geqq 3$ and let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. Assume that $L^k = K_X^{-1}$ for some $k > 0$, where K_X denotes the dualizing sheaf of X. Assume that the locus of irrational singularities,* $\operatorname{Irr}(X)$, *is not empty. Then* $\dim \operatorname{Irr}(X) \geqq k-1$ *with equality if and only if* $\operatorname{Irr}(X)$ *is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$ and X is a cone with* $\operatorname{Irr}(X)$ *as vertex.*

Proof. We prove the above by induction on k.

If $k = 1$, then the assertion that $\dim \operatorname{Irr}(X) \geqq k-1$ is an immediate consequence of the assumption that $\operatorname{Irr}(X)$ is not empty. Since $h^0(K_X \otimes L) = h^0(K_X \otimes K_X^{-1}) = h^0(\mathcal{O}_X) = 1$ the assertion follows from Corollary (1.2).

If $k > 1$, then choose a general $A \in |L|$. By (0.4), we conclude that A is a normal Gorenstein variety on which $\dim \operatorname{Irr}(X) = \dim \operatorname{Irr}(A) + 1$. Since

$$K_A^{-1} = (K_X \otimes L)_A^{-1} = L_A^{k-1}$$

we conclude by the induction hypothesis that $\dim \operatorname{Irr}(A) \geqq k-2$. This gives $\dim \operatorname{Irr}(X) = \dim \operatorname{Irr}(A) + 1 \geqq k-1$.

Further note that $\operatorname{Irr}(X)$ has no isolated points as components. Indeed by the argument of [S], (0.2.2) the number of isolated points in $\operatorname{Irr}(X)$ is bounded by $h^0(K_X \otimes L) = h^0(L^{-(k-1)}) = 0$. From this fact and the fact that $\operatorname{Irr}(A) = A \cap \operatorname{Irr}(X)$ is a linear $\mathbb{P}_{\mathbb{C}}^{k-2}$, it is an easy argument that $\operatorname{Irr}(X)$ is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$. Since any general element $A \in |L|$ is a cone on $A \cap \operatorname{Irr}(X) = \operatorname{Irr}(A)$, elementary arguments of projective geometry show that X has to be a cone on $\operatorname{Irr}(X)$. ■

The following settles a conjecture of Conte and Murre positively (see [C–M], section III).

(1.4) COROLLARY. *Let* $X \subset \mathbb{P}_{\mathbb{C}}$ *be a normal Gorenstein 3-fold with* $K_X^{-1} \cong \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. *If* $\mathrm{Irr}(X)$ *is finite then* $\mathrm{Irr}(X)$ *is a single point and* X *is the cone from this point over a Gorenstein K3-surface* A *with rational singularities. Note that* $\mathrm{Sing}(X)$ *is the cone over* $\mathrm{Sing}(A)$.

§2. Final Remarks

It follows from the results in the last section of [C–M], that if X is a normal Gorenstein 3-fold with K_X^{-1} very ample and $\dim \mathrm{Irr}(X) = 1$, then $\mathrm{Irr}(X)$ is a linear $\mathbb{P}^1_{\mathbb{C}}$.

Let us propose the following

QUESTIONS. Let X be a normal projective Gorenstein variety of dimension $n \geqq 3$. Assume that $L^k = K_X^{-1}$ for some $k > 0$, $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. If $\dim \mathrm{Irr}(X) = k$, is $\mathrm{Irr}(X)$ a linear $\mathbb{P}^k_{\mathbb{C}}$? How far can X deviate from being a cone over $\mathrm{Irr}(X)$ in this case?

REFERENCES

[B] L. Badescu, *Applications of the Grothendieck duality theory to the study of normal isolated singularities*, Revue Roum. Math. Pur. Appl. *26*(5), (1979), 673–689.

[B–S] J. Bingener and U. Storch, *Zur Berechnung der Divisorenklassengruppen kompletter lokaler Ringe* (preprint).

[C–M] A. Conte and J. Murre, *On the definition and on the nature of the singularities of Fano threefolds*, Math. Inst. Univ. of Leiden, The Netherlands, Report n.25 (1985).

[E] R. Elkik, *Singularités rationelles et déformations*, Invent. math., 47 (1978), 139–147.

[S] A. Sommese, *On the Adjunction theoretic structure of projective varieties, Complex Analysis and Algebraic Geometry*, Proceedings, Gottingen 1985, Lectures Notes in Math., 1194 (1986).

Dipartimento di Matematica
Via L. B. Alberti, 4
16132 Genova (Italy)

Department of Mathematics
University of Notre Dame
Notre Dame, Indiana 46556 (U.S.A.)

Received July 11, 1986