SK1 of finite group rings: V.

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SK_1 of finite group rings: V

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We continue here the study of

$$SK_1(\mathbb{Z}G) = \operatorname{Ker}[K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)]$$

for finite G: the group shown by Wall [26] to be precisely the torsion subgroup of $\operatorname{Wh}(G)$. In earlier papers in this series, $SK_1(\mathbb{Z}G)$ has been studied via the extension

$$0 \to Cl_1(\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \to \sum_{p \mid |G|} SK_1(\hat{\mathbb{Z}}_p G) \to 0; \tag{0.1}$$

where $Cl_1(\mathbb{Z}G) \subseteq SK_1(\mathbb{Z}G)$ is the subgroup of elements described via K_2 in localization sequences.

This paper contains the last step in deriving a combinatorial algorithm for describing the odd torsion in $SK_1(\mathbb{Z}G)$. By [17, Theorem 4.8], $SK_1(\mathbb{Z}G)[\frac{1}{2}]$ splits naturally as a sum

$$SK_1(\mathbb{Z}G)[\tfrac{1}{2}] \cong Cl_1(\mathbb{Z}G)[\tfrac{1}{2}] \oplus \sum_{p>2} SK_1(\hat{\mathbb{Z}}_p G).$$

The groups $SK_1(\hat{\mathbb{Z}}_p G)$ (also for $p = 2$) are described by [15, Theorem 3] and [16, Theorem 2], in terms of $H_2(Z_i)$ for certain subgroups $Z_i \subseteq G$. On the other hand, in [17], the problem of describing $Cl_1(\mathbb{Z}G)_{(p)}$ for any odd prime p and any finite G is reduced to the case where G is a p-group (see [17, Theorem 4.8], and the discussion at the end of Section 3 below).

The following theorem is the central result of this paper, and gives a relatively simple way of describing $Cl_1(\mathbb{Z}G)$ when G is a p-group (and p odd). Note that if G is any group, and G acts on $\mathbb{Z}G$ by conjugation, then for any set $S \subseteq G$ of conjugacy class representatives,

$$H_1(G; \mathbb{Z}G) \cong \sum_{h \in S} H_1(Z_G(h)) \otimes \mathbb{Z}\langle h \rangle.$$

(If $X \subseteq G$ is any conjugacy class, and $h \in X$, then $\mathbb{Z}(X) \cong \mathrm{Ind}_{Z_G(h)}^{G}(\mathbb{Z})$ as $\mathbb{Z}G$-modules.) Thus, $H_1(G; \mathbb{Z}G)$ is generated by elements $g \otimes h$ for commuting g, $h \in G$.

THEOREM 3.6. *Fix an odd prime p and a p-group G. Write $\mathbb{Q}G = \prod_{i=1}^{k} B_i$, where each B_i is simple with center F_i and irreducible representation V_i. For each i, let $(\mu_{F_i})_p$ be the group of p-th power roots of unity. Define*

$$\psi_G : H_1(G; \mathbb{Z}G) \to \prod_{i=1}^{k} (\mu_{F_i})_p,$$

where G acts on $\mathbb{Z}G$ by conjugation, by setting

$$\psi_G(g \otimes h) = [\det_{F_i}(g, V_i^h)]_i \quad (g, h \in G, gh = hg, V_i^h = \{x \in V_i : hx = x\}).$$

Then $Cl_1(\mathbb{Z}G) \cong \mathrm{Coker}\,(\psi_G)$.

Examples of computations of $Cl_1(\mathbb{Z}G)$ using Theorem 3.6 for non-abelian G are given in Section 4. For abelian G, the isomorphism $SK_1(\mathbb{Z}G) \cong \mathrm{Coker}\,(\psi_G)$ is proven in [1, Theorem 1.8], and some examples of calculations of $SK_1(\mathbb{Z}G)$ using that are given in Section 5 of the same paper.

Theorem 3.6 (and the other theorems referred to above) are stated, for simplicity, as describing the components of $SK_1(\mathbb{Z}G)$ as abstract groups only. But the proofs also contain enough information so that one can take a specific element in $SK_1(\mathbb{Z}G)$ (e.g., a specific element in $\mathrm{Coker}\,(\psi_G)$ as described above), and represent it by a matrix. The opposite problem, taking a specific matrix over $\mathbb{Z}G$ and deciding how it sits in $SK_1(\mathbb{Z}G)$ (if it does) is harder in general; the study in [20] of the Whitehead transfer homomorphism for oriented S^1-fiber bundles gives one example where this can be done.

In general, for any finite group G, $Cl_1(\mathbb{Z}G)$ is described by localization exact sequences

$$K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p G) \xrightarrow{\varphi} C_p(\mathbb{Q}G) \xrightarrow{\partial} Cl_1(\mathbb{Z}G)_{(p)} \to 0$$

for each prime p; where for any maximal order $\mathfrak{M} \subseteq \mathbb{Q}G$:

$$C_p(\mathbb{Q}G) \cong \varprojlim_n \mathrm{Coker}\,[K_2(\mathfrak{M}) \to K_2(\mathfrak{M}/p^n\mathfrak{M})] \cong \varprojlim_n Cl_1(\mathfrak{M}; p^n\mathfrak{M})$$

$$\cong \mathrm{Coker}\left[K_2\left(\mathfrak{M}\left[\frac{1}{p}\right]\right) \to K_2^{\mathrm{top}}(\hat{\mathbb{Q}}_p G)\right]_{(p)}.$$

The $C_p(\mathbb{Q}G)$ are described by the work of Bak and Rehmann on the congruence subgroup problem [3]. The remaining problem is then to find a set of generators for $K_2^{\text{top}}(\hat{\mathbb{Z}}_pG)$, or at least for its image in $C_p(\mathbb{Q}G)$. In the case of an odd prime p and a p-group G, the formula

$$Cl_1(\mathbb{Z}G) \cong \operatorname{Coker}\left[\psi_G : H_1(G;\mathbb{Z}G) \to \prod_{i=1}^{k} (\mu_{F_i})_p\right]$$

can be explained by noting that norm residue symbols define an isomorphism of $C_p(\mathbb{Q}G)$ with $\prod (\mu_{F_i})_p$, and that $H_1(G;\mathbb{Z}G)(\cong H_1(G;\hat{\mathbb{Z}}_pG))$ and $K_2(\hat{\mathbb{Z}}_pG)$ both are closely related to the cyclic homology group $HC_1(\hat{\mathbb{Z}}_pG)$ (see [21]).

The key new result here about generators for $K_2(\hat{\mathbb{Z}}_pG)$ is:

THEOREM 1.4. *Let p be any prime, and fix a p-group G and an element $z \in Z(G)$. Then*

$$\operatorname{Ker}[K_2^{\text{top}}(\hat{\mathbb{Z}}_pG) \to K_2^{\text{top}}(\hat{\mathbb{Z}}_p[G/z])]$$
$$= \langle \{g, 1 + \lambda(1-z)^i h\} : \lambda \in \hat{\mathbb{Z}}_p, i \geq 1, g, h \in G, gh = hg \rangle.$$

Since $\operatorname{Coker}[K_2^{\text{top}}(\hat{\mathbb{Z}}_pG) \to K_2^{\text{top}}(\hat{\mathbb{Z}}_p[G/z])]$ is also known in the above situation (see Proposition 2.1 below), it should in principle now be possible to inductively construct a set of generators for $K_2^{\text{top}}(\hat{\mathbb{Z}}_pG)$. Unfortunately, it's not always easy to explicitly lift elements from $K_2^{\text{top}}(\hat{\mathbb{Z}}_p[G/z])$ to $K_2^{\text{top}}(\hat{\mathbb{Z}}_pG)$, even where they are known to lift. But such an inductive procedure does work to give generators for $K_2^{\text{top}}(\hat{\mathbb{Z}}_pG)^+$ when p is odd, and this suffices when computing $Cl_1(\mathbb{Z}G)$.

Another consequence of Theorem 1.4 involves a comparison of $Cl_1(RG)$ – when G is any finite group and R the ring of integers is some number field $K \subseteq \mathbb{C}$ – with the "complex Artin cokernel"

$$A_{\mathbb{C}}(G) = \operatorname{Coker}\left[\sum \{R_{\mathbb{C}}(H) : H \subseteq G \text{ cyclic}\} \xrightarrow{\Sigma \operatorname{Ind}} R_{\mathbb{C}}(G)\right].$$

Natural epimorphisms $I_{RG} : A_{\mathbb{C}}(G) \twoheadrightarrow Cl_1(RG)$ are constructed, for such R and G, via localization sequences. Theorem 1.4 can then be applied to show that for any G, I_{RG} is an isomorphism for R large enough. Thus, $A_{\mathbb{C}}(G)$ represents the "largest possible" $Cl_1(RG)$ when G is fixed and R varies. This is the second unexpected appearance of Artin cokernels when studying $K_n(RG)$: it was shown in [18] that $D(\mathbb{Z}G)^+ \cong A_{\mathbb{Q}}(G)$ when G is a p-group and p any odd regular prime.

The obvious remaining question is: what about 2-power torsion in $SK_1(\mathbb{Z}G)$? Unlike the case of odd torsion, this cannot be completely reduced to studying $Cl_1(\mathbb{Z}G)$ for 2-groups G, but the results in [17] show that the main problem is with 2-groups. If G is a p-group (for any p) and $[G, G]$ is central and cyclic, then we can show that $K_2^{\text{top}}(\hat{\mathbb{Z}}_pG)$ is generated by $\{-1, -1\}$ and symbols $\{g, u\}$ for $g \in G$ and $u \in (\hat{\mathbb{Z}}_p[Z_G(g)])^*$; and when $p = 2$ this suffices to get a description of $Cl_1(\mathbb{Z}G)$. But there *are* 2-groups G for which $K_2^{\text{top}}(\mathbb{Z}_2G)$ is not generated by such symbols, and there may not be any simple algorithm for describing $Cl_1(\mathbb{Z}G)$ in general. The best conjecture we have been able to make so far gives upper and lower bounds for $Cl_1(\mathbb{Z}G)$, bounds which differ by exponent two. The question of whether the inclusion $Cl_1(\mathbb{Z}G)_{(2)} \subseteq SK_1(\mathbb{Z}G)_{(2)}$ ever fails to split is also still open.

The paper is organized as follows. Section 1 and 2 deal with the problems of finding generators for $\operatorname{Ker}(K_2(\hat{\mathbb{Z}}_p\alpha))$, and of detecting $\operatorname{Coker}(K_2(\hat{\mathbb{Z}}_p\alpha))$, respectively, when α is a surjection of p-groups whose kernel is central and cyclic. This is applied in Section 3 to prove that $Cl_1(\mathbb{Z}G) \cong \operatorname{Coker}(\psi_G)$ when G is an odd p-group; and ways of using that to compute the odd torsion in $Cl_1(\mathbb{Z}G)$ for arbitrary finite G are discussed. Examples are given in Section 4 to illustrate how Theorem 3.6 works in practice for computing $Cl_1(\mathbb{Z}G)$. Finally, in Section 5, the relationship between $Cl_1(RG)$ and the complex Artin cokernel is studied, and the isomorphism $Cl_1(RG) \cong A_{\mathbb{C}}(G)$ proven for large R.

As for notation, C_n always denotes a (multiplicative) cyclic group of order n, and ζ_n a primitive n-th root of unity. If F is any field, then μ_F denotes the group of roots of unity in F, and $(\mu_F)_p$ the group of p-th power roots of unity.

If R is a $\hat{\mathbb{Q}}_p$-algebra or a $\hat{\mathbb{Z}}_p$-order (e.g., $R = \hat{\mathbb{Q}}_pG$ or $\hat{\mathbb{Z}}_pG$), then $K_2(R)$ *always denotes the topological* K_2. The precise definition of these groups, and their occurrence in localization sequences, is described in [20]: in Theorem 2.1 and the preceding discussion (see also [3]). Here we just note that if R is a $\hat{\mathbb{Z}}_p$-order, then

$$K_2^{\text{top}}(R) \cong \varprojlim_n K_2(R/p^nR).$$

Section 1

If R is a ring, and $I \subseteq R$ is a 2-sided ideal, we define here

$$K_2(R, I) = \operatorname{Ker}[K_2(R) \to K_2(R/I)].$$

A braid diagram analogous to that in [12, Remark 6.6] shows that for any ideals

$\bar{I} \subseteq i \subseteq R$, there is an exact sequence

$$0 \to K_2(R, \bar{I}) \to K_2(R, I) \to K_2(R/\bar{I}, I/\bar{I}) \xrightarrow{\partial} K_1(R, \bar{I}) \to K_1(R, I) \to \cdots$$

The main result of this section is to describe a set of generators for $K_2(\hat{\mathbb{Z}}_pG, (1-z))$; when p is any prime, G is any p-group, and $z \in Z(G)$. Three lemmas will first be needed.

LEMMA 1.1. *Fix a prime p, and a finite ring R of p-power order. Let $J \subseteq R$ be the Jacobson radical, and let $\{\alpha_1, \ldots, \alpha_k\} \subseteq J$ be any set of elements such that $\{p, \alpha_1, \ldots, \alpha_k\}$ generates J (as an ideal). Then for any ideal $I \subseteq J$ of R such that $IJ = JI = 0$, and such that $I \subseteq \langle \alpha_1, \ldots, \alpha_k \rangle_R$ if $p = 2$, $K_2(R, I)$ is generated by symbols of the form*

$$\{1 - \alpha_i, 1 - x\} : 1 \le i \le k, \quad x \in I. \tag{1}$$

Proof. We use the notation and relations for pointed bracket symbols in [25, Proposition 96–97]. By [17, Proposition 2.3], $K_2(R, I)$ is generated by symbols of the form

$$\{1 - \alpha, 1 + x\} = \langle \alpha, 1 + x \rangle = \langle \alpha, x \rangle$$

for $\alpha \in J$ and $x \in I$ $(\alpha x = x\alpha = 0)$. Write $\alpha = pr_0 + \alpha_1 r_1 + \cdots + \alpha_k r_k$; so that

$$\begin{aligned}
\langle \alpha, x \rangle &= \langle pr_0, x \rangle + \sum_{i=1}^{k} \langle \alpha_i r_i, x \rangle = \langle p, r_0 x \rangle + \sum_{i=1}^{k} \langle \alpha_i, r_i x \rangle \\
&= \sum_{i=1}^{k} \{1 - \alpha_i, 1 + r_i x\} + \langle p, r_0 x \rangle + p\langle -1, r_0 x \rangle \\
&= \sum_{i=1}^{k} \{1 - \alpha_i, 1 + r_i x\} + \langle p, r_0 x \rangle + \left\langle -p + \binom{p}{2} r_0 x, r_0 x \right\rangle \quad (x^2 = 0) \\
&= \sum_{i=1}^{k} \{1 - \alpha_i, 1 + r_i x\} + \left\langle \binom{p}{2} r_0 x, r_0 x \right\rangle. \quad (px \in JI = 0).
\end{aligned}$$

If p is odd, then $\binom{p}{2} x = 0$, and we are done. If $p = 2$ and $I \subseteq \langle \alpha_1, \ldots, \alpha_k \rangle_R$, then the same procedure shows that for any $x \in I$, $\langle x, x \rangle$ is a sum of symbols of the form in (1). $\square$

The following technical relation between symbols will be needed in the calculations.

LEMMA 1.2. *Let R be any ring. Fix a, $u \in R^*$ and $n \geq 2$ such that*

$$[a^n, u] = 1 = [a^i u a^{-i}, a^j u a^{-j}]$$

for any i, j. Then

$$\{a, u(aua^{-1})(a^2ua^{-2}) \cdots (a^{n-1}ua^{1-n})\} = \{a^n, u\} + (n-1)\{u, u\} + \sum_{i=1}^{n-1} \{a^i u a^{-i}, u\}.$$

Proof. In $St(R)$, set $x = h_{12}(u)$, $y = h_{13}(a)$, and

$$T = (yxy^{-1})(y^2xy^{-2}) \cdots (y^{n-1}xy^{1-n}).$$

Then

$$\begin{aligned}
\{a, u(aua^{-1}) \cdots (a^{n-1}ua^{1-n})\} &= [y, xT] \\
&= (yxy^{-1})(yTy^{-1})T^{-1}x^{-1} = T(y^nxy^{-n})T^{-1}x^{-1} = [T, y^nxy^{-n}][y^n, x] \\
&= (\operatorname{diag}(aua^{-1} \cdot a^2ua^{-2} \cdots a^{n-1}ua^{1-n}, u^{1-n}) * \operatorname{diag}(u, u^{-1})) + \{a^n, u\} \\
&= \sum_{i=1}^{n-1} \{a^i u a^{-i}, u\} + \{u^{1-n}, u^{-1}\} + \{a^n, u\}.
\end{aligned}$$

Here, for commuting matrices M, $N \in E(R)$, $M^*N \in K_2(R)$ denotes the commutator $[\tilde{M}, \tilde{N}]$ of liftings to $\tilde{M}$, $\tilde{N} \in St(R)$. $\square$

The third lemma will be needed when constructing filtrations of group rings by ideals. By a *p-ring* is meant the ring of integers in any finite extension of $\mathbb{Q}_p$.

LEMMA 1.3. *Fix a prime p, a p-group G, and some $z \in Z(G)$. Let $p^n = |z|$. Then, for any p-ring A, there are isomorphisms*

$$f_k : A/p^n[G/z] \xrightarrow{\cong} \frac{(1-z)^k AG}{(1-z)^{k+1}AG} \quad (k \geq 1)$$

and

$$f'_k : A/p[G/z] \xrightarrow{\cong} \frac{(1-z)^k A/p[G]}{(1-z)^{k+1} A/p[G]}; \quad (1 \le k \le p^n - 1)$$

both induced by sending ξ *to* $(1-z)^k\xi$ *for* $\xi \in AG$.

Proof. Note first that for any $\xi \in AG$, and any $k \ge 1$,

$$(1-z)^k p^n \xi \equiv (1-z)^k (1 + z + z^2 + \cdots + z^{p^n-1})\xi = 0 \quad (\mathrm{mod}\, (1-z)^{k+1} AG). \tag{1}$$

Thus, $(1-z)^k AG/(1-z)^{k+1}AG$ has exponent at most p^n for $k \ge 1$; and is in particular finite. So the map

$$(1-z)^k : (1-z)AG \xrightarrow{\cong} (1-z)^{k+1}AG$$

is an isomorphism: it is clearly onto, and the groups are free A-modules of the same rank.

Thus, for $\xi \in AG$ and $k \ge 1$, if $(1-z)^k\xi = (1-z)^{k+1}\eta$ for some $\eta \in AG$, then $(1-z)(\xi - (1-z)\eta) = 0$, and so

$$\xi \in (1-z)\eta + (1 + z + \cdots + z^{p^n-1})AG \subseteq (1-z)AG + p^n AG.$$

Together with (1), this shows that $(1-z)^k\xi \in (1-z)^{k+1}AG$ if and only if $\xi \in p^n AG + (1-z)AG$. So f_k is well defined and an isomorphism.

If $1 \le k \le p^n - 1$, and $\xi' \in A/p[G]$ is such that $(1-z)^k\xi' \in (1-z)^{k+1}A/p[G]$ then

$$(1 + z + \cdots + z^{p^n-1})\xi' = (1-z)^{p^n-1}\xi' \in (1-z)^{p^n} A/p[G] = 0;$$

and so $\xi' \in (1-z)A/p[G]$. The converse is clear, and so f'_k is a well defined isomorphism. □

The main result of this section can now be shown:

THEOREM 1.4. *Fix a prime p, an unramified p-ring A, a p-group G, and an element* $z \in Z(G)$. *Then*

$$K_2(AG, (1-z)AG) = \mathrm{Ker}\,[K_2(AG) \to K_2(A[G/z])]$$

is a finite group, and is generated by symbols of the form

$$\{g, 1-\lambda(1-z)^i h\} : g, h \in G, [g, h]=1, \lambda \in A, i \geq 1.$$

Proof. Let $H_0 = \langle z \rangle$, and fix a series of subgroups

$$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that for each $i = 1, \ldots, n$,

$$H_i \lhd G \quad \text{and} \quad [H_i : H_{i-1}] = p.$$

For each i, fix $z_i \in H_i \setminus H_{i-1}$. Note that in G/H_{i-1}, z_i is central of order p.
Let $p^m = |z|$. Define

$$S = \{(k; r, i_0, \ldots, i_k) : 0 \leq k \leq n, i_0 \geq 1, 0 \leq r \leq m-1, 0 \leq i_1, \ldots, i_k \leq p-1\}.$$

For each $\sigma = (k; r, i_0, \ldots, i_k) \in S$, set $k(\sigma) = k$, and

$$X(\sigma) = p^r(1-z)^{i_0}(1-z_1)^{i_1} \cdots (1-z_k)^{i_k} \in AG.$$

Define ideals $I'(\sigma) \subseteq I(\sigma) \subseteq AG$ by setting

$$I'(\sigma) = \langle (1-z)^{i_0+1}, p^{r+1}(1-z)^{i_0}; p^r(1-z)^{i_0}(1-z_1)^{i_1} \cdots (1-z_j)^{i_j+1} : 1 \leq j \leq k \rangle$$

and

$$I(\sigma) = I'(\sigma) + \langle X(\sigma) \rangle.$$

The idea now is to use S as a bookkeeping system for filtering the ideal $(1-z)AG$ into "pieces" small enough so thast the theorem can be proven starting with Lemma 1.1. The following diagram gives a visual overview of this filtration in the case where $p=3$, $m=2$, and $n=2$ (i.e., $|z|=9$ and $|G|=81$):

<table>
<tr><th></th><th>$k(\sigma)=0$</th><th>$k(\sigma)=1$</th><th>$k(\sigma)=2$</th></tr>
<tr><td rowspan="18">$(1-z)AG$</td><td rowspan="9">$(0;0,1)$</td><td rowspan="3">$(1;0,1,0)$</td><td>$(2;0,1,0,0)$</td></tr>
<tr><td>$(2;0,1,0,1)$</td></tr>
<tr><td>$(2;0,1,0,2)$</td></tr>
<tr><td rowspan="3">$(1;0,1,1)$</td><td>$(2;0,1,1,0)$</td></tr>
<tr><td>$(2;0,1,1,1)$</td></tr>
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<tr><td>$(2;0,1,2,2)$</td></tr>
<tr><td rowspan="9">$(0;1,1)$</td><td rowspan="3">$(1;1,1,0)$</td><td>$(2;1,1,0,0)$</td></tr>
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<tr><td>$(2;1,1,1,1)$</td></tr>
<tr><td>$(2;1,1,1,2)$</td></tr>
<tr><td rowspan="3">$(1;1,1,2)$</td><td>$(2;1,1,2,0)$</td></tr>
<tr><td>$(2;1,1,2,1)$</td></tr>
<tr><td>$(2;1,1,2,2)$</td></tr>
<tr><td rowspan="6">$(1-z)^2AG$</td><td rowspan="6">$(0;0,2)$</td><td rowspan="3">$(1;0,2,0)$</td><td>$(2;0,2,0,0)$</td></tr>
<tr><td>$(2;0,2,0,1)$</td></tr>
<tr><td>$(2;0,2,0,2)$</td></tr>
<tr><td rowspan="3">$(1;0,2,1)$</td><td>$(2;0,2,1,0)$</td></tr>
<tr><td>$(2;0,2,1,1)$</td></tr>
<tr><td>$(2;0,2,1,2)$</td></tr>
</table>

(1)

The horizontal lines represent ideals in AG, ordered sequentially with the largest at the top. Each box represents some element $\sigma \in S$; the horizontal line at the top of the box represents $I(\sigma)$, while the line at the bottom represents $I'(\sigma)$. That the $I(\sigma)$ and $I'(\sigma)$ actually do correspond with this picture will be shown in Step 2A below.

Step 1. We now show that for any $\sigma \in S$, there is an isomorphism

$$f_\sigma : A/p[G/H_{k(\sigma)}] \xrightarrow{\cong} I(\sigma)/I'(\sigma) \tag{2}$$

defined by setting $f_\sigma([\xi]) = [X(\sigma)\cdot\xi]$ for $\xi \in AG$. This will be proven by induction on $k = k(G)$. If $k = 0$, so $\sigma = (0; r, i)$ for some $i \geq 1$ and $0 \leq r \leq m-1$, then

$$(1-z)^i AG/(1-z)^{i+1}AG \cong A/p^m[G/Z] = A/p^m[G/H_0]$$

by Lemma 1.3; and so

$$I(\sigma)/I'(\sigma) = \frac{p^r(1-z)^i AG + (1-z)^{i+1}AG}{p^{r+1}(1-z)^i AG + (1-z)^{i+1}AG} \cong \frac{p^r A/p^m[G/H_0]}{p^{r+1}A/p^m[G/H_0]} \cong A/p[G/H_0].$$

Now assume $k \geq 1$, and write $\sigma = (k; r, i_0, \ldots, i_k)$. Set

$$\hat{\sigma} = (k-1; r, i_0, \ldots, i_{k-1}) \in S.$$

By induction, we can assume that $I(\hat{\sigma})/I'(\hat{\sigma}) \cong A/p[G/H_{k-1}]$. By definition

$$I'(\sigma) = I'(\hat{\sigma}) + X(\hat{\sigma})(1-z_k)^{i_k+1}AG,$$
$$I(\sigma) = I'(\hat{\sigma}) + X(\hat{\sigma})(1-z_k)^{i_k}AG,$$
$$I(\hat{\sigma}) = I'(\hat{\sigma}) + X(\hat{\sigma})AG.$$

Thus, $I(\hat{\sigma}) \supseteq I(\sigma) \supseteq I'(\sigma) \supseteq I'(\hat{\sigma})$; and by Lemma 1.3:

$$I(\sigma)/I'(\sigma) \cong \frac{(1-z_k)^{i_k}A/p[G/H_{k-1}]}{(1-z_k)^{i_k+1}A/p[G/H_{k-1}]} \cong A/p[G/H_k].$$

(Recall that $H_k = \langle H_{k-1}, z_k \rangle$, and that $0 \leq i_k \leq p-1$.)

Step 2. We next show that for any $\sigma \in S$,

$$K_2(AG/I'(\sigma), I(\sigma)/I'(\sigma)) = \langle \{g, 1 - X(\sigma)\lambda h\} : [g, h] \in H_{k(\sigma)}, \lambda \in A \rangle. \tag{3}$$

This will be proven by downwards induction on $k = k(\sigma)$.

Note first that AG is a local ring with Jacobson radical

$$J(AG) = \langle p, 1-g : g \in G \rangle. \tag{4}$$

If $\sigma \in S$ and $k(\sigma) = n$, then $H_n = G$, and so $I(\sigma)/I'(\sigma) \cong A/p$ by Step 1. In particular,

$$(I(\sigma)/I'(\sigma)) \cdot J(AG/I'(\sigma)) = 0 = J(AG/I'(\sigma)) \cdot (I(\sigma)/I'(\sigma)).$$

So (3) follows in this case from Lemma 1.1 (applied using $\{1-g : g \in G\}$ for the α_i's).

Now fix some $\sigma = (k; r, i_0, \ldots, i_k) \in S$, where $k < n$. For each $0 \le i \le p-1$, set

$$\sigma_i = (k+1; r, i_0, \ldots, i_k, i) \in S.$$

Assume inductively that (3) holds for the σ_i.

Step 2A. We now show that the $I(\sigma_i) \supseteq I'(\sigma_i)$ and $I(\sigma) \supseteq I'(\sigma)$ have the relations implied by diagram (1) above. By definition, $I(\sigma_0) = I(\sigma)$ $(X(\sigma_0) = X(\sigma))$. For any $0 \le i \le p-2$,

$$I'(\sigma_i) = I'(\sigma) + X(\sigma)(1 - z_{k+1})^{i+1} AG = I(\sigma_{i+1}). \tag{5}$$

Furthermore,

$$I'(\sigma_{p-1}) = I'(\sigma) + X(\sigma)(1 - z_{k+1})^p AG = I'(\sigma)$$

by (2): since $X(\sigma)(1 - z_{k+1})^p = f((1 - z_{k+1})^p)$ and

$$(1 - z_{k+1})^p = (1 - z_{k+1}^p) = 0 \in A/p[G/H_k]. \quad (z_{k+1}^p \in H_k).$$

We thus have a filtration

$$I(\sigma) = I(\sigma_0) \supseteq I(\sigma_1) \supseteq \cdots \supseteq I(\sigma_{p-1}) \supseteq I'(\sigma_{p-1}) = I'(\sigma); \tag{6}$$

and $I(\sigma_i) = I'(\sigma_{i-1})$ for $1 \le i \le p-1$.

Step 2B. For shortness in notation, we now write $K_1(I)$, $K_2(I)$ for $K_1(R, I)$, $K_2(R, I)$: R is always a quotient ring of AG. We are assuming that (3) holds for the σ_i; i.e., that

$$K_2(I(\sigma_i)/I'(\sigma_i)) = \langle \{g, 1 - X(\sigma_i)\lambda h\} : [g, h] \in H_{k+1}, \lambda \in A \rangle \tag{7}$$

for each $0 \le i \le p-1$. Let $\{\lambda_1, \ldots, \lambda_s\}$ be a $\hat{\mathbb{Z}}_p$-basis for A. Let $h_1, \ldots, h_t \in G$ be conjugacy class representatives $(\mathrm{mod}\, H_{k+1})$ for those elements such that $[g_l, h_l] \in z_{k+1}H_k$ for some $g_l \in G$; fix also such g_l. Then (7) takes the form

$$K_2(I(\sigma_i)/I'(\sigma_i)) = M_i + \langle \{g_l, 1 - X(\sigma_i)\lambda_j h_l\} : 1 \le j \le s, 1 \le l \le t \rangle; \tag{8}$$

where

$$M_i = \langle \{g, 1 - \lambda X(\sigma_i)h\} : [g, h] \in H_k, \lambda \in A \rangle. \tag{9}$$

Step 2C. Now assume that $i < p-1$; and consider the relative exact sequence

$$K_2(I(\sigma_i)/I'(\sigma_{i+1})) \to K_2(I(\sigma_i)/I'(\sigma_i)) \xrightarrow{\partial} K_1(I(\sigma_{i+1})/I'(\sigma_{i+1}))$$

(recall that $I'(\sigma_i) = I(\sigma_{i+1})$). By (2) (and [24, Corollary 2.6]):

$$K_1(I(\sigma_{i+1})/I'(\sigma_{i+1})) \cong H_0(G; A/p[G/H_{k+1}]), \tag{10}$$

where G acts by conjugation. Furthermore, for $1 \le j \le s$, $1 \le 1 \le t$,

$$\begin{aligned} \partial(\{g_l, 1 - X(\sigma_i)\}\lambda_j h_l) &= [g_l, 1 - X(\sigma_i)\lambda_j h_l] = 1 - X(\sigma_i)\lambda_j(g_l h_l g_l^{-1} - h_l) \\ &= 1 + X(\sigma_i)(1 - z_{k+1})\lambda_j h_l = 1 + X(\sigma_{i+1})\lambda_j h_l \pmod{I'(\sigma_{i+1})} \end{aligned}$$

(recall that $[g_l, h_l] \in z_{k+1}H_k$). By (10), these elements are all independent in $K_1(I(\sigma_{i+1})/I'(\sigma_{i+1}))$. So by (8) and (9),

$$\begin{aligned} &\mathrm{Im}\,[K_2(I(\sigma_i)/I'(\sigma)) \to K_2(I(\sigma_i)I'(\sigma_i))] \\ &\qquad = \mathrm{Im}\,[K_2(I(\sigma_i)/I'(\sigma_{i+1})) \to K_2(I(\sigma_i)/I'(\sigma_i))] \\ &\qquad = M_i = \langle \{g, 1 - \lambda X(\sigma_i)h\} : [g, h] \in H_k, \lambda \in A \rangle: \quad (11) \end{aligned}$$

all elements in M_i lift (using (2)) to $K_2(I(\sigma_i)/I'(\sigma)) \subseteq K_2(I(\sigma)/I'(\sigma))$.

Step 2D. By (8) and (11) (and (6)),

$$K_2(I(\sigma)/I'(\sigma)) = M + \langle \{g, 1-\lambda X(\sigma)(1-z_{k+1})^{p-1}h\} : [g,h] \in z_{k+1}H_k, \lambda \in A\rangle \tag{12}$$

where

$$\begin{aligned} M &= \langle \{g, 1-\lambda X(\sigma)(1-z_{k+1})^i h\} : 0 \le i \le p-1, [g,h] \in H_k, \lambda \in A\rangle \\ &= \langle \{g, 1-\lambda X(\sigma)h\} : [g,h] \in H_k, \lambda \in A\rangle. \end{aligned}$$

(Note that $X(\sigma)^2 = 0$ in $I(\sigma)/I'(\sigma)$.) We want to show that $K_2(I(\sigma)/I'(\sigma)) = M$. Fix $\lambda \in A$ and $g, h \in G$ such that $[g,h] \in z_{k+1}H_k$, and set $u = 1 - X(\sigma)\lambda h$. Then

$$1 - X(\sigma)\lambda(1-z_{k+1})^{p-1}h = \prod_{i=0}^{p-1}(1 - X(\sigma)\lambda z_{k+1}^i h) = \prod_{i=0}^{p-1} g^i u g^{-i} \in AG/I'(\sigma)$$

by (2) ($I(\sigma)/I'(\sigma) \cong A/p[G/H_k]$). So by Lemma 1.2,

$$\begin{aligned} \{g, 1 - x(\sigma)\lambda(1-z_{k+1})^{p-1}h\} &= \{g, u \cdot gug^{-1} \cdots g^{p-1}ug^{1-p}\} \\ &= \{g^p, u\} + (p-1)\{u,u\} + \sum_{j=1}^{p-1}\{g^j u g^{-j}, u\}. \end{aligned}$$

By definition, $\{g^p, u\} \in M$. For any $0 \le j \le p-1$:

$$\begin{aligned} \{g^j u g^{-j}, u\} &= \{1 - X(\sigma)\lambda z_{k+1}^j h, 1 - X(\sigma)\lambda h\} \\ &= \left\{1 - (1-z), 1 - X(\sigma)\frac{X(\sigma)}{1-z}\lambda^2 z_{k+1}^j h^2\right\} \in M \end{aligned}$$

(see [17, Lemma 2.2] for the last step). So from (12) we now get that $K_2(I(\sigma)/I'(\sigma)) = M$; and this finishes the proof of (3).

Step 3. Now fix some $i \ge 1$. For any $0 \le r \le m-1$, (3) applied to $\sigma = (0; r, i)$ says that

$$\begin{aligned} &K_2(p^r(1-z)^i AG/\langle p^{r+1}(1-z)^i, (1-z)^{i+1}\rangle) \\ &\qquad = \langle \{g, 1 - \lambda p^r(1-z)^i h\} : [g,h] \in \langle z\rangle, \lambda \in A\rangle. \end{aligned}$$

For any such g, h, and λ, note that (in AG)

$$[g, 1-\lambda p^r(1-z)^i h] \equiv 0; 1-\lambda p^r(1-z)^i h \equiv (1-\lambda(1-z)^i h)^{p^r} \pmod{(1-z)^{i+1}AG}.$$

It follows that

$$K_2((1-z)^i AG/(1-z)^{i+1}AG) = \langle \{g, 1-\lambda(1-z)^i h\} : [g, h] \in \langle z \rangle, \lambda \in A \rangle. \quad (13)$$

Step 4. The rest of the proof is analogous to Step 2B and 2C. Let $\lambda_1, \ldots, \lambda_s$ be a $\hat{\mathbb{Z}}_p$-basis for A, and let $h_1, \ldots, h_t \in G$ be conjugacy class representatives for G/z. For $1 \le l \le t$, choose $g_1 \in G$ so that $[g_l, h_l] = z^{q_l}$, and $1 \le q_l \le p^m = |z|$ is minimal. Then by (13),

$$K_2((1-z)^i AG/(1-z)^{i+1}AG) = N_i + \langle \{g_l, 1-\lambda_j(1-z)^i h_l\} : 1 \le l \le t, 1 \le j \le s \rangle, \quad (14)$$

where

$$N_i = \langle \{g, 1-\lambda(1-z)^i h\} : [g, h] = 1, \lambda \in A \rangle.$$

Consider the exact sequence

$$K_2\left(\frac{(1-z)^i AG}{(1-z)^{i+2}AG}\right) \to K_2\left(\frac{(1-z)^i AG}{(1-z)^{i+1}AG}\right) \xrightarrow{\partial} K_1\left(\frac{(1-z)^{i+1}AG}{(1-z)^{i+2}AG}\right). \quad (15)$$

For any j, l:

$$\partial(\{g_l, 1-\lambda_j(1-z)^i h_l\}) = [g_l, 1-\lambda_j(1-z)^i h_l] = 1 + q_l\lambda_j(1-z)^{i+1}h_l.$$

By Lemma 1.3, these elements are independent in

$$K_1((1-z)^{i+1}AG/(1-z)^{i+2}AG) \cong H_0(G; A/p^m[G])$$

and have order p^m/q_l (q_l is a power of p). Furthermore, for each j and l, $[g_l^{p^m/q_l}, h_l] = 1$, and so

$$p^m/q_l \cdot \{g_l, 1-\lambda_j(1-z)^i h_l\} \in N_i.$$

So by (14), and the exactness of (15),

$$\operatorname{Im}\left[K_2\left(\frac{(1-z)^i AG}{(1-z)^{i+2}AG}\right) \to K_2\left(\frac{(1-z)^i AG}{(1-z)^{i+1}AG}\right)\right] = \operatorname{Ker}(\partial) = N_i.$$

Every element of N_i lifts to $K_2((1-z)^i AG) \subseteq K_2((1-z)AG)$. Thus, for any $i \geq 1$,

$$K_2((1-z)^i AG) = K_2((1-z)^{i+1}AG) + \langle \{g, 1-\lambda(1-z)^i h\} : gh = hg, \lambda \in A \rangle. \tag{16}$$

By induction, for any $N > 1$,

$$K_2((1-z)AG) = K_2((1-z)^N AG) + \langle \{g, 1-\lambda(1-z)^i h\} : gh = hg, \lambda \in A, 1 \leq i < N \rangle. \tag{17}$$

Let $p^k = \exp(G)$, and recall that $|z| = p^m$. Then $p(1-z) \mid (1-z)^{p^m}$, and so

$$1 + (1-z)^{(k+1)p^m} AG \subseteq 1 + p^{k+1}(1-z)AG \subseteq \{(1+(1-z)\xi)^{p^k} : \xi \in AG\}.$$

Thus, for any commuting $h, g \in G$, any $\lambda \in A$, and any $i \geq (k+1)p^m$:

$$\{g, 1-\lambda(1-z)^i h\} = \{g, (1-(1-z)\xi)^{p^k}\} = \{g^{p^k}, 1-(1-z)\xi\} = 0.$$

(some $\xi \in AG$).

By (16), for any $N > (k+1)p^m$, $K_2((1-z)^{(k+1)p^m} AG) = K_2((1-z)^N AG)$; and so

$$K_2((1-z)^{(k+1)p^m}) = \varprojlim_N K_2((1-z)^{(k+1)p^m} AG/(1-z)^N AG) = 0 \tag{18}$$

Equation (17) now takes the form

$$K_2((1-z)AG) = \langle \{g, 1-\lambda(1-z)^i h\} : gh = hg, \lambda \in A, 1 \leq i < (k+1)p^m \rangle.$$

Furthermore, it suffices to take λ belonging to some $\hat{\mathbb{Z}}_p$-basis for A. This shows that $K_2((1-z)AG)$ is generated by a finite set of elements of finite order, and is hence finite. □

With some more work, one can in fact show that $K_2(AG, (1-z)AG)$ is generated by symbols $\{g, 1-\lambda(1-z)h\}$, where $gh = hg$ in G and λ lies in any fixed $\hat{\mathbb{Z}}_p$-basis for A.

One easy consequence of Theorem 1.4 is:

THEOREM 1.5. *For any prime p, any unramified p-ring A, and any p-group G, $K_2(AG)$ is finite.*

Proof. Fix some $1 \neq z \in Z(G)$. Then $K_2(AG, (1-z)AG)$ is finite by Theorem 1.4. We may assume inductively that $K_2(A[G/z])$ is finite; and so $K_2(AG)$ is also finite. □

In fact, using the results in [17], this can be extended to arbitrary finite G. Whether it is true for arbitrary $\mathbb{Z}_p$-orders, we do not know.

Section 2

Theorem 1.4 gives a set of generators for $\operatorname{Ker}(K_2(A\alpha))$, when $\alpha: \tilde{G} \twoheadrightarrow G$ is a central extension of p-groups with cyclic kernel. In this section, we study $\operatorname{Coker}(K_2(A\alpha))$ when $\operatorname{Ker}(\alpha) \subseteq Z(\tilde{G})$. This problem was studied in [19]: $\operatorname{Coker}(K_2(A\alpha))$ is described there for an arbitrary surjection α, but only up to a mysterious contribution by $H_3(G)$. What we show here is that the $H_3(G)$ contribution vanishes when α is a central extension.

PROPOSITION 2.1. *Let p be any prime, let A be an umramified p-ring, and let $\alpha: \tilde{G} \twoheadrightarrow G$ be any central extension of p-groups (i.e.,* $\operatorname{Ker}(\alpha) \subseteq Z(G)$*). Then there is an exact sequence*

$$0 \to \operatorname{Coker}(H_2(\alpha)) \xrightarrow{T_\alpha} \operatorname{Coker}[K_2(A\alpha): K_2(A\tilde{G}) \to K_2(AG)]$$
$$\xrightarrow{\Gamma_2^*(\alpha)} H_1(G; AG)/\langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1\rangle.$$

Here, T_α is included by the usual inclusion $H_2(G) \to K_2(AG)/\{-1, G\}$, and $\Gamma_2^*(\alpha)$ is induced by the homomorphism

$$\Gamma_2^*(G): K_2(AG) \to H_1(G; AG)/\langle g \otimes \lambda g^n : g \in G, \lambda \in A, n \in \mathbb{Z}\rangle$$

of [19, *Theorem* 3.6]. *In particular, for any $g \in G$, $H = Z_G(g)$, and any*

$u \in (AH)^*$,

$$\Gamma_2^*(\alpha)(\{g, u\}) = g \otimes \Gamma_H(u) \in H_1(G; AG)/\langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1\rangle.$$

Proof. Define the group $\hat{G}$ and the order $\mathfrak{A}$ to be the pullbacks:

$$\begin{array}{ccc} \hat{G} & \xrightarrow{r_1} & \tilde{G} \\ \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle \alpha} \\ \tilde{G} & \xrightarrow{\alpha} & G \end{array} \qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{\hat{r}_1} & A\tilde{G} \\ \downarrow{\scriptstyle \hat{r}_2} & & \downarrow{\scriptstyle A\alpha} \\ A\tilde{G} & \xrightarrow{A\alpha} & AG. \end{array}$$

Set

$$I_1 = \operatorname{Ker}[A\hat{G} \xrightarrow{Ar_1} A\tilde{G}], \qquad I_2 = \operatorname{Ker}[A\hat{G} \xrightarrow{Ar_2} A\tilde{G}], \qquad I = \operatorname{Ker}[A\tilde{G} \xrightarrow{A\alpha} AG].$$

Then $\mathfrak{A} \cong A\hat{G}/(I_1 \cap I_2)$; and so by Lemma 2.4 in [16],

$$\mathfrak{A} \cong A\hat{G}/I_1I_2.$$

Step 1. By [26, Theorem 4.1],

$$\operatorname{tors}(K_1(A\hat{G})) \cong \mu_A \times \hat{G}^{ab} \times SK_1(A\hat{G}); \tag{1}$$

where μ_A denotes the group of roots of unity in A. We first claim that

$$\hat{G}^{ab} \rightarrowtail K_1(A\hat{G}/I_1I_2) \cong K_1(\mathfrak{A}) \tag{2}$$

is injective. To see this, let $I(A\hat{G})$ denote the augmentation ideal of $A\hat{G}$. Then $I(A\hat{G})^2 \supseteq I_1I_2$, and by [19, Proposition 2.2]:

$$A\hat{G}/I(A\hat{G})^2 \cong A \times (A \otimes \hat{G}^{ab}).$$

The isomorphism identifies $g \in \hat{G}^{ab}$ with $(1, 1 \otimes g)$, and so $\hat{G}^{ab} \subseteq K_1(A\hat{G}/I(A\hat{G})^2)$.

Now set $K = \operatorname{Ker}(\alpha) \cong \operatorname{Ker}(r_1)$, and consider the following diagram:

$$\begin{array}{ccccccccc} H_2(\hat{G}) & \longrightarrow & H_2(\tilde{G}) & \xrightarrow{\delta^{r_1}} & K & \longrightarrow & \hat{G}^{ab} & \xrightarrow{H_1(r_1)} & G^{ab} \longrightarrow 0 \\ \downarrow & & \downarrow{\scriptstyle H_2(\alpha)} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle H_1(r_2)} & & \downarrow \\ H_2(\tilde{G}) & \xrightarrow{H_2(\alpha)} & H_2(G) & \xrightarrow{\delta^{\alpha}} & K & \longrightarrow & \tilde{G}^{ab} & \longrightarrow & G^{ab} \longrightarrow 0. \end{array}$$

The rows are the five-term exact sequences for the extensions $r_1 : \hat{G} \twoheadrightarrow \tilde{G}$ and

$\alpha: \tilde{G} \twoheadrightarrow G$ (see [8, Corollary VI. 8.2]). It follows that

$$\operatorname{Ker}[H_1(r_1 \times r_2): \hat{G}^{ab} \to \tilde{G}^{ab} \times \tilde{G}^{ab}] \cong \operatorname{Coker}(H_2(\alpha)). \tag{3}$$

Furthermore, $\delta^{r_1} = \delta^{\alpha} \circ H_2(\alpha) = 0$, so $\operatorname{Ker}(r_1) \cap [\hat{G}, \hat{G}] = 1$, and

$$SK_1(Ar_1): SK_1(A\hat{G}) \to SK_1(A\tilde{G}) \tag{4}$$

is injective by [15, Proposition 7].

Step 2. Now define

$$\Gamma_{AG}: K_1(A\hat{G}) \to H_0(\hat{G}; A\hat{G}); \qquad \Gamma_{A\tilde{G}}: K_1(A\tilde{G}) \to H_0(\tilde{G}; A\tilde{G})$$

as in [20, Theorem 2.7], and recall that they are isomorphisms modulo torsion. By Theorem 1.1 in [19],

$$\Gamma_{A\hat{G}}(1 + I_1 I_2) = \operatorname{Im}[I_1 I_2 \to H_0(\hat{G}; A\hat{G})]. \tag{5}$$

So $\Gamma_{A\hat{G}}$ induces a homomorphism

$$\Gamma_{\mathfrak{A}}: K_1(\mathfrak{A}) \to H_0(\hat{G}; \mathfrak{A}).$$

Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mu_A \times \hat{G}^{ab} \times SK_1(A\hat{G}) \arrow[r] \arrow[d, "f"] & K_1(\mathfrak{A}) \arrow[r, "\Gamma_{\mathfrak{A}}"] \arrow[d, "K_1(r_1 \times r_2)"] & H_0(\hat{G}; \mathfrak{A}) \arrow[d, "H_0(r_1 \times r_2)"] \\
0 \arrow[r] & {[\mu_A \times \tilde{G}^{ab} \times SK_1(A\tilde{G})]^2} \arrow[r] & {[K_1(A\tilde{G})]^2} \arrow[r, "\Gamma_{A\tilde{G}}"] & {[H_0(\tilde{G}; A\tilde{G})]^2}
\end{tikzcd} \tag{6}$$

The bottom row is exact since $\operatorname{Ker}(\Gamma_{A\tilde{G}}) = \operatorname{tors}(K_1(A\tilde{G}))$. The top row is exact at $K_1(\mathfrak{A})$ since by (5), $\operatorname{Ker}(\Gamma_{A\hat{G}}) \to \operatorname{Ker}(\Gamma_{\mathfrak{A}})$ is onto. By (3) and (4), $\hat{G}^{ab} \supseteq \operatorname{Ker}(f) \cong \operatorname{Coker}(H_2(\alpha))$, and this injects into $K_1(\mathfrak{A})$ by (2). So the top row in (6) is exact, and there is an exact sequence

$$0 \to \operatorname{Coker}(H_2(\alpha)) \to \operatorname{Ker}(K_1(r_1 \times r_2)) \to \operatorname{Ker}(H_0(r_1 \times r_2)). \tag{7}$$

By the Mayer–Victoris sequence for a pullback square,

$$\operatorname{Ker}(K_1(r_1 \times r_2)) \cong \operatorname{Coker}[K_2(A\alpha): K_2(A\tilde{G}) \to K_2(AG)]. \tag{8}$$

Step 3. The extension $0 \to I \to \mathfrak{A} \xrightarrow{r_1} AG \to 0$ is $\hat{G}$-equivariantly split by the diagonal map. Thus,

$$\mathrm{Ker}\,[\hat{r}_{1*} : H_0(\hat{G}; \mathfrak{A}) \to H_0(\tilde{G}; A\tilde{G})] \cong H_0(\tilde{G}; I);$$

and so

$$\begin{aligned} \mathrm{Ker}\,(H_0(r_1 \times r_2)) &\cong \mathrm{Ker}\,[H_0(\tilde{G}; I) \to H_0(\tilde{G}; A\tilde{G})] \\ &\cong \mathrm{Coker}\,[H_1(\tilde{G}; A\tilde{G}) \to H_1(\tilde{G}; AG)] \\ &\cong H_1(G; AG)/\langle g \otimes \lambda h : \lambda \in A,\, [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle. \end{aligned} \tag{9}$$

Upon substituting (8) and (9) into (7), we get the exact sequence

$$0 \to \mathrm{Coker}\,(H_2(\alpha)) \xrightarrow{T_\alpha} \mathrm{Coker}\,(K_2(A\alpha)) \xrightarrow{\Gamma_2^*(\alpha)} H_1(G; AG)/ \\ \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle.$$

That $\Gamma_2^*(\alpha)$ is the reduction of the map $\Gamma_2^*(G)$ of [19] follows since the constructions are identical. By diagram chasing, T_α is seen to be the reduction of the standard inclusion $H_2(G) \to K_2(AG)/\{-1, G\}$. □

In fact, in the above situation, $\mathrm{Im}\,(\Gamma_2^*(\alpha))$ can be described precisely with the help of Theorem 3.6 in [19].

Proposition 2.1 will be applied directly in Section 3, when describing $Cl_1(\mathbb{Z}G)$ for odd p-groups G. But we first note one consequence of particular interest. The next theorem is useful when constructing maps

$$\Gamma_2 : K_2(AG) \to H_1(G; AG)/\langle g \otimes \lambda g \rangle$$

for non-abelian p-groups G (compare with [21]).

THEOREM 2.2. *Let $\alpha : \tilde{G} \twoheadrightarrow G$ be any surjection of p-groups such that* $\mathrm{Ker}\,(\alpha) \cap [\tilde{G}, \tilde{G}] = 1$. *Then for any unramified p-ring A, the map*

$$K_2(A\alpha) : K_2(A\tilde{G}) \to K_2(AG)$$

is onto, and its kernel is generated by elements of the form $\{g, 1 + (1 - z)^i h\}$ for $z \in \mathrm{Ker}\,(\alpha)$, $i \geq 1$, and commuting $g, h \in G$.

Proof. Note first that

$$[\mathrm{Ker}\,(\alpha), \tilde{G}] \subseteq \mathrm{Ker}\,(\alpha) \cap [\tilde{G}, \tilde{G}] = 1;$$

so that $\mathrm{Ker}\,(\alpha) \subseteq Z(\tilde{G})$. The exact sequence

$$H_2(\tilde{G}) \xrightarrow{H_2(\alpha)} H_2(G) \xrightarrow{\delta^\alpha} \mathrm{Ker}\,(\alpha) \rightarrowtail \tilde{G}^{ab} \to G^{ab} \to 0$$

(see [8, Corollary VI. 8.2]) shows that $H_2(\alpha)$ is onto. By hypothesis,

$$H_1(G; AG)/\langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1\rangle = 0:$$

commuting elements in G lift to commuting elements in $\tilde{G}$. So $K_2(A\alpha)$ is onto by Proposition 2.1.

Now write α as a composite

$$\alpha : \tilde{G} = G_0 \xrightarrow{\alpha_1} G_1 \xrightarrow{\alpha_2} G_2 \twoheadrightarrow \cdots \xrightarrow{\alpha_n} G_n = G;$$

and so that $\mathrm{Ker}\,(\alpha_j)$ is cyclic for all j. By Theorem 1.4,

$$\mathrm{Ker}\,(K_2(A\alpha_j)) = \langle \{g, 1 + (1-z)^i h\} : z \in \mathrm{Ker}\,(\alpha_j),\ i \geq 1,\ g, h \in G_{j-1},\ gh = hg\rangle$$

for each j. But all such symbols lift to $K_2(A\tilde{G})$; and so $\mathrm{Ker}\,(K_2(A\alpha))$ is generated as described. □

Section 3

We can now derive algorithms for computing the groups $Cl_1(\mathbb{Z}G)[\frac{1}{2}]$ and $SK_1(\mathbb{Z}G)[\frac{1}{2}]$ for finite G. The key extra tool when working with odd torsion is the standard involution on $K_n(\mathbb{Z}G)$ and $K_n(\hat{\mathbb{Z}}_pG)$; for example, this is what was used in [17] to construct natural splittings

$$SK_1(\mathbb{Z}G)[\tfrac{1}{2}] \cong Cl_1(\mathbb{Z}g)[\tfrac{1}{2}] \oplus \sum_{2<p\,|\,|G|} SK_1(\hat{\mathbb{Z}}_pG).$$

Recall that for any group G and any commutative ring R, an antiinvolution $x \to \bar{x}$ on RG is defined by setting

$$\overline{\sum r_i g_i} = \sum r_i g_i^{-1} \quad (r_i \in R,\ g_i \in G).$$

This extends to an antiinvolution on $GL(RG)$ – defined by setting $\overline{(a_{ij})} = (\bar{a}_{ji})$ – and hence an involution on $K_1(RG)$. Similarly, an antiinvolution on $St(RG)$ is induced by setting $\overline{x_{ij}(a)} = x_{ji}(\bar{a})$ $(a \in RG)$; and this restricts to an involution on $K_2(RG)$.

LEMMA 3.1. *For any group ring RG as above, and any commuting units, u,* $v \in (RG)^*$, $\overline{\{u, v\}} = \{\bar{v}, \bar{u}\}$. *In particular, for any* $g \in G$, *and* $u \in (RG)^*$ *such that* $gu = ug$, $\overline{\{g, u\}} = \{g, \bar{u}\}$.

Proof. Recall that $\{u, v\} = [X, Y]$, where $X, Y \in St(RG)$ are arbitrary liftings of $\operatorname{diag}(u, u^{-1}, 1)$ and $\operatorname{diag}(v, 1, v^{-1})$. Then

$$\overline{\{u, v\}} = \overline{[X, Y]} = \bar{Y}^{-1}\bar{X}^{-1}\bar{Y}\bar{X} = \{\bar{v}^{-1}, \bar{u}^{-1}\} = \{\bar{v}, \bar{u}\}.$$

The last statement follows since $\bar{g} = g^{-1}$. □

The importance of the involution for simplifying the computation of $Cl_1(\mathbb{Z}G)$ follows from:

LEMMA 3.2. *For any odd prime p and any p-group G, the involution on* $K_2(\hat{\mathbb{Q}}_p[G])_{(p)}$ *is the identity.*

Proof. By [22, Section 2 and 3], for any p-group G and any irreducible $\mathbb{Q}G$-module V, there are subgroups $K \lhd H \subseteq G$ and a faithful $\mathbb{Q}[H/K]$-representation W such that $V = \operatorname{Ind}_H^G(W)$, $\operatorname{End}_{\mathbb{Q}H}(W) \cong \operatorname{End}_{\mathbb{Q}G}(V)$, and H/K is cyclic. Let $A \subseteq \mathbb{Q}H$ and $B \subseteq \mathbb{Q}G$ denote the corresponding simple summands. Then the induction map restricts to a Morita equivalence from A to B, and hence induces an isomorphism of $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} A)$ to $K_2(\mathbb{Q}_p \otimes_{\mathbb{Q}} B)$. Thus, if

$$S = \{(H, K) : K \lhd H \subseteq G, H/K \text{ cyclic}\},$$

then the map

$$\sum \operatorname{Ind}_{H/K}^G : \sum_{(H,K)\in S} K_2(\hat{\mathbb{Q}}_p[H/K]) \twoheadrightarrow \hat{\mathbb{Q}}_p[G] \tag{1}$$

is onto. Here, $\operatorname{Ing}_{H/K}^G$ is the composite

$$\operatorname{Ind}_{H/K}^G : K_2(\hat{\mathbb{Q}}_p[H/K]) \xrightarrow{\text{incl}} K_2(\hat{\mathbb{Q}}_p H) \xrightarrow{\operatorname{Ind}_H^G} K_2(\hat{\mathbb{Q}}_p G);$$

where the first map is induced by the inclusion of $\hat{\mathbb{Q}}_p[H/K]$ as a direct summand of $\hat{\mathbb{Q}}_pH$.

The $\mathrm{Ind}^G_{K/H}$ commute with the involution, and so by (1) it suffices to prove the lemma when G is cyclic. If $G \cong C_{p^n}$, write $\hat{\mathbb{Q}}_pG \cong \prod_{i=0}^n F_i$, where $F_i \cong \hat{\mathbb{Q}}_p[\zeta_{p^i}]$ (a field). For each i, the involution inverts elements in μ_{F_i}. So from the isomorphism $K_2(F_i) \cong \mu_{F_i}$ and its naturality with respect to automorphisms of F_i, we get that $\{\bar{u}, \bar{v}\} = -\{u, v\} = \{v, u\}$ for $u, v \in F_i^*$. But $\{\bar{n}, \bar{v}\} = \overline{\{v, u\}}$ by Lemma 3.1, and so the involution on $K_2(F_i)$, and hence on $K_2(\hat{\mathbb{Q}}_pG)$, is trivial. □

In fact, Lemma 3.2 also holds for 2-groups, and for arbitrary finite G if $K_2(\hat{\mathbb{Q}}_pG)_{(p)}$ is replaced by $C_p(\mathbb{Q}G)$ (see the definition in the introduction).

The main problem when describing $Cl_1(\mathbb{Z}G)$ for a p-group G is computing the image of $K_2(\hat{\mathbb{Z}}_pG)$ in $K_2(\hat{\mathbb{Q}}_pG)$. Lemma 3.2 shows that when p is odd, it is enough to concentrate attention on $K_2(\hat{\mathbb{Z}}_pG)^+$; and (recall the formula $\overline{\{g, u\}} = \{g, \bar{u}\}$) on $K_1(\hat{\mathbb{Z}}_pG)^+$.

PROPOSITION 3.3. *For any odd prime p, any unramified p-ring A, and any p-group G, Γ_{AG} restricts to an isomorphism*

$$\Gamma^+_{AG}: K_1(AG)^+ \to H_0(G; AG)^+.$$

Proof. By [20, Theorem 2.7], there is an exact sequence

$$0 \to G^{ab} \times SK_1(AG) \to K_1(AG) \xrightarrow{\Gamma_{AG}} H_0(G; AG) \xrightarrow{\omega} G^{ab} \to 0 \tag{1}$$

where $\omega(\sum \lambda_i g_i) = \prod g_i^{\mathrm{Tr}(\lambda_i)}$. These maps all commute with the involution; and $(G^{ab})^+ = 0$ by definition. That $SK_1(AG)^+ = 0$ follows from the definition of the isomorphism

$$\Theta_{AG}: SK_1(AG) \to H_2(G)/H_2^{ab}(G)$$

in [15, diagram on p. 215]. So (1) restricts to an isomorphism

$$\Gamma^+_{AG}: K_1(AG)^+ \to H_0(G; AG)^+. \quad \square$$

If A is an unramified p-ring, and G is an *abelian* p-group, we can now define for any $\lambda \in A$ and $g \in G$ a unit $u(\lambda g) \in (AG)^{*+} \cong K_1(AG)^+$ to be the unique element such that $\Gamma^+_G(u(\lambda g)) = \frac{1}{2}\lambda(g + g^{-1})$. If G is an arbitrary p-group and $g \in G$, we let $u(\lambda g) \in (AG)^*$ be the image of $u(\lambda g) \in (AH)^*$, when $H = \langle g \rangle$.

The results of Sections 1 and 2 can now be used to describe $K_2(\hat{\mathbb{Z}}_pG)^+$:

PROPOSITION 3.4. *For any odd prime p, any unramified p-ring A, and any p-group G,*

$$K_2(AG)^+ = \langle \{g, u(\lambda h)\} : \lambda \in A,\, g, h \in G,\, [g, h] = 1 \rangle.$$

Proof. For any G, define an involution on $H_1(G; AG)$ by setting $\overline{g \otimes \lambda h} = g \otimes \lambda h^{-1}$. Define

$$\Delta_G^+ : H_1(G; AG)^+ \to K_2(AG)^+$$

by setting $\Delta_G^+(g \otimes \tfrac{1}{2}\lambda(h + h^{-1})) = \{g, u(\lambda h)\}$ for any $\lambda \in A$ and commuting g, $h \in G$.

Fix some G, choose $z \in Z(G)$ of order p, set $H = G/z$, and let $\alpha : G \twoheadrightarrow H$ be the projection. Assume inductively that Δ_H^+ is surjective, and consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(H_1(A\alpha))^+ \arrow[r] \arrow[d, "f_1"] & H_1(G; AG)^+ \arrow[r, "H_1(A\alpha)"] \arrow[d, "\Delta_G^+"] & H_1(H; AH)^+ \arrow[r] \arrow[d, "\Delta_H^+"] & \operatorname{Coker}(H_1(A\alpha))^+ \arrow[r] \arrow[d, shift left, "f_2"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(K_2(A\alpha))^+ \arrow[r] & K_2(AG)^+ \arrow[r, "K_2(A\alpha)"] & K_2(AH)^+ \arrow[r] & \operatorname{Coker}(K_2(A\alpha))^+ \arrow[r] \arrow[u, dashed, shift left, "\Gamma_2^+"] & 0
\end{tikzcd} \tag{1}$$

Here, f_1 and f_2 are induced by Δ_G^+ and Δ_H^+, and Γ_2^+ is the restriction of the homomorphism of Proposition 2.1. For any $\lambda \in A$ and commuting g, $h \in G$,

$$\Gamma_2^+ \circ f_2(g \otimes \tfrac{1}{2}\lambda(h + h^{-1})) = \Gamma_2^+(\{g, u(h)\}) = g \otimes \Gamma_{AG}(u(h)) = g \otimes \tfrac{1}{2}\lambda(h + h^{-1});$$

and so f_2 is injective. By Theorem 1.4,

$$\operatorname{Ker}(K_2(A\alpha))^+ = \langle \{g, (1 - \lambda(1 - z)^i h)(1 - \lambda(1 - z^{-1})^i h^{-1})\} : \lambda \in A,\, i \geq 1,\, gh = hg \rangle;$$

and so by Proposition 3.3 (applied to the $K_1(A[Z_G(g)])^+$):

$$\operatorname{Ker}(K_2(A\alpha))^+ \subseteq \langle \{g, u(\lambda h)\} : \lambda \in A,\, [g, h] = 1 \rangle = \operatorname{Im}(\Delta_G^+).$$

By diagram chasing in (1), Δ_G^+ is now seen to be onto. $\square$

It seems quite likely that the homomorphisms Δ_G^+ defined above actually induce isomorphisms

$$HC_1(AG)^+ \cong [H_1(G;AG)/\langle g \otimes \lambda g \rangle]^+ \cong K_2(AG)^+.$$

This is the case at least for abelian p-groups [21, Theorem 3.9].

It remains only to find a description of the image of any $\{g, u(h)\}$ in $K_2(\hat{\mathbb{Q}}_p G)$, when $p > 2$ and G is a p-group. Recall that $K_2(\hat{\mathbb{Q}}_p G)$ is described in terms of norm residue symbol isomorphisms

$$(\, ,\,)_F : K_2(F) \xrightarrow{\cong} \mu_F$$

defined for any finite extension F of $\hat{\mathbb{Q}}_p$ [12, Theorem A.14].

LEMMA 3.5. *Fix an odd prime p and a p-group G; and let $u(g) \in (\hat{\mathbb{Z}}_p G)^*$ for $g \in G$ be defined as above. Write*

$$\hat{\mathbb{Q}}_p G = \prod_{i=1}^{k} B_i; \qquad B_i = M_{r_i}(F_i),$$

where for each i, $F_i \cong \hat{\mathbb{Q}}_p \zeta_{p^m}$ (a field) for some $m \geq 0$ (see [22]). Let

$$\lambda_G : K_2(\hat{\mathbb{Q}}_p G) \to \prod_{i=1}^{k} (\mu_{F_i})_p$$

be the product of the norm residue symbol homomorphisms

$$\lambda_G^i : K_2(B_i) \cong K_2(M_{r_i}(F_i)) \cong K_2(F_i) \xrightarrow{(\, ,\,)} (\mu_{F_i})_p.$$

For each i, let V_i be the irreducible B_i-representation. Then, for any commuting g, $h \in G$,

$$\lambda_G(\{g, u(h)\}) = [\det_{F_i}(g, V_i^h)]_{i=1}^k. \quad (V_i^h = \{x \in V_i : hx = x\}).$$

Proof. Fix some i, set $B = B_i$, $V = V_i$, $F = F_i$, $r = r_i$; and let

$$\alpha : \hat{\mathbb{Q}}_p G \twoheadrightarrow B \cong \mathrm{End}_F(V) \cong M_r(F)$$

be the projection. Let m be such that $F \cong \hat{\mathbb{Q}}_p\zeta_{p^m}$. Set $p^m = \exp(G)$, and let

$$f : B \cong M_r(F) \to M_r(\hat{\mathbb{Q}}_p\zeta_{p^n})$$

be an inclusion. Note that taking norm residue symbols commutes (p is odd) with inclusions of cyclotomic fields: this follows, for example, from the formulas in [2].

Fix commuting $g, h \in G$. Then $\langle g, h\rangle$ is an abelian group of exponent dividing p^n; and so $f\alpha(g)$ and $f\alpha(h)$ are conjugate (simultaneously) to diagonal matrices:

$$f\alpha(g) \sim \operatorname{diag}(u_1, \ldots, u_r), f\alpha(h) \sim \operatorname{diag}(v_1, \ldots, v_r) \quad (u_l, v_l \in \langle \zeta_{p^n}\rangle).$$

with

$$u(h) = \sum_j \lambda_j h^j; \quad (\lambda_j \in \hat{\mathbb{Z}}_p)$$

so that

$$K_2(f\alpha)(\{g, u(h)\}) = \prod_{l=1}^{r} \left\{u_l, \sum_j \lambda_j v_l^j\right\}.$$

By the formulas of Artin and Hasse [2],

$$\lambda_G^i(\{g, u(h)\}) = \prod_{l=1}^{r} \left(u_l, \sum_j \lambda_j v_l^j\right)_F = \prod_{l=1}^{r} u_l^{N_l};$$

where

$$N_l = \frac{1}{p^n}\operatorname{Tr}\left(\log\left(\sum_j \lambda_j v_l^j\right)\right). \quad (\operatorname{Tr} : \hat{\mathbb{Q}}_p\zeta_{p^n} \to \hat{\mathbb{Q}}_p)$$

Recall that $\Gamma_G(u(h)) = \frac{1}{2}(h + h^{-1})$, where $\Gamma_G = (1 - (1/p)\Phi) \circ \log$, and $\Phi(\sum \lambda_i g_i) = \sum \lambda_i g_i^p$. Thus,

$$\begin{aligned}\log(u(h)) &= \left(1 - \frac{1}{p}\Phi\right)^{-1}\left(\tfrac{1}{2}(h + h^{-1})\right) \\ &= \frac{p}{p-1} + \frac{1}{2}\left[(h + h^{-1} - 2) + \frac{1}{p}(h^p + h^{-p} - 2) + \cdots\right].\end{aligned}$$

Hence, for $1 \leq l \leq r$,

$$N_l = \frac{1}{p^n} \operatorname{Tr}\left(\frac{p}{p-1} + \frac{1}{2}\left[(v_l + v_l^{-1} - 2) + \frac{1}{p}(v_l^p + v_l^{-p} - 2) + \cdots\right]\right)$$
$$= \begin{cases} 1 & \text{if } v_l = 1 \\ 0 & \text{if } v_l \neq 1. \end{cases} \quad (v_l \in \langle \zeta_{p^n} \rangle)$$

It follows that

$$\lambda_G^i(\{g, u(h)\}) = \prod_{v_l = 1} u_l = \det_F(g, V^h). \quad \square$$

The main result can now be shown.

THEOREM 3.6. *Let p be an odd prime, and let G be a p-group. Write $\mathbb{Q}G = \prod_{i=1}^k B_i$, where each B_i is a matrix algebra over a field F_i with irreducible representation V_i. Define*

$$\psi_G : H_1(G; \mathbb{Z}G) \to \prod_{i=1}^k (\mu_{F_i})_p$$

by setting, for any commuting $g, h \in G$,

$$\psi_G(g \otimes h) = [\det_{F_i}(g, V_i^h)]_{i=1}^k.$$

Then $Cl_1(\mathbb{Z}G) \cong \operatorname{Coker}(\psi_G)$ and

$$SK_1(\mathbb{Z}G) \cong \operatorname{Coker}(\psi_G) \oplus (H_2(G)/H_2^{ab}(G)).$$

More precisely, there is a commutative square

$$\begin{array}{ccc} K_2(\hat{\mathbb{Q}}_p G)_{(p)} & \xrightarrow[\cong]{\lambda_G} & \prod_{i=1}^k (\mu_{F_i})_p \\ \partial \downarrow & & \downarrow \text{proj} \\ Cl_1(\mathbb{Z}G) & \xrightarrow[\cong]{\Lambda_G} & \operatorname{Coker}(\psi_G); \end{array}$$

where λ_G is induced by the norm residue symbol, and ∂ is the boundary map in the localization sequence.

Proof. By [20, Theorem 2.1 and 2.2], there is an exact sequence

$$K_2(\hat{\mathbb{Z}}_pG) \xrightarrow{\varphi_G} \operatorname{Coker}\left[K_2\left(\mathbb{Z}\left[\frac{1}{p}\right][G]\right) \to K_2(\hat{\mathbb{Q}}_pG)\right] \xrightarrow{\partial} Cl_1(\mathbb{Z}G) \to 0$$

and an isomorphism

$$\operatorname{Coker}\left[K_2\left(\mathbb{Z}\left[\frac{1}{p}\right][G]\right) \to K_2(\hat{\mathbb{Q}}_pG)\right] \cong K_2(\hat{\mathbb{Q}}_pG)_{(p)} \xrightarrow[\cong]{\lambda_G} \prod_{i=1}^{k} (\mu_{F_i})_p.$$

(note that $\mathbb{Z}[1/p][G]$ is a maximal order). Consider the diagram

$$\begin{tikzcd}
H_1(G;\mathbb{Z}G)^+ \arrow[r, "\psi_G^+"] \arrow[d, "\Delta_G^+"] \arrow[dr, phantom, "(1)"] & \prod_{i=1}^{k} (\mu_{F_i})_p \arrow[r, "\mathrm{proj}"] \arrow[dr, phantom, "(2)"] & \operatorname{Coker}(\psi_G) \arrow[r] & 0 \\
K_2(\hat{\mathbb{Z}}_pG)^+ \arrow[r, "\varphi_G^+"] & K_2(\hat{\mathbb{Q}}_pG)_{(p)} \arrow[r, "\partial"] \arrow[u, "\lambda_G", "\cong"'] & Cl_1(\mathbb{Z}G) \arrow[r] \arrow[u, dashed, "\Lambda_G"] & 0.
\end{tikzcd}$$

By Lemma 3.2, $\operatorname{Im}(\varphi_G^+) = \operatorname{Im}(\varphi_G)$; and $\operatorname{Im}(\psi_G^+) = \operatorname{Im}(\psi_G)$ since $\psi_G(g \otimes h) = \psi_G(g \otimes h^{-1})$ by definition. So the rows above are exact. The map Δ_G^+, defined by setting $\Delta_G^+(g \otimes h) = \{g, u(h)\}$, is onto by Proposition 3.4, and (1) commutes by Lemma 3.5. So there is a unique isomorphism

$$\Lambda_G : Cl_1(\mathbb{Z}G) \xrightarrow{\cong} \operatorname{Coker}(\psi_G)$$

which makes (2) commute.

The exact sequence

$$0 \to Cl_1(\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \to SK_1(\hat{\mathbb{Z}}_pG) \to 0$$

is naturally split by [17, Theorem 4.8], and

$$SK_1(\hat{\mathbb{Z}}_pG) \cong H_2(G)/H_2^{ab}(G) \cong H_2(G)/\langle g \wedge h : g, h \in G, gh = hg \rangle$$

by [15, Theorem 3]. So

$$SK_1(\mathbb{Z}G) \cong \operatorname{Coker}(\psi_G) \oplus (H_2(G)/H_2^{ab}(G)). \quad \square$$

In [17, Theorem 4.8], the computation of $Cl_1(\mathbb{Z}G)_{(p)}$ for odd p and arbitrary finite G was reduced to the case of a p-group. More precisely, if $C_1, \ldots, C_k$ are conjugacy class representatives for cyclic subgroups in G of order prime to p, and $N_i = N_G(C_i)$, $Z_i = Z_G(C_i)$, and $\mathfrak{P}(Z_i)$ is the set of p-subgroups, then

$$Cl_1(\mathbb{Z}G)_{(p)} \cong \sum_{i=1}^{k} H_0\Big(N_i/Z_i; \varinjlim_{H\in\mathfrak{P}(Z_i)} Cl_1(\mathbb{Z}H)\Big). \tag{3.7}$$

Here, the limits are taken with respect to inclusion and conjugation among subgroups.

This direct sum decomposition is somewhat awkward, and hence a more direct description of $Cl_1(\mathbb{Z}G)_{(p)}$ seems also desirable. In fact, one can define homomorphisms

$$\psi_G : H_1(G; \mathbb{Z}G) \to \prod_{i=1}^{k} (\mu_{F_i}), \qquad \Big(\mathbb{Q}G \cong \prod_{i=1}^{k} B_i,\ F_i = Z(B_i)\Big)$$

for arbitrary finite G, such that $Cl_1(\mathbb{Z}G)[\frac{1}{2}] \cong \operatorname{Coker}(\psi_G)[\frac{1}{2}]$. But alone the definition of ψ_G become quite complicated as soon as we start working with non-p-groups; and the most efficient way of describing $Cl_1(\mathbb{Z}G)[\frac{1}{2}]$ for concrete G does seem to be by means of (3.7) above, together with Theorem 3.6. Some techniques for calculating with the help of (3.7) are presented in [17, Section 5].

Section 4

Theorem 3.6 reduces the calculation of $Cl_1(\mathbb{Z}G)$, for an odd order p-group G, to a straightforward combinatorial algorithm. We now give some examples to illustrate how this works in practice. Examples of calculations for abelian G are presented in [1]; and for non-abelian G of order p^3, $Cl_1(\mathbb{Z}G)$ is calculated in [19, Theorem 7.5] using a weaker form of the theorem. So here we take some non-abelian groups of order p^4 to give a sample of some of the techniques which can be used. Throughout this section, p denotes a fixed odd prime.

Note first that for any p-group G and any commuting g, $h \in G$,

$$\psi_G(g \otimes g) = 0; \qquad \psi_G(g \otimes h^n) = \psi_G(g \otimes h) \quad (\text{if } p \nmid n);$$

and

$$\psi_G(aga^{-1} \otimes aha^{-1}) = \psi_G(g \otimes h) \quad (\text{any } a \in G).$$

Thus, when describing $\mathrm{Im}(\psi_G)$, it suffices to consider $\psi_G(g \otimes h)$ as h runs through a set of $\mathbb{Q}$-conjugacy class representatives in G, and g a set of generators for $Z_G(h)/h$.

An irreducible representation V of G will be described by listing eigenvalues for the actions of various group elements on V – or, when necessary, by describing the irreducible components of $V \mid H$ for some appropriate $H \subseteq G$.

Finally, note that when $|G| = p^4$, then $SK_1(\hat{\mathbb{Z}}_p G) = 0$ by [15, Proposition 23]. So $SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G)$ in this case.

PROPOSITION 4.1. *Assume* $G \cong H \times C_p$, *where* H *is non-abelian,* $|H| = p^3$, *and* $\exp(H) = p$. *Then*

$$SK_1(\mathbb{Z}G) \cong (\mathbb{Z}/p)^{(p^2+3p-6)/2}.$$

Proof. Fix generators a, $b \in H$ and $c \in C_p$; and set $z = [a, b]$. Then $Z(G) = \langle z, c\rangle$, and for any $g \in G \setminus Z(G)$, $Z_G(g) = \langle Z(G), g\rangle$. Set $\zeta = \zeta_p$, and note that

$$\mathbb{Q}[G] \cong \mathbb{Q} \times \prod^{p^2+p+1} Q[\zeta] \times \prod^{p} M_p(\mathbb{Q}[\zeta]).$$

The following table describes $\psi = \psi_G$. Here, $(H^{ab})^*$ denotes the set of irreducible complex characters of H^{ab}, and $(*)$ for eigenvalues means that all powers of ζ occur.

Representation Indexed by E'val (a, b, c, z)	$U \cong \mathbb{Q}\zeta$ — $(\zeta, 1, 1, 1)$	$V_m \cong \mathbb{Q}\zeta$ $0 \le m < p$ $(\zeta^m, \zeta, 1, 1)$	$W_\chi \cong \mathbb{Q}\zeta$ $\chi \in (H^{ab})^*$ $(\chi(a), \chi(b), \zeta, 1)$	$X_m \cong (\mathbb{Q}\zeta)^p$ $0 \le m < p$ $(*, *, \zeta^m, \zeta)$
$\psi(a \otimes cz^{-i})$	ζ	ζ^m	1	1
$\psi(b \otimes cz^{-i})$	1	ζ	1	1
$\psi(a \otimes (1-c))$	1	1	$\chi(a)$	1
$\psi(b \otimes (1-c))$	1	1	$\chi(b)$	1
$\psi(c \otimes 1)$	1	1	ζ	1
$\psi(G \otimes (1-z))$	1	1	1	1
$\psi(z \otimes gc^{-i})$	1	1	1	ζ
$\psi(c \otimes gc^{-i})$ $(g \in H \setminus \langle z\rangle)$	1	1	ζ(if $\chi(g) = \zeta^i$) 1(if $\chi(g) \ne \zeta^i$)	ζ^m
$\psi(c \otimes \xi)$	1	1	1	ζ^m

Here, in the last line, $\xi = a(1+b+\cdots+b^{p-1}) - b(c+c^2+\cdots+c^{p-1})$. By inspection,

$$SK_1(\mathbb{Z}G) \cong \operatorname{Coker}(\psi) \cong (\mathbb{Z}/p)^{p-1} \oplus (\mathbb{Z}/p[C_p \times C_p]/I) \oplus (\mathbb{Z}/p)^{p-2}, \tag{1}$$

where $I \subseteq \mathbb{Z}/p[C_p \times C_p]$ is the ideal generated by elements $(\sum_{g\in K} g)$ for subgroups $K \subseteq C_p \times C_p$ of order p.

Write $C_p \times C_p = \langle g \rangle \times \langle h \rangle$, and let $J = \langle 1-g, 1-h \rangle \subseteq \mathbb{Z}/p[C_p \times C_p]$ denote the Jacobson radical. Then

$$I = \langle (1-g)^{p-1}, (1-h)^{p-1}; (1-g^ih)^{p-1}\colon 1 \le i \le p-1 \rangle.$$

Furthermore, for any $1 \le i \le p-1$:

$$(1-g^ih) = 1 - [1-(1-g)]^i[1-(1-h)] \equiv i(1-g) + (1-h) \pmod{J^2}$$

and so

$$\begin{aligned}(1-g^ih)^{p-1} &\equiv \sum_{k=0}^{p-1} \binom{p-1}{k} i^k(1-g)^k(1-h)^{p-1-k} \\ &= \sum_{k=0}^{p-1} (-i)^k(1-g)^k(1-h)^{p-1-k} \pmod{J^p}.\end{aligned}$$

The determinant of $[(-i)^k]_{i,k=1}^{p-2}$ is invertible over $\mathbb{Z}/p$ (a van der Monde determinant), and so

$$I + J^p = \langle (1-g)^k(1-h)^{p-1-k}\colon 0 \le k \le p-1 \rangle = J^{p-1}.$$

But $J^{2p-1} = 0$, and hence this implies that $I = J^{p-1}$. So as a group,

$$\begin{aligned}&\mathbb{Z}/p[C_p \times C_p]/I \cong (\mathbb{Z}/p)^{1/2p(p-1)} \text{ with basis} \\ &\{(1-g)^i(1-h)^j\colon i, j \ge 0,\ i+j < p-1\}.\end{aligned}$$

The result now follows from (1). $\square$

In the above example, the fact that $[G, G]$ was central helped to keep the description of ψ_G simple. The next example illustrates additional complexities which can arise when this is no longer the case. First, a lemma is needed.

LEMMA 4.2. *Let G be cyclic of order $p^n (n \ge 1)$ with generator $g \in G$.*

Then, for any

$$0 \neq \alpha = \sum_{i=0}^{p^n-1} a_i g^i \in \mathbb{Z}/p[G] \qquad (a_i \in \mathbb{Z}/p);$$

$\mathbb{Z}/p[G]/(\alpha) \cong (\mathbb{Z}/p)^k$ *(as groups), where*

$$k = \min\left\{m \geq 0\colon \sum_{i=0}^{p^n-1} \binom{i}{m} a_i \neq 0 \text{ in } \mathbb{Z}/p\right\}.$$

Proof. By direct calculation,

$$\alpha = \sum_{i=0}^{p^n-1} a_i g^i = \sum_{i=0}^{p^n-1} a_i (1 + (g-1))^i = \sum_{i=0}^{p^n-1} a_i \sum_{m=0}^{i} \binom{i}{m} (g-1)^m$$

$$= \sum_{m=0}^{p^n-1} \left(\sum_{i=0}^{p^n-1} \binom{i}{m} a_i \right) (g-1)^m. \tag{1}$$

Recall that $\mathbb{Z}/p[G]$ is a local ring with maximal ideal generated by $(g-1)$. So if k is defined as above, then $\alpha = (g-1)^k u$ for some unit u in $\mathbb{Z}/p[G]$, and

$$rk[\mathbb{Z}/p[G]/(\alpha)] = rk[\mathbb{Z}/p[G]/(g-1)^k] = k. \quad \square$$

PROPOSITION 4.3. *Set $H = \langle a \rangle \times \langle b \rangle \times \langle c \rangle = C_p^3$, $K = \langle x \rangle \cong C_p$, and let G be any extension of the form*

$$1 \to H \to G \to K \to 1$$

such that

$$xax^{-1} = ab, \qquad xbx^{-1} = bc, \qquad xcx^{-1} = c.$$

Then

$$SK_1(\mathbb{Z}G) \cong Cl_1(\mathbb{Z}G) \cong (\mathbb{Z}/p)^{3(p-1)/2}.$$

Proof. The action of x on $\mathbb{Q}H$ fixes $\mathbb{Q}[H/\langle b, c \rangle]$, and permutes the other $p^2 + p$ summands freely. Thus,

$$\mathbb{Q}G \cong \mathbb{Q}[G^{ab}] \times \prod^{p+1} M_p(\mathbb{Q}[\zeta]);$$

where $\zeta = \zeta_p$. The following table presents ψ_G, where the nonabelian representations are described by their restrictions to H:

Representation	$U \cong \mathbb{Q}\zeta$	$V_m \cong \mathbb{Q}\zeta$	$W \cong (\mathbb{Q}\zeta)^p$	$X_m \cong (\mathbb{Q}\zeta)^p$
Indexed by	—	$0 \le m < p$	—	$0 \le m < p$
E'val $(a, b, c; x)$	$(1, 1, 1; \zeta)$	$(\zeta, 1, 1; \zeta^m)$	$(\zeta^r, \zeta, 1)$ $(1 \le r \le p)$	$(\zeta^{m+1/2r(r-1)}, \zeta^r, \zeta)$ $(1 \le r \le p)$
$\psi(a \otimes c)$	1	ζ	1	1
$\psi(x \otimes c)$	ζ	ζ^m	1	1
$\psi(a \otimes (1-c))$	1	1	1	ζ^T
$\psi(x \otimes (1-c))$	1	1	1	$\zeta^u (x^p = c^u)$
$\psi(a \otimes (b-c))$	1	1	1	ζ^m
$\psi(c \otimes b)$	1	1	1	ζ
$\psi(b \otimes ac^{-i})$	1	1	ζ	$\zeta^{R(i-m)}$
$\psi(c \otimes ac^{-i})$	1	1	1	$\zeta^{S(i-m)}$
$\psi(c \otimes gx)(g \in H)$	1	1	1	$\begin{cases} \zeta & ((gx)^p = 1) \\ 1 & ((gx)^p \ne 1). \end{cases}$

Here, $T = \sum_{r=1}^{p} \frac{1}{2} r(r-1)$;

$R(i) = \sum \{r \colon 1 \le r \le p, \frac{1}{2} r(r-1) \equiv i \pmod{p}\}$;

$S(i) = \#\{r \colon 1 \le r \le p, \frac{1}{2} r(r-1) \equiv i \pmod{p}\}$.

Note that solutions to $\frac{1}{2} r(r-1) \equiv i$ come in pairs $\{r, p+1-r\}$ (unless $r = (p+1)/2$). This shows that for all i,

$$R(i) = \frac{p+1}{2} S(i) \equiv \tfrac{1}{2} S(i) \pmod{p}.$$

Identify $\prod_{X_m} \langle \zeta \rangle$ with $\mathbb{Z}/p[C_p]$, by identifying X_m with g^m for some generator g of C_p. Then

$$\begin{aligned} SK_1(\mathbb{Z}G) &\cong \operatorname{Coker}(\psi_G) \\ &\cong (\mathbb{Z}/p)^{p-1} \oplus \left(\mathbb{Z}/p[C_p] \Big/ \left\langle \sum_m g^m, \sum_m m g^m, \sum_m S(i-m) g^m \quad (\text{any } i) \right\rangle \right) \\ &\cong (\mathbb{Z}/p)^{p-1} \oplus \mathbb{Z}/p[C_p]/I, \end{aligned}$$

where I is the ideal generated by

$$\alpha = \sum_m S(m)g^{-m} = \sum_{k=1}^{p} g^{-1/2k(k-1)}.$$

By Lemma 4.2, we will be done upon showing that

$$\sum_{k=1}^{p} \binom{\frac{1}{2}k(k-1)}{n} \begin{cases} \equiv 0 & \text{for} \quad 0 \le n < \frac{p-1}{2} \\ & (\operatorname{mod} p) \\ \not\equiv 0 & \text{for} \quad n = \frac{p-1}{2} \end{cases} \tag{1}$$

But the sum is a polynomial in k (over $\mathbb{Z}/p$) of degree exactly $2n$; and (1) follows since

$$p - 1 = \min \left\{ m \ge 0 \colon \sum_{k=1}^{p} k^m \not\equiv 0 \pmod p \right\}. \quad \square$$

The groups covered above turn out to be the most difficult cases for computing $SK_1(\mathbb{Z}G)$ when $|G| = p^4$. In fact, all other groups of order p^4 are covered by the following proposition (this can easily be checked directly, but also follows from the classificastion in [9, section III.12]).

PROPOSITION 4.4. *Assume that G is non-abelian of order p^4, and that there is a subgroup $H \lhd G$ such that $H \cong C_{p^3}$ or $H \cong C_{p^2} \times C_p$. Then*

$$\begin{aligned} SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G) &\cong (\mathbb{Z}/p)^{p-1} && \textit{if} \quad G^{ab} \cong C_p \times C_p \\ &\cong (\mathbb{Z}/p)^{2(p-1)} && \textit{if} \quad G^{ab} \cong C_{p^2} \times C_p \\ &\cong (\mathbb{Z}/p)^{(p+2)(p-1)/2} && \textit{if} \quad G^{ab} \cong C_p \times C_p \times C_p. \end{aligned}$$

Proof. Write

$$\mathbb{Q}G = \mathbb{Q}[G^{ab}] \times M \quad \text{and} \quad \mathbb{Q}H = \mathbb{Q}[H/[G, G]] \times M';$$

where M is a product of rank p matrix algebras over fields. Then the inclusion $M' \subseteq M$ is a sum of inclusions of the form

$$\prod^{p} \mathbb{Q}\zeta_{p^r} \subseteq M_p(\mathbb{Q}\zeta_{p^r}); \qquad Q\zeta_{p^{r+1}} \subseteq M_p(\mathbb{Q}\zeta_{p^r}) \quad (r = 1, 2).$$

In particular, $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M')_{(p)}$ surjects onto $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M)_{(p)}$. Since $Cl_1(\mathbb{Z}H)=0$ [10, Theorems 4.4.1 and 5.1.1], this shows that

$$K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M)_{(p)} \subseteq \operatorname{Im}[\varphi_G : K_2(\hat{\mathbb{Z}}_p G) \to K_2(\hat{\mathbb{Q}}_p G)_{(p)}].$$

In other words, if $\mathbb{Q}[G^{ab}] \cong \prod_{i=1}^{k} F_i$, then

$$SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G) \cong \operatorname{Coker}\left[\operatorname{proj}\circ\psi_G\colon H_1(G;\mathbb{Z}G) \to \prod_{i=1}^{k} (\mu_{F_i})_p\right].$$

If $G^{ab} \cong C_p \times C_p$, with basis $\{a, b\}$, then $\operatorname{Im}(\operatorname{proj}\circ\psi_G)$ is generated by the images of $a\otimes 1$ and $b\otimes 1$, and so $SK_1(\mathbb{Z}G)$ hs rank $(p+1)-2=p-1$. If $G^{ab}\cong C_p^3$, then there are generators a, b, c such that $c\in Z(G)$, and the computation follows from the table in the proof of Proposition 5.1. The proof when $G^{ab}\cong C_{p^2}\times C_p$ is similar. □

It is interesting to note that for each of these classes of p-groups, the rank of $Cl_1(\mathbb{Z}G)$ is a polynomial in p. This has already been remarked in the case of abelian p-groups (see [1, Conjecture 5.8]); but is harder to formulate as a precise conjecture in the non-abelian case.

Section 5

As another application of Theorem 1.4, we now study the relationship between the complex Artin cokernel

$$A_{\mathbb{C}}(G) = \operatorname{Coker}\left[\sum \{R_{\mathbb{C}}(H)\colon H\subseteq G \text{ cyclic}\} \xrightarrow{\text{Ind}} R_{\mathbb{C}}(G)\right]$$

of a finite group G, and $Cl_1(RG)$ for large R.

First, epimorphisms

$$I_{RG} : A_{\mathbb{C}}(G) \twoheadrightarrow Cl_1(RG)$$

are constructed, for G any finite group and R the ring of integers in any number field $K\subseteq\mathbb{C}$ (the identiification of K as a subfield of $\mathbb{C}$ is needed when defining I_RG). The I_{RG} are shown to be natural with respect to homomorphisms and transfer maps, and then shown to be isomorphisms for sufficiently large R.

The following lemma on norm residue symbols will be needed.

LEMMA 5.1. *Fix a prime p, fix extensions $E \supseteq F \supseteq \hat{\mathbb{Q}}_p$, and let $\hat{\mu} \subseteq E^*$ and $\mu \subseteq F^* \cap \hat{\mu}$ be groups of roots of unity. Then the diagram*

$$\begin{array}{ccc} K_2(E) & \xrightarrow{(,)_{\hat{\mu}}} & \hat{\mu} \\ \Big\downarrow \mathrm{trf}_F^E & & \Big\downarrow [\hat{\mu}:\mu] \\ K_2(F) & \xrightarrow{(,)_{\mu}} & \mu \end{array} \tag{1}$$

commutes; where $(,)_{\hat{\mu}}$ and $(,)_{\mu}$ are the norm residue symbol homomorphisms.

Proof. Set $n = |\hat{\mu}|$ and $m = |\mu|$. Fix $u \in F^*$ and $v \in E^*$, and let $E(\alpha)/E$ be an extension such that $\alpha^n = u$. The diagram

$$\begin{array}{ccc} E^* & \xrightarrow{\hat{s}} & \mathrm{Gal}\,(E(\alpha)/E) \\ \Big\downarrow N_{E/F} & & \Big\downarrow \mathrm{res} \\ F^* & \xrightarrow{s} & \mathrm{Gal}\,(F(\alpha^{n/m})/F) \end{array}$$

commutes by [23, Section XI.3]; where $\hat{s}$ and s are the reciprocity maps and res is induced by restriction. By [23, Proposition XIV.6],

$$\begin{aligned} (u, N_{E/F}(v))_{\mu} &= s(N_{E/F}(v))(\alpha^{n/m})/\alpha^{n/m} \\ &= [\hat{s}(v)(\alpha)/\alpha]^{n/m} = ((u, v)_{\hat{\mu}})^{n/m}. \end{aligned} \tag{2}$$

Since $\mathrm{trf}_F^E(\{u, c\}) = \{u, N_{E/F}(v)\}$ for $u \in F^*$ and $v \in E^*$, this shows that (1) commutes on the subgroup $\{F^*, E^*\} \subseteq K_2(E)$. Furthermore,

$$\mathrm{trf}_F^E(\{F^*, E^*\}) = \{F^*, N_{E/F}(E^*)\} = K_2(F):$$

the last equality is shown in [14, Lemma] when $\mathrm{Gal}\,(E/F)$ is cyclic, and follows from [6, Chapter VI, §2.2] ($N_{E/F}$ is onto) when $\mathrm{Gal}\,(E/F)$ is non-abelian simple. Since $K_2(E) \cong \mu_E$ and $K_2(F) \cong \mu_F$ are cyclic [12, Theorem A.14], it follows that $\{F^*, E^*\} \supseteq K_2(E)_{(p)}$ for any prime $p \mid |K_2(F)|$, and hence any $p \mid |\mu|$. So (1) commutes. $\square$

Now fix a finite group G, and let $K \subseteq \mathbb{C}$ be any splitting field for G: i.e., KG is a product of matrix algebras over K. As in [20, Section 2], we define for each prime p:

$$\begin{aligned} C_p(KG) &= \mathrm{Coker}\left[K_2\left(\mathfrak{M}\left[\frac{1}{p}\right]\right) \to K_2(\hat{K}_p G)\right] \quad (\hat{K}_p = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} K) \\ &\cong \mathrm{Coker}\,[K_2(\mathfrak{M}) \to K_2(\hat{\mathfrak{M}}_p)] \qquad (\hat{\mathfrak{M}}_p = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathfrak{M}) \end{aligned}$$

where $\mathfrak{M} \subseteq KG$ is any maximal order. Then $C_p(KG)$ is a p-group for all p (since $K_2(\mathfrak{M}_p)$ is a p-group). Finally, set

$$C(KG) = \sum_p C_p(KG).$$

Write $KG = \prod_{i=1}^k B_i$, where $B_i \cong \mathrm{End}_K(V_i)$ for each i, and $V_1, \ldots, V_k$ are the irreducible KG-modules. By results going back to Bass, Milnor, and Serre [5], $C(KG) = 0$ if K has a real embedding. If K is purely imaginary, then there is an isomorphism

$$\lambda_{KG} : C(KG) \xrightarrow{\cong} \prod_{i=1}^{k} (\mu_K)$$

such that for any prime $\mathfrak{p} \subseteq R$, and any units $u \in K^*$ and $v \in (\hat{K}_{\mathfrak{p}}[G])^*$,

$$\lambda_{KG}(\{u, v\}_{\mathfrak{p}}) = [(u, \det_K(v, V_i))]_{i=1}^k$$

Here, $\{u, v\}$ denotes the image of

$$\{u, v\} \in K_2(\hat{K}_{\mathfrak{p}}[G]) \to C(KG);$$

and

$$(\,,\,)_{\mathfrak{p}} : (\hat{K}_{\mathfrak{p}})^* \times (\hat{K}_{\mathfrak{p}})^* \to \mu_K$$

denotes the norm residue symbol with values in μ_K. See [20, Theorem 2.2] for more details.

Thus, when $K \subseteq \mathbb{C}$ is a splitting field for G and has no real embedding and $KG \cong \prod_{i=1}^k B_i$ as above, an isomorphism $\tilde{I}_{KG}$ from $R_{\mathbb{C}}(G)$ to $C(KG)$ can be defined as the composite

$$\tilde{I}_{KG} : R_{\mathbb{C}}(G) \cong \prod_{i=1}^{k} \mathbb{Z} \xrightarrow{\prod[1 \mapsto \exp(2\pi i/m)]} \prod_{i=1}^{k} \mu_K \underset{\cong}{\xrightarrow{\lambda_{KG}^{-1}}} C(KG) \qquad (m = |\mu_K|).$$

In other words, for each $1 \leq i \leq k$, we set

$$I_{KG}([V_i]) = \lambda_{KG}^{-1}([\exp(2\pi i/m)]_i);$$

where $[V_i] \in R_{\mathbb{C}}(G)$ denotes the class of $\mathbb{C} \otimes_K V_i$.

If K is a splitting field for G but has a real embedding, we set $\tilde{I}_{KG}=0$ ($C(KG)=0$). If $K\subseteq\mathbb{C}$ is a number field which does not split G, set $n=\exp(G)$ and $L=K(\zeta_n)$, and define

$$\tilde{I}_{KG}=\mathrm{trf}_{KG}^{LG}\circ\tilde{I}_{LG}:R_{\mathbb{C}}(G)\to C(LG)\to C(KG).$$

(Note that L is a splitting field for G by [5, Theorem 4.1.1].) This definition of the I_{KG} seems rather artificial; but the following proposition shows that these maps do have all desired naturality properties.

PROPOSITION 5.2. *For any number field $K\subseteq\mathbb{C}$ and any finite group G, $\tilde{I}_{KG}$ is surjective. The $\tilde{I}_{KG}$ are natural in that for any homomorphism $\alpha:\tilde{G}\to G$ of finite groups, for any $H\subseteq G$, and for any pair $K\subseteq L\subseteq\mathbb{C}$ of number fields, the followng diagrams all commute*:

$$\begin{tikzcd} & R_{\mathbb{C}}(G) \arrow[dl] \arrow[dr] \arrow[d, phantom, "(1)"] & \\ C(LG) \arrow[rr, "\mathrm{trf}_{KG}^{LG}"] & & C(KG) \end{tikzcd}
\qquad
\begin{tikzcd} R_{\mathbb{C}}(\tilde{G}) \arrow[r, "R_{\mathbb{C}}(\alpha)"] \arrow[d, "\tilde{I}_{K\tilde{G}}"] \arrow[dr, phantom, "(2)"] & R_{\mathbb{C}}(G) \arrow[r, "\mathrm{Res}_H^G"] \arrow[d, "\tilde{I}_{KG}"] \arrow[dr, phantom, "(3)"] & R_{\mathbb{C}}(H) \arrow[d, "\tilde{I}_{KH}"] \\ C(K\tilde{G}) \arrow[r, "C(K\alpha)"] & C(KG) \arrow[r, "\mathrm{trf}_{KH}^{KG}"] & C(KH) \end{tikzcd}$$

Proof. The proposition will be proven in four steps. For finite G and arbitrary $K\subseteq\mathbb{C}$, we regard $K_0(KG)=R_K(G)$ as a subring of $R_{\mathbb{C}}(G)$ in the usual fashion (identifying $[V]\in R_K(G)$ with $[C\otimes_K V]\in R_{\mathbb{C}}(G)$).

Step 1. By construction, $\tilde{I}_{KG}$ is surjective if K splits G. To see that $\tilde{I}_{KG}$ is surjective in general, we must show for any G, and any number fields $K\subseteq L$, that the transfer map

$$\mathrm{trf}_{KG}^{LG}:K_2(\hat{L}_pG)\to K_2(\hat{K}_pG)$$

is onto for each prime p ($\hat{L}_p=\hat{\mathbb{Q}}_p\otimes_{\mathbb{Q}}L$, etc.).

Write $\hat{K}_pG\cong\prod_{i=1}^k M_{n_i}(D_i)$, where the D_i are division algebras. For each i, set $F_i=Z(D_i)$, the center, and let $E_i\subseteq D_i$ be a maximal subfield. By [3, Corollary 4.15], $K_2(D_i)$ is generated by symbols $\{F_i^*, D_i^*\}$; and hence $K_2(E_i)\to K_2(D_i)$ is onto by [14, Proposition].

Consider the following square, for each $1\le i\le k$:

$$\begin{tikzcd} K_1(L\otimes_K E_i) \arrow[r, "\mathrm{incl}"] \arrow[d, "t_i'"] & K_2(L\otimes_K D_i) \arrow[d, "t_i"] \\ K_2(E_i) \arrow[r, two heads] & K_2(D_i). \end{tikzcd}$$

Here t_i and t_i' are the transfer maps. The square commutes since the two sides are induced by tensoring with the bimodules

$$D_i \otimes_{E_i} (L \otimes_K E_i) \cong L \otimes_K D_i.$$

The map t_i' is the product of the transfer homomorphisms for the field summands of $L \otimes_K E_i$, each of which is onto by [12, Corollary A.15]. So t_i is also onto. But trf_{KG}^{LG} is isomorphic to the sum of the t_i, and is hence surjective.

Step 2. Fix K and G such that K is a totally imaginary splitting field for G. In particular, $K_0(KG) = R_{\mathbb{C}}(G)$. For any finite dimensional (left) KG-module V, define

$$f_V : C(K) \to C(KG)$$

to be the homomorphism induced by the functor

$$V \otimes_K : K\text{-}\mathbf{mod} \to KG\text{-}\mathbf{mod}.$$

If V is irredicible, then f_V is just the Morita equivalence identifying $C(B)$ with $C(K)$, where $B \subseteq KG$ is the simple summand with irreducible representation V. So by definition,

$$\tilde{I}_{KG}([V]) = f_V(\lambda_K^{-1}(\exp(2\pi i/m))); \qquad (m = |\mu_K|) \tag{4}$$

where $\lambda_K : C(K) \xrightarrow{\cong} \mu_K$ is induced by the norm residue symbol. Both sides of (4) are additive $(f_{V \oplus W} = f_V + f_W)$, so (4) holds for arbitrary V.

Step 3. We can now show the commutativity of triangle (1) above: that $\tilde{I}_{KG} = \mathrm{trf}_{KG}^{LG} \circ \tilde{I}_{LG}$ for any G and any number fields $K \subseteq L \subseteq \mathbb{C}$. It suffices to do this when K and L both are totally imaginary splitting fields for G. In particular, $K_0(KG) = R_{\mathbb{C}}(G)$.

By (4), for any finite dimensional KG-module V,

$$\tilde{I}_{KG}([V]) = f_V(\lambda_K^{-1}(\exp(2\pi i/m))), \qquad (m = |\mu_K|)$$
$$\mathrm{trf}_{KG}^{LG}(\tilde{I}_{LG}([V])) = \mathrm{trf}_{KG}^{LG} \circ f_{L \otimes_K V}(\lambda_L^{-1}(\exp(2\pi i/n))); \qquad (n = |\mu_L|)$$

and it remains to check the commutativity of the following diagram:

$$\begin{tikzcd}
\mu_L \arrow[r, "\lambda_L^{-1}", "\cong"'] \arrow[d, "n/m"] \arrow[dr, phantom, "(5)"] & C(L) \arrow[r, "f_{L\otimes V}"] \arrow[d, "\mathrm{trf}_K^L"] \arrow[dr, phantom, "(6)"] & C(LG) \arrow[d, "\mathrm{trf}_{KG}^{LG}"] \\
\mu_K \arrow[r, "\lambda_K^{-1}"] & C(K) \arrow[r, "f_V"] & C(KG).
\end{tikzcd}$$

But (5) commutes by Lemma 5.1, while (6) commutes since the two composites are induced by tensoring with the bimodules

$$KG^{LG} \otimes_{LG} (L \otimes_K V)_L = {}_{KG}V \otimes_K L_L.$$

Step 4. Now fix a homomorphism $\alpha: \tilde{G} \to G$ of finite groups, and a subgroup $H \subseteq G$. We must show that (2) and (3) commute for any number field $K \leq \mathbb{C}$. If $L \supseteq K$ is any pair of number fields, then the squares

$$\begin{tikzcd}
C(L\tilde{G}) \arrow[r, "C(L\alpha)"] \arrow[d, "\mathrm{trf}_{K\tilde{G}}^{L\tilde{G}}"] & C(LG) \arrow[r, "\mathrm{trf}_{LH}^{LG}"] \arrow[d, "\mathrm{trf}_{KG}^{LG}"] & C(LH) \arrow[d, "\mathrm{trf}_{KH}^{LH}"] \\
C(K\tilde{G}) \arrow[r, "C(K\alpha)"] & C(KG) \arrow[r, "\mathrm{trf}_{KH}^{KG}"] & C(KH)
\end{tikzcd}$$

commute (just compare bimodules). So by (1) (Step 3), it suffices to prove the commutativity of (2) and (3) when K is a splitting field for $\tilde{G}$, G and H (and totally imaginary).

Fix such a K; in particular, $K_0(K\tilde{G}) = R_{\mathbb{C}}(\tilde{G})$ and $K_0(KG) = R_{\mathbb{C}}(G)$. Fix finite dimensional modules V over $K\tilde{G}$ and W over KG. *Set*

$$x = \lambda_K^{-1}(\exp(2\pi i/|\mu_k|)) \in C(K).$$

Then, by (4),

$$\tilde{I}_{KG}(R_{\mathbb{C}}(\alpha)([V])) = f_{KG\otimes_{K\tilde{G}}V}(x),$$

$$C(K\alpha)\circ\tilde{I}_{K\tilde{G}}([V]) = C(K\alpha)\circ f_V(x);$$

$$\tilde{I}_{KH}(\mathrm{Res}_H^G([W])) = f_{W|H}(x),$$

and

$$\mathrm{trf}_{KH}^{KG}\circ\tilde{I}_{KG}([W]) = \mathrm{trf}_{KH}^{KG}\circ f_W(x).$$

So we will be done upon showing that the following triangles commute:

$$\begin{tikzcd}[column sep=small] & C(K\tilde{G}) \arrow[dd, "C(K\alpha)"] \\ C(K) \arrow[ur, "f_V"] \arrow[dr, "f_{KG}\otimes_{K\tilde{G}}V"'] & \\ & C(KG) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & C(KG) \arrow[dd, "\mathrm{trf}_{KH}^{KG}"] \\ C(K) \arrow[ur, "f_W"] \arrow[dr, "f_{W|H}"'] & \\ & C(KH) \end{tikzcd}$$

But they are induced by the following pairs of isomorphic bimodules:

$${}_{KG}(KG\otimes_{K\tilde{G}}V)_K \cong {}_{KG}KG\otimes_{K\tilde{G}}V_K \quad \text{and} \quad {}_{KH}W_K \cong {}_{KH}KG\otimes_{KG}W_K;$$

and we are done. □

Again fix a finite group G and a number field $K\subseteq\mathbb{C}$, and let $R\subseteq K$ be the ring of integers. Then $Cl_1(RG)$ is described by a localization sequence

$$\sum_p K_2(\hat{R}_pG)\to C(KG)\xrightarrow{\partial_{RG}} Cl_1(RG)\to 0$$

(see [20, Theorem 2.1] for details). We now consider the composite

$$R_{\mathbb{C}}(G)\xrightarrow{\tilde{I}_{KG}} C(KG)\xrightarrow{\partial_{RG}} Cl_1(RG).$$

Both maps are natural with respect to induction from subgroups of G. Hence, since $Cl_1(RH)=SK_1(RH)=0$ for any cyclic $H\subseteq G$ by [1, Theorem 3.3], $\partial_{RG}\circ I_{KG}$ vanishes on any element of $R_{\mathbb{C}}(G)$ induced up from a cyclic subgroup. Thus, $\partial_{RG}\circ\tilde{I}_{KG}$ factors through a homomorphism

$$I_{RG}:A_{\mathbb{C}}(G)\to Cl_1(RG),$$

where $A_{\mathbb{C}}(G)$ is the Artin cokernel.

THEOREM 5.3. *For any finite group G, and any number field $K\subseteq\mathbb{C}$ with ring of integers R,*

$$I_{RG}:A_{\mathbb{C}}(G)\to Cl_1(RG)$$

is surjective. The I_{RG} are natural in that for any homomorphism $\alpha:\tilde{G}\to G$ of finite groups, any $H\subseteq G$, and any pair $R\subseteq S$ of rings of integers in number fields, the *following diagrams all commute*:

$$\begin{tikzcd} & A_{\mathbb{C}}(G) \arrow[dl, "I_{SG}"'] \arrow[dr, "I_{RG}"] & \\ Cl_1(SG) \arrow[rr, "\mathrm{trf}^{SG}_{RG}"] & & Cl_1(RG) \end{tikzcd} \qquad \begin{tikzcd} A_{\mathbb{C}}(\tilde{G}) \arrow[r, "A_{\mathbb{C}}(\alpha)"] \arrow[d, "I_{R\tilde{G}}"] & A_{\mathbb{C}}(G) \arrow[r, "\mathrm{Res}^G_H"] \arrow[d, "I_{RG}"] & A_{\mathbb{C}}(H) \arrow[d, "I_{RH}"] \\ Cl_1(R\tilde{G}) \arrow[r, "Cl_1(A\alpha)"] & Cl_1(RG) \arrow[r, "\mathrm{trf}^{RG}_{RH}"] & Cl_1(RH). \end{tikzcd}$$

Proof. For any R and G, I_{RG} is surjective since $\tilde{I}_{KG}$ and ∂_{RG} both are surjective. The naturality properties follow from the corresponding properties for the $\tilde{I}_{KG}$ (Proposition 5.2), and the naturality of the boundary maps ∂_{RG} in the localization sequences. □

Now that the I_{RG} have been constructed, we can finally apply Theorem 1.4 to show that they are isomorphisms for sufficiently large R. For any finite G, $a_{\mathbb{C}}(G)$ will denote the complex Artin exponent: the order of $1 \in R_{\mathbb{C}}(G)$ in $A_{\mathbb{C}}(G)$. By Frobenius reciprocity,

$$a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G));$$

i.e., $a_{\mathbb{C}}(G) \cdot x$ is induced from cyclic subgroups for any $x \in R_{\mathbb{C}}(G)$. By the Artin induction theorem [7, Theorem 39.1], $a_{\mathbb{C}}(G) \mid |G|$.

THEOREM 5.4. *Let G be any finite group, and set $n = a_{\mathbb{C}}(G) \cdot \exp(G)$. Let K be any number field such that $\zeta_n \in K$, and let $R \subseteq K$ be the ring of integers. Then I_{RG} is an isomorphism*: $Cl_1(RG) \cong A_{\mathbb{C}}(G)$.

Proof. This will be shown first for p-groups, then for p-elementary groups, and finally for arbitrary finite groups.

Step 1. Let G be a p-group, and set $p^k = a_{\mathbb{C}}(G)$, $p^m = \exp(G)$, and $q = p^{k+m}$. By Theorem 5.3(1), it will suffice to show that I_{RG} is an isomorphism when $K = \mathbb{Q}\zeta_q$ and $R = \mathbb{Z}\zeta_q$.

Let C_q be a (multiplicative) cyclic group of order q with generator z. Consider the pullback square

$$\begin{tikzcd} \hat{\mathbb{Z}}_p[C_q \times G] \arrow[r, "\alpha"] \arrow[d] & \hat{\mathbb{Z}}_p\zeta_q[G] \arrow[d] \\ \hat{\mathbb{Z}}_p[(C_q/z^{p^{n+m-1}}) \times G] \arrow[r, "\beta"] & \mathbb{Z}/p[(C_q/z^{p^{n+m-1}}) \times G]; \end{tikzcd} \tag{1}$$

where α is induced by: $\alpha(z)=\zeta_q$. Then $K_2(\beta)$ is onto by [17, Lemma 1.7] if $p>2$; or if $p=2$ since the only torsion in $K_1(\hat{\mathbb{Z}}_2[(C_q/z^{p^{n+m-1}})\times G], 2)$ is (-1) (see [15, Proposition 2]). So by the Mayer–Vietoris sequence for (1), $K_2(\alpha)$ is onto.

Now consider the following commutative diagram:

$$\begin{tikzcd}
K_2(\hat{\mathbb{Z}}_p[C_q\times G]) \arrow[r] \arrow[d, "K_2(\alpha)"] & C(\mathbb{Q}[C_q\times G])_{(p)} \arrow[d, "\alpha_*"'] & R_{\mathbb{C}}(G)\otimes\mathbb{Z}/q \arrow[dl, "\tilde{I}_{KG}"] \arrow[d, "\partial\circ\tilde{I}_{KG}"] & \\
K_2(\hat{\mathbb{Z}}_p\zeta_q[G]) \arrow[r, "\varphi_{RG}"] & C_p(\mathbb{Q}\zeta_q[G]) \arrow[r, "\partial"] & Cl_1(RG)_{(p)} \arrow[r] & 0
\end{tikzcd} \tag{2}$$

where the bottom row is exact [20, Theorem 2.1]. By Theorem 1.4,

$$\begin{aligned}
K_2(\hat{\mathbb{Z}}_p[C_q\times G]) &= K_2(\hat{\mathbb{Z}}_pG)\oplus K_2(\hat{\mathbb{Z}}_p[C_q\times G],(1-z)) \\
&= K_2(\hat{\mathbb{Z}}_pG)\oplus\langle\{h, 1-(1-z)^ig\}, \{z, 1-(1-z)^ig\}: \\
&\qquad\qquad h\in G, g\in C_q\times G, hg=gh, i\geq 1\rangle.
\end{aligned}$$

It follows that

$$K_2(\hat{\mathbb{Z}}_p\zeta_p[G]) = K_2(\hat{\mathbb{Z}}_pG)+X+Y;$$

where with $\zeta=\zeta_q$:

$$X=\langle\{h, 1-(1-\zeta)^i\zeta^jg\}: h, g\in G, hg=gh, i\geq 1, j\in\mathbb{Z}\rangle$$

and

$$Y=\langle\{\zeta, 1-(1-\zeta)^i\zeta^jg\}: g\in G, i\geq 1, j\in\mathbb{Z}\rangle.$$

Recall that $p^m=\exp(G)$. Then

$$\exp(X)\mid p^m \quad\text{and}\quad \exp(\varphi_{RG}(K_2(\hat{\mathbb{Z}}_pG)))\mid\exp(C(\mathbb{Q}G)_{(p)})\mid p^m.$$

Furthermore, by definition,

$$\varphi_{RG}(Y)\subseteq\operatorname{Im}\Big[\sum\{C(\mathbb{Q}\zeta_q[H]): H\subseteq G \text{ cyclic}\}\to C(\mathbb{Q}\zeta_q[G])\Big].$$

So by diagram (2) (recalling that $q=p^{k+m}$):

$$\begin{aligned}
\operatorname{Ker}(\partial\circ\tilde{I}_{KG}) &\subseteq p^kR_{\mathbb{C}}(G)+\operatorname{Im}\Big[\sum\{R_{\mathbb{C}}(H): H\subseteq G \text{ cyclic}\}\to R_{\mathbb{C}}(G)\Big] \\
&= p^kR_{\mathbb{C}}(G)+\operatorname{Ker}[R_{\mathbb{C}}(G)\to A_{\mathbb{C}}(G)].
\end{aligned}$$

Since $p^k = a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G))$,

$$\operatorname{Ker}[I_{RG}\colon A_{\mathbb{C}}(G) \twoheadrightarrow Cl_1(RG)] \subseteq p^k A_{\mathbb{C}}(G) = 0;$$

and so I_{RG} is an isomorphism.

Step 2. Now assume that G is p-elementary: $G \cong C_m \times H$ where $p \nmid m$ and H is a p-group. Set $n = a_{\mathbb{C}}(G) \cdot \exp(G)$, fix a number field $K \subseteq \mathbb{C}$ containing ζ_n, and let R be the ring of integers of K. Then

$$\begin{aligned} A_{\mathbb{C}}(G) &\cong \operatorname{Coker}\left[\sum \{R_{\mathbb{C}}(C_m) \otimes R_{\mathbb{C}}(H_0)\colon H_0 \subseteq H \text{ cyclic}\} \to R_{\mathbb{C}}(C_m) \otimes R_{\mathbb{C}}(H)\right] \\ &\cong R_{\mathbb{C}}(C_m) \otimes A_{\mathbb{C}}(H) \cong \prod^m A_{\mathbb{C}}(H). \end{aligned}$$

On the other hand, the identification $K[G] \cong \prod^m K[H]$ (each factor corresponding to a character of C_m) induces an inclusion $RG \subseteq \prod^m R[H]$ of index prime to p; and hence an isomorphism

$$Cl_1(RG)_{(p)} \cong \prod^m Cl_1(RH)_{(p)} \cong \prod^m A_{\mathbb{C}}(H) \cong A_{\mathbb{C}}(G).$$

(see [17, Proposition 1.2]). Since I_{RG} is onto, it must be an isomorphism.

Step 3. Now let G be an arbitrary finite group, set $n = a_{\mathbb{C}}(G) \cdot \exp(G)$, and let R be any ring of integers containing ζ. Let $\mathscr{E}$ be the set of elementary subgroups of G. For any $H \in \mathscr{E}$, $\exp(H) \mid \exp(G)$ and $a_{\mathbb{C}}(H) \mid a_{\mathbb{C}}(G)$, so I_{RH} is an isomorphism by Step 2. Consider the following square, which commutes by Theorem 5.3:

$$\begin{tikzcd} A_{\mathbb{C}}(G) \arrow[r, "I_{RG}"] \arrow[d, "\sum \operatorname{Res}_H^G"] & Cl_1(RG) \arrow[d, "\sum \operatorname{trf}_H^G"] \\ \sum_{H \in \mathscr{E}} A_{\mathbb{C}}(H) \arrow[r, "\sum I_{RH}", "\cong"'] & \sum_{H \in \mathscr{E}} Cl_1(RH) \end{tikzcd}$$

In the language of [10], $A_{\mathbb{C}}(-)$ is a module over the Frobenius functor $R_{\mathbb{C}}(-)$, and hence is detected by restriction to elementary subgroups. So $\sum \operatorname{Res}_H^G$ is injective in the above square, and I_{RG} is an isomorphism. □

By [4, Theorem XI.4.7], for any finite G,

$$a_{\mathbb{C}}(G) = \prod_{p \mid |G|} a_{\mathbb{C}}(G_p),$$

where G_p is a p-Sylow subgroup. Thus, the description of

$$a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G)) = \max_R (\exp(Cl_1(RG)))$$

reduces immediately to the p-group case.

If G is a non-cyclic p-group, then there is a surjection $G \twoheadrightarrow C_p \times C_p$ and an induced surjection of $A_{\mathbb{C}}(G)$ onto $A_{\mathbb{C}}(C_p \times C_p)$. This last group is easily checked to be non-zero (see [1, Lemma 5.5] for details). Thus, for any finite G, $A_{\mathbb{C}}(G)$ is p-torsion free if and only if G_p is cyclic, $A_{\mathbb{C}}(G) = 0$ if and only if G is metacyclic, and these in turn imply similar statements about the $Cl_1(RG)$ (and $SK_1(RG)$). In fact, for fixed p and R such that $\zeta_p \in R$ (or $\zeta_4 \in R$ if $p = 2$), and any G, $Cl_1(RG)_{(p)} = 0$ if and only if G_p is cyclic (see [1, Theorem 3.5]).

A general description of $a_{\mathbb{C}}(G)$ has been given by Gluck [27]. The formula is much more complicated than that for the rational Artin exponent $a_{\mathbb{Q}}(G)$ given by Lam [11]. If G is non-cyclic, and abelian or of exponent p, then $a_{\mathbb{C}}(G) = a_{\mathbb{Q}}(G) = (1/p)\,|G|$. On the other hand, if G is a semidihedral 2-group, then $a_{\mathbb{C}}(G) = 2$ ($a_{\mathbb{Q}}(G) = 4$); and if p is odd and G a non-abelian group of order p^3 and exponent p^2, then $a_{\mathbb{C}}(G) = p$ ($a_{\mathbb{Q}}(G) = p^2$).

To end, we note that Theorem 5.3 allows a new interpretation of the following result in [13] (Theorem 1).

COROLLARY 5.5. *Let G be a finite group, and let R be the ring of integers in some number field. Then $Cl_1(RG)$ is generated by induction from elementary subgroups of G.*

Proof. By the Brauer induction theorem, $R_{\mathbb{C}}(G)$, and hence $A_{\mathbb{C}}(G)$ are generated in induction from elementary subgroups of G. The result follows since $I_{RG}\colon A_{\mathbb{C}}(G) \to Cl_1(RG)$ is natural and surjective. □

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