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A uniqueness theorem for $\mathbf{Z}^n$-periodic variational problems

V. BANGERT

1. Introduction

We consider a variational problem of the following type: Given an integrand $F:\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}$ which is $\mathbf{Z}$-periodic in the first $n+1$ variables we look for functions $u:\mathbf{R}^n\to\mathbf{R}$ which minimize $\int F(x, u, u_x)\,dx$ with respect to all compactly supported variations of u. Under appropriate growth conditions on F, Moser [9] succeeds to construct a large set of so-called "*minimal solutions without selfintersections*" to this problem. These solutions are natural generalizations of the affine functions $u(x)=\alpha\cdot x+u_0$, $\alpha\in\mathbf{R}^n$, which are minima for integrands F depending on u_x only. In particular, Moser's solutions have a uniquely determined "*slope*" or "*rotation vector*" $\alpha\in\mathbf{R}^n$. For irrational α these minimal solutions are obtained as limits of minimal solutions with rational slope. Obviously this procedure leads to a uniqueness problem: To what extent do these limit sets of minimal solutions with irrational slope depend on the approximating sequence? Here we show that for every $\alpha\in\mathbf{R}^n\setminus\mathbf{Q}^n$ this limit set is essentially unique. Though the problem at hand is elliptic the minimality and the boundary conditions at infinity allow us to state this uniqueness in a form which is reminiscent of a 1-dimensional initial value problem:

THEOREM. *Suppose $\bar\alpha=(-\alpha, 1)\in\mathbf{R}^{n+1}$ is rationally independent and $x_0\in\mathbf{R}^n$. Then for every $u_0\in\mathbf{R}$ there exists at most one minimal solution u without selfintersections such that $u(x_0)=u_0$ and u has slope α.*

From Moser's work [9] it is known that the set of $u_0\in\mathbb{R}$ for which such solution does exist contains at least a Cantor set. For more details and for the case that α is irrational but $\bar\alpha=(-\alpha, 1)$ is rationally dependent see Section 5.

For integrands F depending on u_x only Moser ([9], Theorem (2.3)) proved that every minimal solution without selfintersections is affine. This implies the theorem stated above in this particular case.

In our proof we use Moser's estimates which are based on the delicate regularity theory for minima of such problems, see e.g. the book by Ladyzhenskaya and Ural'tseva [6]. Apart from this the methods are elementary. They rely on the interplay of the $\mathbf{Z}^{n+1}$-action and the maximum principle.

All this work is inspired by results on the case $n=1$, in particular by Hedlund's investigations [5] on geodesics on a 2-dimensional Riemannian torus and by Mather's [7] and Aubry/LeDaeron's [1] work on invariant subsets of monotone twist maps. In particular, the uniqueness theorem stated above is due to Aubry/LeDaeron in the case $n = 1$. For a survey of these topics see [10].

A direct generalization of Hedlund's ideas would consist in considering hypersurfaces in $\mathbf{R}^{n+1}$ which are (homotopically or absolutely) area minimizing with respect to a $\mathbf{Z}^{n+1}$-periodic Riemannian metric on $\mathbf{R}^{n+1}$. This entails the wellknown difficulty that one has to handle parametric hypersurfaces instead of graphs. For $n=2$ such generalization to homotopically area minimizing surfaces is possible and we intend to describe this in a subsequent paper. The starting point is the work by Freedman/Hass/Scott [3] and Schoen/Yau [11] in which the periodic case is treated.

The paper is organized as follows: In Section 2 we fix the notation and present Moser's variational problem. Section 3 gives a survey of Moser's results [9]. At the end of Section 3 we can state our goal in a precise form. In Section 4 we describe a new approach to the basic invariant, the rotation vector of a minimal solution without self-intersections. Section 5 contains the statements of our results which will be proved in Section 6. In Section 7 we indicate some directions for future research and open problems.

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2. The variational problem

In this section we describe the setting of the problem which is due to J. Moser [9]. Moreover we briefly survey the results on partial differential equations that we need.

The coordinates of a point in $\mathbf{R}^{2n+1}$ will be denoted by $(x, x_{n+1}, p) = (\bar{x}, p)$ where $x \in \mathbf{R}^n$, $x_{n+1} \in \mathbf{R}$, $\bar{x} = (x, x_{n+1}) \in \mathbf{R}^{n+1}$ and $p \in \mathbf{R}^n$. The integrand of our variational problem is a function $F: \mathbf{R}^{2n+1} \to \mathbf{R}$ with the following properties, cf. [9], (3.1).

(F_1) $F \in C^{2,\varepsilon}(\mathbf{R}^{2n+1})$ for some $\varepsilon > 0$.
(F_2) F is $\mathbf{Z}^{n+1}$-periodic in $\bar{x}$, i.e. $F(\bar{x} + \bar{k}, p) = F(\bar{x}, p)$ for all $\bar{k} \in \mathbf{Z}^{n+1}$.
(F_3) $\delta\,|\xi|^2 \leq \sum_{\mu,\nu=1}^{n} F_{p_\mu p_\nu}(\bar{x}, p)\xi_\mu\xi_\nu \leq \delta^{-1}\,|\xi|^2$ for some $\delta \in (0, 1)$.
(F_4) $|F_{\bar{x}p}(\bar{x}, p)| \leq c(1 + |p|)$ and $|F_{\bar{x}\bar{x}}(\bar{x}, p)| \leq c(1 + |p|^2)$ for some $c > 0$.

According to (F_2) we can consider F as a function on $T^{n+1} \times \mathbf{R}^n$ where $T^{n+1} = \mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ denotes the $(n+1)$-torus. (F_3) ensures the ellipticity of the Euler equation corresponding to F. Obviously (F_3) implies that F grows quadratically in $|p|$, i.e. there exist $\delta_0 \in (0, 1)$, $c_0 \geq 0$ such that

(2.1) $$\delta_0\,|p|^2 - c_0 \leq F(\bar{x}, p) \leq \delta_0^{-1}\,|p|^2 + c_0$$

We want to consider integrals like

$$\int_\Omega F(x, u(x), u_x(x))\,dx$$

where $\Omega \subseteq \mathbf{R}^n$ is open and $u : \Omega \to \mathbf{R}$. According to (2.1) it is reasonable to work in the Sobolev space $W^{1,2}_{\text{loc}}(\Omega)$ of all $u \in L^2_{\text{loc}}(\Omega)$ with (distributional) first derivatives in $L^2_{\text{loc}}(\Omega)$. The subspace of all $\phi \in W^{1,2}_{\text{loc}}(\Omega)$ with compact support will be denoted by $W^{1,2}_{\text{comp}}(\Omega)$.

We are interested in functions u which minimize $\int F(x, u, u_x)\,dx$ in the following global sense:

(2.2) DEFINITION. A function $u \in W^{1,2}_{\text{loc}}(\mathbf{R}^n)$ is a minimal solution of the variational problem with integrand F (briefly: u is minimal) if for all $\phi \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} (F(x, u + \phi, u_x + \phi_x) - F(x, u, u_x))\,dx \geq 0.$$

If $\Omega \subseteq \mathbf{R}^n$ is open we say that $u \in W^{1,2}_{\text{loc}}(\Omega)$ is minimal in Ω if for all $\phi \in W^{1,2}_{\text{comp}}(\Omega)$

$$\int_\Omega (F(x, u + \phi, u_x + \phi_x) - F(x, u, u_x))\,dx \geq 0.$$

According to (F_2) we have a $\mathbf{Z}^{n+1}$-action T on the set of minimal solutions: If $u \in W^{1,2}_{\text{loc}}(\mathbf{R}^n)$ and $\bar{k} = (k, k_{n+1}) \in \mathbf{Z}^{n+1}$ let $T_{\bar{k}}u \in W^{1,2}_{\text{loc}}(\mathbf{R}^n)$ be defined by

(2.3) $$(T_{\bar{k}}u)(x) = u(x - k) + k_{n+1}$$

Obviously (F_2) implies that $T_{\bar{k}}u$ is minimal if u is minimal. The action of $T_{\bar{k}}$ corresponds to translation of graph (u) by $\bar{k} \in \mathbf{Z}^{n+1}$. As a consequence of ($F_3$) and ($F_4$) every $u \in W^{1,2}_{\text{loc}}(\Omega)$ which is minimal in Ω inherits regularity from the integrand F, i.e. if $F \in C^{l,\varepsilon}(\mathbf{R}^{2n+1})$ and $l \geq 2$ then $u \in C^{l,\varepsilon}(\Omega)$.

This regularity result is proved in three steps:
First one uses minimality to show Hölder continuity, in particular local boundedness of u. For this we only need property (2.1). Then one can use the regularity theory for bounded solutions of quasi-linear elliptic equations (the Euler equation of our problem) to obtain $u \in C^{1,\varepsilon}(\Omega)$. Here one uses ($F_3$) and ($F_4$). Now $C^{1,\varepsilon}$-regularity allows us to reduce the regularity problem to the wellknown linear case. For details we refer to the books by Ladyzhenskaya/Ural'tseva [6] and by Giaquinta [4].

The preceding discussion shows that every minimal solution is a classical solution of the quasi-linear elliptic Euler equation of our problem. So we have the following maximum principle:

(2.4) LEMMA. *Suppose $u, v \in W^{1,2}_{\text{loc}}(\Omega)$ are minimal in the connected open set $\Omega \subseteq \mathbf{R}^n$. If $u \leq v$ then either $u = v$ or $u < v$.*

A detailed proof of (2.4) is given in [9], Section 4.

3. Moser's results

We survey those notions and results from [9] whch are necessary to understand this paper. For the convenience of the reader we freely weaken or omit statements from [9].

In the present noncompact situation the existence of minimal solutions poses a nontrivial problem. Moser's work deals with existence and properties of a distinguished class of minimal solutions which we define below.

Note that $C^0(\mathbf{R}^n)$ is partially ordered by setting $u < v$ if and only if $u(x) < v(x)$ for all $x \in \mathbf{R}^n$. The $\mathbf{Z}^{n+1}$-action (2.3) preserves this order.

(3.1) DEFINITION. A function $u \in C^0(\mathbf{R}^n)$ is said not to have selfintersections if the T-orbit of u is totally ordered, i.e. for all $\bar{k} \in \mathbf{Z}^{n+1}$ we have $T_{\bar{k}}u < u$ or $T_{\bar{k}}u = u$ or $T_{\bar{k}}u > u$.

Obviously this condition means that any two translates of graph (u) by integer vectors are either identical or disjoint. Put differently, the projection of graph (u) into $T^{n+1} = \mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ does not have non-trivial selfintersections.

(3.2) NOTATION: We let $\mathcal{M} = \mathcal{M}(F)$ denote the set of minimal solutions without selfintersections.

EXAMPLE. For the Dirichlet integrand $F(\bar{x}, p) = |p|^2$ the affine functions $u(x) = \alpha \cdot x + u_0$, $\alpha \in \mathbf{R}^n$, $u_0 \in \mathbf{R}$ constitute $\mathcal{M}(F)$.

This example shows that for $n \geqslant 2$ one cannot expect that every minimal solution does not have selfintersections: Every harmonic function on $\mathbf{R}^n$ which is not affine is a minimal solution for the Dirichlet integrand and has selfintersections, for details cf. [9], Section 2. For $n = 1$, however, minimal solutions (for arbitrary F) do not have selfintersections. This can be deduced from [1], see also [2]. The set $\mathcal{M}(F)$ is thought to be the natural generalization of the set of affine minimal solutions for the Dirichlet integrand (or, more generally, for integrands only depending on p).

Now suppose that $u \in C^0(\mathbf{R}^n)$ does not have selfintersections. It is not difficult to show that there exists a unique $\alpha \in \mathbf{R}^n$ such that $|u(x) - \alpha \cdot x|$ is bounded. For reasons to be discussed in Section 4 this $\alpha \in \mathbf{R}^n$ is called the rotation vector of u. In particular, $\mathcal{M}$ decomposes into the disjoint union

$$(3.3) \quad \mathcal{M} = \bigcup_{\alpha \in \mathbf{R}^n} \mathcal{M}_\alpha$$

where $\mathcal{M}_\alpha$ denotes the T-invariant set of those $u \in \mathcal{M}$ with rotation vector α.

The most important properties of minimal solutions without selfintersections can be stated as follows, cf. [9], Theorems (2.1) and (3.1):

THEOREM. *There exist constants c_1 and γ_1 such that for all $u \in \mathcal{M}_\alpha$:*

$$(3.4) \quad |u(x+y) - u(x) - \alpha \cdot y| \leqslant c_1\sqrt{1 + |\alpha|^2}$$

$$(3.5) \quad |u_x|_{C^\varepsilon} < \gamma_1$$

Here c_1 only depends on F while γ_1 depends on F and $|\alpha|$.

Geometrically (3.4) says that the distance from graph(u) to the affine hyperplane in $\mathbf{R}^{n+1}$ through $(0, u(0))$ with normal $\bar{\alpha} = (-\alpha, 1)$ is bounded uniformly for all $u \in \mathcal{M}$. By (3.5) all $u \in \mathcal{M}_\alpha$ are Lipschitz with constant γ_1. (3.5) has the following consequence, cf. [9], Corollary (3.3):

(3.6) THEOREM. *Every sequence u_i with $u_i \in \mathcal{M}_{\alpha_i}$ and both $|u_i(0)|$ and $|\alpha_i|$*

bounded contains a subsequence which is C^1-convergent on compact sets to some $u \in \mathcal{M}$.

In particular, $\mathcal{M}$ and the $\mathcal{M}_\alpha$ are closed with respect to C^1-convergence on compact sets.

Now we discuss Moser's results on the existence of minimal solutions. For $\alpha \in \mathbf{Q}^n$ set

$$\mathcal{M}_\alpha^{\text{per}} = \{u \in \mathcal{M}_\alpha \mid T_{\bar{k}} u = u \text{ for all } \bar{k} \in \mathbf{Z}^{n+1} \text{ with } k_{n+1} = k \cdot \alpha\}$$

So $u \in \mathcal{M}_\alpha$ is in $\mathcal{M}_\alpha^{\text{per}}$ if graph (u) is translated into itself by every vector $\bar{k} \in \mathbf{Z}^{n+1}$ orthogonal to $\bar{\alpha} = (-\alpha, 1)$. Hence the projection of graph (u) to T^{n+1} is an n-torus representing a prime $\mathbf{Z}$-homology class. Moser's basic existence result says:

(3.7) THEOREM. *For all $\alpha \in \mathbf{Q}^n$ we have $\mathcal{M}_\alpha^{\text{per}} \neq \varnothing$.*

Given $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ we can choose a sequence $\alpha_i \in \mathbf{Q}^n$ with $\lim \alpha_i = \alpha$. Using (3.7), the T-invariance of $\mathcal{M}_{\alpha_i}^{\text{per}}$ and (3.6) we see that there exist $u_i \in \mathcal{M}_{\alpha_i}^{\text{per}}$ such that a subsequence of u_i converges to some $u \in \mathcal{M}$. (3.4) implies $u \in \mathcal{M}_\alpha$, hence $\mathcal{M}_\alpha \neq \varnothing$. For $u \in \mathcal{M}_\alpha$, let

$$\mathcal{M}(u) \subseteq \mathcal{M}_\alpha$$

denote the closure (with respect to C^1-convergence on compact sets) of the T-orbit of u. By the maximum principle (2.4) the set $\mathcal{M}(u)$ is totally ordered. It is not difficult to conclude that for $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ the order-preserving $\mathbf{Z}^{n+1}$-action T on $\mathcal{M}(u)$ has a unique minimal set (i.e. a unique smallest closed and non-empty T-invariant set) which we denote by $\mathcal{M}^{\text{rec}}(u)$, for details see Section 4. These minimal sets are constructed and discussed in [9], Section 6. It is easily proved that one of the following alternatives is true ($\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$, $u \in \mathcal{M}_\alpha$):

(3.8) *The graphs of the minimal solutions $v \in \mathcal{M}^{\text{rec}}(u)$ form a foliation of $\mathbf{R}^{n+1}$, i.e. for every $\bar{x} \in \mathbf{R}^{n+1}$ there exists a unique $v \in \mathcal{M}^{\text{rec}}(u)$ such that $\bar{x} = (x, v(x))$.*

(3.9) *The graphs of the minimal solutions $v \in \mathcal{M}^{\text{rec}}(u)$ form a lamination of $\mathbf{R}^{n+1}$, i.e. the order preserving homeomorphism*

$$H: v \in \mathcal{M}^{\text{rec}}(u) \to v(0) \in \mathbf{R}$$

maps $\mathcal{M}^{\text{rec}}(u)$ onto a ($\mathbf{Z}$-periodic) Cantor set in $\mathbf{R}$.

Our principal aim is to prove that for all $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ and all $u, v \in \mathcal{M}_\alpha$ we have $\mathcal{M}^{\text{rec}}(u) = \mathcal{M}^{\text{rec}}(v)$.

4. The rotation vector

We describe an alternative approach to the rotation vector of a nonselfintersecting $u \in C^0(\mathbf{R}^n)$ as defined in [9]. This leads to some results on the the T-action on $\mathcal{M}_\alpha$ which will be used in the sequel. One of the advantages of the approach presented here is that it generalizes easily to the case of parametric hypersurfaces.

For every $u \in C^0(\mathbf{R}^n)$ we can consider semigroups

$$G_+(u) = \{\bar{k} \in \mathbf{Z}^{n+1} \mid T_{\bar{k}}u \geq u\}$$

and

$$G_-(u) = \{\bar{k} \in \mathbf{Z}^{n+1} \mid T_{\bar{k}}u \leq u\} = -G_+(u).$$

If u does not have selfintersections in the sense of (3.1) we have $G_+(u) \cup G_-(u) = \mathbf{Z}^{n+1}$. The subgroup

$$G_+(u) \cap G_-(u) = \{\bar{k} \in \mathbf{Z}^{n+1} \mid T_{\bar{k}}u = u\}$$

cannot have rank $n+1$ since $i \cdot \bar{e}_{n+1} \notin G_+(u) \cap G_-(u)$ for all $i \in \mathbf{Z} \setminus \{0\}$. Using these facts we will prove:

(4.1) LEMMA. *If $u \in C^0(\mathbf{R}^n)$ does not have selfintersections there exists a unique $\bar{\alpha} = (-\alpha, 1) \in \mathbf{R}^{n+1}$ such that*

$$\{\bar{k} \in \mathbf{Z}^{n+1} \mid \bar{k} \cdot \bar{\alpha} > 0\} \subseteq G_+(u) \subseteq \{\bar{k} \in \mathbf{Z}^{n+1} \mid \bar{k} \cdot \bar{\alpha} \geq 0\}$$

Remark. Alternatively one can characterize $\bar{\alpha} = (-\alpha, 1)$ by the following statement which we will mostly use:

(4.2) If $\bar{k} \in \mathbf{Z}^{n+1}$ and $\bar{k} \cdot \bar{\alpha} > 0$ then $T_{\bar{k}}u > u$.

Note, however, that $T_{\bar{k}}u > u$ will only imply $\bar{k} \cdot \bar{\alpha} \geq 0$ in general. According to (4.2) the order of the T-orbit $\{T_{\bar{k}}u \mid \bar{k} \in \mathbf{Z}^{n+1}\}$ is closely related to the order of the real numbers $\bar{k} \cdot \bar{\alpha}$, cf. the remarks following (4.7).

From (4.2) one can easily derive that $|u(x) - \alpha \cdot x|$ is bounded.

Proof of (4.1). Let $\rho:\mathbf{R}^{n+1}\backslash\{0\}\to S^n$ denote the radial projection, $\rho(\bar{x})=|\bar{x}|^{-1}\cdot\bar{x}$. For a semigroup $G\subseteq\mathbf{Z}^{n+1}$ let $C(G)\subseteq S^n$ denote the closure of the set $\rho(G\backslash\{0\})$. Then $C(G)$ is locally convex, i.e. every great-circle segment of length $<\pi$ with endpoints in $C(G)$ is completely contained in $C(G)$. We want to show that $C(G_+(u))$ is a hemisphere. It is wellknown and easy to prove that a locally convex subset C of S^n is contained in some hemisphere unless $C=S^n$. Since $C(\mathbf{Z}^{n+1})=S^n$ and $G_+(u)\cup(-G_+(u))=\mathbf{Z}^{n+1}$ we have

$$C(G_+(u))\cup(-C(G_+(u)))=S^n$$

So either $C(G_+(u))$ is a hemisphere or $C(G_+(u))=S^n$. We want to show that the assumption $C(G_+(u))=S^n$ leads to a contradiction:
Since rank $(G_+(u)\cap G_-(u))<n+1$ the set $C(G_+(u)\cap G_-(u))$ is contained in a great $(n-1)$-sphere. Let H denote one of the corresponding open hemispheres. Since $C(G_+(u))=S^n$ there exist linearly independent vectors $\bar{k}_i\in G_+(u)$, $1\leq i\leq n+1$, such that $\rho(\bar{k}_i)\in H$. Denote the semigroup generated by the $\bar{k}_i$ by $G\subseteq G_+(u)$. Then $C(G)$ has non-empty interior $\operatorname{Int}(C(G))\subseteq H$. By our asssumption we have $C(G_-(u))=S^n\supseteq C(G)$. Hence there exists $\bar{k}\in G_-(u)$ with $\rho(k)\in\operatorname{Int}(C(G))\subseteq H$. But this implies $\bar{k}=\sum t_i\bar{k}_i$ for rational numbers $t_i>0$. So there exists $m\in\mathbf{N}$ such that $m\bar{k}\in G_+(u)\cap G_-(u)$ and $\rho(m\bar{k})=\rho(\bar{k})\in H$ which contradicts $C(G_+(u)\cap G_-(u))\cap H=\varnothing$. Thus $C(G_+(u))$ is indeed a hemisphere. Obviously $C(G_+(u))$ contains the coordinate vector $\bar{e}_{n+1}$ in its interior. So there exists a unique $\bar{\alpha}=(-\alpha,1)\in\mathbf{R}^{n+1}$ such that

$$C(G_+(u))=\{\bar{x}\in S^n\mid\bar{x}\cdot\bar{\alpha}\geq 0\}$$

In particular, $\bar{k}\in G_+(u)$ implies $\bar{k}\cdot\bar{\alpha}\geq 0$. Since $G_+(u)\cup(-G_+(u))=\mathbf{Z}^{n+1}$ we conclude that $\bar{k}\cdot\bar{\alpha}>0$ implies $\bar{k}\in G_+(u)$. This proves our claim.

The vector $\alpha\in\mathbf{R}^n$ is called the rotation vector of u in [9] since its components α_i, $1\leq i\leq n$, are the rotation numbers of the "generalized circle maps" τ_i which map the set $\{u(k)+k_{n+1}\mid\bar{k}\in\mathbf{Z}^{n+1}\}\subseteq\mathbf{R}$ onto itself by

$$\tau_i(u(k)+k_{n+1})=u(k+e_i)+k_{n+1},$$

cf. [9], Appendix to Section 2. In this approach property (4.2) corresponds to [9], Lemma (6.1).

(4.3) DEFINITION. Suppose $\alpha\in\mathbf{R}^n\backslash\mathbf{Q}^n$, $u\in\mathcal{M}_\alpha$ and $\bar{\alpha}=(-\alpha,1)$. We say that u can be approximated from above (resp. from below) if

$$u=\inf\{T_{\bar{k}}u\mid\bar{k}\cdot\bar{\alpha}>0\}$$

(resp. if $u = \sup\{T_{\bar{k}}u \mid \bar{k} \cdot \bar{\alpha} < 0\}$). If u can be approximated from above or from below u is called recurrent.

We let $\mathcal{M}_\alpha^{\mathrm{rec}}$ denote the set of recurrent elements of $\mathcal{M}_\alpha$, $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$. For $\alpha \in \mathbf{Q}^n$ it is reasonable to define $\mathcal{M}_\alpha^{\mathrm{rec}} := \mathcal{M}_\alpha^{\mathrm{per}}$. We set $\mathcal{M}^{\mathrm{rec}} = \bigcup_{\alpha \in \mathbf{R}^n} \mathcal{M}_\alpha^{\mathrm{rec}}$.

Remarks. 1. If $u \in \mathcal{M}_\alpha$ then the set

$$\mathcal{M}(u) = \text{closure of the } T\text{-orbit of } u$$

is totally ordered and every sequence in $\mathcal{M}(u)$ which is decreasing (resp. increasing) and bounded below (resp. above) C^1-converges on compact sets, cf. (2.4) and (3.6). So u can be approximated from above if and only if there exists a sequence $\bar{k}_i \in \mathbf{Z}^{n+1}$ such that $\bar{k}_i \cdot \bar{\alpha}_i > 0$ and $T_{\bar{k}_i}u$ C^1-converges to u on compact sets.

2. If $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ but $(-\alpha, 1) = \bar{\alpha}$ is rationally dependent then our definition of recurrence is more restrictive than [9], Definition (6.4).

The following lemma implies that for all $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ and all $u \in \mathcal{M}_\alpha$ the set $\mathcal{M}^{\mathrm{rec}}(u) := \mathcal{M}^{\mathrm{rec}} \cap \mathcal{M}(u)$ is the unique minimal set of the $\mathbf{Z}^{n+1}$-action T on $\mathcal{M}(u)$. This accounts for the term "recurrent".

(4.4) LEMMA. *Suppose $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ and $u \in \mathcal{M}_\alpha$ can be approximated from above (resp. from below). Then for every $v \in \mathcal{M}(u)$ we have*

$$u = \inf\{T_{\bar{k}}v \mid T_{\bar{k}}v > u\}$$

(resp. $u = \sup\{T_{\bar{k}}v \mid T_{\bar{k}}v < u\}$*).*

Note. More generally (4.4) holds for all $v \in \mathcal{M}_\alpha$ for which $\mathcal{M}(v) \cup \mathcal{M}(u)$ is totally ordered.

Proof. Suppose $u \in \mathcal{M}_\alpha$ can be approximated from below and $\mathcal{M}(u) \cup \mathcal{M}(v)$ is totally ordered. Define $\bar{v} = \sup\{T_{\bar{k}}v \mid T_{\bar{k}}v < u\}$ and assume $\bar{v} < u$. Since $\mathcal{M}(u) \cup \mathcal{M}(v)$ is totally ordered and since u can be approximated from below there exists $\bar{h} \in \mathbf{Z}^{n+1}$ such that $\bar{h} \cdot \bar{\alpha} < 0$ and $\bar{v} < T_{\bar{h}}u < u$. Setting $\bar{k} = -\bar{h}$ we obtain $T_{\bar{k}}\bar{v} < u$ and $\bar{k} \cdot \bar{\alpha} > 0$. But this contradicts the definition of $\bar{v}$: If $\bar{v} = \lim T_{\bar{k}_i}v$ with $T_{\bar{k}_i}v \le \bar{v}$ we have $\bar{v} < T_{\bar{k}+\bar{k}_i}v < u$ for all sufficiently large $i \in \mathbf{N}$. So $\bar{v} < u$ is not true. Now the definition of $\bar{v}$ and the maximum principle (2.4) show that $\bar{v} = u$.

At this stage it is easy to prove that $\mathcal{M}^{\mathrm{rec}}(u)$ corresponds either to a foliation (3.8) or to a lamination (3.9) of $\mathbf{R}^{n+1}$. In the second case $H: \mathcal{M}^{\mathrm{rec}}(u) \to \mathbf{R}$,

$H(v)=v(0)$ maps $\mathcal{M}^{\rm rec}(u)$ homeomorphically onto a Cantor set $C\subseteq\mathbf{R}$. The endpoints of intervals in $\mathbf{R}\backslash C$ correspond to elements in $\mathcal{M}^{\rm rec}(u)$ which can be approximated only from above or only the from below. The – uncountably many – other elements of C correspond to minimal solution in $\mathcal{M}^{\rm rec}(u)$ which can be approximated both from above and from below.

If $\bar\alpha$ is rationally independent (i.e. $\bar\alpha\cdot\bar k=0$ and $\bar k\in\mathbf{Z}^{n+1}$ imply $\bar k=0$) "neighboring" recurrent solutions converge to each other for $|x|\to\infty$ (cf. [9], Section 6):

(4.5) LEMMA. *Suppose $\bar\alpha=(\alpha,-1)$ is rationally independent and we have $v_0, v_1\in\mathcal{M}^{\rm rec}(u)\subseteq\mathcal{M}_\alpha$ such that $v_0<v_1$ and there does not exist $w\in\mathcal{M}^{\rm rec}(u)$ with $v_0<w<v_1$. Then*

$$\int_{\mathbf{R}^n}(v_1-v_0)\,dx\leqslant 1.$$

Proof. Let $\sigma=\{(x,x_{n+1})\in\mathbf{R}^{n+1}\mid v_0(x)<x_{n+1}<v_1(x)\}$. Then σ does not intersect any of its translates $\sigma+\bar k$, $\bar k\in\mathbf{Z}^{n+1}\backslash\{0\}$. Hence σ projects injectively to T^{n+1}, so that $\mathrm{vol}_{n+1}(\sigma)=\int_{\mathbf{R}^n}(v_1-v_0)\,dx\leqslant\mathrm{vol}\,(T^{n+1})=1$.

Recurrent minimal solutions are as periodic as possible:

(4.6) LEMMA. *Suppose $\alpha\in\mathbf{R}^n\backslash\mathbf{Q}^n$ and $u\in\mathcal{M}^{\rm rec}_\alpha$. Then $T_{\bar k}u=u$ for all $\bar k\in\mathbf{Z}^{n+1}$ with $\bar k\cdot\bar\alpha=0$.*

Proof. Suppose u can be approximated from below. If $\bar h\in\mathbf{Z}^{n+1}$ and $\bar h\cdot\bar\alpha=0$ then $T_{\bar h}u\geqslant u$. Otherwise there would exist $\bar k\in\mathbf{Z}^{n+1}$ with $\bar k\cdot\bar\alpha<0$ and $T_{\bar h}u<T_{\bar k}u$, hence $T_{(\bar h-\bar k)}u<u$. This contradicts $(\bar h-\bar k)\cdot\bar\alpha=-\bar k\cdot\bar\alpha>0$. Replacing $\bar h$ by $-\bar h$ we obtain $T_{\bar h}u=u$.

The following lemma will be crucial since it allows us to use compactness arguments in $\mathcal{M}^{\rm rec}_\alpha$, cf. the proof of (6.4) and (6.6).

(4.7) LEMMA. *Suppose $\alpha\in\mathbf{R}^n\backslash\mathbf{Q}^n$ and $u, v\in\mathcal{M}^{\rm rec}_\alpha$ can both be approximated from below. Let sequences $\bar k_i\in\mathbf{Z}^{n+1}$, $\bar h_i\in\mathbf{Z}^{n+1}$ be given such that*

$$\lim T_{\bar k_i}u=\bar u,\ \lim T_{\bar k_i}v=\bar v\quad\text{and}\quad T_{\bar h_i}\bar u<u,\ \lim T_{\bar h_i}\bar u=u.$$

Then we have

$$\lim T_{\bar h_i}\bar v=v.$$

Note. An analogous result holds if u and v can both be approximated from above.

Proof. Fix some $i \in \mathbf{N}$. Because of $T_{\bar{h}_i}\bar{u} < u$ and $\lim T_{\bar{k}_j}u = \bar{u}$ we have $T_{(\bar{h}_i+\bar{k}_j)}u < u$ for almost all $j \in \mathbf{N}$. According to (4.6) this implies $(\bar{h}_i + \bar{k}_j) \cdot \bar{\alpha} < 0$ and hence $T_{\bar{h}_i}\bar{v} = \lim_{j\to\infty} T_{(\bar{h}_i+\bar{k}_j)}v \leqslant v$. The sequence $(T_{\bar{h}_i}\bar{v})(0)$ is easily seen to be bounded. So, by (3.6), we may assume that $T_{\bar{h}_i}\bar{v}$ converges to some $w \leqslant v$ and we have to prove that $w = v$. If $w < v$ our hypothesis on v says that there exists $\bar{k} \in \mathbf{Z}^{n+1}$ such that $\bar{k} \cdot \bar{\alpha} < 0$ and $w < T_{\bar{k}}v < v$. Now $T_{(-\bar{k})}w < v$ implies $T_{(\bar{h}_i-\bar{k})}\bar{v} < v$ for almost all $i \in \mathbf{N}$. The arguments used above show $T_{(\bar{h}_i-\bar{k})}\bar{u} \leqslant u$ and hence $T_{(-\bar{k})}u \leqslant u$. But this contradicts $(-\bar{k}) \cdot \bar{\alpha} > 0$.

Statement and proof of Lemma (4.7) may look somewhat mysterious. For readers familiar with Denjoy theory the following observation may clarify things: The map $h_u : \mathcal{M}(u) \to \mathbf{R}$, $h_u(\bar{u}) = \sup\{\bar{k} \cdot \bar{\alpha} \mid T_{\bar{k}}u \leqslant \bar{u}\}$, is continuous, non-decreasing and satisfies $h_u(T_{\bar{k}}\bar{u}) = h_u(\bar{u}) + \bar{k} \cdot \bar{\alpha}$. If h_u is strictly increasing ($\Leftrightarrow \mathcal{M}^{\text{rec}}(u)$ foliates $\mathbf{R}^{n+1}$) then h_u conjugates the $\mathbf{Z}^{n+1}$-action on $\mathcal{M}(u)$ to the action of $\mathbf{Z}^{n+1}$ on $\mathbf{R}$ given by $t \in \mathbf{R} \to t + \bar{k} \cdot \bar{\alpha}$. So, if both $\mathcal{M}^{\text{rec}}(u)$ and $\mathcal{M}^{\text{rec}}(v)$ foliate $\boldsymbol{R}^{n+1}$ then the actions of $\mathbf{Z}^{n+1}$ on $\mathcal{M}(u)$ and $\mathcal{M}(v)$ are conjugate, so that (4.7) is obvious (actually we will later see that in this case $\mathcal{M}(u) = \mathcal{M}^{\text{rec}}(u) = \mathcal{M}^{\text{rec}}(v) = \mathcal{M}(v)$). But even in the general case one can use h_u and h_v to show that the $\mathbf{Z}^{n+1}$-action on $\mathcal{M}^{\text{rec}}(u)$ and $\mathcal{M}^{\text{rec}}(v)$ are similar to a certain extent and this is precisely the meaning of (4.7).

5. Statement of the results

The most interesting feature of the minimal solutions without selfintersections is that they are natural generalizations of the affine minimal solutions $u(x) = \alpha \cdot x + u_0$ of a variational problem with integrand $F = F(p)$ not depending on $\bar{x}$. Now "how natural" these solutions in $\mathcal{M}_\alpha$, $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$, really are depends on an answer to the following question which remains open in [9].

Is $\mathcal{M}^{\text{rec}}(u)$ independent of $u \in \mathcal{M}_\alpha$?

If the answer is "no" there will be many disjoint minimal sets of type $\mathcal{M}^{\text{rec}}(u)$ in $\mathcal{M}_\alpha$. These will contain functions whose graphs intersect and $\mathcal{M}_\alpha^{\text{rec}}$ will be a very complicated set unlike the set of affine functions with fixed slope α. However, this complicated situation does not occur:

(5.1) THEOREM. *For every $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ the set $\mathcal{M}_\alpha^{\text{rec}}$ is totally ordered.*

So, as in the case of affine functions of fixed slope, if $u, v \in \mathcal{M}_\alpha^{\text{rec}}$ and $u(0) < v(0)$ then $u(x) < v(x)$ for all $x \in \mathbf{R}^n$. As a simple consequence of (5.1) the $\mathbf{Z}^{n+1}$-action on $\mathcal{M}_\alpha$ has a unique minimal set:

(5.2) COROLLARY. *If* $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ *and* $u \in \mathcal{M}_\alpha$ *then* $\mathcal{M}^{\text{rec}}(u) = \mathcal{M}_\alpha^{\text{rec}}$.

Proof that (5.1) *implies* (5.2). According to the note following (4.4) we can apply (4.4) to all $u, v \in \mathcal{M}_\alpha^{\text{rec}}$ since $\mathcal{M}_\alpha^{\text{rec}}$ is totally ordered. Hence $\mathcal{M}_\alpha^{\text{rec}} = \mathcal{M}^{\text{rec}}(v)$ for all $v \in \mathcal{M}_\alpha^{\text{rec}}$. If $w \in \mathcal{M}_\alpha \setminus \mathcal{M}_\alpha^{\text{rec}}$ and $v \in \mathcal{M}^{\text{rec}}(w)$ then $\mathcal{M}^{\text{rec}}(v) = \mathcal{M}^{\text{rec}}(w)$. Hence $\mathcal{M}_\alpha^{\text{rec}} = \mathcal{M}^{\text{rec}}(w)$ for all $w \in \mathcal{M}_\alpha$.

The methods used in the proof of (5.1) easily yield the following stronger version of (5.1) for generic α.

(5.3) THEOREM. *If* $\bar{\alpha} = (-\alpha, 1)$ *is rationally independent then* $\mathcal{M}_\alpha$ *is totally ordered.*

To put (5.1)–(5.3) into perspective we compare with the corresponding facts for $\mathcal{M}_\alpha^{\text{per}} =: \mathcal{M}_\alpha^{\text{rec}}$ if $\alpha \in \mathbf{Q}^n$. The analogue of (5.1) is true, cf. [9], Theorem (5.2). This will also follow from the proof of (5.1). The analogue of (5.2) will not always be true: The set $\{u \mid u(x) = \alpha \cdot x + u_0\}$ consists of uncountably many discrete T-orbits if $\alpha \in \mathbf{Q}^n$. However, for "generic" F there will only be one T-orbit in $\mathcal{M}_\alpha^{\text{per}}$. Finally, Morse's work [8] shows that for $n = 1$ and $\alpha \in \mathbf{Q}$ the set $\mathcal{M}_\alpha$ will in general not be totally ordered, cf. also [2], Section 5.

As a simple consequence of Corollary (5.2) we obtain:

(5.4) COROLLARY. *Every recurrent* $u \in \mathcal{M}^{\text{rec}}$ *can be approximated by periodic minimal solutions without selfintersections.*

Proof. As always we talk about the topology of C^1-convergence on compact sets. For $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ let $\tilde{\mathcal{M}}_\alpha$ denote the set of $v \in \mathcal{M}_\alpha$ which can be approximated by periodic solutions in $\mathcal{M}$. Then $\tilde{\mathcal{M}}_\alpha \neq \varnothing$ is closed and invariant under the $\mathbf{Z}^{n+1}$-action. Since $\mathcal{M}_\alpha^{\text{rec}}$ is the unique minimal set of this action restricted to $\mathcal{M}_\alpha$ we have $\mathcal{M}_\alpha^{\text{rec}} \subseteq \tilde{\mathcal{M}}_\alpha$.

6. Proofs of the theorems

We introduce the following abbreviations:
For open sets $\Omega \subseteq \mathbf{R}^n$ and $u \in W^{1,2}_{\text{loc}}(\Omega)$ we set

$$I(u, \Omega) := \int_\Omega F(x, u, u_x)\, dx$$

$$D(u, \Omega) := \sup\left\{\int_\Omega (F(x, u, u_x) - F(x, u + \phi, u_x + \phi_x))\, dx \mid \phi \in W^{1,2}_{\text{comp}}(\Omega)\right\}$$

provided $I(u, \Omega)$ and $D(u, \Omega)$ exist as extended real numbers. Obviously $D(u, \Omega) = 0$ if u is minimal in Ω and $D(u, \Omega) > 0$ otherwise.

The maximum principle (2.4) implies:

(6.1) LEMMA. *If $u \neq v : \Omega \to \mathbf{R}$ are minimal in a connected open set $\Omega \subseteq \mathbf{R}^n$ and $u(x) = v(x)$ for some $x \in \Omega$ then*

$$D(\max(u, v), \Omega) > 0 \quad \textit{and} \quad D(\min(u, v), \Omega) > 0.$$

Before we start with the details we outline the proof of Theorem (5.1): We assume that $u \neq v$ are in $\mathcal{M}_\alpha^{\mathrm{rec}}$, $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$, and that $u(x) = v(x)$ for some $x \in \mathbf{R}^n$. We want to show that this contradicts the minimality of u and v. Lemma (6.1) can be used to prove this in the special case that $v > u$ holds only on a bounded set $B \subseteq \mathbf{R}^n$. This case is of no particular importance for the rest of the proof but it can be used to illustrate the general idea and the difficulties that we have to overcome: In this case $\max(u, v)$ is a compactly supported variation of u and $\min(u, v)$ is a compactly supported variation of v. Hence the minimality of u and v implies that for all connected open sets $\Omega \supseteq \bar{B}$:

$$(6.2) \quad \begin{aligned} I(\max(u, v), \Omega) &= I(u, \Omega) + D(\max(u, v), \Omega) \\ I(\min(u, v), \Omega) &= I(v, \Omega) + D(\min(u, v), \Omega) \end{aligned}$$

On the other hand the following equation is true quite generally

$$(6.3) \quad I(\max(u, v), \Omega) + I(\min(u, v), \Omega) = I(u, \Omega) + I(v, \Omega)$$

Obviously (6.2) and (6.3) contradict (6.1).

In general we have to cope with the difficulty that every component of the set $\{x \in \mathbf{R}^n \mid u(x) \neq v(x)\}$ might be non-compact. In this case there are two effects which work against each other and we want to show that the balance is in our favor:

a) First, and this is favorable, Lemma (6.1) says that we can reduce $I(\max(u, v), \Omega)$ and $I(\min(u, v), \Omega)$ by compactly supported variations.
b) However, in order to use the minimality of u and v we have to change $\max(u, v)$ resp. $\min(u, v)$ so that they coincide with u resp. v outside some large compact set. This has the negative effect to increase the integrals on the left hand sides of (6.2).

We will show that the increase on the left hand side of (6.2) can be estimated above by const. $\cdot\, r^{n-1}$ if we change $\max(u, v)$ and $\min(u, v)$ only outside the ball $B(0, r)$ of radius r about $0 \in \mathbf{R}^n$. On the other hand, a quantitative version of (6.1) will show that $D(\max(u, v), B(0, r))$ and $D(\min(u, v), B(0, r))$ can be estimated below by $\delta \cdot r^n$ for $r \geqslant r_0$ and for some $\delta > 0$. So, for large $r \geq r_0$ the balance $2\delta \cdot r^n - \text{const.} \cdot r^{n-1}$ will be positive and this will contradict the minimality of u and v.

Now we start with the details. First note that we may assume that both u and v can be approximated from below: Since $u, v \in \mathcal{M}_\alpha^{\text{rec}}$ we can approximate u resp. v by sequences u_i resp. v_i where all the u_i and v_i can be approximated from below, cf. (4.4). Using the maximum principle we see that graph $(u_i) \cap$ graph $(v_i) \neq \emptyset$ if i is large enough. So we can replace u, v by u_i, v_i.

Next we use (4.7) to show that graph $(u) \cap$ graph (v) is actually a large set:

(6.4) LEMMA. *There exists $r_0 > 0$ such that every ball of radius $r \geqslant r_0$ contains a point x with $u(x) = v(x)$.*

Proof. Otherwise there exists a sequence of balls $B(x_i, i)$ where $x_i \in \mathbf{R}^n$, $i \in \mathbf{N}$ such that $u - v$ does not change sign on $B(x_i, i)$, say $u > v$ on $B(x_i, i)$. Choose a sequence $\bar{k}_i = (k_i, j_i) \in \mathbf{Z}^n \times \mathbf{Z}$ such that $x_i + k_i \in [0, 1)^n$ and such that (a subsequence of) $T_{\bar{k}_i}u$ and $T_{\bar{k}_i}v$ converge, say $\lim T_{\bar{k}_i}u = \bar{u}$, $\lim T_{\bar{k}_i}v = \bar{v}$. This is possible by (3.6). Since $T_{\bar{k}_i}u > T_{\bar{k}_i}v$ on $B(x_i + k_i, i)$ we have either $\bar{u} = \bar{v}$ or $\bar{u} > \bar{v}$. But then (4.4) and (4.7) imply that $u = v$ or $u > v$ which contradicts our hypothesis on u and v.

We need the following semi-continuity property of $D(u, \Omega)$:

(6.5) LEMMA. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded and let $w_i : \Omega \to \mathbf{R}$ be a sequence of functions with uniform Lipschitz constant L. Suppose the w_i converge with their first derivatives almost everywhere to $w : \Omega \to \mathbf{R}$. Then $D(w, \Omega) \leqslant$ $\liminf D(w_i, \Omega)$.*

Proof. By Lebesgue's theorem on dominated convergence we have

$$I(w, \Omega) = \lim I(w_i, \Omega)$$

So it remains to prove that for every $\phi \in W^{1,2}_{\text{comp}}(\Omega)$ there exists a sequence $\phi_i \in W^{1,2}_{\text{comp}}(\Omega)$ such that

$$(*) \quad I(w + \phi, \Omega) \geqslant \limsup I(w_i + \phi_i, \Omega)$$

We choose $\delta > 0$ such that $\phi(x) = 0$ if $x \in \Omega$ and dist $(x, \partial\Omega) \leqslant 2\delta$. There exists a Lipschitz function $\lambda : \Omega \to [0, 1]$ such that $\lambda(x) = 0$ if dist $(x, \partial\Omega) \leqslant \delta$ and $\lambda(x) = 1$ if dist $(x, \partial\Omega) \geqslant 2\delta$. We set $\phi_i := \lambda(w - w_i + \phi)$ so that $\phi_i \in W^{1,2}_{\text{comp}}(\Omega)$. Since $w_i + \phi_i = \lambda w + (1 - \lambda)w_i$ and λ is Lipschitz the $w_i + \phi_i$ converge with their first derivatives almost everywhere to $w + \phi$ and the dominated convergence theorem applies. So

$$I(w + \phi, \Omega) = \lim I(w_i + \phi_i, \Omega)$$

and this proves $(*)$.

Now we return to $u, v \in \mathcal{M}_\alpha^{\mathrm{rec}}$ given above and prove a uniform version of (6.1):

(6.6) LEMMA. *There exist* $\varepsilon > 0$ *and* $r_1 > 0$ *such that for all* $x \in \mathbf{R}^n$ *with* $u(x) = v(x)$:

$$D(\max(u, v), B(x, r_1)) > \varepsilon \quad \textit{and} \quad D(\min(u, v), B(x, r_1)) > \varepsilon$$

Proof. We use a similar argument as in the proof of (6.4). If (6.6) is not true there exists a sequence $x_i \in \mathbf{R}^n$ such that $u(x_i) = v(x_i)$ and, e.g.,

$$\lim (D(\max(u, v), B(x_i, i))) = 0$$

We choose a sequence $\bar{k}_i = (k_i, j_i) \in \mathbf{Z}^n \times \mathbf{Z}$ such that $y_i = x_i + k_i \in [0, 1)^n$ and such that (a subsequence of) $T_{\bar{k}_i}u$, $T_{\bar{k}_i}v$ converge, say $\lim T_{\bar{k}_i}u = \bar{u}$, $\lim T_{\bar{k}_i}v = \bar{v}$. By $\mathbf{Z}^{n+1}$-invariance (F_2) we have

$$(*) \quad \lim D(\max(T_{\bar{k}_i}u, T_{\bar{k}_i}v), B(y_i, i)) = 0$$

By (3.5) we know that the sequence $\max(T_{\bar{k}_i}u, T_{\bar{k}_i}v)$ is uniformly Lipschitz continuous. It is easy to see that $\max(T_{\bar{k}_i}u, T_{\bar{k}_i}v)$ converges together with the first derivatives almost everywhere to $\max(\bar{u}, \bar{v})$.

So (6.5) and $(*)$ imply

$$D(\max(\bar{u}, \bar{v}), B(0, r)) = 0$$

for all $r > 0$. Since $\bar{u}$ and $\bar{v}$ coincide at every accumulation point of the sequence y_i Lemma (6.1) yields $\bar{u} = \bar{v}$. As in (6.4) we can use (4.4) and (4.7) to conclude that $u = v$, contrary to our hypothesis.

Lemmas (6.4) and (6.6) combine to complete the first part of the proof of (5.1):

(6.7) LEMMA. *There exist* $\delta > 0$ *and* $r_2 > 0$ *such that for all* $r \geqslant r_2$:

$$D(\max(u, v), B(0, r)) > \delta \cdot r^n \quad \textit{and} \quad D(\min(u, v), B(0, r)) > \delta \cdot r^n.$$

We prove the first inequality: According to (6.4) and (6.6) there exists a constant $c = c(n, r_0, r_1) > 0$ such that for every $r \geqslant r_2 := r_0 + r_1$ the ball $B(0, r)$ contains at least $c \cdot r^n$ disjoint balls of radius r_1 such that u and v coincide at the centers of these balls. Now our claim follows from (6.6) with $r_2 = r_0 + r_1$ and $\delta = \varepsilon \cdot c$.

In the second and last part of the proof of (5.1) we construct functions $w_r^+, w_r^- \in W^{1,2}_{\text{loc}}(\mathbf{R}^n)$ which coincide with $\max(u, v)$ resp. $\min(u, v)$ on $B(0, r)$ and with u resp. v outside some compact set. If we can achieve this so that there exists $A > 0$ such that for all $r \geqslant 1$

$$\int_{\mathbf{R}^n} (F(x, w_r^+, (w_r^+)_x) - F(x, u, u_x))\, dx \leqslant A \cdot r^{n-1}$$

and

$$\int_{\mathbf{R}^n} (F(x, w_r^-, (w_r^-)_x) - F(x, v, v_x))\, dx \leqslant A \cdot r^{n-1}$$

our proof will easily be completed. Here is the general construction of such $w_r^{\pm}$:

(6.8) LEMMA. *Let $w_1, w_2 : \mathbf{R}^n \to \mathbf{R}$ have Lipschitz constant L and suppose $0 \leqslant w_2 - w_1 \leqslant C$. Then for all $r \geqslant 1$ there exists $w : \mathbf{R}^n \to \mathbf{R}$ such that*

(a) *w is Lipschitz with constant* $2L + 1$,
(b) $w \mid B(0, r) = w_2 \mid B(0, r)$,
(c) *$w = w_1$ outside some compact set,*
(d) $\operatorname{vol}_n(\{x \in \mathbf{R}^n \mid |x| \geqslant r$ *and* $w(x) \neq w_1(x)\}) \leqslant (1 + C)^{n-1} \int_{\partial B(0,r)} (w_2 - w_1)\, d\sigma$

Here $d\sigma$ denotes the volume element of $S^{n-1}(r) = \partial B(0, r)$.

Proof. We define $w \mid B(0, r) := w_2 \mid B(0, r)$ so that (b) is satisfied. To define w outside $B(0, r)$ let $x_t := (t + r)x/|x|$ be the radial line starting at $x_0 = r(x/|x|)$ and parameterized by arclength. We define for $x \neq 0$, $t \geqslant 0$:

$$w(x_t) := \max\{w_2(x_0) - (L + 1)t,\ w_1(x_t)\}$$

Since the radial lines are orthogonal to $\partial B(0, r)$ one easily proves that w is Lipschitz with constant $2L + 1$. Since w_1 has Lipschitz constant L we have

$$w_1(x_t) \geqslant w_2(x_0) + (w_1(x_0) - w_2(x_0)) - Lt$$

and hence

$$w(x_t) = w_1(x_t) \quad \text{if} \quad t \geqslant w_2(x_0) - w_1(x_0).$$

Since $w_2 - w_1 \leqslant C$ the functions w and w_1 coincide outside $B(0, r + C)$. Moreover, $|x| \geqslant r$ and $w(x) \neq w_1(x)$ imply

$$|x| \leqslant r + (w_2 - w_1)\left(r\frac{x}{|x|}\right) \leqslant r + c$$

Now integration in polar coordinates and a simple estimate yield (d).

As a simple consequence of (6.8) we obtain:

(6.9) LEMMA. *Under the hypotheses of* (6.8) *there exists a constant* $\tilde{A} = \tilde{A}(n, C, L, F)$, *independent of* r, *such that*:

$$\left|\int_{\mathbf{R}^n \setminus B(0,r)} (F(x, w, w_x) - F(x, w_1, (w_1)_x))\, dx\right| \leqslant \tilde{A}\int_{\partial B(0,r)} (w_2 - w_1)\, d\sigma$$

Note. We obtain the same estimate with w_1 replaced by w_2 if we require w to coincide with w_1 on $B(0, r)$ and with w_2 outside some compact set.

Proof. Since $F(x, w(x), w_x(x))$ and $F(x, w_1(x), (w_1)_x(x))$ are uniformly bounded for all $x \in \mathbf{R}^n$ our claim follows from (6.8)(d).

Finally, we complete the proof of (5.1): For given $r \geq 1$ we apply (6.9) to $w_1 = u$, $w_2 = \max(u, v)$ and obtain $w =: w_r^+$. If we write $w_r^+ = u + \phi_r^+$ then, by (6.8)(a) and (c), $\phi_r^+ \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$ and $u + \phi_r^+ = \max(u, v)$ on $B(0, r)$, by (6.8)(b). Moser's estimates (3.4) and (3.5) show that the assumptions on $w_1 = u$ and $w_2 = \max(u, v)$ are satisfied. Hence (6.9) implies that there exists $A > 0$ such that for all $r \geqslant 1$:

$$\left|\int_{\mathbf{R}^n \setminus B(0,r)} (F(x, u + \phi_r^+, (u + \phi_r^+)_x) - F(x, u, u_x))\, dx\right| \leqslant A \cdot r^{n-1}$$

If $\phi \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$ and $\Omega \subseteq \mathbf{R}^n$ we abbreviate

$$\Delta(u, \phi, \Omega) := \int_\Omega (F(x, u + \phi, (u + \phi)_x) - F(x, u, u_x))\, dx$$

So the inequality above takes the form

(6.10) $|\Delta(u, \phi_r^+, \mathbf{R}^n - B(0, r))| \leqslant A \cdot r^{n-1}$

Similarly there exists $\phi_r^- \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$ such that $v + \phi_r^- = \min(u, v)$ on $B(0, r)$ and

$$(6.10)' \quad |\Delta(v, \phi_r^-, \mathbf{R}^n \setminus B(0, r))| \leq A \cdot r^{n-1}$$

On the other hand (6.3) implies

$$(6.11) \quad I(u + \phi_r^+, B(0, r)) + I(v + \phi_r^-, B(0, r)) = I(u, B(0, r)) + I(v, B(0, r))$$

Adding (6.10), (6.10)′ and (6.11) we obtain

$$(6.12) \quad |\Delta(u, \phi_r^+, \mathbf{R}^n) + \Delta(v, \phi_r^-, \mathbf{R}^n)| \leq 2A \cdot r^{n-1}.$$

On $B(0, r)$ we have $u + \phi_r^+ = \max(u, v)$, $v + \phi_r^- = \min(u, v)$ so that (6.7) implies for all $r \geq r_2$:

$$(6.13) \quad D(u + \phi_r^+, B(0, r)) + D(v + \phi_r^-, B(0, r)) \geq 2\delta \cdot r^n.$$

Now (6.13) says that by compactly supported variations of $u + \phi_r^+$ and $v + \phi_r^-$ we can reduce the corresponding integrals by $\delta \cdot r^n$ while (6.12) says that the sum of these integrals exceeds the sum of the integrals for u and v by at most $2A \cdot r^{n-1}$. So, if $r > \max\{A/\delta, r_2\}$ we find compactly supported variations of u and v such that the sum of their integrals is reduced. This contradicts our hypothesis that both u and v are minimal and completess the proof of (5.1).

Finally we present a proof for Theorem (5.3): If $\bar{\alpha}$ is rationally independent then $\mathcal{M}_\alpha$ is totally ordered.

We argue by contradiction. In view of (5.1) we are left with the case that $u \in \mathcal{M}_\alpha \setminus \mathcal{M}_\alpha^{\text{rec}}$ and $v \in \mathcal{M}_\alpha$ coincide for some $x \in \mathbf{R}^n$, but $u \neq v$. We set

$$u^+ := \inf\{\tilde{u} \in \mathcal{M}^{\text{rec}}(u) \mid \tilde{u} > u\}$$
$$u^- := \sup\{\tilde{u} \in \mathcal{M}^{\text{rec}}(u) \mid \tilde{u} < u\}$$

Since $u \notin \mathcal{M}^{\text{rec}}(u)$ we have $u^- < u < u^+$ and, by (4.5),

$$(6.14) \quad \int_{\mathbf{R}^n} (u^+ - u^-)\, dx \leq 1.$$

We want to show that we may assume that $u^- < v < u^+$. According to (5.1) this is true if $v \in \mathcal{M}_\alpha^{\text{rec}}$. If $v \in \mathcal{M}_\alpha \setminus \mathcal{M}_\alpha^{\text{rec}}$ and $v(y) = u^-(y)$ for some $y \in \mathbf{R}^n$ the preceding arguments can be used with u replaced by v and v replaced by u^-. But $u^- \in \mathcal{M}_\alpha^{\text{rec}}$, hence $v^- < u^- < v^+$. So we obtain the same situation as above, i.e. we may assume $u^- < v < u^+$ right away.

Now our argument is similar to the proof of (5.1), but simpler: By (6.1) there exist $\varepsilon > 0$ and $r_0 > 0$ such that for $r \geqslant r_0$

$$(6.15) \quad D(\max(u, v), B(0, r)) > \varepsilon \quad \text{and} \quad D(\min(u, v), B(0, r)) > \varepsilon.$$

On the other hand (6.14) and $u^- < u < u^+$, $u^- < v < u^+$ imply

$$\int_{\mathbf{R}^n} (\max(u, v) - u)\, dx \leqslant 1$$

$$\int_{\mathbf{R}^n} (v - \min(u, v))\, dx \leqslant 1$$

Hence there exists a sequence $r_i \to \infty$, $r_i \geqslant 1$, such that

$$\varepsilon_i^+ := \int_{\partial B(0, r_i)} (\max(u, v) - u)\, d\sigma$$

and

$$\varepsilon_i^- := \int_{\partial B(0, r_i)} (v - \min(u, v))\, d\sigma$$

both converge to 0. Now we apply (6.9) with $w_2 := \max(u, v)$, $w_1 := u$ and $r := r_i$ and obtain $\phi_i^+ \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$ such that $u + \phi_i^+ = \max(u, v)$ on $B(0, r_i)$ and

$$(6.16) \quad |A(u, \phi_i^+, \mathbf{R}^n \setminus B(0, r_i))| \leqslant \tilde{A} \cdot \varepsilon_i^+.$$

Similarly there exists $\phi_i^- \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$ such that $v + \phi_i^- = \min(u, v)$ on $B(0, r_i)$ and

$$(6.16)' \quad |A(v, \phi_i^-, \mathbf{R}^n \setminus B(0, r_i))| \leqslant \tilde{A} \cdot \varepsilon_i^-.$$

If we choose i so large that $r_i \geqslant r_0$ and $\tilde{A}(\varepsilon_i^+ + \varepsilon_i^-) < 2\varepsilon$ then (6.15), (6.16) and (6.16)′ contradict our hypothesis that u and v are minimal.

7. Concluding remarks

For rationally independent $\bar{\alpha} = (-\alpha, 1) \in \mathbf{R}^{n+1}$ Theorem (5.3) provides a qualitative description of $\mathcal{M}_\alpha$ which is as complete as we can reasonably expect it to be: $\mathcal{M}_\alpha$ is totally ordered, i.e. the graphs of functions in $\mathcal{M}_\alpha$ laminate $\mathbf{R}^{n+1}$. The elements of $\mathcal{M}_\alpha^{\text{rec}}$ are those which can be approximated (with respect to C^1-convergence on compact sets) by their own translates (and hence by translates of any $v \in \mathcal{M}_\alpha$). Either the graphs of functions in $\mathcal{M}_\alpha^{\text{rec}}$ foliate $\mathbf{R}^{n+1}$ or they form a Cantor set. In the second case neighboring elements $u^- < u^+$ in $\mathcal{M}_\alpha^{\text{rec}}$ satisfy

$$\int_{\mathbf{R}^n} (u^+ - u^-)\, dx \leqslant 1$$

Any $u \in \mathcal{M}_\alpha \backslash \mathcal{M}_\alpha^{\text{rec}}$ determines two neighboring elements $u^-, u^+ \in \mathcal{M}_\alpha^{\text{rec}}$ such that $u^- < u < u^+$. It is not difficult to show that for given rationally independent $\bar{\alpha}$ we will have $\mathcal{M}_\alpha = \mathcal{M}_\alpha^{\text{rec}}$ for "generic" integrands F.

Theorem (5.1) and Lemma (4.6) give a similarly complete picture for $\mathcal{M}_\alpha^{\text{rec}}$ if $\alpha \in \mathbf{R}^n \backslash \mathbf{Q}^n$ and $\bar{\alpha} = (-\alpha, 1)$ is not necessarily rationally independent. For $\alpha \in \mathbf{Q}^n$ Moser's results [9], (5.2)–(5.4) answer the basic qualitative questions for $\mathcal{M}_\alpha^{\text{per}} = \mathcal{M}_\alpha^{\text{rec}}$. Contrary to the rationally independent case, however, we do not generically have $\mathcal{M}_\alpha = \mathcal{M}_\alpha^{\text{rec}}$ in these cases. This can be proved by considering limits of sequences $u_i \in \mathcal{M}_{\alpha_i}$ where the $\bar{\alpha}_i = (-\alpha_i, 1)$ are rationally independent while $\bar{\alpha} = \lim \bar{\alpha}_i$ is rationally dependent. The interesting structure of $\mathcal{M}_\alpha$ for $n = 1$ and $\alpha \in \mathbf{Q}$, cf. [2], Section 5, indicates that it is worthwhile to study $\mathcal{M}_\alpha \backslash \mathcal{M}_\alpha^{\text{rec}}$ in the case $n > 1$. This is one of the subjects of a forthcoming paper. A related problem is the following:

The discussion following (3.2) shows that for $n > 1$ the set $\mathcal{M}$ can be properly contained in the set of all minimal solutions. Now it is desirable to characterize $\mathcal{M}$ by properties which are weaker than the condition "no selfintersections". In analogy to the Liouville theorem for harmonic functions one might ask if not every minimal solution u with linear growth (i.e. $|u(x)| \leqslant C(|x| + 1)$ for some $C > 0$) is in $\mathcal{M}$, i.e. does not have selfintersections. A weaker conjecture is that a minimal solution u is in $\mathcal{M}$ if $|u(x) - \alpha \cdot x|$ is bounded for some $\alpha \in \mathbf{R}^n$. Actually this is true if $\bar{\alpha} = (-\alpha, 1)$ is rationally independent but it is an open question as soon as

$$\operatorname{rank}\,(\mathbf{Z}^{n+1} \cap \{\bar{x} \mid \bar{x} \cdot \bar{\alpha} = 0\}) \geqslant 2.$$

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Mathematisches Institut
Universität Bern
Sidlerstrasse 5
CH-3012 Bern

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