New link invariants and applications.

Autor(en): **Orr, Kent E.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **62 (1987)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-47360

Comment. Math. Helvetici 62 (1987) 542–560 0010-2571/87/040542-19\$01.50 + 0.20/0

New link invariants and applications

KENT E. ORR

§0. Introduction

This paper presents a new sequence of higher dimensional link cobordism invariants which vanish on boundary links. These extend invariants of J. Levine and N. Sato. The Sato–Levine invariants are shown to vanish on higher dimensional links with simply connected components. These new invariants employ the lower central series of the link group and should be viewed as the correct higher dimensional analogues of the classical link cobordism invariants defined by J. Milnor in [Mi], answering question (b) from the end of his paper.

An m-link in the $(n+2)$-sphere S^{n+2} is an oriented, locally flat, codimension two and ordered m-component submanifold $L = \Sigma_1 \cup \cdots \cup \Sigma_m \subset S^{n+2}$. This paper is concerned with m-links $L \subset S^{n+2}$ with $n \geq 2$. Such a link is spherical if each component Σ_i of L has the homotopy type of the n-sphere. A cobordism of a link L_0 to a link L_1 is a manifold $C \subset S^{n+2} \times [0, 1]$ piecewise linearly homeomorhic to $L_0 \times [0, 1]$ and meeting $S^{n+2} \times \{0, 1\}$ transversely with $C \cap (S^{n+2} \times i) = L_i$; $i = 0, 1$. A boundary m-link $L \subset S^{n+2}$ is an m-link which has m pairwise disjoint, oriented, locally flat $(n+1)$-manifolds (Seifert manifolds) $V_1 \cup \cdots \cup V_m \subset S^{n+2}$ such that for each i, $\partial V_i = \Sigma_i$, the i^{th} component of L.

Classifying higher dimensional spherical link cobordism yields two problems:

(i) Classify cobordism of boundary links, and

(ii) Determine the relationship under cobordism between the collection of spherical links and the subcollection of boundary links.

This division arises naturally since all knots (1-links) bound Seifert manifolds making the first problem accessible to the techniques used to study knot cobordism. The first problem was solved for knots in [L] and for links in [CS]. (See also [Ker], [Kea], [Ko].) In contrast, most fundamental questions arising from the second problem remain unanswered. (For partial results, see [C1] and [D]. A classification of links in S^4 is obtained in [C1].) Although methods exist for constructing non-boundary links *it is not known if all higher dimensional spherical links are cobordant to boundary links.*

Early progress on (ii) was made independently by J. Levine and N. Sato (see also [C1], [C2], [C3], [D]). To an m-link $L \subset S^{n+2}$ they associate an element

$\beta(L) \in \pi_{n+2}(S^m)$ called the Sato–Levine invariant of L. $\beta(L)$ is a well defined cobordism invariant of L, defined on a class of links containing all spherical links. Sato gives homological conditions characterizing links L for which $\beta(L)$ is defined and calls these links semi-boundary links. His constructions realize elements in $\pi_{n+2}(S^2)$, $n \geq 2$ by non-spherical 2-links [Sa]. (See also [R].) In [C2], T. Cochran proves that if $L \subset S^4$ is a spherical 2-link with one unknotted component, $\beta(L) = 0$. He displays other large classes of spherical 2-links in S^4 with $\beta(L) = 0$.

The invariants defined in this paper fully explain the failure of the Sato–Levine invariant to detect spherical links not cobordant to boundary links but still provide considerable evidence that such links may exist. We prove the following theorems.

THEOREM 4.1. *Let $L \subset S^{n+2}$ be an m-link with $H_1(L) = 0$, then $\beta(L) = 0$.*

THEOREM 3.5. *Let $L \subset S^{n+2}$ be any link with at least four components for which $\beta(L)$ is defined, then $\beta(L) = 0$.*

This has a nice corollary.

COROLLARY 3.6. *Let $\{M_i\}$ be a collection of at least four locally flat, oriented, codimension one submanifolds of S^{n+2} with transverse intersection. Assume $\partial M_i \cap M_j = \emptyset$, $i \neq j$. (Of course, ∂M_i may be empty.) Then $\bigcap_{i=1}^m M_i$ is a framed boundary.*

Let $F(m)$ be the free group on m generators and let $F_k(m)$ be its k^{th} lower central subgroup. The quotient homomorphism

$$\psi_k^m : F(m) \to \frac{F(m)}{F_k(m)}$$

induces an inclusion of Eilenberg–MacLane spaces

$$\psi_k^m : \bigvee^m S^1 = K(F(m), 1) \to K\left(\frac{F(m)}{F_k(m)}, 1\right).$$

Let K_k^m be the mapping cone of ψ_k^m. K_k^m is simply connected. The homomorphisms ψ_k^m induce a homomorphism

$$\psi_\omega^m : F(m) \to \lim_{\overleftarrow{k}} \frac{F(m)}{F_k(m)}$$

and an inclusion of the spaces

$$\psi^m_\omega : \overset{m}{\bigvee} S^1 = K(F(m), 1) \to K\left(\varprojlim_k \frac{F(m)}{F_k(m)}, 1\right).$$

Let K^m_ω be the mapping cone of ψ^m_ω. The quotient homomorphism

$$\rho^m_{k,l} : \frac{F(m)}{F_k(m)} \to \frac{F(m)}{F_l(m)}, \; k > l$$

induces a map

$$\psi^m_{k,l} : K^m_k \to K^m_l.$$

If $j > k > l$ then $\psi^m_{k,l} \circ \psi^m_{j,k}$ is homotopic to $\psi^m_{j,l}$. We abuse notation and also write $\psi^m_{k,l}$ for the induced homomorphism on homotopy groups

$$\psi^m_{k,l} : \pi_{n+2}(K^m_k) \to \pi_{n+2}(K^m_l).$$

We have a homotopy commutative diagram of maps and spaces.

$$\begin{tikzcd}
 & K^m_\omega \arrow[d, "\cdots\psi^m_{\omega,k+1}"'] \arrow[dr] \arrow[drr] \arrow[drrrr, "\psi^m_{\omega,3}"] \arrow[drrrrr, "\psi^m_{\omega,2}"] & & & & & \\
\cdots \arrow[r] & K^m_{k+1} \arrow[r, "\psi^m_{k+1,k}"'] & K^m_k \arrow[r, "\psi^m_{k,k-1}"'] & K^m_{k-1} \arrow[r] & \cdots \arrow[r] & K^m_3 \arrow[r, "\psi^m_{3,2}"'] & K^m_2
\end{tikzcd}$$

To obtain the aforementioned theorems it is necessary to consider links with additional structure. By a based m-link (L, τ) we mean an m-link $L \subset S^{n+2}$ and a choice of meridians τ for L. Similarly, we define a notion of based cobordism between two based m-links. (See §1 for the precise definitions.) Our main theorem presents a framework for studying problem (ii). It holds for based links in any higher dimension with any number of components.

THEOREM 2.1. *Let (L, τ) be a based m-link with $L \subset S^{n+2}$ and $H_1(L) = 0$. There is an infinite sequence of elements $\{\theta^m_k(L, \tau)\}$, $2 \le k \le \omega$, with $\theta^m_k(L, \tau) \in \pi_{n+2}(K^m_k)$ and a collection of homomorphisms $\{\psi^m_{k,l}\}_{k>l}$ where $\psi^m_{k,l} : \pi_{n+2}(K^m_k) \to$*

$\pi_{n+2}(K_l^m)$ *with the following properties:*

(i) $\theta_k^m(L, \tau)$ *is an invariant of the based cobordism class of* (L, τ).

(ii) *If* $k > l$ *then* $\psi_{k,l}^m(\theta_k^m(L, \tau)) = \theta_l^m(L, \tau)$.

(iii) *If* $j > k > l$ *then* $\psi_{k,l}^m \circ \psi_{j,k}^m = \psi_{j,l}^m$.

(iv) $\theta_k^m(L, \tau) = 0$ *for a choice* τ *of meridians for* L *if and only if* $\theta_k^m(L, \tau') = 0$ *for any other choice* τ' *of meridians for* L.

(v) *If* L *is cobordant to a boundary link then* $\theta_k^m(L, \tau) = 0$ *for all* $2 \le k \le \omega$ *and all choices of meridians for* L.

(vi) $\theta_2^m(L, \theta) = \theta_2^m(L)$ *does not depend on the choice* τ *of meridians for* L.

These invariants can be defined for a more general class of links and, in this broader context, are all realizable (see §5.)

In §2 we define these invariants and discuss their properties. §3 conducts a detailed analysis of the invariant $\theta_2^m(L) \in \pi_{n+2}(K_2^m)$ and reveals its relation to the Sato–Levine invariants. This section ends with a proof of Theorem 3.5. In §4 we apply Theorem 2.1 to obtain Theorem 4.1. The paper concludes with a brief discussion of how these invariants can be generalized to study classical links and some applications to appear in a future paper.

The results presented here constitute my Ph.D. thesis written at Rutgers University under the supervision of Professor Julius L. Shaneson. I thank him for his unfailing encouragement, patience and insight.

§1. Notation and definitions

For an m-link $L \subset S^{n+2}$, $T(L)$ will denote a tubular neighborhood of L in S^{n+2}. The exterior of L, the closure of $S^{n+2} - T(L)$, will be denoted E_L. π_L will represent the fundamental group of the link exterior E_L. We write π when the link L is understood. Similarly, we write π_C for the group of the exterior E_C of a cobordism $C \subset S^{n+2} \times [0, 1]$.

By a choice of meridians for L we mean an embedding

$$\tau : \bigvee^m S^1 \to E_L$$

such that τ restricted to the i^{th} copy of S^1 will be homotopic relative the base point to a fiber of the tubular neighborhood of Σ_i joined by a path to the base point of E_L. We will alternatively speak of a choice of meridians for L as the homomorphism induced by τ on fundamental groups,

$$\tau : \pi_1\Big(\bigvee^m S^1\Big) = F(m) \to \pi_L.$$

Note that τ maps the generators of $F(m)$ to a set in π_L whose normal closure is all of π_L.

A based m-link (L, τ) is an m-link $L \subset S^{n+2}$ and a choice of meridians τ for L. A based cobordism between two based links (L_0, τ_0) and (L_1, τ_1) is a pair (C, T) where C is a cobordism from L_0 to L_1 and $T: F(m) \to \pi_C$ is a homomorphism making the following diagram commute:

$$\begin{tikzcd} & \pi_{L_0} \arrow[dr, "\iota_0"] & \\ F(m) \arrow[ur, "\tau_0"] \arrow[rr, "T"] \arrow[dr, "\tau_1"'] & & \pi_C \\ & \pi_{L_1} \arrow[ur, "\iota_1"'] & \end{tikzcd}$$

ι_0 and ι_1 are homomorphisms induced by inclusions of spaces. Based cobordism is an equivalence relation on based links.

All homology and cohomology is taken with untwisted integer coefficients and we write $H_i(X)$ and $H^i(X)$ for $H_i(X, \mathbf{Z})$ and $H^i(X, \mathbf{Z})$, respectively. $H^*(G)$ is $H^*(K(G, 1))$ where $K(G, 1)$ is an Eilenberg–MacLane space, i.e., a CW-complex with $\pi_1(K(G, 1)) = G$ and $\pi_i(K(G, 1)) = 0$, $i \geq 2$. For a group G, G_k is the k^{th} lower central series subgroup of G defined recursively by $G_1 = G$ and $G_k = [G, G_{k-1}]$.

§2. New invariants of link cobordism

Let (L, τ) be a based m-link with $H_1(L) = 0$; $L \subset S^{n+2}$. The Mayer–Vietoris sequence implies $H_1(E_L) = \mathbf{Z}^m$ and, along with the condition that $H_1(L) = 0$, implies $H_2(E_L) = 0$. By Hopf's theorem [B], $H_2(\pi_L) = 0$.

Our choice of meridians $\tau: F(m) \to \pi_L$ induces an isomorphism $H_1(F(m)) \to H_1(\pi_L)$. Since $H_2(\pi_L) = 0$, Stalling's theorem [St] implies $F(m)/F_k(m) \cong \pi/\pi_k$ for all finite k. The induced homotopy equivalence $K(F(m)/F_k(m), 1) \to K(\pi/\pi_k, 1)$ will be denoted $\bar{\tau}$.

Let $\phi_k: E_L \to K(F(m)/F_k(m), 1)$ be the composition of the map $E_L \to K(\pi/\pi_k, 1)$ realizing $\pi_L \to \pi/\pi_k$ with the map $(\bar{\tau})^{-1}$. Let $\Sigma_i \times S^1 \to E_L$ be the inclusion in E_L of the boundary of a tubular neighborhood of Σ_i in S^{n+2}; $\Sigma_i \times S^1 = \partial T(\Sigma_i) \subset E_L$. Choosing paths from the base point of E_L to the components of $\partial T(L)$ in E_L defines a homomorphism

$$\pi_1(\Sigma_i \times S^1) = \pi_1(\Sigma_i) \times \pi_1(S^1) \to \pi_L \to \frac{F(m)}{F_k(m)}.$$

Since $H_1(L) = 0$, $(\pi_1(\Sigma_i))_k \cong \pi_1(\Sigma_i)$ so that $\pi_1(\Sigma_i) \to F(m)/F_k(m)$ is the trivial homomorphism for all i. This implies, for suitably chosen paths, the composition

$$\Sigma_i \times S^1 \to E_L \to K\left(\frac{F(m)}{F_k(m)}, 1\right)$$

is homotopic to a map sending the first factor to a point and the second factor to the image of the i^{th} generator in $K(F(m)/F_k(m), 1)$. Since $K(F(m)/F_k(m), 1)$ is an Eilenberg–MacLane space, we have a homotopy commutative diagram of maps and spaces.

$$\begin{array}{ccccc} \partial E_L = L \times S^1 & \longrightarrow & E_L & \xrightarrow{\phi_k} & K\left(\frac{F(m)}{F_k(m)}, 1\right) \\ \downarrow & & & & \cup \\ \coprod^m S^1 & & \longrightarrow & & \bigvee^m S^1 \end{array}$$

The left vertical map is projection on the second factor. Thus ϕ_k extends canonically to a map $\bar{\phi}_k : S^{n+2} \to K_k^m$. Define $\theta_k^m(L, \tau) = [\bar{\phi}_k] \in \pi_{n+2}(K_k^m)$.

Similarly, the composition $\pi_L \to \varprojlim_k(\pi/\pi_k) \xrightarrow{\cong} \varprojlim_k(F(m)/F_k(m))$ determines a map $\phi_\omega : E_L \to K(\varprojlim_k(F(m)/F_k(m)), 1)$ extending to a map $\bar{\phi}_\omega : S^{n+2} \to K_\omega^m$. Define $\theta_\omega^m(L, \tau) = [\bar{\phi}_\omega] \in \pi_{n+2}(K_\omega^m)$. Note that $\theta_k^m(L, \tau)$, $2 \le k \le \omega$, is well defined if we assume ϕ_k is induced from ϕ_ω.

THEOREM 2.1. *The elements $\{\theta_k^m(L, \tau)\}$, $2 \le k \le \omega$ and the homomorphisms $\psi_{k,l}^m$ have the following properties*

(i) *$\theta_k^m(L, \tau)$ is an invariant of the based cobordism class of (L, τ).*
(ii) *If $k > l$ then $\psi_{k,l}^m(\theta_k^m(L, \tau)) = \theta_l^m(L, \tau)$.*
(iii) *If $j > k > l$ then $\psi_{k,l}^m \circ \psi_{j,k}^m = \psi_{j,l}^m$.*
(iv) *$\theta_k^m(L, \tau) = 0$ for a choice τ of meridians for L if and only if $\theta_k^m(L, \tau') = 0$ for any other choice τ' of meridians for L.*
(v) *If L is cobordant to a boundary link then $\theta_k^m(L, \tau) = 0$ for all $2 \le k \le \omega$ and all choices of meridians for L.*
(vi) *$\theta_2^m(L, \tau) = \theta_2^m(L)$ does not depend on the choice τ of meridians for L.*

Proof. The elements $\theta_k^m(L, \tau)$ are defined above and the homomorphisms $\{\psi_{k,l}^m\}_{k>l}$ in the introduction. Properties (ii) and (iii) are clear from construction.

Let (C, T) be a based cobordism between based links (L_0, τ_0) and (L_1, τ_1). Since $H_1(C) \cong H_1(L_0) = 0$, $T : F(m) \to \pi_C$ induces isomorphisms

$$\frac{F(m)}{F_k(m)} \xrightarrow{\cong} \frac{\pi_C}{(\pi_C)_k}$$

for all k. We have a commutative diagram.

$$\begin{tikzcd} & \pi_{L_0} \arrow[dr] \arrow[drr] & & \\ F(m) \arrow[ur, "\tau_0"] \arrow[dr, "\tau_1"'] \arrow[rr, "T"] & & \pi_C \arrow[r] & \dfrac{F(m)}{F_k(m)} \\ & \pi_{L_1} \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

As in the construction of $\theta_k^m(L, \tau)$, we have a map $E_C \to K(F(m)/F_k(m), 1)$ extending to a homotopy $S^{n+2} \times [0, 1] \to K_k^m$ between $\theta_k^m(L_0, \tau_0)$ and $\theta_k^m(L_1, \tau_1)$. This demonstrates (i).

Let τ and $\tau' : \bigvee^m S^1 \to E_L$ be two choices of meridians for L. Let ϕ_k and $\phi_k' : E_L \to K(F(m)/F_k(m), 1)$ be the maps corresponding to the choices of meridians τ and τ', respectively. The map

$$(\bar{\tau}')^{-1} \circ \bar{\tau} : K\left(\frac{F(m)}{F_k(m)}, 1\right) \to K\left(\frac{F(m)}{F_k(m)}, 1\right)$$

sends ϕ_k to ϕ_k'. Clearly, $(\bar{\tau}')^{-1} \circ \bar{\tau}$ induces a map $h : K_k^m \to K_k^m$ which is a homology equivalence of simply connected spaces and, therefore, is a homotopy equivalence. If $h_* : \pi_{n+2}(K_k^m) \to \pi_{n+2}(K_k^m)$ is the homomorphism induced by h, then h_* is an isomorphism and $h_*(\theta_k^m(L, \tau)) = \theta_k^m(L, \tau')$ which proves (iv).

When $k = 2$ it is easily observed that the map $(\bar{\tau}')^{-1} \circ \bar{\tau}$ induces the identity automorphism on the abelian group $F(m)/F_2(m)$. Thus, it induces the identity map on K_2^m. Therefore, (vi) is true.

Lastly, assume L is a boundary link. Then a theorem of Gutierrez [Gu] gives a choice of meridians $\tau : \bigvee^m S^1 \to E_L$ which splits. Let $\alpha : E_L \to \bigvee^m S^1$ be a splitting map. The epimorphism $\pi_L \to \pi/\pi_k$ is realized by a map $E_L \to K(\pi/\pi_k, 1)$ factoring through $\bigvee^m S^1 \subset K(\pi/\pi_k, 1)$ representing the image of a suitable choice of meridians for L in $K(\pi/\pi_k, 1)$. So, $\phi_k : E_L \to K(F(m)/F_k(m), 1)$ factors through $\bigvee^m S^1$ and $\theta_k^m(L, \tau) = 0$. This proves (v) and completes the proof of Theorem 2.1.

§3. $\theta_2^m(L)$ and the Sato–Levine invariant

Let $L = \Sigma_1 \cup \cdots \cup \Sigma_m \subset S^{n+2}$ be an m-link such that each Σ_i bounds a Seifert manifold V_i with $V_i \cap \Sigma_j = \varnothing$, $i \neq j$. N. Sato [Sa] finds homological conditions on $L \subset S^{n+2}$ necessary and sufficient to insure that manifolds $\{V_i\}$ exist and calls such links semi-boundary links. In particular, any link $L \subset S^{n+2}$ with $H_1(L) = 0$ is a semi-boundary link. However, $H_1(L) = 0$ is not a necessary condition and all results in this section hold for arbitrary semiboundary links.

A choice of a normal one-frame for each manifold V_i determines a framing for $V=\bigcap_{i=1}^m V_i$. $V\subset S^{n+2}$ is a framed $(n+2-m)$-manifold and by the Pontrjagin–Thom construction (see e.g. [Sto]) yields an element $\beta(L)\in\pi_{n+2}(S^m)$. $\beta(L)=0$ if $\bigcap_{i=1}^m V_i$ is a framed boundary. $\beta(L)$, the Sato–Levine invariant of L, is a well defined cobordism invariant of L which vanishes on boundary links. Note that if $L'\subset L$ is a sublink of L then $\beta(L')$ is a cobordism invariant of L.

For a semi-boundary m-link $L\subset S^{n+2}$ with given Seifert manifolds $\{V_i\}$ we will construct an explicit realization of the Hurewicz homomorphism $\pi_1(E_L)\to H_1(E_L)$. This construction utilizes the Seifert manifolds for L and allows a detailed investigation of $\theta_2^m(L)\in\pi_{n+2}(K_2^m)$. The relationship between $\theta_2^m(L)$ and the Sato–Levine invariant will become apparent.

LEMMA 3.1 (compare [C2, Lemma 3.1]). *For a semi-boundary m-link L there is a map $\rho_L: E_L\to\prod^m S^1=T^m$ with the following properties*

(i) *ρ_L realizes the Hurewicz homomorphism, i.e., ρ_L induces an isomorphism $(\rho_L)_*: H_1(E_L)\to H_1(T^m)$.*

(ii) *There is a regular point $*\in T^m$ for ρ_L such that $\rho_L^{-1}(*)=\bigcap_{i=1}^m V_i$.*

Proof. Assume $V_i\cap T(\Sigma_i)$ is a collar for the manifold V_i. Apply the Pontrjagin–Thom construction to $(V_i\cap E_L,\partial(V_i\cap E_L))\subset(E_L,\partial E_L)$ in a relative form to obtain a map $\rho_i: E_L\to S_i^1$. Let

$$\rho_L=\prod_{i=1}^m \rho_i: E_L\to\prod_{i=1}^m S_i^1=T^m$$

be the product map. Alexander duality implies $H_1(E_L)\cong\mathbf{Z}^m$ generated by meridians. Since ρ_i sends the i^{th} meridian to a generator in $H_1(S_i^1)$ and all other meridians to $0\in H_1(S_i^1)$, it follows that ρ_L induces an isomorphism on the first homology groups, showing (i).

Let $*_i$ be a regular point for ρ_i, then $*=(*_i)_{i=1}^m$ is a regular point for ρ_L. By construction, $\rho_i^{-1}(*_i)=V_i$ so that

$$\rho_L^{-1}(*)=\left(\prod_{i=1}^m\rho_i\right)^{-1}(*)=\bigcap_{i=1}^m\rho_i^{-1}(*_i)=\bigcap_{i=1}^m V_i$$

confirming property (ii) and proving the lemma.

$\rho_L|_{\partial E_L}:\partial E_L=L\times S^1\to\bigvee^m S^1\subset T^m$. So ρ_L extends to a map $\bar\rho_L$ from S^{n+2} to the mapping cone of the inclusion

$$\bigvee^m S^1=K(F(m),1)\to K\left(\frac{\pi}{\pi_2},1\right)=T^m.$$

This mapping cone is K_2^m. Since ρ_L realizes the Hurewicz map, this extension represents $\theta_2^m(L) \in \pi_{n+2}(K_2^m)$.

COROLLARY 3.2. *$\theta_2^m(L)$ is defined whenever L is a semi-boundary link.*

Note. It is not hard to construct Seifert manifolds from any map $\rho: E_L \to T^m$ if ρ realizes the Hurewicz map and extends to a map $S^{n+2} \to K_2^m$. This implies $\theta_2^m(L)$ is defined if and only if L is a semi-boundary link.

Let $\eta^m: K_2^m \to S^m$ be the degree one map defined by collapsing everything outside a ball in K_2^m to a point. By construction, there is a regular point $* \in S^m$ for the map $\eta^m \circ \bar{\rho}_L$ such that

$$(\eta^m \circ \bar{\rho}_L)^{-1}(*) = \bigcap_{i=1}^{m} V_i \subset S^{n+2}.$$

This proves the following proposition.

PROPOSITION 3.3. *For each $m \geq 2$ there is a degree one map $\eta^m: K_2^m \to S^m$ inducing $\eta_*^m: \pi_{n+2}(K_2^m) \to \pi_{n+2}(S^m)$ such that $\eta_*^m(\theta_2^m(L)) = \beta(L)$ for every semi-boundary m-link L.*

We conclude this section by proving Theorem 3.5. We need a lemma which will also be used in the proofs of Propositions 4.2 and 4.3.

LEMMA 3.4. *Let ξ be a bundle over S^q with fiber S^r and let $p: \xi \to S^m$ be any map. If $q \leq r < m$ then*

$$p_*: \pi_{n+2}(\xi) \to \pi_{n+2}(S^m)$$

is the zero homomorphism for all n.

Proof. The obstructions to a section of ξ lie in $H^q(S^q; H^{q-1}(S^r)) = 0$ since $q \leq r$. Let $s: S^q \to \xi$ be a section and $i: S^r \to \xi$ be the inclusion of a fiber. The exact sequence of a fibration implies

$$i_* + s_*: \pi_{n+2}(S^r) \oplus \pi_{n+2}(S^q) \to \pi_{n+2}(\xi)$$

is an isomorphism. $p_* \circ (i_* + s_*) = p_* \circ i_* + p_* \circ s_*$ and since $r < m$ both these homomorphisms are zero.

THEOREM 3.5. *Let $L \subset S^{n+2}$ be a semi-boundary m-link with $m \geq 4$, then* $\beta(L) = 0$.

Proof. Let $p : T^2 \to S^2$, $q : T^{m-2} \to S^{m-2}$ and $c : S^2 \times S^{m-2} \to S^m$ be degree one maps. Then $p \times q : T^2 \times T^{m-2} \to S^2 \times S^{m-2}$ has degree one. Since $(m-2) \geq 2$, $S^2 \times S^{m-2}$ is simply connected and $p \times q$ factors through K_2^m giving the following homotopy commutative diagram of spaces. The horizontal map is η^m since it is a

$$\begin{tikzcd} K_2^m \arrow[rr, "\eta^m"] \arrow[dr] & & S^m \\ & S^2 \times S^{m-2} \arrow[ur, "c"'] & \end{tikzcd}$$

composition of degree one maps and thus has degree one. By lemma 3.4,

$$c_* : \pi_{n+2}(S^2 \times S^{m-2}) \to \pi_{n+2}(S^m)$$

is the zero homomorphism implying

$$\eta_*^m : \pi_{n+2}(K_2^m) \to \pi_{n+2}(S^m)$$

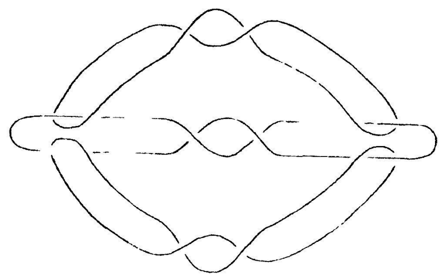
factors through zero. By proposition 3.3, $\beta(L) = \eta_*^m(\theta_2^m(L)) = 0$.

COROLLARY 3.6. *Let $\{M_i\}$ be a collection of at least four locally flat, oriented, codimension one submanifolds of S^{n+2} with transverse intersection. Assume $\partial M_i \cap M_j = \emptyset$, $i \neq j$. (Of course, ∂M_i may be empty.) Then $\bigcap_{i=1}^m M_i$ is a framed boundary.*

Proof. If M_i is any closed manifold then remove a disk from M_i in the complement of $\bigcup_{j \neq i} M_j$ to obtain a manifold $\bar{M}_i$. If ∂M_i is non-empty, let $M_i = \bar{M}_i$. The result is a semi-boundary m-link $L = \amalg^m \partial \bar{M}_i \subset S^{n+2}$. $\bigcap_{i=1}^m M_i = \bigcap_{i=1}^m \bar{M}_i$ is a framed manifold representing $\beta(L) \in \pi_{n+2}(S^m)$. Since $m \geq 4$, Theorem 3.5 implies $\beta(L) = 0$ and $\bigcap_{i=1}^m M_i$ is a framed boundary.

$\pi_{n+2}(S^m) \neq \pi_{n+2}(K_2^m)$ unless $m = 2$. For example, $K_2^3 \simeq S^2 \vee S^2 \vee S^2 \vee S^3$ and by the Hilton–Milnor theorem [WG] $\pi_{n+2}(K_2^3)$ has summands generated by Whitehead products. A homotopy equivalence $K_2^3 \to S^2 \vee S^2 \vee S^2 \vee S^3$ can be chosen so that each sphere represents the Sato–Levine invariant of L or one of its sublinks. The remaining linearly independent summands of $\pi_{n+2}(K_2^3)$ are new invariants of higher dimensional link cobordism and can be interpreted geometrically. Any element in $\pi_{n+2}(K_2^m)$ can be realized for some suitably chosen semi-boundary m-link. (See §5.)

Moreover, many classical links are semi-boundary links. A classical link is a semi-boundary link if all of its pairwise linking numbers vanish. $\theta_2^m(L)$ is defined for classical semi-boundary links. (See §5.) The following, which we offer without proof, is an example of a semi-boundary 3-link with vanishing Sato–Levine invariants and non-vanishing $\theta_2^3(L) \in \pi_3(K_2^3)$.



§4. More applications of Theorem 2.1: the vanishing of the Sato–Levine invariants for spherical links

THEOREM 4.1. *Let $L \subset S^{n+2}$ be an m-link with $m = 2$ or 3 and $H_1(L) = 0$. Then $\beta(L) = 0 \in \pi_{n+2}(S^m)$.*

An analysis of the spaces K_k^m, $k = 2, 3, 4$ and $m = 2, 3$ will imply this result. It is a consequence of $\theta_4^m(L, \tau)$ being defined for links L with $H_1(L) = 0$. This suggests the viewpoint that $\theta_2^m(L)$ is an obstruction to $\theta_4^m(L, \tau)$ being defined. We will exploit this perspective in a future paper. (See §5.)

To prove Theorem 4.1 we need two propositions.

PROPOSITION 4.2. *$\psi_{4,2}^2 : \pi_{n+2}(K_4^2) \to \pi_{n+2}(K_2^2)$ is the zero homomorphism for all n.*

PROPOSITION 4.3. *$\eta_*^3 \circ \psi_{3,2}^3 : \pi_{n+2}(K_3^3) \to \pi_{n+2}(S^3)$ is the zero homomorphism for all n.*

Proof of Theorem 4.1.

Case $m = 2$. By Theorem 2.1, $H_1(L) = 0$ implies that for any choice τ of meridians for L, $\theta_4^2(L, \tau) \in \pi_{n+2}(K_4^2)$ is defined and $\psi_{4,2}^2(\theta_4^2(L, \tau)) = \theta_2^2(L)$. By Proposition 4.2, $\psi_{4,2}^2$ is the zero homomorphism. Hence $\theta_2^2(L) = 0$. By Proposition 3.3,

$$\beta(L) = \eta_*^2(\theta_2^2(L)) = \eta_*^2(0) = 0.$$

Case $m = 3$. Since $H_1(L) = 0$, Theorem 2.1 implies that a choice of meridians τ for L gives an element $\theta_3^3(L, \tau)$ with $\psi_{3,2}^3(\theta_3^3(L, \tau)) = \theta_2^3(L)$. By Proposition 3.3 there is a homomorphism $\eta_*^3 : \pi_{n+2}(K_2^3) \to \pi_{n+2}(S^3)$ with $\eta_*^3(\theta_2^3(L)) = \beta(L)$. Thus,

$$\beta(L) = \eta_*^3(\theta_2^3(L)) = (\eta_*^3 \circ \psi_{3,2}^3)(\theta_3^3(L, \tau))$$

which by Proposition 4.3 is zero.

The rest of this section is devoted to proving Propositions 4.2 and 4.3. This will involve proving several lemmas which we fear may obscure the central ideas of the proof. So we give a brief outline of the approach. We seek sufficient homotopy theoretic information concerning the spaces K_k^m for k small and $m = 2$, 3 to determine their homotopy type. This is done by relating $H^*(K_k^m)$ to $H^*(F(m)/F_k(m))$ in the obvious manner. Results of Magnus [MKS] allow the calculation of $H^*(F(m)/F_k(m))$ for k small. we then apply the following well known theorem of Whitehead [WJ].

THEOREM 4.4 (Whitehead). *Let X be a finite simply connected* CW*-complex of dimension less than five with no cohomological torsion. Then X is determined up to homotopy type by its cohomology ring $H^*(X)$.*

The spaces K_k^m have dimensions which increase rapidly with k. For example, $\dim K_4^2 = 5$ and $\dim K_3^3 = 6$; dimensions already too large to apply the above theorem. This difficulty is effaced by finding four dimensional spaces K_E and K_G (related to appropriately chosen group extensions) through which the maps $\psi_{4,3}^2 : K_4^2 \to K_3^2$ and $\psi_{3,2}^3 : K_3^3 \to K_2^3$ factor. Theorem 4.4 then supplies ample information to analyze the maps in question. This results in a situation where lemma 3.4 proves the desired propositions. Recall some results of Magnus and some facts about central extensions of groups.

Let $0 \to A \to E \to G \to 1$ be a group extension with A central in E. This centrality implies the induced action of G on $H^*(A)$ is trivial [GR]. Such extensions are classified by elements in $H^2(G; A)$ as follows. The Hochschild–Serre cohomology spectral sequence for this extension yields a five term sequence.

$$0 \to H^1(G; A) \to H^1(E; A) \to H^1(A; A) \xrightarrow{d_2^{0,1}} H^2(G; A) \to H^2(E; A).$$

If 1_A represents the element in $\hom(A, A) \cong H^1(A; A)$ corresponding to the identity homomorphism on A then $d_2^{0,1}(1_A)$ classifies the extension, i.e. central extensions of A by G lie in one-to-one correspondence to elements in $H^2(G; A)$

where the correspondence takes the extension $0 \to A \to E \to G \to 1$ to the element $d_2^{0,1}(1_A) \in H^2(G; A)$.

In [Ma], Magnus studied the central extensions

$$\frac{F_k(m)}{F_{k+1}(m)} \to \frac{F(m)}{F_{k+1}(m)} \to \frac{F(m)}{F_k(m)}$$

and showed that $F_k(m)/F_{k+1}(m)$ is free abelian of finite rank equal to

$$N_k = \frac{1}{k} \sum_{d \mid k} \mu(d) m^{k/d}.$$

Here $\mu(d)$ is the Mobius function evaluated on d; $\mu(1) = 1$, $\mu(d) = (-1)^l$ if d is a product of l distinct primes and $\mu(d) = 0$ otherwise. Notice that this implies $K(F(m)/F_{k+1}(m), 1)$ is a bundle over $K(F(m)/F_k(m), 1)$ with fiber a torus T^{N_k}. Since $K(F(m)/F_2(m), 1)$ is a torus, this implies the spaces $K(F(m)/F_k(m), 1)$ have the homotopy type of manifolds. Hence K_k^m is a finite complex for any k and m. Hopf's formula [Gr] implies

$$H_2\left(\frac{F(m)}{F_k(m)}\right) \cong \frac{F_k(m)}{F_{k+1}(m)} \cong \mathbf{Z}^{N_k}.$$

Now consider the Hochschild–Serre spectral sequence for $0 \to F_k(m)/F_{k+1}(m) \to F(m)/F_{k+1}(m) \to F(m)/F_k(m) \to 1$, where the coefficients lie in any finitely generated free abelian group A.

$$E_2^{p,q} \cong H^p\left(\frac{F(m)}{F_k(m)}\right) \otimes H^q\left(\frac{F_k(m)}{F_{k+1}(m)}; A\right).$$

$H^1(F(m)/F_k(m); A) \cong hom\,(\mathbf{Z}^m, A) \cong E_\infty^{1,0}$ and $d_2^{0,1}$ must be injective. By the Universal Coefficient theorem $H^2(F(m)/F_{k+1}(m); A) \cong hom\,(\mathbf{Z}^{N_{k+1}}, A)$ which has no torsion implying $E_3^{2,0} = E_\infty^{2,0}$ is torsionless. One easily checks that $E_2^{0,1} \cong E_2^{2,0}$ so that $d_2^{0,1}$ is an injection of isomorphic $\mathbf{Z}$-modules with torsionless cokernel and must be an isomorphism.

LEMMA 4.5. *$K_3^2 \simeq S^2 \vee S^2 \vee S^3$ and $H^2(K_2^2) \to H^2(K_3^2)$ is the zero homomorphism.*

Proof. The group extension

$$\mathbf{Z}\cong\frac{F_2(2)}{F_3(2)}\to\frac{F(2)}{F_3(2)}\to\frac{F(2)}{F_2(2)}\cong\mathbf{Z}^2$$

along with a standard spectral sequence argument provides a calculation of $H^*(F(2)/F_3(2))$. ($d_2^{0,1}$ is an isomorphism by the above discussion.)

$$\tilde{H}^*(K_3^2)\cong H^*\left(K\left(\frac{F(2)}{F_3(2)},1\right),S^1\vee S^1\right)\cong H^*(S^2\vee S^2\vee S^3)$$

Theorem 4.4 finishes the proof.

We obtained above the following calculation of $\tilde{H}^*(F(2)/F_3(2))$:

$$\tilde{H}^*\left(\frac{F(2)}{F_3(2)}\right)\cong\begin{cases}\mathbf{Z}^2, & \text{generated by } a_1 \text{ and } a_2, \quad i=1\\ \mathbf{Z}^2, & \text{generated by } b_1 \text{ and } b_2, \quad i=2\\ \mathbf{Z}, & \text{generated by } c, \quad\quad\quad\;\; i=3\end{cases}$$

where $a_ib_j=\pm\delta_{ij}c$ and all other products are zero. Let $0\to\mathbf{Z}\to E\to F(2)/F_3(2)$ be the extension classified by $b_1\in H^2(F(2)/F_3(2))$. The next lemma follows from observing that since $H_1(F(2)/F_3(2))$ is free abelian and $H_2(F(2)/F_3(2))\cong F_3(2)/F_4(2)$, then $0\to\mathbf{Z}^2\to F(2)/F_4(2)\to F(2)/F_3(2)$ is the maximal stem cover over $F(2)/F_3(2)$. (See [Gr] for details.) However we include the following easy proof for completeness.

LEMMA 4.6. *E is a quotient of $F(2)/F_4(2)$.*

Proof. In the Hochschild–Serre cohomology spectral sequence for the extension $\mathbf{Z}^2\to F(2)/F_4(2)\to F(2)/F_3(2)$, let

$$x=(d_2^{0,1})^{-1}(b_1)\in E_2^{0,1}\cong H^1\left(\frac{F_3(2)}{F_4(2)}\right).$$

Let $\varepsilon: F_3(2)/F_4(2)\cong\mathbf{Z}^2\to\mathbf{Z}$ be a homomorphism such that if $1\in H^1(\mathbf{Z})$ corresponds to the identity homomorphism of $\mathbf{Z}$ and $\varepsilon^*: H^1(\mathbf{Z})\to H^1(F_3(2)/F_4(2))$, then $\varepsilon^*(1)=z\in H^1(F_3(2)/F_4(2))$. If $H\subset F_3(2)/F_4(2)$ is the kernel of ε and G the cokernel of the inclusion homomorphism $H\to F(2)/F_4(2)$, then we have the following commutative diagram where the rows are extensions and the left two vertical homomorphisms are onto.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & G & \longrightarrow & \dfrac{F(2)}{F_3(2)} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \dfrac{F_3(2)}{F_4(2)} & \longrightarrow & \dfrac{F(2)}{F_4(2)} & \longrightarrow & \dfrac{F(2)}{F_3(2)} & \longrightarrow & 1 \end{array}$$

This yields the following diagram where the top and bottom horizontal homomorphisms are from the spectral sequence for $\mathbf{Z}\to G\to F(2)/F_3(2)$ and $F_3(2)/F_4(2)\to F(2)/F_4(2)\to F(2)/F_3(2)$ respectively.

$$\begin{array}{ccc} H^1(\mathbf{Z}) = E_2^{0,1} & \xrightarrow{d_2^{0,1}} & E_2^{2,0} = H^2\left(\dfrac{F(2)}{F_3(2)}\right) \\ \Big\downarrow{\scriptstyle \iota^*} & & \| \\ H^1\left(\dfrac{F_3(2)}{F_4(2)}\right) = E_2^{0,1} & \xrightarrow{d_2^{0,1}} & E_2^{2,0} \end{array}$$

Thus, $d_2^{0,1}(1) = d_2^{0,1}(\varepsilon^*(1)) = d_2^{0,1}(x) = b_1$ and $G \cong E$.

We now have a sequence of homomorphisms with $\psi_{E,3}\circ\psi_{4,E} = \psi_{4,3}^2$

$$F(2)\to\frac{F(2)}{F_4(2)}\xrightarrow{\psi_{4,E}} E\xrightarrow{\psi_{E,3}}\frac{F(2)}{F_3(2)}\longrightarrow\frac{F(2)}{F_2(2)}$$

Let K_E be the mapping cone of the inclusion map $K(F(2), 1) = S^1 \vee S^1 \to K(E, 1)$ induced from the quotient homomorphism $F(2)\to E$.

LEMMA 4.7. *$K_E \simeq \xi \vee S^3 \vee S^3$ where ξ is a bundle over S^2 with fiber S^2 and $H^3(K_3^2)\to H^3(K_E)$ is the zero homomorphism.*

Proof. In the cohomology spectral sequence for $\mathbf{Z}\to E\to F(2)/F_3(2)$, $d_2^{0,1}(1) = b_1 \in H^2(F(2)/F_3(2))$. The remaining differentials are computed easily using the ring structure of $H^*(F(2)/F_3(2))$. Here is $E_\infty^{*,*}$:

$q=1$	0	$\mathbf{Z}$	$\mathbf{Z}^2$	$\mathbf{Z}$
$q=0$	$\mathbf{Z}$	$\mathbf{Z}^2$	$\mathbf{Z}$	0
	$p=0$	$p=1$	$p=2$	$p=3$

After choosing generators $\beta_1, \beta_2 \in H^2(E)$ and $\delta \in H^4(E)$ representing generators in $E_\infty^{2,0}$ and $E_\infty^{1,1}$, respectively, one can calculate products in $H^*(E)$ using the ring structure of $E_\infty^{*,*}$. $\beta_1^2 = 0$, $\beta_1\beta_2 = \delta$ and $\beta_2^2 = \lambda\delta$ for some integer λ. From the exact sequence of the pair $(K(E, 1), S^1 \vee S^1)$ we obtain the following computation of $\tilde{H}^*(K_E) \cong H^*(K(E, 1), S^1 \vee S^1)$.

$$\tilde{H}^*(K_E) \cong \begin{cases} 0 & i=1 \\ \mathbf{Z}^2 \quad \text{generated by } \beta_1 \text{ and } \beta_2, & i=2 \\ \mathbf{Z}^2 & i=3 \\ \mathbf{Z} \quad \text{generated by } \delta, & i=4 \end{cases}$$

Here $\beta_1\beta_2 = \delta$, $\beta_1^2 = 0$ and $\beta_2^2 = \lambda\delta$. All other products are zero. By Theorem 4.4, $K_E \simeq S^2 \times S^2 \vee S^3 \vee S^3$ if λ is an even integer and $K_E \simeq \mathbb{C}P(2)\# - \mathbb{C}P(2) \vee S^3 \vee S^3$ if λ is an odd integer. It is well known (and easy to show) that $S^2 \times S^2$ and $\mathbb{C}P(2)\# - \mathbb{C}P(2)$ are both S^2 bundles over S^2 (see e.g. [Ma].)

We have a commutative diagram where the vertical arrows are excision isomorphisms.

$$\begin{array}{ccc} H^3\left(\dfrac{F(2)}{F_3(2)}\right) & \to & H^3(E) \\ \cong\uparrow & & \uparrow\cong \\ H^3(K_3^2) & \to & H^3(K_E) \end{array}$$

Since $H^3(F(2)/F_3(2)) \to H^3(E)$ factors through $E_\infty^{3,0} = 0$, $H^3(K_3^2) \to H^3(K_E)$ is zero.

The methods of constructing K_E and proving Lemmas 4.6 and 4.7 can be used to show Lemma 4.8.

LEMMA 4.8. *There is an extension* $\mathbf{Z} \to G \to F(3)/F_2(3)$, *a space* K_G, *and homomorphisms*

$$\psi_{3,G} : \pi_{n+2}(K_3^3) \to \pi_{n+2}(K_G) \quad \text{and} \quad \psi_{G,2} : \pi_{n+2}(K_G) \to \pi_{n+2}(K_2^3)$$

such that $\psi_{G,2} \circ \psi_{3,G} = \psi_{3,2}^3 : \pi_{n+2}(K_3^3) \to \pi_{n+2}(K_2^3)$ *and the following holds*:
(i) $K_G \simeq \xi_1 \# \xi_2 \vee S^3 \vee S^3 \vee S^3$ *where each* ξ_i *is a bundle over* S^2 *with fiber* S^2.
(ii) $H^3(K_2^3) \to H^3(K_G)$ *is the zero homomorphism.*

We proceed to prove Propositions 4.2 and 4.3.

Proof of Proposition 4.2. By Lemma 4.5, $K_3^2 \simeq S^2 \vee S^2 \vee S^3$ and $H^2(K_2^2) \to H^2(K_3^2)$ is the zero homomorphism. So $K_3^2 \to K_2^2$ factors through S^3 by a map

$\alpha_1 : K_3^2 \to S^3$ inducing an isomorphism $\alpha_1^* : H^3(S^3) \to H^3(K_3^2)$. Lemma 4.7 implies

$$\psi_{E,3} \circ \alpha_1^* : H^3(S^3) \to H^3(K_E)$$

is the zero homomorphism. By Lemma 4.7, $K_E \simeq \xi \vee S^3 \vee S^3$. Thus $\alpha_1 \circ \psi_{E,3} : K_E \to S^3$ factors through ξ by a map $\alpha_4 : K_E \to \xi$ and we have a homotopy commutative diagram as follows.

$$\begin{tikzcd}
K_4^2 \arrow[r, "\psi_{4,E}"] & K_E \arrow[r, "\psi_{E,3}"] \arrow[d, "\alpha_4"'] & K_3^2 \arrow[r, "\psi_{3,2}^2"] \arrow[dr, "\alpha_1"] & K_2^2 \\
& \xi \arrow[rr, "\alpha_3"'] & & S^3 \arrow[u, "\alpha_2"']
\end{tikzcd}$$

$\psi_{4,2}^2 \simeq \alpha_2 \circ \alpha_3 \circ \alpha_4 \circ \psi_{4,E}$. By Lemma 3.4, $(\alpha_3)_* : \pi_{n+2}(\xi) \to \pi_{n+2}(S_2^3)$ is the zero homomorphism implying $\psi_{4,2}^2 : \pi_{n+2}(K_4^2) \to \pi_{n+2}(K_2^2)$ is also.

Proof of Proposition 4.3. By Lemma 4.8 (ii), $H^3(K_2^3) \to H^3(K_G)$ is the zero homomorphism. So the composition $\psi_{G,2} \circ (\eta^3)^* : H^3(S^3) \to H^3(K_G)$ is zero. By Lemma 4.8 (i), $K_G \simeq \xi_1 \# \xi_2 \vee S^3 \vee S^3 \vee S^3$ so that $\eta^3 \circ \psi_{G,2} : K_G \to S^3$ factors through $\xi_1 \# \xi_2$ by some map $\alpha_1 : K_G \to \xi_1 \# \xi_2$. Let $\alpha_2 : \xi_1 \# \xi_2 \to S^3$ be a map such that $\alpha_2 \circ \alpha_1 : K_G \to S^3$ is homotopic to $\eta^3 \circ \psi_{G,2}$. ξ_2 has a cell decomposition with two 2-cells and one 4-cell. Since S^3 is two connected we can identify these 2-cells to a point obtaining a map $\alpha_3 : \xi_1 \# \xi_2 \to \xi_1$ which factors $\alpha_2 : \xi_1 \# \xi_2 \to S^3$ through ξ_1. We have a homotopy commutative diagram of maps and spaces:

$$\begin{tikzcd}
K_3^3 \arrow[r, "\psi_{3,G}"] & K_G \arrow[r, "\psi_{G,2}"] \arrow[d, "\alpha_1"'] & K_2^3 \arrow[r, "\eta^3"] & S^3 \\
& \xi_1 \# \xi_2 \arrow[rr, "\alpha_3"'] \arrow[urr, "\alpha_2"] & & \xi_1 \arrow[u, "\alpha_4"']
\end{tikzcd}$$

Hence $\eta^3 \circ \psi_{3,2}^3 \simeq \alpha_4 \circ \alpha_3 \circ \alpha_1 \circ \psi_{3,G}$. By Lemma 3.4,

$$(\alpha_4)_* : \pi_{n+2}(\xi_1) \to \pi_{n+2}(S^3)$$

is zero implying

$$\eta_*^3 \circ \psi_{3,2}^3 : \pi_{n+2}(K_3^3) \to \pi_{n+2}(S^3)$$

is zero.

§5. Further remarks

The condition $H_1(L)=0$ is considerably stronger than is necessary to show $\beta(L)$ vanishes. If, for a based semi-boundary 2-link, the compositions $\pi_1(\Sigma_i)\to\pi_L\to\pi/\pi_4$ are trivial, one can show that $\pi/\pi_4\cong F(2)/F_4(2)$ and that no obstructions exist to defining $\theta_4^2(L,\tau)\in\pi_{n+2}(K_4^2)$. Thus, $\theta_2^2(L)=\beta(L)=0$. This viewpoint enables one to study classical link cobordism. If the longitudes of a classical link $L\subset S^3$ lie "deeply" in the lower central series for π_L then $\theta_k^m(L,\tau)$ is defined. One can show, for example, that $\beta(L)=\theta_2^2(L)=0$ if and only if the longitudes lie in $(\pi_L)_4$; a fact first observed by T. Cochran in [C3]. This lays the groundwork for the relation between the invariants $\{\theta_k(L)\}$ for classical links and the $\bar{\mu}$-invariants defined by J. Milnor in [Mi]. A future paper will examine this viewpoint and offer new results about Milnor's invariants.

Milnor's invariants have been identified with Massey products, Cochran's derived invariants, and invariants of Murasugi [P], [C4], [Mu], [T]. Massey products and some of Cochran's derived invariants are invariants of higher dimensional links. It would be interesting to understand their relation to the θ_k-invariants.

The strength of our invariants for studying link cobordism is illustrated by the following realization theorem whose statement and proof will appear in a future paper.

THEOREM. *Let $x\in\pi_{n+2}(K_k^m)$, $n\geq 1$, k finite, then there is a based m-link (L,τ), (L may not be spherical) with $\theta_k^m(L,\tau)$ definable and equal to x.*

Notice this is true in the classical dimension where all links are spherical. As a corollary, this implies Sato–Levine invariants are realizable for two and three component semi-boundary links. Of course, for 2-links in any dimension $\beta(L)$ has been realized by N. Sato in [Sa]. (See also [R].) The proof of this theorem is an easy transversality argument which suggests a geometrical interpretation for two links to have the same θ_k-invariants. This takes the form of a cobordism of links as manifolds in $S^{n+2}\times[0,1]$ with certain homotopy theoretic conditions on the complement.

Lastly, the invariant $\theta_\omega^m(L,\tau)$ is presently the only known invariant which can even potentially detect a classical link whose longitudes lie in the intersection of the lower central series of its link group but which is not cobordant to a boundary link.

REFERENCES

[B] BROWN, K. S. *Graduate Texts in Mathematics*. Volume 87: *Cohomology of Groups*. Springer-Verlag, 1982.

[C1] COCHRAN, T. *Slice Links in S^4*. Trans. Amer. Math. Soc. *285*: 389–401, 1984.

[C2] COCHRAN, T. *On an Invariant of Link Cobordism in Dimension* 4. Topology and Its Applications *18* (2, 3): 97–108, 1984.

[C3] COCHRAN, T. *Geometric Invariants of Link Cobordism*. Comment. Math. Helv. *60:* 291–311, 1985.

[C4] COCHRAN, T. *Derivatives of Links:* Milnor's Concordance Invariants and Massey's Products. Preprint.

[CS] CAPPELL, S. and SHANESON, J. L. *Link Cobordism*. Comment. Math. Helv. *55:* 20–49, 1980.

[D] DEMEO, R. *Cobordisms of Even Dimensional Links*. Duke Math Journal 48(1): 23–33, 1981.

[Gr] GRUENBERG, K. W. *Cohomological Topics in Group Theory*. Lecture Notes in Mathematics. Springer-Verlag, 1970.

[Gu] GUTIERREZ, M. *Boundary Links and an Unlinking Theorem*. Trans. A.M.S. *171:* 491–499, 1972.

[Kea] KEARTON, C. *Cobordism of Knots and Blanchfield Duality*. J. London Mathematical Soc. *10*(2): 406–408, 1975.

[Ker] KERVAIRE, M. *Les Noeuds De Dimensions Superieure*. Bull. Soc. Math. France *93:* 225–271, 1965.

[Ko] KO, K. H. *Siefert Matrices and Boundary Links*. PhD thesis, Brandies University, 1984.

[L] LEVINE, J. *Knot Cobordism in Codimension Two*. Comment. Math. Helv. *44:* 229–224, 1969. MR 39 #7618.

[Ma] MANDELBAUM, R. *Four Dimensional Topology: An Introduction*. Bulletin Amer. Math. Soc. *2*(1): 1–260, 1980.

[Mi] MILNOR, J. *Isotopy of Links. Algebraic Geometry and Topology*. Princeton U. Press, Princeton, N.J., 1954, pages 280–306.

[MKS] MAGNUS, W., KARRASS, A., and SOLITAR, R. *Combinatorial Group Theory*. Interscience Publishers, 1976.

[Mu] MURASUGI, K. *Nilpotent Coverings of Links and Milnor's Invariant*. To appear in Proceedings of Sussex Conference, 1982. London Maths. Society, 95.

[P] PORTER, R. *Milnor's $\bar{\mu}$-Invariants and Massey Products*. Trans. A.M.S. *257* (1), 1980.

[R] RUBERMAN, D. *Concordance of Links in S^4*. Four Manifold Theory (Durham, N.H., 1982) p. 481–483. Contemp. Math., 35, Amer. Math. Soc., Providence, R.I., 1984.

[Sa] SATO, N. *Cobordism of Semi-Boundary Links*. Topology and Its Applications *18* (2, 3): 225–234, 1984.

[St] STALLINGS, J. *Homology and Lower Series of Groups*. J. Alg. *2:* 170–181, 1965.

[Sto] STONG, R. *Notes on Cobordism Theory*. Princeton Univ. Press, 1968.

[T] TURAEV, V. G. *The Milnor Invariants and Massey Products*. J. Soviet Math *12*(1), 1979. (translation).

[WG] WHITEHEAD, G. W. *Graduate Texts in Mathematics*. Volume 61: *Elements of Homotopy Theory*. Springer-Verlag, New York, 1978.

[WJ] WHITEHEAD, J. H. C. *On Simply Connected 4-Dimensional Polyhedra*. Comment. Math. Helv. *22:* 48–92, 1949.

The University of Chicago
Dept of Mathematics
5734 University Avenue
Chicago Illinois 60637/USA

Received October 10, 1985/December 9, 1986