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Gaps and bands of one dimensional periodic Schrödinger operators, II

JOHN GARNETT and EUGENE TRUBOWITZ

§1. Introduction

Let $q(x) \in L^2_{\mathbf{R}}[0, 1]$, the Hilbert space of square integrable real valued functions on the unit interval. Extend $q(x)$ to the whole line $\mathbf{R}$ by $q(x+1) = q(x)$. The spectrum of the Schrödinger operator $-d^2/dx^2 + q(x)$, acting on $L^2(\mathbf{R})$, is the set of λ such that

$$-y'' + q(x)y = \lambda y \tag{1.1}$$

has a nontrivial solution bounded on $\mathbf{R}$. The spectrum is contained in $\mathbf{R}$ and it is the union of a sequence of closed intervals $[\lambda_{2n-2}, \lambda_{2n-1}]$, where $\lambda_n = \lambda_n(q)$, $n \geq 0$, satisfies

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots.$$

These intervals are called *bands* and the intervening, possibly void, open intervals are called *gaps*. The possible arrangements of gaps and bands were investigated in [1]. This paper continues that study and includes some applications and simplifications.

Let $\gamma_n(q) = \lambda_{2n}(q) - \lambda_{2n-1}(q)$ be the n-th gap length. It is well known that $\gamma_n(q) \in (l^2)^+$, the space of nonnegative sequences with $\sum \gamma_n^2 < \infty$. Two of the three main results of [1] are:

(a) *Whenever* $\gamma_n \in (l^2)^+$, *there exists* $q \in L^2_{\mathbf{R}}([0, 1])$ *such that* $\gamma_n(q) = \gamma_n$, $n = 1, 2, \ldots$. *Moreover,* q *can be chosen from the even subspace* E *of* $q \in L^2_{\mathbf{R}}[0, 1]$ *such that*

$$q(1-x) = q(x)$$

(b) *the spectrum is determined, up to a translation, by the gap lengths* $\gamma_n(q)$.

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Let $\mu_n(q)$, $n \geq 1$, be the Dirichlet spectrum of q, that is, the spectrum of (1.1) for the boundary condition

$$y(0) = y(1) = 0,$$

and let $\nu_n(q)$, $n \geq 0$, be it's Neumann spectrum, i.e. the spectrum of (1.1) with boundary condition

$$y'(0) = y'(1) = 0.$$

Then $q \in E$ if and only if $\{\mu_n(q), \nu_n(q)\} = \{\lambda_{2n-1}(q), \lambda_{2n}(q)\}$, so that for q even,

$$\gamma_n(q) = |\mu_n(q) - \nu_n(q)|.$$

As functions on $L^2_{\mathbf{R}}[0, 1]$, $\mu_n(q)$ and $\nu_n(q)$ are real analytic (while λ_{2n} is not analytic at a q for which $\lambda_{2n}(q) = \lambda_{2n-1}(q)$) and hence the *signed gap length* $\sigma_n(q) = \mu_n(q) - \nu_n(q)$, $n \geq 1$, is real analytic in q. Furthermore, the map $\sigma : L^2_{\mathbf{R}}[0, 1] \to \ell^2$ defined by $\sigma(q) = (\sigma_n(q))$, $n \geq 1$, is a real analytic mapping from the Hilbert space $L^2_{\mathbf{R}}[0, 1]$ to the Hilbert space ℓ^2. The third main result of [1] is:

(c) *Let E_0 be the space of even potentials in $L^2_{\mathbf{R}}[0, 1]$ satisfying $\int_0^1 q(x)\, dx = 0$. Then the map*

$$E_0 \ni q \to \sigma(q) = (\sigma_1(q), \sigma_2(q), \ldots)$$

is a real analytic isomorphism between E_0 and ℓ^2, that is, σ is one-to-one and onto and both σ and σ^{-1} are real analytic maps of Hilbert space.

Of course, since $\gamma_n(q) = |\sigma_n(q)|$, $q \in E_0$, result (c) included result (a).

The proof of (a), (b) and (c) in [1] applied harmonic measure arguments to the identification, due to Marčenko and Ostrovskii [3], of band configurations with certain slit quarter planes. In Section 2 we give a direct proof, using analysis in Hilbert space, that the Jacobian

$$d_q\sigma : E_0 \to \ell^2$$

is invertible. From this it follows easily that σ is one-to-one, and that, if σ is onto, than by the Inverse Function Theorem, σ^{-1} is real analytic. Consequently, result (c) can be proved without the intricate Section 6 of [1]. We cannot prove σ is onto ℓ^2 using only the method of Section 2 without a still unknown estimate of $\|q\|_2$ in terms of $\|\sigma(q)\|_{\ell^2}$. However, in Sobolev space such an estimate is available and thus we show in Section 2 that σ is an isomorphism from $E_0 \cap H^k = \{q \in E_0 : q$ has k derivatives periodic and in $L^2([0, 1])\}$ onto $\ell^2_k = \{(\sigma_n) : \sum n^{2k}\sigma_n^2 < \infty\}$.

In Section 3 result (c) is used to prove Marčenko's theorem [2] that the finite band potentials (those q with $\gamma_n(q) = 0$ for large n) are norm dense in L^2, and that q has primitive period $1/k$ if and only if $\gamma_n(q) = 0$ when k does not divide n.

In Section 4 we give some inequalities that band lengths must satisfy and we show that for real analytic potentials the band lengths determine the spectrum up to a translation. Here the harmonic measure methods of [1] reappear.

§2. Signed gap lengths

We need a general interpolation lemma.

LEMMA 2.1. *Suppose $\phi(\lambda)$ is an entire function satisfying*

$$\sup_{|\lambda|=(n+1/2)^2\pi^2} \left| \phi(\lambda) \Big/ \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \right| = o(1)$$

as $n \to \infty$. Then

$$\phi(\lambda) = \sum_{n \geq 1} \phi(\xi_n) \prod_{\substack{m \geq 1 \\ m \neq n}} \frac{\xi_m - \lambda}{\xi_m - \xi_n}$$

for any sequence ξ_n, $n \geq 1$, of distinct complex numbers satisfying $\xi_n = n^2\pi^2 + o(1)$.

Proof. If ξ_m, $m \geq 1$, is a distinct sequence with $\xi_m = m^2\pi^2 + o(1)$, then

$$\prod \frac{\xi_m - z}{m^2\pi^2} = \frac{\sin\sqrt{z}}{\sqrt{z}}\left(1 + o\left(\frac{\log n}{n}\right)\right)$$

unifomly on the circles $\Gamma_n = \{|z| = (n + 1/2)^2\pi^2\}$. Hence the meromorphic function

$$f(z) = \frac{\phi(z)}{z - \lambda} \prod_{m \geq 1} \frac{m^2\pi^2}{\xi_m - z}$$

satisfies $\sup_{\Gamma_n} |f(z)| = o(n^{-2})$, $n \to \infty$; and the sum of its residues inside Γ_n has limit 0 as $n \to \infty$. But $f(z)$ has simple poles of λ and at ξ_n, $n \geq 1$, and $f(z)$ is regular elsewhere. Summing the residues, we obtain

$$0 = \phi(\lambda) \prod_{m \geq 1} \frac{m^2\pi^2}{\xi_m - \lambda} - \sum_{n=1}^{\infty} \phi(\xi_n) \frac{n^2\pi^2}{\lambda_n - \lambda} \prod_{m \neq n} \frac{m^2\pi^2}{\xi_m - \xi_n},$$

which is the assertion of the lemma. □

We turn to the main result of this section.

THEOREM 2.2. *For all $q \in E_0$, the Jacobian $d_q\sigma : E_0 \to \ell^2$ is an isomorphism onto ℓ^2.*

Proof. See Chapter 2 of [6] for the facts used in this proof.
The components of $d_q\sigma$ are

$$d_q\sigma_n = d_q\mu_n - d_q\nu_n = g_n^2 - h_n^2,$$

where

$$g_n^2(t) = 2\sin^2 n\pi t + 0\left(\frac{1}{n}\right)$$

and

$$h_n^2(t) = 2\cos^2 n\pi t + 0\left(\frac{1}{n}\right)$$

are the respective squares of the n-th Dirichlet and Neumann eigenfunctions. Hence the operator $d_q\sigma$ is the sum of the isomorphic Fourier series operator

$$E_0 \ni f \to (-2\langle \cos 2n\pi t, f \rangle, n \geq 1)$$

and the compact operator

$$E_0 \ni f \to \left(\left\langle 0\left(\frac{1}{n}\right), f \right\rangle, n \geq 1\right),$$

and $d_q\sigma : E_0 \to \ell^2$ is a Fredholm operator.

When q is even the vectors $g_m^2 - 1$, $m \geq 1$, form a basis for E_0 with dual basis $-2a_m'(x)$, where

$$a_m(x) = y_1(x, \mu_m) y_2(x, \mu_m)$$

and where $y_1(x, \mu_m)$, $y_2(x, \mu_m)$ are the fundamental solutions of (1.1) for $\lambda = \mu_m$ with

$$\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

That is, if $f, g \in E_0$, then

$$\langle f, -2a_m' \rangle \in l^2$$
$$\langle g_m^2 - 1, g \rangle \in l^2$$

and

$$\int fg\, dx = \sum_{m=1}^{\infty} \langle f, -2a_m' \rangle \langle g_m^2 - 1, g \rangle.$$

Therefore it is sufficient to prove that the matrix

$$a_{n,m} = \langle g_n^2 - h_n^2, -2a_m' \rangle$$

is invertible in $B(l^2, l^2)$.

We have $\langle g_n^2, -2a_m' \rangle = \delta_{n,m}$, and because $a_m(0) = a_m(1) = 0$,

$$\langle -h_n^2, -2a_m' \rangle = 2\int h_n^2 a_m'\, dx = -4\int h_n' h_n y_1(x, \mu_m) y_2(x, \mu_m)\, dx$$
$$= \int y_1 h_n [h_n, y_2] + y_2 h_n [h_n, y_1]\, dx$$

where $[f, g] = fg' - f'g$. But by (1.1),

$$\frac{d}{dx}[h_n, y_j] = (\nu_n - \mu_m) h_n y_j.$$

So if $\nu_n \neq \mu_m$, then

$$\langle -h_n^2, -2a_m' \rangle = \frac{1}{\nu_n - \mu_m}([h_n, y_1][h_n, y_2])\Big|_0^1$$
$$= \frac{1}{\nu_n - \mu_m} h_n^2(1) y_1'(1) y_2'(1) = \frac{(-1)^m}{\nu_n - \mu_m} h_n^2(1) y_1'(1, \mu_m)$$

since $h_n'(0)=h_n'(1)=0$, since $y_1'(0)=0$ and since, when q is even, $y_2'(1,\mu_m)=(-1)^m$. Also

$$h_n^2(1)=\frac{y_1^2(1,\nu_n)}{\|y_1(\cdot,\nu_n)\|_2^2}=\frac{(-1)^{n+1}}{\dot{y}_1'(1,\nu_n)}$$

where $\dot{y}=\partial y/\partial\lambda$, because $y_1(1,\nu_n)=(-1)^n$ when q is even and because $\|y_1(\cdot,\nu_n)\|_2^2=-\dot{y}_1'(1,\nu_n)y_1(1,\nu_n)$. From the product formulas

$$y_1'(1,\mu_m)=(\nu_0-\mu_m)\prod_{k\geq 1}\frac{\nu_k-\mu_m}{k^2\pi^2}$$

$$\dot{y}_1'(1,\nu_n)=\frac{-(\nu_0-\nu_n)}{n^2\pi^2}\prod_{1\leq k\neq n}\frac{\nu_k-\nu_n}{k^2\pi^2}$$

we conclude that

$$\langle -h_n^2, -2a_m'\rangle=(-1)^{n+m}\prod_{0\leq k\neq n}\frac{\nu_k-\mu_m}{\nu_k-\nu_n} \tag{2.1}$$

when $\nu_n\neq\mu_m$. If $\nu_n=\mu_m$, then $n=m$ and $[h_n,y_j]=\|y_1\|_2^{-1}\delta_{j,2}$ because the Wronskian $[y_1,y_2]=1$. Consequently $\langle -h_n^2,-2a_m'\rangle=1$ and (2.1) also holds when $\nu_n=\mu_m$. Thus our matrix is

$$a_{n,m}=\delta_{n,m}+(-1)^{n+m}\prod_{0\leq k\neq n}\frac{\nu_k-\mu_m}{\nu_k-\nu_n},$$

and $(a_{n,m})$ is Fredholm because $d_q\sigma$ is a Fredholm operator.

By the Fredholm alternative, $d_q\sigma$ is an isomorphism of E_0 onto ℓ^2 if the transpose $(a_{m,n})$ is one-to-one. Now suppose $\tau=(\tau_n,n\geq 1)\in\ell^2$ lies in the kernel of $(a_{m,n})$. Then

$$0=(-1)^n\frac{\tau_n}{\nu_0-\mu_n}+\sum_{m\geq 1}(-1)^m\frac{\tau_m}{\nu_0-\nu_m}\prod_{1\leq k\neq m}\frac{\nu_k-\mu_n}{\nu_k-\nu_m}.$$

Consider the function

$$\phi(\lambda)=\sum_{m\geq 1}(-1)^m\frac{\tau_m}{\nu_0-\nu_m}\prod_{1\leq k\neq m}\frac{\nu_k-\lambda}{\nu_k-\nu_m}.$$

We will show in a moment that $\phi(\lambda)$ is a entire function of λ satisfying

$$\left|\phi(\lambda)\Big/\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right| = o(1) \tag{2.2}$$

uniformly on the circles $|\lambda|(n+1/2)^2\pi^2$ as $n\to\infty$. But since

$$\phi(\mu_n) = (-1)^{n+1}\frac{\tau_n}{\nu_0-\mu_n}, \qquad \phi(\nu_n) = (-1)^n\frac{\tau_n}{\nu_0-\nu_n},$$

$\phi(\xi_n)=0$ at some point ξ_n in the n-th gap. Consequently $\phi\equiv 0$ by Lemma 2.1 and $\tau_n=0$, $n\geq 1$. That means the transpose $(a_{m,n})$ is one-to-one and $d_q\sigma$ is an isomorphism.

It remains to prove (2.2). Since $\nu_n - n^2\pi^2\in \ell^2$,

$$\begin{aligned}\prod_{1\leq k\neq m}\frac{\nu_n-\lambda}{\nu_k-\nu_m} &= \left(\prod_{1\leq k\neq m}\frac{k^2\pi^2-\lambda}{k^2\pi^2-m^2\pi^2}\right)\left(1+o\left(\frac{\log n}{n}\right)\right)\\ &= \frac{m^2\pi^2}{m^2\pi^2-\lambda}\left(\prod_{1\leq k\neq m}\frac{k^2}{k^2-m^2}\right)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\left(1+o\left(\frac{\log n}{n}\right)\right)\\ &= 2(-1)^{m+1}\frac{m^2\pi^2}{m^2\pi^2-\lambda}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\left(1+o\left(\frac{\log n}{n}\right)\right)\end{aligned}$$

on $|\lambda| = (n-1/2)^2\pi^2$. Hence for such λ,

$$\left|\phi(\lambda)\Big/\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right| \leq \text{Const.}\sum_m\frac{\tau_m}{|m^2-(n+1/2)^2|} = o(1). \qquad \square$$

It will be convenient to replace E_0 by

$$\mathscr{E}_0 = \{q\in E : \lambda_0(q)=0\}.$$

Since even potentials are determined by their Dirichlet spectra and since $\mu_n(q+c)=\mu_n(q)+c$, and $\nu_n(q+c)=\nu_n(q)+c$, the map $q\to q-[q]$, where $[q]=\int_0^1 q(x)\,dx$, is an isomorphism from $\mathscr{E}_0$ to E_0 preserving signed gap lengths. Let

$$\mathscr{E}^n = \{q\in\varepsilon_0 : \gamma_m(q)=\sigma_m(q)=0,\ m>n\}.$$

Because, by Theorem 2.2, σ is local analytic isomorphism on $\mathscr{E}_0$, $\mathscr{E}^n$ is a real analytic submanifold of $\mathscr{E}_0$ of dimension n.

COROLLARY 2.3. *For each $n \geq 1$, the signed gap length map is a real analytic isomorphism of $\mathscr{E}^n$ onto $\mathbf{R}^n$.*

Proof. The image $\sigma(\mathscr{E}^n)$ is an open subset $\mathbf{R}^n$ because $\sigma: \mathscr{E}_0 \to \ell^2$ is a local homeomorphism and $\mathscr{E}^n = \sigma^{-1}(\mathbf{R}^n) \cap \varepsilon_0$. We next show $\sigma(\mathscr{E}^n)$ is closed. The identity from [7],

$$q(t) = \lambda_0 + \sum_{m \geq 1} \{\lambda_{2m} + \lambda_{2m-1} - 2\mu_m(T_t q)\},$$

where $T_t q(x) = q(x+t)$, yields

$$|q(t)| \leq \sum_{m=1}^{n} \gamma_m(q), \quad q \in \mathscr{E}^n \tag{2.3}$$

Hence the preimage in $\mathscr{E}^n$ of any compact subset of $\mathbf{R}^n$ is bounded in L^2. It is also weakly closed because the functions $\sigma_m(q) = \mu_m(q) - \nu_m(q)$ are weakly continuous. Thus the preimage of a compact subset of $\mathbf{R}^n$ is a weakly compact subset of $\mathscr{E}^n$, and it follows that the map $\sigma: \mathscr{E}^n \to \mathbf{R}^n$ is proper and that $\sigma(\mathscr{E}^n)$ is a nonempty, closed subset of $\mathbf{R}^n$. Therefore σ maps $\mathscr{E}^n$ onto $\mathbf{R}^n$.

Now let M be the set of points in $\mathbf{R}^n$ having more than one preimage. Then M is open because σ is a local homeomorphism. But M is also closed. Indeed, if there are distinct points q_j and p_j in $\mathscr{E}^n$ such that $\sigma(p_j) = \sigma(q_j) \to \sigma \in \mathbf{R}^n$, then because the map is proper there are subsequences such that $p_j \to p \in \mathscr{E}^n$ and $q_j \to q \in \mathscr{E}^n$. If $p = q$ then $p_j = q_j$ for j large because the map σ is homeomorphic on a neighborhood of p. So $p \neq q$ and M is closed. But $0 \notin M$ by (2.3). Thus $M \neq \varnothing$ and the mapping is one-to-one.

The map $\sigma: \mathscr{E}^n \to \mathbf{R}^n$ is real analytic because μ_m and ν_m are real analytic on $L^2_{\mathbf{R}}[0, 1]$. The inverse map is real analytic because $d_q\sigma$ is invertible. □

It is now easy to show that the map σ is one-to-one on $\mathscr{E}_0$ (and hence on E_0).

COROLLARY 2.4. *The signed gap length map in one-to-one on $\mathscr{E}_0$.*

Proof. Suppose not. Then some point $\tau \in \ell^2$ has at least two preimages. Since σ is a local homeomorphism, the same is true for each point in some neighborhood of τ, so it is also true at

$$\tau^{(N)} = (\tau_1, \ldots, \tau_N, 0, 0, \ldots)$$

for N sufficiently large. But that contradicts Corollary 2.3. □

Write ℓ_k^2 for the space of sequences (a_n) with $\sum n^{2k} |a_n|^2 < \infty$. From the asymptotics for $y_2(1, \lambda, q)$ and $y_1'(1, \lambda, q)$ we have

$$\mu_n(q) = n^2\pi^2 + [q] - \langle \cos 2n\pi x, q \rangle + \ell_1^2$$
$$\nu_n(q) = n^2\pi^2 + [q] + \langle \cos 2n\pi x, q \rangle + \ell_1^2.$$

Hence for $q \in E_0$, $\sigma_n(q) \in \ell_1^2$ if and only if $\langle \cos 2n\pi x, q \rangle + \ell_1^2$; i.e. if and only if q is in the Sobolev space

$$H^1 = \{q \in L^2_{\mathbf{R}}[0, 1] : q' \in L^2_{\mathbf{R}}[0, 1]\}.$$

THEOREM 2.5. *The signed gap length map from $E_0 \cap H^1$ to ℓ_1^2 is one-to-one and onto.*

Proof. By Corollary 2.4 σ is one-to-one. To prove it is onto fix $\tau \in \ell_1^2$ and let $\tau^{(N)} = (\tau_1, \tau_2, \ldots, \tau_N, 0, 0, \ldots)$. By Corollary 2.3 there is $q_N \in \varepsilon^N$ such that $\sigma(q_N) = \tau^{(N)}$, and by (2.3)

$$|q_N(t)| \leq \sum_{n=1}^{N} |\tau_n| \leq \left(\sum_{n=1}^{N} n^2\tau_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^{-2}\right)^{1/2},$$

so that $\|q_N\|_2 \leq \text{Const.}\, \|\tau\|_{\ell_1^2}$. Let $q \in \varepsilon_0$ be a weak limit of the sequence $\{q_N\}$. Then

$$\sigma_n(q - [q]) = \tau_n$$

for all n, and $q - [q] \in H^1 \cap E_0$ since $\tau \in \ell_1^2$. $\square$

Remark 2.6. We are unable to prove the full result that σ maps E_0 onto ℓ^2 by this method. What is needed is an estimate of $\|q\|_2$ in terms of $\gamma_n(q)$ more powerful than (2.3). Such an estimate should be useful for other problems.

Remark 2.7. It is possible, by refining the proof of Theorem 2.2, to show that $\sigma : E_0 \cap H^1 \to \ell_1^2$ is an analytic isomorphism. We omit the details.

Remark 2.8. It is known [3, p. 534] that $\gamma_n(q) \in \ell_k^2$ if and only if $q \in H^k$, i.e. if and only if q has k derivatives which are periodic and lie in $L^2_{\mathbf{R}}[0, 1]$. Thus the proof of Theorem 2.5 shows that

$$\sigma : E_0 \cap H^k \to \ell_k^2$$

is one-to-one and onto. We have not verified the likely statement that this map is bianalytic.

§3. Two applications

The potential $q \in L^2_{\mathbf{R}}[0, 1]$ is called a *finite band* potential if $\gamma_n(q) = 0$ for all but finitely many n. Marčenko [2, p. 258] proved that the set of finite band potentials is norm dense in $L^2_{\mathbf{R}}[0, 1]$. Here we derive that Theorem from result (c), stated in the introduction.

THEOREM 3.1 (Marčenko). *The set of finite band potentials is norm dense in* $L^2_{\mathbf{R}}[0, 1]$.

For $q \in E$, Theorem 3.1 is immediate from results (c). To prove it for arbitrary q we need two additional theorems. Define

$$\kappa_n(q) = \log((-1)^n y_2'(1, \mu_u, q)).$$

In [6] it is proved that $\kappa_n(q) \in \ell^2_1$, i.e. that $\sum n^2 \kappa_n^2(q) < \infty$, and that the correspondence

$$q \to (\mu_n(q) - [q], \kappa_n(q))$$

is a homeomorphism from $L^2_{\mathbf{R}}[0, 1]$ onto $\ell^2 \times \ell^2_1$. That is the first theorem.

The second theorem is the description of the isospectral manifold

$$L(q) = \{p \in L^2_{\mathbf{R}}[0, 1] : \lambda_n(p) = \lambda_n(q), \text{ all } n\}$$

given in [4]. The parameters

$$\mu_n(p) \in [\lambda_{2n-1}, \lambda_{2n}]$$

and

$$\text{sign } \kappa_n(p)$$

uniquely determine $p \in L(q)$. Although true generally, this theorem will only be used for finite band potentials, and such potentials satisfy the smoothness assumptions of [4].

Proof of Theorem 3.1. Fix $q \in L^2_{\mathbf{R}}[0,1]$. Since $\lambda_n(q+c) = \lambda_n(q)$, we may suppose $\lambda_0(q) = 0$. By result (c) there exist, for $N = 1, 2, \ldots$, $e_N \in \varepsilon_0$ such that

$$\gamma_n(e_0) = \begin{cases} \gamma_n(q) & n \leq N \\ 0 & n > N \end{cases}$$

and

$$\mu_n(e_N) = \lambda_{2n-1}(e_N), \quad n = 1, 2, \ldots.$$

Since $\mu_n(q) \in [\lambda_{2n-1}(q), \lambda_{2n}(q)]$ there exists $t_n \in [0, 1]$ such that

$$\mu_n(q) = t_n\lambda_{2n}(q) + (1 - t_n)\lambda_{2n-1}(q),$$

and by the second theorem just cited there exists $q_N \in L(e_N)$ such that for all n,

$$\mu_n(q_N) = t_n\lambda_{2n}(e_N) + (1 - t_n)\lambda_{2n-1}(e_N)$$

and

$$\operatorname{sign} \kappa_n(q_N) = \operatorname{sign} \kappa_n(q).$$

By the first cited theorem $\|q_N - q\|_2 \to 0$ if

$$\|\mu(q_N) - \mu(q)\|_{\ell^2} \to 0 \tag{3.1}$$

and

$$\|\kappa_n(q_N) - \kappa_n(q)\|_{\ell^2_1} \to 0. \tag{3.2}$$

By the second theorem there exists $e \in \mathscr{E}_0$ such that for all n,

$$\lambda_n(e) = \lambda_n(q)$$
$$\mu_n(e) = \lambda_{2n-1}(q).$$

Then $\|\gamma_n(e_N) = \gamma_n(e)\|_{\ell^2} \to 0$ and $\sigma_n(e_N)$ and $\sigma_n(e)$ have the same sign, so that

$$\|\sigma_n(e_N) - \sigma_n(e)\|_{\ell^2} \to 0. \tag{3.3}$$

Hence by result (c), $\|e_N - e\|_2 \to 0$ and by the first theorem

$$\|\mu_n(e_N) - \mu_n(e)\|_{\ell^2} \to 0. \tag{3.4}$$

But then by the choices of q_N, e_N and e,

$$\mu_n(q_N) - \mu_n(q) = -t_n(\sigma_n(e_N) - \sigma_n(e)) + \mu_n(e_N) - \mu_n(e),$$

and (3.3) and (3.4) imply (3.1).

To prove (3.2) we use the identity

$$2\cosh \kappa_n(q) = (-1)^n \Delta(\mu_n(q), q),$$

where $\Delta(\lambda, q)$ is the discriminant function

$$\Delta(\lambda, q) = y_1(1, \lambda, q) + y_2'(1, \lambda, q)$$

and the inequality

$$|x - y|^2 \leq 2\,|\cosh x - \cosh y|,$$

valid when x and y have the same sign. They give

$$\begin{aligned} n^2\,|\kappa_n(q_N) - \kappa_n(q)|^2 &\leq n^2\,|\Delta(\mu_n(q_N), q_N) - \Delta(\mu_n(q), q)| \\ &= n^2\,|\Delta(\mu_n(q_N), e_N) - \Delta(\mu_n(q), e)|. \end{aligned}$$

Since $\|e_N - e\|_2 \to 0$, $\Delta(\lambda, e_N) \to \Delta(\lambda, e)$ uniformly on compact sets. Thus by (3.1)

$$|\Delta(\mu_n(q_N), e_N) - \Delta(\mu_n(q), e)| \to 0$$

for each n. Moreover,

$$\begin{aligned} &|\Delta(\mu_n(q_N), e_N) - \Delta(\mu_n(q), e)| \\ &\quad \leq |\Delta(\mu_n(q_N), e_N) - 2(-1)^n| + |\Delta(\mu_n(q), e) - 2(-1)^n| \end{aligned}$$

since $\Delta(\lambda_{2n-1}, e_N) = \Delta(\lambda_{2n-1}, e) = 2(-1)^n$. Because $\dot{\Delta}(\lambda) = \partial\Delta/\partial\lambda$ is an entire function of order $\frac{1}{2}$, having one zero $\dot{\lambda}_n$ in each gap $[\lambda_{2n-1} \leq \lambda \leq \lambda_{2n}]$ and no other zeros, the product representation

$$\dot{\Delta}(\lambda) = \prod_{n \geq 1} \frac{\dot{\lambda}_n - \lambda}{n^2\pi^2}$$

shows that

$$\sup_{\lambda_{2n-1} \leq \lambda \leq \lambda_{2n}} |\dot{\Delta}(\lambda, q)| \leq c\,\frac{\gamma_n(q)}{n^2}.$$

Hence by (3.5)

$$n^2 |\Delta(\mu_n(q_N), e_N) - \Delta(\mu_n(q), e)| \le c\gamma_n^2(q)$$

and by dominated convergence

$$\lim_{N\to\infty} \sum_n n^2 |\kappa_n(q_N) - \kappa_n(q)|^2 = 0. \quad \square$$

Our second application concerns the subspace $L_k^2 \subset L_R^2[0, 1]$ of all functions whose periodic extensions have primitive period $1/k$, $k = 1, 2, \ldots$. First of all, if $q \in L_k^2$, $\gamma_n(q) = 0$ whenever $k \nmid n$. To see this, recall from [7] that $\mu_n(t) = \mu_n(T_t q)$, where $T_t q(x) = q(x - t)$, then as t runs from 0 to 1, $\mu_n(t)$ makes n complete trips between $\lambda_{2n-1}(q)$ and $\lambda_{2n}(q)$, when $\gamma_n(q) \neq 0$. By assumption, $\mu_n(t + 1/k) = \mu_n(t)$. Therefore, n is equal to k times the number of complete trips in time $1/k$, and so k divides n, $k \mid n$.

Now let $E_0^k = E_0 \cap L_k^2$. It follows from the observation above that the restriction of σ to E_0^k maps into $\ell^2(k) = \{\sigma \in \ell^2 : \sigma_n = 0 \text{ whenever } k \nmid n\}$. Without any change in the argument of [1] or that of Section 2, one can show that σ is a real analytic isomorphism between E_0^k and $\ell^2(k)$ or between $E_0^k \cap H^1$ and $\ell_1^2(k)$.

Suppose $q \in E$ and $\sigma(q) \in \ell^2(k)$. Then there is a $p \in E_0^k$ such that $\sigma(p) = \sigma(q)$. However, σ is globally one-to-one on E_0 so that $p = q - [q]$. In other words, $q \in E_0$ has primitive period $1/k$ if and only if $\gamma_n(q) = 0$ whenever $k \nmid n$.

It is easy to extend this observation to all of $L_R^2[0, 1]$. Let $q \in L_R^2[0, 1]$ and $L(q) = \{r \in L_R^2[0, 1] \mid \lambda_i(r) = \lambda_i(q) i \ge 0\}$, i.e., the isospectral set of q. It is not hard to see that $L(q) \cap E \neq \phi$ and that all points in $L(q)$ have the same primitive period. See [4]. Thus we have proved

THEOREM 3.2. *The potential q has primitive period $1/k$ if and only if $\gamma_n(q) = 0$ whenever k does not divide n.*

§4. Band lengths

Let $\alpha_n(q) = \alpha_n = \lambda_{2n-1} - \lambda_{2n}$ be the length of the n-th band. It is well known that

$$\alpha_n(q) - (2n-1)\pi^2 \in \ell^2 \tag{4.1}$$

and in [1] and [5] it was shown that

$$\tilde{\alpha}_n = (2n-1)\pi^2 - \alpha_n \geq 0, \tag{4.2}$$

with equality holding for some n if and only if q is constant.

THEOREM 4.1. *For all n and all q,*

$$\alpha_n - \alpha_{n-1} + \alpha_{n-2} \mp \cdots > 0, \tag{4.3}$$

and

$$\beta_n = \tilde{\alpha}_n - \tilde{\alpha}_{n-1} + \tilde{\alpha}_{n-2} \mp \cdots \geq 0. \tag{4.4}$$

Moreover, if $\beta_n = 0$ for some n, then q is constant and $\beta_k = 0$ for all k.

Note that by (4.1), (4.3) has content only for small n. By (4.4) $\beta_n \leq \tilde{\alpha}_n$, so that by (4.1)

$$\beta_n \in \ell^2_+$$

and by (4.4) and (4.3),

$$0 \leq \beta_n < n\pi^2. \tag{4.5}$$

We shall show that (4.5) is sharp for every n and that, properly interpreted, the Jacobian $d_g\beta_n : E_0 \to \ell^2$ is invertible at $q = 0$. A simple characterization of band lengths thus seems unlikely.

Proof. Recall from [1] that there exists $h_n = h_n(q) \geq 0$, such that $\sum n^2 h_n^2 < \infty$, and such that

$$\delta(\lambda, q) = \cos^{-1}\left(\frac{\Delta(\lambda, q)}{2}\right)$$

is a conformal mapping from the half plane $\{\operatorname{Im} \lambda > 0\}$ onto the slit quarter plane

$$\Omega(h) = \{x > 0, y > 0\} \setminus \bigcup_{n=1}^{\infty} T_n$$

where

$$T_n = \{n\pi + iy : 0 < y \leq h_n\}.$$

Under $\delta(\lambda, q)$ the n-th band is mapped onto the segment

$$B_n = [(n-1)\pi-, n\pi+] \subset \partial\Omega(h),$$

and if

$$u_n(z, h) = \omega(z, B_n, \Omega(h)) = u_n(z)$$

is the harmonic measure of B_n in $\Omega(h)$, then

$$\alpha_n = \lim_{x\to\infty} 2\pi x^2 u_n(x+ix, h). \tag{4.6}$$

Let $k \le n$ and let $z = x + ix$ with $x > n\pi$. Then $u_k(z)$ is the probability that a Brownian path starting at z makes its first exit from $\Omega(h)$ through B_k. Letting S_k be the set of such paths, we write

$$u_k(z) = P_z(S_k)$$

Brownian paths can be assumed continuous. Thus every path in S_k must cross the half line $J_k = \{x = k\pi, y > 0\}$ before it leaves $\Omega(h)$. Let R_k be those paths in S_k which, before leaving $\Omega(h)$, last meet $J_k \cup J_{k-1}$ in J_k, and let L_k be those whose last contact with $J_k \cup J_{k-1}$, before departing from $\Omega(h)$, is in J_{k-1}. Then R_k and L_k are P_z measurable, $R_k \cap L_k = \phi$ and

$$P_z(S_k) = P_z(R_k) + P_z(L_k).$$

But

$$P_z(L_k) = P_z(R_{k-1})$$

by a reflection. Since $L_1 = \varnothing$, we conclude that

$$u_n(z) - u_{n-1}(z) \pm \cdots = P_z(R_n) > 0$$

which by (4.6) yields (4.3). To prove (4.4), let

$$V_n(z) = u_n(z, 0) - u_n(z, h).$$

Then

$$\tilde{\alpha}_n = \lim_{x\to\infty} 2\pi x^2 V_n(x+ix).$$

On $\partial\Omega(h)$,

$$V_n(\zeta) = \sum_{k=1}^{\infty} u_n(\zeta, 0)\chi_{T_k}(\zeta), \tag{4.7}$$

and the argument above shows

$$\sum_{k=1}^{n} (-1)^{n-k} u_k(\zeta, 0) > 0$$

on $\bigcup T_k$. Hence for x large

$$\sum_{k=t}^{n} (-1)^{n-k} V_k(x + ix) \geq 0$$

and (4.4) holds. If equality holds in (4.4) then by (4.7), $\bigcup T_k$ has zero harmonic measure in $\Omega(h)$. That means all gap lengths are zero and q is constant. □

To see that (4.5) is sharp, note that $q \to h_n(q)$ maps onto ℓ_1^2 and that by (4.6),

$$\lim_{h_n \to \infty} \alpha_k = 0, \qquad 1 \leq k \leq n.$$

For $q \in E$, define

$$a_n(q) = \mu_n(q) - \nu_{n-1}(q), \quad n \geq 1$$

and

$$b_n(q) = \sum_{k=1}^{n} (-1)^{n-k}((2n-1)\pi^2 - a_n(q)) = n\pi^2 - \sum_{k=1}^{n} (-1)^{n-k} a_n(q).$$

Then for $q \in E$

$$b_n(q) = \beta_n(q) + \operatorname{Max}(\sigma_n(q), 0),$$

and for each potential $q \in L^2$ there is $q^+ \in E$ with $\mu_n(q^+) \leq \nu_n(q^+)$ and $\lambda_n(q^+) = \lambda_n(q)$, so that $b_n(q^+) = \beta_n(q)$.

THEOREM 4.2. *At $q = 0$ the Jacobian $d_q(b_n): E_0 \to \ell^2$ is an isomorphism onto ℓ^2.*

Proof. At $q=0$, $f\in E_0$,

$$\langle d_q a_n, f\rangle = \langle 2\sin^2 n\pi t - 2\cos^2(n-1)\pi t, f\rangle$$

and

$$\langle d_q b_n, f\rangle = -2\langle \sin^2 n\pi t, f\rangle = \langle \cos 2n\pi t, f\rangle,$$

and $(\cos 2n\pi t)_{n\geq t}$ is a complete orthonormal system in E_0. □

THEOREM 4.3. (a) *If q and $\tilde{q}$ are finite band potentials and if $\alpha_n(q)=\alpha_n(\tilde{q})$ for infinitely many n, then the periodic spectra of q and $\tilde{q}$ agree up to a translation.*

(b) *If q and $\tilde{q}$ are real analytic, and if $\alpha_n(q)=\alpha_n(\tilde{q})$ for all large n, then q and $\tilde{q}$ have the same periodic spectrum up to a translation.*

Proof. Let $\phi(z,q)$ be the inverse of the mapping $\delta(\lambda,q)$. If q is a finite band potential then $h_n(q)=0$, $n>N$ and $\phi(z,q)$ reflects to be analytic in the complement of the finite union of vertical slits $\{|x|=n\pi,\ |y|\leq h_n(q),\ 1\leq n\leq N\}$. For z large we have

$$\phi(z,q)=z^2+o\left(\frac{1}{z}\right).$$

By the hypothesis of (a),

$$\phi(z+\pi,q)-\phi(z,q)=\phi(z+\pi,\tilde{q})-\phi(z,\tilde{q}) \tag{4.8}$$

holds for an infinite sequence of integers tending to ∞. Hence (4.8) holds for all z, and $\phi(z,q)$ and $\phi(z,\tilde{q})$ have the same singularities. Therefore $h_n(q)=h_n(\tilde{q})$ for all n, which means the spectra of q and $\tilde{q}$ differ by at most a translation.

To prove (b), set $f(z)=\phi(z,q)-\phi(z,\tilde{q})$. By reflection $f(z)$ is analytic in

$$\Omega^* = \mathbf{C}\Big/\bigcup_{n=1}^{\infty}(S_n\cup S\_S_{-n})$$

where $S_n=\{x=n\pi,\ |y|\leq \operatorname{Max}(h_{|n|}(q),\ h_{|n|}(\tilde{q}))\}$, and by the asymptotics for $\Delta(\lambda,q)$, $f(z)$ is bounded on Ω^*. Since q and $\tilde{q}$ are real analytic, we have by [7],

$$\begin{aligned}\operatorname{Max}(h_n(q), h_n(\tilde{q}) &\leq C\operatorname{Max}(\gamma_n(q), \gamma_n(\tilde{q}))\\ &\leq Ce^{-an}\end{aligned}$$

for constants a and C. Viewing S_n as two-sided, we see that $f(z)$ has continuous boundary values on S_n and that for $n \geq 1$,

$$\sup_{z \in S_n} |f(z) - f(n\pi+)| \leq \gamma_n(q) + \gamma_n(\tilde{q}) \leq Ce^{-an}.$$

By hypothesis there is N so that

$$f((n+1)\pi-) - f(n\pi+) = \alpha_n(q) - \alpha_n(\tilde{q}) = 0$$

for $n \geq N$, and hence

$$\sup_{z \in S_n} |f(z)| \leq Ce^{-an}, \quad n \geq N. \tag{4.9}$$

Set $h^* = \sup_n \{h_n(q), h_n(\tilde{q})\}$. We shall prove

$$|f(x + Ch^*)| \leq Ce^{-a'x}, \quad x > x_0 \tag{4.10}$$

First assume (4.10). Then because $f(z)$ is bounded and analytic in $\{y > h^*\}$,

$$\log |f(z)| \leq C_1 + C_2 \int_{x > x_0} \frac{-a'x}{1 + x^2}\, dx = -\infty$$

on $|z - i(h^* + 1)| < \frac{1}{2}$. Therefore $f = 0$ and (b) is proved.

We turn to the proof of (4.10). Let Δ_n be the disc $\{|z - n\pi| < 2Ae^{-a|n|}\}$ with A so large that $\operatorname{dist}(S_n, \partial\Delta_n) \geq Ae^{-an}$ for all $n \geq 1$. Then by (4.9) and the three circles theorem,

$$\sup_{\partial\Delta_n} |f(z) \leq C_2 e^{-an} \tag{4.11}$$

Now let $\Omega = \mathbf{C} \setminus \bigcup_{n=1}^{\infty} (\bar{\Delta}_n \cup \bar{\Delta}_{-n})$ and for $\delta > 0$ fixed and x large, set

$$E_x = \cup \{\Delta_n : |n\pi - x| < \delta x\}.$$

LEMMA 4.4. *There is $C(h^*, a)$ such that for x large,*

$$\omega(x + ih^*, E_x, \Omega) \geq C(h^*, a)$$

Note that by the subharmonicity of $\log|f|$, this lemma and (4.11) imply (4.10) and hence the theorem.

Proof of Lemma 4.9. Fix δ_1, $0<\delta_1<\delta$, to be determined later, and let $N_x \sim 2\delta_1 x/\pi$ be the number of n such that $|n\pi - x| < \delta_1 x$. Set

$$u(z) = \frac{1}{N_x} \sum_{|n\pi - x| < \delta_1 x} \log \frac{1}{|z - n\pi|}.$$

Then $u(z)$ is harmonic and bounded above in Ω, and

$$\sup_{z \in \partial\Omega \setminus E_x} u(z) \leq \log \frac{1}{(\delta - \delta_1)x} + c = \alpha.$$

But if $z \in E_x$ then

$$\begin{aligned} u(z) &\leq \frac{1}{N_x} \log(Ae^{-an}) + \frac{1}{N_x} \sum_{1=2}^{N_x/2} \log \frac{1}{v\pi} \\ &\leq \log \frac{1}{\delta_1 x} + c'(a) = \beta, \end{aligned}$$

and by a similar calculation,

$$u(x + ih^*) \geq \beta + c(h^*, a).$$

We choose δ_1 so that $\beta - \alpha = c'' > 1$. Then by the maximum principle,

$$\omega(z, E_x, \Omega) \geq \frac{u(z) - \alpha}{\beta - \alpha}$$

and

$$\omega(x + ih^*, E_x, \Omega) \geq \frac{c(h^*, a)}{c''} \quad \square.$$

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