Normal subgroups of classical groups over Banach algebras.

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Introduction

All subgroups H of the general linear group GL_nA, $n \geq 3$, over a Banach algebra A with 1, which are normalized by elementary matrices were described in [2] (when $n = 2$, [2] shows that the situation becomes more complicated and gives partial results). In this paper, we obtain similar result for unitary groups. In the next two paragraphs we give Wall's [4] definition of unitary groups which include symplectic and orthogonal groups.

Let A be an associative ring with 1, $*$: $A \to A$ an anti-automorphism of A (i.e. $*$ is a bijection on A such that $(x-y)^* = x^* - y^*$ and $(xy)^* = y^*x^*$ for all x and y in A). We assume that $x^{**} = \varepsilon x \varepsilon^*$ for some unit $\varepsilon = \varepsilon^{*-1}$ of A and every x in A.

Set $F_n = e_{1,2} + \varepsilon e_{2,1} + \cdots + e_{2n-1,2n} + \varepsilon e_{2n,2n-1} \in GL_{2n}A$, where $e_{i,j}$ are the matrix units. Let $U_{2n}(A, *, \varepsilon)$ or just $U_{2n}A$ for short denote the group of all g in $GL_{2n}A$ such that $g^*F_ng = F_n$ ($*$ is extended to anti-automorphisms of the matrix rings in the usual way: $(g^*)_{i,j} = (g_{j,i})^*$). Since $F_n^* = F_n^{-1}$, $F_n \in U_{2n}A$ and $U_{2n}A$ is invariant under $*$.

Here are classical examples.

EXAMPLE 1. $*$ is identical on A and $\varepsilon = -1$. Then A is commutative and $U_{2n}A$ is the standard symplectic group $Sp_{2n}A$.

EXAMPLE 2. $*$ are as in Example 1, but $\varepsilon = 1$. If 2 is not a zero divisor in A, then $U_{2n}A$ is the orthogonal group $O_{2n}A$ of a quadratic form in $2n$ variables of Witt index n.

EXAMPLE 3. A be the complex numbers,$*$ the complex conjugation, $\varepsilon = 1$ or -1. Then $U_{2n}A$ is isomorphic to the standard unitary group.

EXAMPLE 4. $A = D \times D^{\mathrm{op}}$, where D is an associative ring with 1, D^{op} is the opposite ring, and $(d, d')^* = (d', d)$ for any (d, d') in A. Then, for any ε, $U_{2n}A$ is isomorphic to $GL_{2n}D$.

The unitary group $U_{2n}A$ is a normal subgroup of the group $GU_{2n}A$ of unitary *similitudes*, i.e. of all matrices g in $GL_{2n}A$ such that $g^*F_ngF_n^{-1}$ is a scalar matrix over the center of A (the scalar matrix depends on g, and it is 1_{2n} if and only if g is in $U_{2n}A$). For any ideal $B = B^*$ of A, let $GU_{2n}(A, B)$ denote the group of g in $GU_{2n}A$ which reduce to scalar matrices over the center of A/B modulo B. This is a normal subgroup of $GU_{2n}A$.

Now we will define elementary unitary matrices. First, we define a bijection $'$ of the natural numbers by $(2i)' = 2i - 1$ and $(2i-1)' = 2i$ for any integer $i \geqq 1$. For any a in A and any integers i, j such that $1 \leqq i \neq j \leqq 2n$ we set $E_{i,j}(a) = 1_{2n} + ae_{i,j} - \varepsilon^c a^* \varepsilon^{c'} e_{j',i'}$, where $c = ((-1)^i - 1)/2$ and $c' = (1 - (-1)^j)/2$. It is easy to check that all these $E_{i,j}(a)$ belong to $U_{2n}A$. In particular, $E_{2k-1,2k}(a) = 1_{2n} + (a - \varepsilon^* a^*)e_{2k-1,2k}$ and $E_{2k,2k-1}(a) = 1_{2n} + (a - a^*\varepsilon)e_{2k,2k-1}$.

For any ideal $B = B^*$ of A, let $EU_{2n}B$ denote the subgroup of $GU_{2n}B$ generated by all $E_{i,j}(b)$ with b in B, and let $EU_{2n}(A, B)$ denote the normal subgroup of $EU_{2n}A$ generated by $EU_{2n}B$.

THEOREM. *Suppose that A is a Banach algebra with* 1 *and $n \geq 2$. Then,*

(1) $EU_{2n}(A, B) = [EU_{2n}A, EU_{2n}B] = [EU_{2n}A, GU_{2n}(A, B)] = [EU_{2n}(A, B), GU_{2n}A]$ *for any ideal $B = B^*$ of A.*

So for every subgroup H of $GU_{2n}(A, B)$ containing $EU_{2n}(A, B)$ we have $[H, EU_{2n}A] = EU_{2n}(A,B)$, hence H is normalized by $EU_{2n}A$. Conversely, if $n \geq 3$, then

(2) *for any subgroup H of $GU_{2n}A$ which is normalized by $EU_{2n}A$ there is an ideal $B = B^*$ such that $EU_{2n}(A, B) \subset H \subset GU_{2n}(A, B)$.*

Remarks. 1. When $n = 1$, (2) fails in the case of ordinary orthogonal groups, because then the group EU_2A is trivial. The conclusions (1) with $n = 1$ and (2) with $n = 2$ hold under additional conditions on A, $*$, ε (for example, when A is commutative and $\varepsilon = \pm 1$), but the situation in general is unclear.

2. When $*$ is the identity (so A is commutative) and $\varepsilon = 1$ or -1, our theorem is contained in results of [3].

3. When A has no proper ideals $B = B^*$ (for example, A is simple), our theorem says that the group $EU_{2n}A$ modulo its center $GU_{2n}(A, 0) \cap EU_{2n}A$ is simple for $n \geq 3$. Compare this with results of [1] about simplicity of unitary groups over some factors A.

4. The group $EU_{2n}(A, B)$ is contained in the identity component $GU_{2n}(A, B)^0$ of $GU_{2n}(A, B)$. On the other hand, this component is contained in the subgroup $GEU_{2n}(A, B)$ of $GU_{2n}(A, B)$ generated by $EU_{2n}(A, B)$ and diagonal matrices. Therefore, when $n \geq 2$, $EU_{2n}(A, B) = [(GU_{2n}A)^0, GU_{2n}(A, B)^0]$.

Proof of (1)

Evidently, $EU_{2n}(A, B) \supset [EU_{2n}A, EU_{2n}B]$. Let us prove the inverse inclusion, i.e. that every elementary unitary matrix $E_{i,j}(a)$ in $EU_{2n}B$ belongs to the commutator subgroup $[EU_{2n}A, EU_{2n}B]$. When $i \neq j'$, $E_{i,j}(a)$ is the image of an elementary matrix under a monomorphism $s: GL_nA \to U_{2n}A$ such that $s(E_nA) \subset EU_{2n}A$ and $s(E_nB) \subset EU_{2n}B$. (For a monomial matrix f in GL_nA, depending on i and j, with a non-zero entry 1 or ε in each row and column, we have

$$F_n = f^*\begin{pmatrix} 0 & 1_n \\ \varepsilon 1_n & 0 \end{pmatrix} f \quad \text{and} \quad s(g) = f^{-1}\begin{pmatrix} g & 0 \\ 0 & g^{*-1} \end{pmatrix} f \quad \text{for all } g \text{ in } GL_nA.$$

Since $E_nB \subset [E_nA, E_nB]$ (see [2]), $E_{i,j}(a) \in s([E_nA, E_nB]) \subset [EU_{2n}A, EU_{2n}B]$.)

When $i = j'$, we pick an integer k in the interval $1 \leq k \leq 2n$ such that $k \neq i$ and $k + i$ is even. Then $E_{i,j}(a) = E_{k,j}(x)[E_{k,k'}(-a), E_{i,k}(-1)] \in [EU_{2n}A, EU_{2n}B]$, where $x := a - \varepsilon^* a^*$. (By our definition, $[g, h] := ghg^{-1}h^{-1}$, so $[E_{k,k'}(-a), E_{i,k}(-1)] = 1_{2n} - xe_{k,i'} - xe_{i,k'} + xe_{i,i'}$.)

The first equality in (1) is proved.

To prove the second one, pick an arbitrary g in $GU_{2n}(A, B)$. For any elementary $E_{i,j}(a)$ in $EU_{2n}A$ any rational number r, we set $h(r) = [g, E_{i,j}(ra)]$. We want to prove that $h(1) \in EU_{2n}(A, B)$. When r is close to 0, $h(r)$ is close to $h(0) = 1_{2n}$, hence it is the product of a diagonal matrix from $U_{2n}B$ and a matrix from $EU_{2n}B$. Let $GEU_{2n}B$ be the group of all such products. Evidently, it is normalized by $EU_{2n}A$. So $h(1) \in GEU_{2n}B$. Hence $\varphi(h') = [g, h'] \in GEU_{2n}B$ for any h' in $EU_{2n}A$. Note that $h' \to \varphi(h')EU_{2n}(A, B)$ is a group homomorphism from $EU_{2n}A$ to $GEU_{2n}B/EU_{2n}(A, B)$. Since the first group here is perfect (see above) and the second group is commutative (because it is a factor group of the group $GE_nB/E_n(A, B)$ which is commutative by the Whitehead lemma that allows us to permute diagonal matrices modulo elementary matrices provided that $n \geq 2$), we conclude that the homomorphism is trivial. That is, $\varphi(h') \in EU_{2n}(A, B)$ for all h' in $EU_{2n}A$. In particular, when $h' = E_{i,j}(a)$, we obtain that $h(1)$ is in $EU_{2n}(A, B)$. The second equality in (1) is proved.

Using this with $B = A$, we conclude that $EU_{2n}A = EU_{2n}(A, A)$ is a normal subgroup of $GU_{2n}A = GU_{2n}(A, A)$. Since $GU_{2n}(A, B)$ is also a normal subgroup of $GU_{2n}A$, we conclude that $EU_{2n}(A, B) = [EU_{2n}A, GU_{2n}(A, B)]$ is normal in $GU_{2n}A$ too. That is, we obtain the third equality in (1).

Proof of (2)

Let H be a subgroup of $GU_{2n}A$, and let H be normalized by $EU_{2n}A$. For any integers i and j such that $1 \leq i \neq j \leq 2n$, we set $X_{i,j} = \{a \in A : E_{i,j}(a) \in H\}$. Clearly, they are additive subgroups of A. The identity of the form $[E_{i,j}(a), E_{j,k}(b)] = E_{i,k}(ab)$, where $i \neq j \neq i'$, $k \neq j \neq k'$, and $k \neq j \neq k'$, show that $X_{i,j} = X_{1,3}$ whenever $i \neq j'$ and that $X_{1,3} =: B$ is an ideal of A. Since $X_{1,3} = (X_{4,2})^*$, $B = B^*$. Now it is easy to check that $X_{2i-1,2i} = (X_{2i,2i-1})^* = \{b \in B : b^* = -\varepsilon b\}$ for $i = 1, \ldots, n$.

Let us show now that the image of H modulo B belongs to the center of $GU_{2n}(A/B)$. Otherwise, an element g of H does not commute with an elementary unitary matrix (note that the centralizer of $EU_{2n}(A/B)$ in $GU_{2n}(A/B)$ consists of scalar matrices over the center of A/B). Then g does not commute with an elementary unitary matrix $E_{i,j}(a)$ modulo $GU_{2n}(A, B)$ (otherwise we would obtain a non-trivial homomorphism from the perfect group $EU_{2n}A$ to the commutative group $GU_{2n}(A, B)/GU_{2n}B$). So $h = [g, E_{i,j}(a/N)] \in H$ and h is outside of $GU_{2n}(A, B)$ for any natural number N. Taking a large N, we obtain a matrix h in H outside $GU_{2n}(A, B)$ which is arbitrarily close to the identity matrix 1_{2n}.

Now, after a permutation of the basis, we will think about $U_{2n}A$ as $U_2(M_nA)$, where M_nA is the ring of n by n matrices over A. Since h is close to the identity, we can write $h = E_{2,1}(c)\,\mathrm{diag}(d, d^{*-1})E_{1,2}(b)$ where $d \in GL_nA$, $B = -\varepsilon^* b^* \in M_nA$, $c = -c^*\varepsilon \in M_nA$. If $c \notin M_nB$, we can replace h by $b'^{1,2}h(-b')^{1,2}$ with a matrix $b' = -\varepsilon^* b'^*$ in M_nA (which results in replacing d by $d + b'c$) to get a non-diagonal entry of d outside B. Moreover the matrix b' above can be taken to be small, so the d-entry stays invertible. Similarly, if $b \notin M_nB$, we can replace h by $(-c')^{2,1}hc'^{2,1}$ with $c' = -c'^*\varepsilon$ in M_nA to reach the same objective.

Thus, we can assume that H contains an element of the form $h = E_{2,1}(c)\,\mathrm{diag}(d, d^{*-1})E_{1,2}(b)$ with $d \in GL_nA \backslash G_n(A, B)$ (i.e. the image of d in $GL_n(A/B)$ is not a scalar matrix over the center of A/B).

To complete our proof we need to show that H contains an elementary unitary matrix outside $GU_{2n}(A, B)$.

Consider the set T of all triples $(c', d', b') \in M_nA \times GL_nA \times M_nA$ such that $t(c', d', b') := E_{2,1}(c')\,\mathrm{diag}(d', d'^{*-1})E_{1,2}(b') \in H$. Note that if (c', d', b'), $(c'', d'', b'') \in T$, then $(c' - c'', d'd''^{-1}, d''(b' - b'')d''^*)$, $(d''^{-1}(c' - c'')d''^{*-1}, d''^{-1}d', d' - d'') \in T$. Indeed, $E_{2,1}(c'')^{-1}(t(c', d', b')t(c'', d'', b'')^{-1}E_{2,1}(c'') = t((c' - c'', d'd''^{-1}, d''(b' - b'')d''^*) \in H$ and $E_{2,1}(c'')(t(c'', d'', b'')^{-1}t(c', d', b'))E_{2,1}(c'')^{-1} = t(d''^{-1}(c' - c'')d''^{*-1}, d''^{-1}d', d' - d'') \in H$. Therefore, the projection of $T \subset M_nA \times GL_nA \times M_nA$ on each of 3 factors is a subgroup there.

Since the set $t(T)$ is normalized by all elements of the form $\mathrm{diag}(u, u^{-1})$, where $u \in E_nA$, the group E_nA acts on T. We use this action to define an operation $[\,,\,]' : E_nA \times T \to T$ as follows: $[u, (c', d', b')]' = (u^{*-1}c'u^{-1} - c', [u, d'], d'(ub'u^* - b')d'^*)$, where $u \in E_nA$ and $(c', d', b') \in T$.

Note that the second projection H' of T is a subgroup of GL_nA and H' is normalized by E_nA. By [2], H' contains an elementary matrix $z^{1,2} = 1_n + ze_{1,2}$ with z outside of B. So we can assume that $d = z^{1,2}$ with z in $A \backslash B$. We will need no conditions on A anymore.

We have $[1^{3,1}, (c, d, b)]' = (c', z^{3,2}, \cdot) = x_1 \in T$ with the matrix $c' = c - (-1)^{1,3}c(-1^{3,1})$ in M_nA having non-zero entries only in the first column and row.

Next we consider the triple $x_2 = [(-1)^{2,1}x_1]' = (c'', z^{3,1}, \cdot)$ in T. Here $c'' = 1^{1,2}c'(1^{2,1}) - c'$ can have non-zero entries only at position $(1, 1)$.

Set $x_3 = [1^{2,3}, x_2]' = (0, z^{2,1}, \cdot) \in H$. Then

$$
\begin{aligned}
[t(x_3), (e_{1,3} - \varepsilon^* e_{3,1})^{1,2}] &= [\operatorname{diag}(z^{2,1}, (-z^*)^{1,2}), (e_{1,3} - \varepsilon^* e_{3,1})^{1,2}] \\
&= (z^{2,1}(e_{1,3} - \varepsilon^* e_{3,1})z^{*1,2} - (e_{1,3} - \varepsilon^* e_{3,1})^{1,2} \\
&= (ze_{2,3} - \varepsilon^* z^* e_{3,2})^{1,2} \in H.
\end{aligned}
$$

Thus, H contains the elementary unitary matrix $t(0, 1_n, ze_{2,3} - \varepsilon^* z^* e_{3,2}) = (ze_{2,3} - e^* z^* e_{3,2})^{1,2}$ (which is $E_{3,4}(z)$ in the original notation) outside $GU_{2n}(A, B)$. Our proof is completed.

Here are some entries of the first 3 columns of the matrices h, $t(x_1)$, $t(x_2)$, $t(x_3)$ (namely, the positions (i, j) of the c- and d-parts with $1 \le i, j \le 3$).

1	z	0
0	1	0
0	0	1
·	·	·
·	·	·
·	·	·

1	0	0
0	1	0
0	z	1
·	·	·
·	0	0
·	0	0

1	0	0
0	1	0
z	0	1
·	0	0
0	0	0
0	0	0

1	0	0
z	1	0
0	0	1
0	0	0
0	0	0
0	0	0

REFERENCES

[1] PIERE DE LA HARPE, *Simplicity of the projective unitary groups defined by simple factors*, Comm. Math. Helv. **54** (1979), 334–345.

[2] L. N. VASERSTEIN, *Normal subgroups of the general linear groups over Banach algebras*, J. Pure Appl. Algebra **41** (1986), 99–112.

[3] ——, *On normal subgroups of Chevalley groups over commutative rings*, Tôhoku Math. J. **38** (1986), 219–230.

[4] C. T. C. WALL, *On the axiomatic foundations of the theory of Hermitian forms*, Proc. Cambridge Phil. Soc. **67** (1970), 243–250.

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