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The geometry and spectrum of the one holed torus

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1. Introduction

This paper deals with the question whether isospectral Riemann surfaces can occur accidentally. Two Riemann surfaces are called isospectral if their length spectra, i.e. the lists of lengths of all closed geodesics, are the same. It had been conjectured by Gel'fand [5] and was later proved in the generic case by Wolpert [9] that a compact Riemann surface with the Poincaré metric of constant curvature -1 is determined up to isometry by its length spectrum. But many isospectral non isometric pairs have been found, even with quite small genus [8], [7], [3], [2]. All these examples are obtained by gluing together building blocks in two combinatorially different ways. However, isospectral pairs might also occur, in some sense, accidentally.

Here we make a first attempt to prove that accidental pairs do not exist, by investigating how the length spectrum behaves as a function of the parameters in Teichmüller space. This function is extremely complex, and we shall confine ourselves to the simplest non trivial case, the Riemann surfaces of signature $(1, 1)$, or one holed tori.

The boundary is supposed to be a simple closed geodesic, the Teichmüller space has dimension 3. The length spectrum refers to the *smooth* closed geodesics, including the boundary geodesic. Geodesics which bounce back at the boundary are not considered. The result is

THEOREM A. *Two Riemann surfaces of signature* $(1, 1)$ *which have the same length spectrum are isometric.*

Under the additional hypothesis that the two surfaces have the same boundary length, Theorem A has already been proved by Haas [6]. The other length spectrum, which includes the geodesics which are reflected at the boundary, will be considered in a forthcoming paper where the technique is extended to closed Riemann surfaces of genus 2. We conjecture that Theorem A holds also for the spectrum of the Laplacian with respect to Dirichlet or Neumann boundary conditions.

As a byproduct we obtain in section 4 an explicit solution of the moduli problem for signature (1, 1):

THEOREM B. *The set of isometry classes of one holed tori is in a natural 1–1-correspondence with the set*

$$F = \{(c, b, m) \in \mathbb{R}^3 \mid 1 < c \leq b \leq m \leq bc;\ c^2 + b^2 + m^2 < 2cbm\}$$

In sections 2 and 3 we shall use a $\mathbb{Z} \times \mathbb{Z}$-holed covering to provide various length estimates. These will be brought together in section 5 to show how isospectrality is ruled out. In the final section we shall use the $\mathbb{Z} \times \mathbb{Z}$-holed covering again to characterize the simple closed geodesics in terms of the canonical generators of the fundamental group. This characterization has been a guide in our study of the length spectrum and, although no longer used here, is of some interest by itself.

2. The one holed torus and its $\mathbb{Z} \times \mathbb{Z}$-holed covering

A one holed torus is a compact bordered surface of signature (1, 1) carrying a hyperbolic metric of curvature -1 in which the boundary is a closed geodesic. To simplify the language, T will always stand for a one holed torus. We start by constructing T via the standard gluing procedure [4]. At the same time we shall obtain the Fenchel–Nielsen parameters and the $\mathbb{Z} \times \mathbb{Z}$-holed covering plane. All segments will be geodesic, and to simplify notation, we shall use the same symbol for a segment and for its length.

Let β_0, η, γ be suitable positive numbers and glue together four identical rectangular hyperbolic pentagons with sides $\beta_0/2$, *, $\eta/4$, *, $\gamma/2$ as shown in Fig. 1 to obtain a one holed hyperbolic rectangle Q.

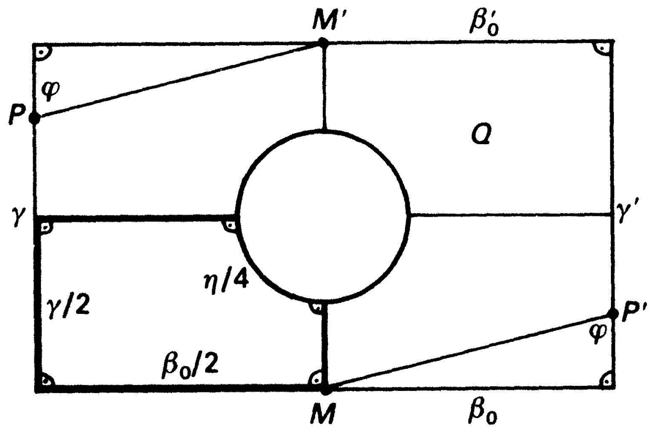

Fig. 1

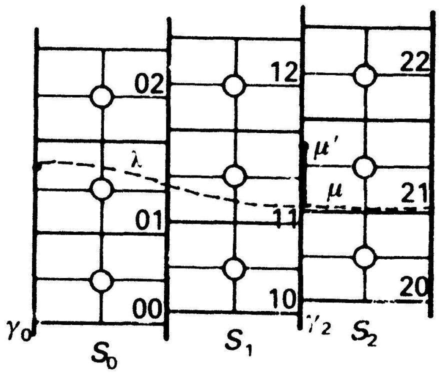

Fig. 2

As is well known (cf. [1]) such pentagons exist for any positive values of γ, η and the following trigonometric formula holds

$$\operatorname{sh}\frac{\beta_0}{2}\cdot\operatorname{sh}\frac{\gamma}{2}=\operatorname{ch}\frac{\eta}{4}. \tag{2.1}$$

It follows from standard hyperbolic geometry that the redecomposition of Q into rectangular pentagons is unique. Hence Q is determined up to isometry by γ and η, which range freely in the interval $(0, \infty)$. We shall denote by β_0', γ, β_0, γ' the sides of Q (Fig. 1). The "hole" is η.

Now think of Q as being realized by an ordinary one holed rectangle in $\mathbb{R}^2$ which is parallel to the standard coordinate axes. Translate Q in the vertical direction to tesselate a $\mathbb{Z}$-holed strip S_0 with copies Q_{0j} $(j \in \mathbb{Z})$ of Q. Then translate S_0 to tesselate a $\mathbb{Z}\times\mathbb{Z}$-holed plane $\mathscr{X}$ with copies S_i of S_0 $(i \in \mathbb{Z})$ resp with copies Q_{ij} of Q as indicated in Fig. 2, where each S_i is shifted against S_{i-1} by $\alpha \cdot \gamma/2$, $\alpha \in \mathbb{R}$ being a free parameter, called *twist parameter.* $\mathscr{X}$ carries the hyperbolic structure of the Q_{ij} and

$$\Gamma = \mathbb{Z}\times\mathbb{Z}$$

has a natural action on $\mathscr{X}$ by isometries such that the Q_{ij} are fundamental domains.

We define $T(\eta, \gamma, \alpha)$ to be the *marked* surface $\mathscr{X}/\Gamma$ where the marking consists of recalling the parameters η, γ, α which have lead to the construction of $T(\eta, \gamma, \alpha)$. Sides γ and γ' of Q yield a closed geodesic on $T(\eta, \gamma, \alpha)$ which we also denote γ. The set

$$\mathscr{T} = \{T(\eta, \gamma, \alpha) \mid \eta, \gamma \in \mathbb{R}^+, \alpha \in \mathbb{R}\}$$

is the Fenchel–Nielsen-model of the Techmüller space of Riemann surfaces of signature $(1, 1)$ [4].

In order to express the twist parameter α in terms of the length of a closed geodesic, consider $P \in \gamma \subseteq Q$ and $P' \in \gamma' \subseteq Q$ with the *oriented* distance

$$\operatorname{dist}(P, \beta_0') = \operatorname{dist}(P', \beta_0) = \frac{\alpha}{2}\gamma$$

(later on we shall restrict ourselves to $0 \leqslant \alpha \leqslant \frac{1}{2}$).

Connecting P and P' with the midpoints M' and M of β_0' resp β_0 we obtain two isometric rectangular triangles with base $\beta/2$ and angle φ at P resp. P' for which the following formulae hold

$$\operatorname{ch}\frac{\beta}{2} = \operatorname{ch}\frac{\beta_0}{2} \cdot \operatorname{ch}\frac{\alpha\gamma}{2} \tag{2.2}$$

$$\operatorname{sh}\frac{\beta_0}{2} = \operatorname{sh}\frac{\beta}{2} \cdot \sin\varphi. \tag{2.3}$$

Since each S_i is shifted against S_{i-1} by $\alpha\gamma/2$, point P' of $Q_{i-1,j+1}$ matches with point P of Q_{ij}, and it follows that segments PM' and MP' together define a smooth closed geodesic β on $T(\eta, \gamma, \alpha)$ whose length is determined by (2.2).

The two geodesics γ and β will be used as canonical generators of the fundamental group. Now define

$$\mathscr{F} = \{T(\eta, \gamma, \alpha) \in \mathscr{T} \mid 0 \leqslant \alpha \leqslant \tfrac{1}{2}: \gamma \leqslant \beta\}, \tag{2.4}$$

where β is calculated in terms of (η, γ, α) via (2.1-2).

It is easy to see that $\mathscr{F}$ contains each one holed torus T. In fact, it suffices to cut open T along the *shortest* non boundary geodesic and then drop common perpendiculars between the boundary geodesics to find the above one holed rectangle Q. That, in fact, two different tori in $\mathscr{F}$ are not isometric will be proven in section 4.

Some of our arguments will use $\mathrm{PSL}(2, \mathbb{R})$. Consider the transformations of the upper half plane $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ which are given by the matrices

$$B^m = \begin{pmatrix} \operatorname{ch}\frac{m\beta}{2} + \cos\varphi \cdot \operatorname{sh}\frac{m\beta}{2} & -\sin^2\varphi \cdot \operatorname{sh}\frac{m\beta}{2} \\ -\operatorname{sh}\frac{m\beta}{2} & \operatorname{ch}\frac{m\beta}{2} - \cos\varphi \cdot \operatorname{sh}\frac{m\beta}{2} \end{pmatrix} \tag{2.5}$$

$$C^n = \begin{pmatrix} e^{n\gamma/2} & 0 \\ 0 & e^{-n\gamma/2} \end{pmatrix}; \qquad m, n \in \mathbb{Z}$$

where φ is the angle between γ and β at P (Fig. 1). B^m is the m-th power of $B=B^1$ and C^n is the n-th power of $C=C^1$. The traces are $\operatorname{tr} B = 2\operatorname{ch}\frac{\beta}{2}$, $\operatorname{tr} C = 2\operatorname{ch}\frac{\gamma}{2}$. The fixed points of C are $0,\infty$ and those of B are $\pm 1\text{-}\cos\varphi$. It follows that the axes of B and C intersect at an angle φ, and there exists a universal covering $\tilde{H} \to T(\eta, \gamma, \alpha)$ for which C and B are the generators of the deck group which correspond to the generators γ and β of the fundamental group of $T = T(\eta, \gamma, \alpha)$; $\tilde{H}$ a simply connected domain in H.

Any closed geodesic ω on T is known to be freely homotopic to a word $\beta^{m_1}\gamma^{n_1}\cdots\beta^{m_k}\gamma^{n_k}$ which is determined up to conjugation in the fundamental group. Moreover, since geodesics which differ by parametrization are not distinguished, *two such words represent the same geodesic if and only if one is conjugate to the other or to the inverse of the other.* The length of ω is given by

$$\operatorname{ch}\frac{1}{2}\omega = \frac{1}{2}\left|\operatorname{tr}\left(B^{m_1}C^{n_1}\cdots B^{m_k}C^{n_k}\right)\right|. \tag{2.6}$$

We shall use the notation

$$\|\beta^{m_1}\gamma^{n_1}\cdots\beta^{m_k}\gamma^{n_k}\| = \|\omega\| = \operatorname{ch}\frac{1}{2}\omega$$

and abbreviate

$$b=\|\beta\|,\ c=\|\gamma\|,\ e=\|\eta\|,\ m=\|\beta\gamma^{-1}\|$$

$$\Delta = \left(\operatorname{sh}^2\frac{\beta}{2}\cdot\operatorname{sh}^2\frac{\gamma}{2} - \operatorname{ch}^2\frac{\eta}{4}\right)^{1/2} = \operatorname{sh}\frac{\beta}{2}\cdot\operatorname{sh}\frac{\gamma}{2}\cdot\cos\varphi$$

(cf. (2.1), (2.3)). We shall compute some lengths via matrices:

2.7 LEMMA. *Let $T = T(\eta, \gamma, \alpha) \in \mathscr{F}$. Then*

(i) $m = \|\beta\gamma^{-1}\| = bc - \Delta \geqslant b.$

(ii) $\|\beta\gamma\| = bc + \Delta \leqslant \|\beta^2\| \leqslant \|\beta\gamma^{-1}\beta\gamma^{-1}\|.$

(iii) $\|\beta\gamma\| \leqslant \|\beta^2\gamma^{-1}\|.$

(iv) $\|\beta\gamma^n\| = \operatorname{ch}\left(\frac{n+\alpha}{2}\gamma\right)\cdot\operatorname{ch}\frac{1}{2}\beta_0;\qquad \|\beta\gamma^n\| \geqslant \|\beta\gamma\|\ \text{ if }\ n \neq 0, -1.$

(v) $\|\beta\gamma\beta^{-1}\gamma\| = e + 2c^2.$

(vi) $\|\beta\gamma^2\beta^{-1}\gamma^{-1}\| = \|\beta\gamma^{-1}\beta^{-1}\gamma^2\| = c + 2ec$

and the two words in (vi) *represent different closed geodesics.*

Proof. From (2.2), (2.3) we obtain $\mathrm{sh}\frac{\beta}{2}\cdot\cos\varphi=\mathrm{ch}\frac{\beta}{2}\cdot\mathrm{th}\frac{\alpha\gamma}{2}$. Since $0\leqslant\alpha\leqslant\frac{1}{2}$, this implies

$$\Delta\leqslant b\cdot\mathrm{th}\frac{\gamma}{4}\cdot\mathrm{sh}\frac{\gamma}{2}=b(c-1).$$

This yields the inequality in (i). The remaining statements follow from direct computation with (2.5), (2.6).

The next lemma will play a crucial role in the study of the length spectrum.

2.8 LEMMA

(i) $ch^2\frac{\eta}{4}=\frac{1}{2}(e+1)=2cbm+1-c^2-b^2-m^2.$

(ii) *If* $T(\eta,\gamma,\alpha)\in\mathscr{F}$ *then* $1<c\leqslant b\leqslant m\leqslant bc$.

(iii) *In the domain* $1<c\leqslant b\leqslant m\leqslant bc$ *the function* $(c,b,m)\to 2\,cbm+1-c^2-b^2-m^2$ *is monotone increasing in all* 3 *variables.*

Proof. (i) Write $\mathrm{ch}^2\frac{\eta}{4}=\mathrm{sh}^2\frac{\beta}{2}\cdot\mathrm{sh}^2\frac{\gamma}{2}-\Delta^2$, where $\Delta^2=(m-bc)^2$, (2.7(i)), and check. (ii) $c\leqslant b$ is from the definition of $\mathscr{F}$; the remaining inequalities are from 2.7(i). (iii) The partial derivatives are positive.

3. The bottom of the spectrum

We *define* $\mathrm{Sp}\,(T)$ to be the increasing sequence of all $\|\omega\|$, where ω runs through the *primitive* closed geodesics of T. We split $\mathrm{Sp}\,(T)$ into the disjoint subsequences

$$\mathrm{Sp}^A\,(T)=\{\|\omega\|\in\mathrm{Sp}\,(T)\mid\omega\cap\gamma=\varnothing\}$$

$$\mathrm{Sp}^B\,(T)=\{\|\omega\|\in\mathrm{Sp}\,(T)\mid\omega\cap\gamma\neq\varnothing\}.$$

3.1 LEMMA A. *The values* e, $e+2c^2$, $c+2ec$ *occur in* $\mathrm{Sp}^A\,(T)$, *and* $c+2ec$ *at least twice. If* $\|\omega\|$ *occurs in* $\mathrm{SP}^A\,(T)$ *and* $\omega\neq\eta$ *then* $\|\omega\|\geqslant\min\{e+2c^2,c+2ec\}$. $(T\in\mathscr{F})$.

Proof. Clearly $e=\|\eta\|\in\mathrm{Sp}^A\,(T)$. From 2.7 $e+2c^2=\|\beta\gamma\beta^{-1}\gamma\|$ and $c+2ec=\|\beta\gamma^2\beta^{-1}\gamma^{-1}\|=\|\beta\gamma^{-1}\beta^{-1}\gamma^2\|$. Looking at the $\mathbb{Z}\times\mathbb{Z}$-holed covering $\mathscr{X}$ of T

we see that each of the three words has a lift in $\mathscr{X}$ which stays in one of the strips S_i, say in S_0. By standard hyperbolic geometry the lift of the corresponding geodesic stays in S_0 as well, and we obtain the first statement.

As for the second, let $\|\omega\| \in \mathrm{Sp}^A(T)$, $\omega \neq \eta$. As before, ω has a lift $\hat{\omega}$ in $\mathscr{X}$ which stays in S_0. We think of S_0 as being realized by a vertical strip in $\mathbb{R}^2$. For each Q_{0j} in S_0 we denote by σ_j, σ_j' the common perpendicular from the hole in Q_{0j} to the left resp. the right vertical boundary geodesic of S_0, and by β_j the side of Q_{0j} which is a lift of β_0. There are two cases:

a) $\hat{\omega}$ has local maxima (like e.g. point M in Fig. 3a). Then $\hat{\omega}$ also has local minima and therefore $\hat{\omega}$ has two subarcs whose projections in T intersect in only finitely many points, where each arc λ looks as follows (Fig. 3a): λ is contained in some Q_{0j}. It starts on β_j or β_{j+1}, crosses one of the perpendiculars σ_j, σ_j' and ends on the other.

Let the endpoints of λ move freely on β_j resp. β_{j+1} and on σ_j resp. σ_j', and let $\lambda' \subset Q_{0j}$ be the unique minimal curve in this homotopy class. We obtain a rectangular hexagon with sides of length β_0, $\gamma/2$, *, λ', *, γ, for which

$$\operatorname{ch} \lambda' = \operatorname{sh} \gamma \cdot \operatorname{sh} \frac{\gamma}{2} \cdot \operatorname{ch} \beta_0 - \operatorname{ch} \gamma \cdot \operatorname{ch} \frac{\gamma}{2}. \tag{3.2}$$

([1]) For the hexagon with sides β_0, $\gamma/2$, *, $\eta/2$, *, $\gamma/2$ (half of Q_{0j}) we have similarly

$$\operatorname{ch} \frac{\eta}{2} = \operatorname{sh}^2 \frac{\gamma}{2} \cdot \operatorname{ch} \beta_0 - \operatorname{ch}^2 \frac{\gamma}{2}.$$

The two formulae together yield $\operatorname{ch} \lambda \geq \operatorname{ch} \lambda' = c + 2ec$ which proves Lemma A in this first case.

b) $\hat{\omega}$ has no local maxima and minima. Since $\omega \neq \gamma^n$, $\hat{\omega}$ intersects some of the σ_j as well as some of the σ_j' and we find two subarcs of $\hat{\omega}$ with projections in T

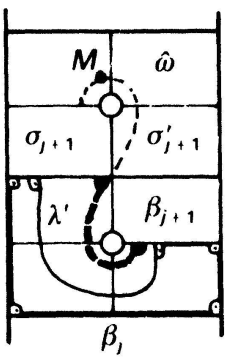

Fig. 3a

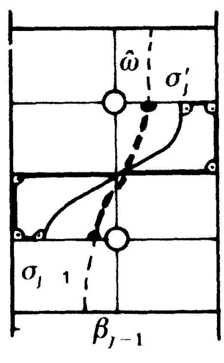

Fig. 3b

intersecting in only finitely many points, where each arc λ looks as follows (Fig. 3b): For some j, λ starts on σ_{j-1} (or σ'_{j-1}) crosses β_j and ends on σ'_j (resp. σ_j). Let again the endpoints of λ move freely on σ_{j-1}, σ'_j (resp. σ'_{j-1}, σ_j) and take the minimal curve λ' in this homotopy class. For symmetry reasons we obtain two isometric trirectangles with sides of length $\lambda'/2$, *, $\gamma/2$, $\beta_0/2$ (Fig. 3b). By trigonometry ([1])

$$\operatorname{sh}\frac{\lambda'}{2}=\operatorname{ch}\frac{\beta_0}{2}\cdot\operatorname{sh}\frac{\gamma}{2}. \tag{3.3}$$

Together with (2.1) we obtain

$$\begin{aligned}\operatorname{ch}\lambda' &= 1+2\operatorname{sh}^2\frac{\lambda'}{2}=1+2\operatorname{sh}^2\frac{\gamma}{2}\cdot\left(1+\operatorname{sh}^2\frac{\beta_0}{2}\right)\\ &= 2c^2+2\operatorname{ch}^2\eta/4-1=2c^2+e\end{aligned}$$

Hence $\|\omega\|\geqslant 2c^2+e$ which proves Lemma A also in the second case.

3.4 LEMMA B. *Let $T\in\mathscr{F}$. If $\|\omega\|$ occurs in* $\mathrm{Sp}^B(T)$ *and if ω is not representable by γ, β, $\beta\gamma^{-1}$, then $\|\omega\|\geqslant\|\beta\gamma\|=bc+\Delta$.*

Proof. Let $\hat{\omega}$ be a lift of ω in $\mathscr{X}$. We consider two cases.

a) $\hat{\omega}$ has locally left most points (like e.g. point L in Fig. 4a). Then $\hat{\omega}$ also has locally right most points and, since ω intersects γ we find two subarcs of $\hat{\omega}$ whose projections in T intersect in only finitely many points, and where each arc λ looks as follows (Fig. 4a): λ is contained in one of the strips S_i. It starts on one of the perpendiculars v which connect neighbouring holes in S_i, then crosses another such perpendicular and ends on one of the two vertical boundary geodesics of S_i, say on γ_i.

Let the endpoints of λ move freely on v and γ_i, and take λ' minimal in this homotopy class. We obtain a rectangular hexagon with sides of length β_0, *, λ', *, $\frac{\beta_0}{2}$, $k\gamma$, where $k\in\mathbb{N}\setminus\{0\}$. By the same formula as in (3.2) and (2.1) we have

$$\begin{aligned}\operatorname{ch}\lambda' &= \operatorname{ch}k\gamma\cdot\operatorname{sh}\beta_0\cdot\operatorname{sh}\frac{\beta_0}{2}-\operatorname{ch}\beta_0\cdot\operatorname{ch}\frac{\beta_0}{2}\\ &= \operatorname{ch}\frac{\beta_0}{2}\cdot\left(2\operatorname{sh}\frac{\beta_0}{2}\cdot\operatorname{ch}k\gamma-2\operatorname{sh}^2\frac{\beta_0}{2}-1\right)\\ &\geqslant \operatorname{ch}\frac{\beta_0}{2}\cdot\left(4\operatorname{ch}^2\frac{\eta}{4}-1\right).\end{aligned}$$

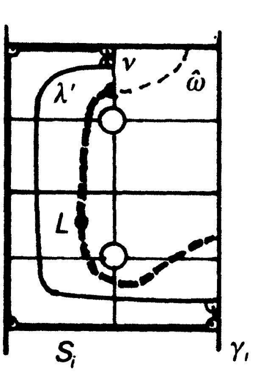

Fig. 4a

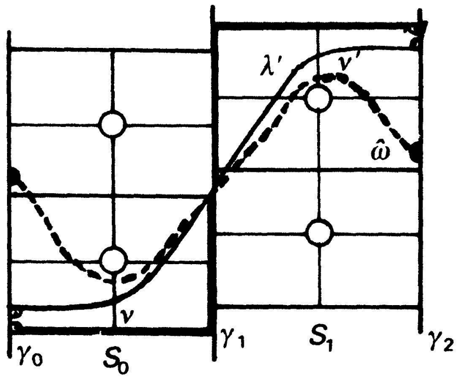

Fig. 4b

By Lemma 2.8

$$\mathrm{ch}^2 \frac{\eta}{4} \geqslant \max\{1, 2c^3 - 3c^2 + 1\} \tag{3.5}$$

and by (2.7(ii)) and (2.2) $\|\beta\gamma\| \leqslant b(2c-1) \leqslant \mathrm{ch}\frac{\beta_0}{2} \cdot (2c-1)(c+1)/2$. Altogether we find $\mathrm{ch}\frac{\omega}{2} \geqslant \mathrm{ch}\,\lambda \geqslant \mathrm{ch}\,\lambda' \geqslant \|\beta\gamma\|$ which proves Lemma B in the first case.

b) $\hat{\omega}$ has no locally left and right most points. Then ω is representable by a word of the form $\beta\gamma^{n_1}\beta\gamma^{n_2}\cdots\beta\gamma^{n_k}$. In view of Lemma 2.7 and the hypothesis in Lemma B we may restrict ourselves to the case that $k \geqslant 2$. If $k = 2$ we may suppose in addition that $n_1 \notin \{0, -1\}$ or $n_2 \notin \{0, -1\}$. We consider the particular case $k = 2$ first and assume w.l.o.g. that $n_1 \notin \{0, -1\}$.

Here $\hat{\omega}$ can be chosen to have the following properties (Fig. 4b): $\hat{\omega}$ is contained in $S_0 \cup S_1$ with its endpoints on the vertical boundary geodesics γ_0 and γ_2 of $S_0 \cup S_1$. It connects the perpendicular v between the holes of Q_{00} and Q_{01} in S_0 with the perpendicular v' between the holes of Q_{1,n_1} and Q_{1,n_1+1} in S_1 and intersects no further such perpendicular. Let the endpoints of $\hat{\omega}$ move freely on γ_0 and γ_2, and let λ' be minimal in this homotopy class.

By symmetry, λ' forms two isometric trirectangles with sides of length $\frac{1}{2}|n_1 + \alpha|\,\gamma$, β_0, $*$, $\lambda'/2$. By the same formula as in (3.3) and by (2.7(iv))

$$\begin{aligned}
\mathrm{sh}\frac{\lambda'}{2} &= \mathrm{ch}\left(\frac{(n_1+\alpha)\gamma}{2}\right) \cdot \mathrm{sh}\,\beta_0 \\
&\geqslant 2\,\mathrm{ch}\left(\frac{(1+\alpha)\gamma}{2}\right) \cdot \mathrm{sh}\frac{\beta_0}{2} \cdot \mathrm{ch}\frac{\beta_0}{2} \\
&\geqslant 2\,\mathrm{sh}\frac{\beta_0}{2} \cdot \|\beta\gamma\|.
\end{aligned}$$

By (2.1) and (3.5)

$$\mathrm{sh}^2\frac{\beta_0}{2} \geqslant \max\left\{\frac{1}{(c-1)(c+1)}, \frac{2c^2-c-1}{c+1}\right\} > \frac{1}{4}.$$

Since $\|\omega\| \geqslant \mathrm{ch}\frac{\lambda'}{2}$ we have thus proved that

$$\|\beta\gamma^{n_1}\beta\gamma^{n_2}\| \geqslant \|\beta\gamma\| \quad \text{for all} \quad n_1, n_2 \in \mathbb{Z}.$$

It remains to consider the geodesics represented by $\beta\gamma^{n_1}\cdots\beta\gamma^{n_k}$, $k \geqslant 3$. Here $\hat{\omega}$ contains two subarcs λ, μ whose projections in T intersect in at most finitely many points and which have the following properties: λ is contained in $S_0 \cup S_1$ connecting γ_0 with γ_2, and μ is contained in S_2 connecting the vertical boundary geodesics γ_2, γ_3 (Fig. 2). Clearly $\mu \geqslant \beta_0$. And by (2.2), since $T \in \mathcal{F}$,

$$\mathrm{ch}\frac{\beta_0}{2} \geqslant \left(\mathrm{ch}\frac{\gamma}{2}\right)\Big/\mathrm{ch}\frac{\gamma}{4} > \mathrm{ch}\frac{\gamma}{4}$$

i.e. $\beta_0 \geqslant \gamma/2$. We may therefore replace μ by an arc $\mu' \leqslant \mu$ on γ_2 such that $\lambda \cup \mu'$ projects to a closed curve in T which is representable in the form $\beta\gamma^{u_1}\beta\gamma^{u_2}$. Since

$$\|\omega\| \geqslant \lambda + \mu \geqslant \lambda + \mu' \geqslant \|\beta\gamma^{u_1}\beta\gamma^{u_2}\| \geqslant \|\beta\gamma\|,$$

Lemma B is now fully proved.

4. The space of one holed tori

As remarked in section 2, every one holed torus T occurs in $\mathcal{F}$. We shall now use Lemma B to prove that the elements in $\mathcal{F}$ are pairwise non isometric.

To this end observe that every *simple* closed geodesic in $T \in \mathcal{F}$ except η intersects γ. Hence it follows from Lemma B that the three smallest members of the sequence $\mathrm{Sp}(T)$ which belong to simple closed non boundary geodesics are precisely c, b, m where $c \leqslant b \leqslant m$. Therefore, if T, $T' \in \mathcal{F}$ are isometric then T and T' have the same values of c, b, m. Since the parameters η, γ, α are uniquely determined by c, b, m (via (2.1), (2.2), (2.8(i))), T *is* T'. This proves Theorem B.

In view of (2.1), (2.2) we may rewrite $\mathcal{F}$ in the form

$$\mathcal{F} = \left\{T(\eta, \gamma, \alpha) \mid \eta > 0; 0 \leqslant \alpha \leqslant \frac{1}{2}; 0 < f(\alpha, \gamma) \leqslant \mathrm{ch}\frac{\eta}{4}\right\}$$

where

$$f(\alpha, \gamma) = \operatorname{sh}\frac{\gamma}{2} \cdot \left(\frac{\operatorname{ch}^2 \frac{\gamma}{2}}{\operatorname{ch}^2 \frac{\alpha\gamma}{2}} - 1 \right)^{1/2}.$$

This function is monotone increasing in γ if $0 \leq \alpha \leq \frac{1}{2}$. An even simpler form is obtained if c, b, m are used as parameters: By Lemma 2.8 we have

$$1 < c \leq b \leq m \leq bc,$$

and since $\operatorname{ch}\frac{\eta}{4} > 1$ also

$$c^2 + b^2 + m^2 < 2cbm.$$

Conversely, if three numbers c, b, m satisfy these inequalities then $2b^2(c-1) \leq 2cbm - m^2 \leq b^2c^2$ by the monotonicity of the function $m \to 2cbm - m^2$ in this domain. Hence, (2.8(i)), (2.1) and (2.2) allow to solve for η, γ, α with, in fact, $\eta, \gamma > 0$; $0 \leq \alpha \leq \frac{1}{2}$ and $\gamma \leq \beta$. Thus we have 1–1 correspondences between

$$F = \{(c, b, m) \in \mathbb{R}^3 \mid 1 < c \leq b \leq m \leq bc; c^2 + b^2 + m^2 < 2\,cbm\}$$

and $\mathcal{F}$ and the set of isometry classes of one holed tori. (Theorem B).

5. Proof of Theorem A

For $T \in \mathcal{F}$, the spectrum is an *increasing* sequence $L = \operatorname{Sp}(T)$ of "spectral lines" where four of them are labeled e, c, b, m and the others will be considered unlabeled. L has the following properties.

(i) *All lines smaller than* $\min\{e + 2c^2; c + 2ec; bc\}$ *are labeled.*

(ii) *If* $b \leq e$ *then* $m \leq \min\{e + 2c^2; c + 2ec; bc\}$

(iii) $1 < c \leq b \leq m \leq bc$.

(iv) $e = 4cbm + 1 - 2c^2 - 2b^2 - 2m^2$.

(v) *L splits into two disjoint increasing subsequences* $\operatorname{Sp}^A(T)$, $\operatorname{Sp}^B(T)$, *where* $\operatorname{Sp}^B(T) = \{c, b, m, \ldots\}$ *and where* $\operatorname{Sp}^A(T) = \{e, e + 2c^2, \ldots, c + 2ec, c + 2ec\}$ *if* $c < e$ *resp.* $\operatorname{Sp}^A(T) = \{e, c + 2ec, c + 2ec, \ldots, e + 2c^2, \ldots\}$ *if* $e \leq c$.

(vi) $\operatorname{Sp}^A(T)$ *is a function of c and e alone.*

Proof. (i) and (v) are from Lemmata A, B. (vi) holds because $\mathrm{Sp}^A(T)$ lives on the 3-holed sphere obtained by cutting open T along γ, whose geometry is determined entirely by γ and η. (iii) is (2.8(ii)) and (iv) is (2.8(i)). To prove (ii) suppose on the contrary $bc \geqslant e + 2c^2 \geqslant b + 2c^2$. It follows that $b(c-1) \geqslant 2c^2$. Since $m \geqslant b$ we conclude (cf. 2.8(iii))

$$bc \geqslant e + 2c^2 = 4cbm + 1 - 2m^2 - 2b^2 \geqslant 4b^2(c-1) \geqslant 8bc^2$$

which is impossible for $c > 1$.

In view of (i)–(iii) the labeling in the sequence L has one of the following forms (L is ordered by size)

(1) $L = \{c, b, m, \ldots, e, \ldots\}$ *with* $m < e$.
(2) $L = \{c, b, e, m, \ldots\}$ *with* $b < e$,
(3) $L = \{c, e, \ldots, b, \ldots, m, \ldots\}$ *with* $c < e$,
(4) $L = \{e, c, \ldots, b, \ldots, m, \ldots\}$.

(The dots indicate that there may or may not be unlabeled lines in between). Now let $T' \in \mathcal{F}$ have the same spectrum as T. Then some other four lines in L are labeled e', c', b', m', (i)–(vi) hold for T', too, and the new labeling also has one of the forms (1)–(4). We shall prove that

$$(*) \quad c = c', \qquad b = b', \qquad m = m'.$$

By section 4 this implies $T = T'$ and will prove Theorem A. Our main argument is the following consequence of (iv).

(vii) *If* $c \leqslant c'$, $b \leqslant b'$, $m \leqslant m'$ *where at least one inequality is strict, then* $e < e'$.

(For the proof use increasing functions $t \to c(t)$, $b(t)$, $m(t)$ satisfying $1 < c(t) \leqslant b(t) \leqslant m(t) \leqslant b(t) \cdot c(t)$; $0 \leqslant t \leqslant 1$ with boundary values c, b, m and c', b', m', and apply 2.8(iii)).

We proceed by cases. If the labeling for T has form (1), then (vii) implies that the labeling for T' has the same form, and (*) is obvious. The same argument works if the labeling for T has form (2). If both labelings have the *same* form (3) or (4), then (vi) implies $\mathrm{Sp}^A(T) = \mathrm{Sp}^A(T')$ so that $\mathrm{Sp}^B(T) = \mathrm{Sp}^B(T')$ and (*) follows from (v). It remains the case that one labeling has form (3) and the other has form (4) where we may assume w.l.o.g. $c < e$ so that simultaneously

$$L = \{c, e, \ldots, e + 2c^2, \ldots, c + 2ec, c + 2ec, \ldots\} \qquad \textit{with } c < e$$

$$L = \{e', c', \ldots, \quad * \quad, \ldots, \quad * \quad, \quad * \quad, \ldots, c' + 2e'c', \ldots\}.$$

We shall lead this to a contradiction. First observe that $c' + 2ec' \leq e' + 2c'^2$ so that by (i)

$$b'c' \leq c + 2ec$$

(since line $c + 2ec$ has no '-label). Hence $b' \leq \frac{c}{c'} + 2c < e + 2c^2$, and we obtain from (v) that $L = \{c, e, b, \ldots\} = \{e', c', b', \ldots\}$ with $b = b' < e + 2c^2$. Next observe by (iii) that $m \leq bc = b'c' \frac{c}{e} \leq (c + 2ec) \cdot \frac{c}{e} < e + 2c^2$ so that by (v) $L = \{c, e, b, m, \ldots\} = \{e', c', b', m'\}$. This contradicts (vii), and Theorem A is proved.

6. Simple closed curves

6.1 DEFINITION. *A finite sequence of nonzero integers $N_1, \ldots, N_p$ is said to have small variation if sums of m consecutive elements never differ by more than* ± 1:

$$\left| \sum_{i=1}^{m} N_{\lambda+i} - \sum_{i=1}^{m} N_{\kappa+i} \right| \leq 1$$

for all m, λ, κ (indices $\bmod p$).

Thus the sequences 5, 5, 5, 4 and 5, 5, 5, 4, 5, 5, 5, 4, 5, 5, 4 have small variation, but 5, 5, 5, 3 and 5, 5, 5, 4, 5, 5, 5, 4, 5, 4 have not. Observe also that no change of sign occurs.

6.2 THEOREM. *Let $\gamma, \beta \in \pi_1(T)$ be canonical generators of the fundamental group of a one holed torus T. Then every non trivial simple closed curve ω on T can, after suitably renaming the generators, be represented by one of the following words.*

(i) $\omega^* = \gamma$

(ii) $\omega^* = \gamma\beta\gamma^{-1}\beta^{-1}$

(iii) $\omega^* = \gamma\beta^{N_1}\gamma\beta^{N_2} \cdots \gamma\beta^{N_p}$

where the sequence $N_1, \ldots, N_p$ has small variation.

Conversely, each of these words is homotopic to a multiple of a simple closed curve.

In the above, "renaming" means that we interchange γ, β and/or replace a generator by its inverse.

Proof. We use the $\mathbb{Z}\times\mathbb{Z}$-holed covering $\mathscr{X}$ which we visualize as a small ε-neighbourhood in $\mathbb{R}^2$ of the grid

$$G=\{(x_1,x_2)\in\mathbb{R}^2\mid x_1\in\mathbb{Z} \text{ or } x_2\in\mathbb{Z}\}$$

In this model of $\mathscr{X}$, the vertical lines $\{(x_1,x_2)\mid x_1=m\}$, $m\in\mathbb{Z}$ are lifts of γ and the horizontal lines $\{(x_1,x_2)\mid x_2=n\}$, $n\in\mathbb{Z}$ are lifts of β. the deck transformation group Γ consists of the mappings $(x_1,x_2)\mapsto(x_1+k,x_2+l)$, k, $l\in\mathbb{Z}$.

Now let ω^* represent a simple closed curve ω on T and assume ω^* cannot be put into form (i) or (ii) by renaming the generators. Then

$$\omega^*=\gamma^{M_1}\beta^{N_1}\cdots\gamma^{M_p}\beta^{N_p}$$

with all exponents $\neq 0$. Since the space $\gamma\cup\beta$ is a deformation retract of T and since ω is simple it follows that the curve ω^* $(\subseteq\gamma\cup\beta)$ has only inessential self intersections in the sense that arbitrarily small homotopies in T remove all self intersections of ω^*. Accordingly each lift $\tilde{\omega}^*\subset G$ of ω^* has but inessential self intersections and any two lifts intersect but inessentially. (Otherwise the intersection projects to a self-intersection of ω). An inessential intersection is shown in Fig. 5b, the intersections of the bold and dotted lines in Fig. 5a, 5c are essential.

We proceed by eliminating cases. By "subsequence of ω" we also mean sequences like $\gamma^{M_p}\beta^{N_p}\gamma^{M_1}\beta^{N_1}$ etc.

STEP 1. *All* $|N_i|=1$ *or all* $|M_i|=1$

Proof. If for some i and j, $|N_i|\geqslant 2$, a given lift of ω^* in G contains a horizontal *and* a vertical segment of length $\geqslant 2$. Using a suitable deck transformation we find two lifts of ω^* which, at some point, intersect like in Fig. 5a. This is impossible if ω is simple.

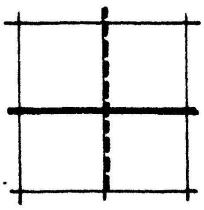

Fig. 5a

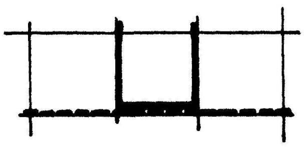

Fig. 5b

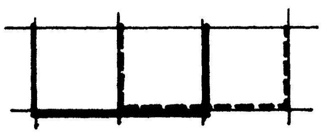

Fig. 5c

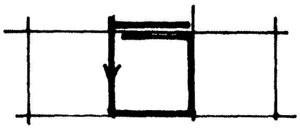

Fig. 6a

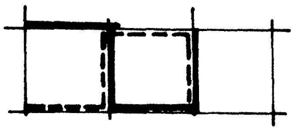

Fig. 6b

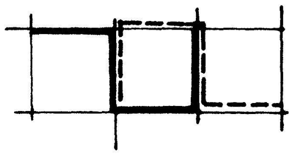

Fig. 6c

We may from now on assume that, say all $|M_i| = 1$

STEP 2. *ω^* contains no sequence $\gamma^\varepsilon\beta^N\gamma^{-\varepsilon}$ with $|N| \geqslant 2$, $\varepsilon = \pm 1$.*

Proof. Using deck transformations again, we would find two lifts of ω^* which intersect like in Fig. 5c.

STEP 3. *ω^* contains no sequence $\gamma^\varepsilon\beta^\delta\gamma^{-\varepsilon}$ with ε, $\delta \in \{-1, 1\}$.*

Proof. Let such a sequence, say $\gamma^{-1}\beta\gamma$ be given. Select a lift of ω^*. It contains a "local minimum" as shown in Fig. 6a, b, c and consequently also a local maximum.

Figure 6a shows a local minimum which corresponds to the sequence $\beta^{-1}\gamma^{-1}\beta\gamma\beta^{-1}$. If we try to continue this curve without introducing essential intersections and without allowing to write $\cdots\gamma\gamma^{-1}\cdots$ or $\cdots\beta\beta^{-1}\cdots$ we would have to circle around the same square infinitely often. Hence the configuration of Fig. 6a does not occur (recall that ω^* is already assumed to be not of type (ii)). Accordingly, the local minimum must be "casserole shaped" like in Figs 6b, 6c, and moreover, because of Step 2, the local maximum is also a casserole, upside down. A deck transformation brings the two casseroles together like in Fig 6b or 6c. In Fig. 6b, the intersection is essential, in Fig. 6c we have the same continuation problem as in Fig. 6a. Hence casseroles do not occur. This proves Step 3.

Because of Step 2 and 3 we may from now on assume that

$$\omega^* = \gamma\beta^{N_1}\gamma\beta^{N_2}\cdots\gamma\beta^{N_p}$$

STEP 4. *The sequence $N_1, \ldots, N_p$ has small variation.*

Proof. Assume on the contrary that $|N_1 + \cdots N_m - (N_{l+1} + \cdots + N_{l+m})| \geqslant 2$ for some l and m (indices mod p). Choose again a lift of ω^*.

The parts of the lift in G which correspond to $\gamma\beta^{N_1}\cdots\gamma\beta^{N_m}$ resp. $\gamma\beta^{N_{l+1}}\cdots\gamma\beta^{N_{l+m}}$ have height m, the directed horizontal differences of their endpoints differ by at least 2. Hence a decktransformation may bring the parts together such that they start and end both at the same altitude but such that one of them starts on the left and ends on the right of the other. This causes an essential intersection. The first part of the theorem is now proved.

As for the second part let two lifts of ω^* be given where ω is primitive and of type (iii). If ω does not lie in the homotopy class of a simple closed curve, these lifts can be found such that they have an essential intersection. This is only possible if two stair like parts occur with the same height but with horizontal lengths differing by at least 2. Theorem 6.2 is now proved.

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