Comment. Math. Helvetici 63 (1988) 305–311

0010-2571/88/020305-07\$01.50 + 0.20/0

Every rational surface is separably split

KEVIN R. COOMBES*

Let F be a field with algebraic closure $\bar{F}$. A surface X is called rational over F if $\bar{X} = X \times_F \bar{F}$ is birational to $\mathbb{P}^2$. An extension E/F is said to split X if the surface X_E is birational with $\mathbb{P}^2_E$ by a sequence of monoidal transformations centered at E-points.

When Bloch wrote the paper [1] which renewed the study of zero cycles on rational surfaces, he made an unpleasant technical assumption. His techniques only worked for separably split rational surfaces; i.e., surfaces admitting a splitting field E/F which is a separable extension of F. Those who followed needed the same assumption [2, 3, 10].

This unfortunate circumstance arose because one did not yet know the validity of

THEOREM 1. *Every rational surface is separably split.*

This note will rectify matters. Consequently, the results of [1, 2, 3, 10] actually hold for all rational surfaces. For instance, the exact sequence of [1, Thm. 0.1] actually exists for every rational surface, and Thms. 0.3 and 0.4 (*op. cit.*) hold for all conic bundle surfaces. Similarly, one can generalize [10, Thm. 6.1] by removing the restriction to prime-to-p torsion.

COROLLARY [10, Thm. 6.1]. *Let X be a rational surface over either a local field or a field of cohomological dimension ≤ 1. Then the group $A_0(X)$, of zero cycles of degree zero modulo rational equivalence, is finite.*

Not surprisingly, the principal tool used is the work of Iskovskih [7] classifying minimal rational surfaces over any field. However, his argument relies on the "Adjunction Lemma" [7, p. 20, Lem. 2], whose proof has only appeared in print in the case when the ground field is perfect [Manin, 8]. In the first section of this paper, the Adjunction Lemma is proved for separably closed fields. Iskovskih's results are then used in the second section to prove Theorem 1.

* Partially supported by the NSF.

I would like to thank the referee for reading this paper so carefully and for numerous suggestions and improvements, not the least of which was the necessity of including the first section.

§1. The adjunction lemma

Throughout this section, let X be a smooth projective surface over a separably closed field F. Recall that X is said to be F-minimal if the only F-morphisms $X \to Y$ which are birational equivalences to a smooth projective F-surface Y are actually isomorphisms.

THEOREM 2 (The Adjunction Lemma). *Let X be an F-minimal surface with $q = P_2 = 0$. For any invertible sheaf $\mathscr{L}$ on X, there exists an integer $n_0 \geq 0$ such that $n \geq n_0$ implies*

$$H^0(X, \mathscr{L} \otimes \omega_X^n) = 0.$$

Proof. Because X is smooth and proper over F, the language of divisors can be used in place of the language of invertible sheaves. Thus, it is necessary to show, for any divisor D and for sufficiently large n, that there are no effective representatives of the divisor class of $D + nK$.

Since Fulton's [4] version of the Hirzebruch–Riemann–Roch (HRR) Theorem holds for a projective variety over any field, one has

$$\chi(\mathcal{O}_X) = 1 - q + p_g = 1 + p_g > 0.$$

Because X is smooth over F, Serre duality [5] holds with dualizing sheaf $\omega_X = \Omega^2_{X/F} = \mathcal{O}_X(-K)$. Therefore,

$$h^2(-K) = h^0(2K) = P_2 = 0.$$

So, applying HRR to the anticanonical divisor gives

$$h^0(-K) - h^1(-K) = \chi(\mathcal{O}_X(-K)) = \tfrac{1}{2}(-K) \cdot (-2K) + \chi(\mathcal{O}_X) = K^2 + 1 + p_g.$$

If $K^2 \geq 0$, then the last chain of equations implies $h^0(-K) > 0$. So, $-K$ has an effective representative, which is not zero because $P_2 = 0$. Thus, there exists a hyperplane section H with $K \cdot H < 0$. Furthermore, if n is large, then $(D + nK) \cdot H = D \cdot H + nK \cdot H < 0$. Thus, $D + nK$ cannot be linearly equivalent to an effective divisor.

So, assume $K^2<0$. For n large, $(D+nK)\cdot K<0$. If $D+nK$ is linearly equivalent to an effective divisor for arbitrarily large n, then at some point n_1 we have both $(D+n_1K)\cdot K<0$ and $D+n_1K\sim\sum m_iC_i$ with all $m_i\geq 0$. In particular, there is a component $C=C_i$ such that $C\cdot K<0$. Then n large implies $C\cdot(D+nK)<0$. But if $C^2\geq 0$ and $D+nK$ has an effective representative, then $C\cdot(D+nK)\geq 0$. Since this is impossible, the proof of the theorem has been reduced to:

LEMMA 3. *Let X be a smooth projective surface over a separably closed field F. Let C be an integral curve on X with $C^2<0$ and $C\cdot K<0$. Then C is an exceptional curve of the first kind defined over F.*

Proof. Define $\omega_C=\mathcal{O}_C(C+K)$. By definition, there are exact sequences

$$0\to\mathcal{O}_X(-C)\to\mathcal{O}_X\to\mathcal{O}_C\to 0$$
$$0\to\mathcal{O}_X\to\mathcal{O}_X(K+C)\to\omega_C\to 0.$$

By Serre duality on X and the additivity of the Euler characteristic, one has

$$\chi(\omega_C)=\chi(\mathcal{O}_X(K+C))-\chi(\mathcal{O}_X)=\chi(\mathcal{O}_X(-C))-\chi(\mathcal{O}_X)=-\chi(\mathcal{O}_C).$$

By HRR [4, Ex. 18.3.4] on the projective curve C over F, one has

$$\deg(\omega_C)=\chi(\omega_C)-\chi(\mathcal{O}_C)=-2\chi(\mathcal{O}_C).$$

But by [4, Ex. 2.4.9], one knows $\deg(\omega_C)=C\cdot(C+K)$.

Now consider the natural morphism $\pi:\bar{X}\to X$. Because F is separably closed, the pullback $\pi^*(C)=qD$, where q is a power of the characteristic of F and D is an integral curve on $\bar{X}$. Since the formation of relative differentials is stable under base change [6], one also knows that $\pi^*(K)=\bar{K}$ is a canonical divisor on $\bar{X}$. Therefore,

$$C^2=q^2D^2 \quad\text{and}\quad C\cdot K=qD\cdot\bar{K}.$$

In particular, both D^2 and $D\cdot\bar{K}$ are negative. By the Adjunction Formula over the algebraically closed field $\bar{F}$, D is an exceptional curve of the first kind. Hence

$$C^2=-q^2 \quad\text{and}\quad C\cdot K=-q.$$

Let E be the algebraic closure of F in the function field $F(C)$. Then E is a purely inseparable extension of F of degree q and D is already defined and isomphic to $\mathbb{P}^1$ over E. In fact, $F(C) \approx E(t)$ is a rational function field in one variable, and $D \to C$ is the normalization. Thus, there is an exact sequence

$$0 \to \mathcal{O}_C \to \phi_*(\mathcal{O}_D) \to \text{Torsion} \to 0.$$

Therefore,

$$\begin{aligned} q^2 + q = -C \cdot (C + K) = -\deg(\omega_C) = 2\chi(\mathcal{O}_C) \\ \leq 2h^0(\mathcal{O}_C) \leq 2h^0(\mathcal{O}_D) = 2[E:F] = 2q. \end{aligned}$$

In other words, $q^2 \leq q$ or $q \leq 1$. But this is only possible if $q = 1$ and therefore $E = F$.

§2. Proof of Theorem 1

The proof will proceed through a series of reductions. First, Theorem 1 is clearly equivalent to

THEOREM 4. *Let F be a separably closed field. Every rational surface over F is split by F.*

It is in this form that the theorem will be proved. Assume hereafter that F is separably closed.

PROPOSITION 5. *Let $f: X \to Y$ be a birational morphism of smooth projective surfaces over F. Then f factors as a sequence of monoidal transformations centered at F-points.*

Proof. Because f factors over $\bar{F}$ as a sequence of blowups of points, it must factor over F as a sequence of blowups of closed points. It suffices to show that the blowup of a closed point whose residue field is a nontrivial purely inseparable extension of F can never give rise to a smooth surface.

Let Q be a closed point of Y with residue field E a purely inseparable extension of degree $p^n > 1$. Since smoothness is local, one may replace Y by an open affine $\operatorname{Spec} A$ where Q is defined by a maximal ideal $\mathfrak{m} = (x, y)$. Let $R = A_{\mathfrak{m}}$ be the (regular) local ring of Q on Y. The Second Exact Sequence of

[Matsumura, 9]

$$\begin{array}{ccccc} \mathfrak{m}/\mathfrak{m}^2 \to & \Omega_{R/F} \otimes E & \to & \Omega_{E/F} & \to 0 \\ & \wr\wr & & \cup & \\ & E \oplus E & & E & \end{array}$$

shows that there is at least one linear relation between dx and dy over $E \approx A/\mathfrak{m}$. Without loss of generality, one may assume there exists $t \in A$ such that $dx \equiv t\,dy \bmod \mathfrak{m}$.

By shrinking the affine Y if necessary, the blowup can be defined locally by $X = \operatorname{Spec} B$ where $B = A[T]/(xT - y)$. The module of differentials of this ring is

$$\Omega_{B/F} = (B\,dT \otimes (\Omega_{A/F} \otimes B))/(T\,dx + x\,dT - dy).$$

At the point defined by $\mathfrak{n} = (x, y, T - t)$, one has

$$T\,dx + x\,dT - dy \equiv t\,dx - dy \equiv 0 \bmod \mathfrak{n}.$$

Therefore, $\Omega_{B/F} \otimes B/\mathfrak{n}$ is three-dimensional and $\mathfrak{n}$ is not a smooth point on the blowup X.

Remark. Let $E = F(\alpha)$ be a purely inseparable extension of degree p. Let X be the blowup of $\mathbb{P}^2$ at the F-scheme defined by the E-point $(\alpha:0:1)$. Although the proposition shows that X is not a smooth F-surface, it is a regular projective F-surface which becomes birationally equivalent to $\mathbb{P}^2$ over the algebraic closure. It is not separably split.

By Prop. 5, one may assume that X is an F-minimal rational surface. By Theorem 2, one may use Iskovskih's classification of such surfaces [7]. One property that all such surfaces share is that the rank of the Picard group is small.

LEMMA 6. *Let X be a rational surface over a separably closed field F. Then* $\operatorname{Pic}(X)$ *is a subgroup of finite index in* $\operatorname{Pic}(\bar{X})$.

Proof. Let $C_1, \ldots, C_r$ be a finite set of curves generating $\operatorname{Pic}(\bar{X})$. Each C_i is defined over some purely inseparable extension of F of finite degree. So, for some $n \geq 0$, each $p^n C_i$ is an F-rational divisor. Therefore

$$p^n \operatorname{Pic}(\bar{X}) \subset \operatorname{Pic}(X) \subset \operatorname{Pic}(\bar{X}).$$

PROPOSITION 7. *The only possible minimal rational surfaces over F are*
(I) *Severi–Brauer surfaces*
(II) *smooth conic bundle surfaces* $X \to \mathbb{P}^1$.

Proof. Iskovskih [7] has shown that the minimal surfaces are either (i) del Pezzo surfaces of Picard rank 1 over F or (ii) generically smooth conic bundles of Picard rank 2, possibly with singular fibres, over a smooth genus zero curve.

By Lem. 6, the rank of the Picard group is unchanged by passage to the algebraic closure. The only del Pezzo surfaces with Picard rank 1 are the Severi–Brauer surfaces (del Pezzo surfaces of degree 9). Since smooth projective curves of genus zero always have points over a separably closed field, they are all isomorphic to $\mathbb{P}^1$. The only conic bundles over $\mathbb{P}^1$ with Picard rank 2 are the smooth conic bundles.

Now it is well-known that every Severi–Brauer surface has points over a separably closed field, and is therefore separably split. See, for instance [11]. So, it only remains to consider conic bundles.

PROPOSITION 8. *All smooth conic bundles* $X \to \mathbb{P}^1$ *are separably split.*

Proof. By [7, Thm. 3], every smooth conic bundle is associated to a rank 2 vector bundle. But vector bundles on $\mathbb{P}^1$ split as a sum of line bundles, and their projectivization is unchanged by twisting by line bundles [6]. So, it is enough to consider the surfaces

$$X_n = \mathbb{P}(\mathcal{O} \otimes \mathcal{O}(n)) \qquad n \geq 0$$

over $\mathbb{P}^1$.

When $n = 0$, $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is clearly separably split. When $n > 0$, the surface X_n contains a unique F-curve B_n of self intersection $-n$. Now X_n can be split over F by choosing $n-1$ points

$$t_1, \ldots, t_{n-1} \in \mathbb{P}^1(F),$$

writing $Q_i = f^{-1}(t_i) \cap B_n$, and then blowing up all the Q_i and blowing down the proper transform of B_n.

This completes the proof of Theorems 1 and 4.

REFERENCES

[1] BLOCH, S., *On the Chow groups of certain rational surfaces.* Ann. Sci. Éc. Norm. Sup. *14* (1981) 41–59.

[2] COLLIOT-THÉLÈNE, J.-L., *Hilbert's theorem 90 for K_2 with application to the Chow groups of rational surfaces*. Invent. Math. *71* (1983) 1–20.
[3] COLLIOT-THÉLÈNE, J.-L., Letter to Murthy and Roy, 5 March 1983.
[4] FULTON, W., *Intersection Theory*. Springer-Verlag: Berlin/New York, 1984.
[5] HARTSHORNE, R., *Residues and Duality*, Lect. Notes in Math. 20. Springer-Verlag: Berlin/New York, 1966.
[6] HARTSHORNE, R., *Algebraic Geometry*. Springer-Verlag: Berlin/New York, 1977.
[7] ISKOVSKIH, V. A., *Minimal models of rational surfaces over arbitrary fields*. Math USSR Izvestia *14* (1980) 17–39.
[8] MANIN, Y. I., *Rational surfaces over perfect fields II*. Math. USSR Sbornik *1* (1967) 141–168.
[9] MATSUMURA, H., Commutative Algebra, Benjamin/Cummings, Reading MA, 1980.
[10] MURTHY, M. P. and ROY, A., *Torsion in K_2 of field and 0-cycles on rational surfaces*. Comm. Math. Helv. *59* (1984) 165–186.
[11] SERRE, J.-P., *Local Fields*, Springer-Verlag: Berlin/New York, 1979.

Dept of Mathematics
University of Michigan
Ann Arbor, MI 48105 USA

Received April 1, 1987