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On weakly positive unit forms

HANS-JOACHIM VON HÖHNE

In this paper we examine integral quadratic forms $q : \mathbf{Z}^n \to \mathbf{Z}$ of the shape

$$q(x) = \sum_{a=1}^{n} x_a^2 + \sum_{a<b} q_{ab} x_a x_b.$$

We call such a q a *unit form* and say that q is *weakly positive* if $q(x) > 0$ whenever $x > 0$ (i.e. whenever $x \neq 0$ and $x_a \geqq 0$ for $a = 1, \ldots, n$). Our main purpose is to work out a criterion for weak positivity of unit forms.

This question arises variously in representation theory, in that of quivers [G1], partially ordered sets [D], matrix problems [Ro] and certain classes of finite-dimensional algebras [Br], [HR], [Bo1, Bo4]: In each of these articles, the considered structure Ω has up to isomorphism only finitely many indecomposable representations iff some unit form q_Ω, naturally attached to Ω, is weakly positive; if this is the case, the indecomposable representations correspond bijectively to the positive roots of q_Ω, i.e. to the $x \in \mathbf{N}^n$ such that $q_\Omega(x) = 1$.

Given a unit form q on $\mathbf{Z}^n$ and a nonempty subset $I = \{i_1 < \cdots < i_m\} \subset \{1, \ldots, n\}$, we denote by q^I the quadratic form gd_I where $d_I : \mathbf{Z}^m \to \mathbf{Z}^n$ maps $e_k \in \mathbf{Z}^m$ (the k-th natural basis vector) onto $e_{i_k} \in \mathbf{Z}^n$; we shall call such a form q^I a *restriction* of q. According to Ovsienko [Ov], q is called a *critical form* if q^I is weakly positive for each $I \subsetneqq \{1, \ldots, n\}$ though q itself is not. With this definition, a unit form q is weakly positive iff q has no critical restriction q^I.

So it remains to examine the critical forms. In §1 we show that they can be constructed by certain elementary transformations from the unit forms attached to the extended Dynkin graphs $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$. This has already been observed by Ovsienko [Ov], but his proof is scarcely accessible (for partial results see [HV], [Ri]). However, the construction is practicable only in case $\tilde{A}_n$ and $\tilde{D}_n$. To complete the classification of critical forms of type $\tilde{E}_n$, $6 \leqq n \leqq 8$, we introduce in §2 another construction, which proceeds by induction on the number of variables and so relates critical forms of different extended Dynkin types.

As an application in representation theory, we obtain in §3 the criterion of Kleiner–Nazarova–Roiter [K, NR] characterizing representation-finiteness for partially ordered sets and, for certain classes of finite-dimensional algebras, a variant of the criterions of Bongartz [Bo3] and Ringel [Ri]. In particular, we have a new approach to the lists of Bongartz [Bo2] and Happel–Vossieck [HV], avoiding computers.

The results of this paper are parts of the authors doctoral thesis. This version was written while the author was visiting the University of Zürich supported by the Deutsche Forschungsgemeinschaft. He would like to thank P. Gabriel for his suggestions concerning the presentation.

1. Critical forms and extended Dynkin graphs

1.1 With each unit form $q:\mathbf{Z}^n \to \mathbf{Z}$ with coefficients $q_{ab}(=q(e_a+e_b)-q(e_a)-q(e_b))$ we associate the *weighted graph* Q_q given by the vertex set $\{1, \ldots, n\}$ and an 'integrally weighted' edge $a \overset{q_{ab}}{\text{---}} b$ for any two vertices $a \neq b$. We visualize Q_q by drawing

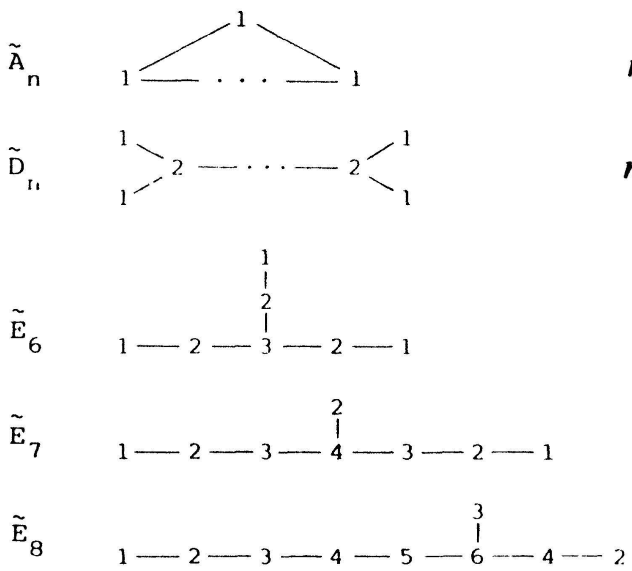
$-q_{ab}$ full edges $a \vdots b$ if $q_{ab} \leqq 0$ and

q_{ab} broken edges $a \vdots b$ if $q_{ab} > 0$.

Conversely, a weighted graph Q with vertex set $Q_0 = \{1, \ldots, n\}$ and weighted edges $a \overset{Q_{ab}}{\text{---}} b$ for $a \neq b$ gives rise to the unit form $q_Q : \mathbf{Z}^n \to \mathbf{Z}$, $q_Q(x) = \sum_{a=1}^n x_a^2 + \sum_{a<b} Q_{ab} x_a x_b$. So unit forms and weighted graphs correspond bijectively. We call a unit form q *indecomposable* if any two vertices $a \neq b$ are connected by a sequence $a = a_0, a_1, \ldots, a_s = b$ such that $q_{a_{i-1}a_i} \neq 0$ for $i = 1, \ldots, s$.

In this paper, a special rôle is played by the unit forms q_Δ where Δ is an extended Dynkin graph (Figure 1). The q_Δ can be characterized as those indecomposable unit forms which are nonnegative (i.e. $q_\Delta(x) \geqq 0$ for all x), degenerated and satisfy $q_{\Delta ab} \leqq 0$ for all $a \neq b$. Moreover, each q_Δ has a radical of the shape rad $q_\Delta = \mathbf{Z} y_\Delta$ where $y_\Delta \gg 0$ (i.e. $y_{\Delta a} > 0$ for all a); the coordinates of y_Δ are the numbers given in Fig. 1 [Bou]. In particular, each q_Δ is critical.

1.2 In order to investigate the remaining critical forms, consider an arbitrary unit form q on $\mathbf{Z}^n$ and a pair (a, b) of distinct indices such that $q_{ab} = 1$. Denoting by $T_{ab} \in \mathrm{Gl}\,(n, \mathbf{Z})$ the elementary transformation which fixes e_c for $c \neq b$ and maps e_b onto $e_b - e_a$ (= reflection of e_b at the hyperplane orthogonal to e_a), we obtain a new unit form $q' = qT_{ab}$ with $q'_{ab} = -1$; we shall call T_{ab} the *inflation* of q with respect to (a, b).



$n+1$ vertices, $n \geqq 1$,

$n+1$ vertices, $n \geqq 4$,

Figure 1

Clearly, there is no critical form on $\mathbf{Z}^n$ if $n \leqq 1$. In case $n=2$ the critical forms are those of the shape $x_1^2+x_2^2+mx_1x_2$ where $m \leqq -2$.

THEOREM (Ovsienko). *Let q be a critical form in $n \geqq 3$ variables. Then there exist an extended Dynkin graph Δ and an iterated inflation T of q such that $q_\Delta = qT$. Up to isomorphism, Δ is uniquely determined by q.*

Proof. Set $V=\{x \in \mathbf{R}^n \mid x \gg 0 \text{ or } 0 \gg x\}$, extend q to $\mathbf{R}^n$ in the obvious way and denote $q(x, y)=q(x+y)-q(x)-q(y)$.

1) We claim that $q(x)>0$ whenever $0 \neq x \in \mathbf{R}^n \backslash V$. First let $x \in \partial V$ ($=$boundary of V). If $x \in \mathbf{Z}^n$ or $x \in \mathbf{Q}^n$, the assertion is clear. In general, $q(x) \geqq 0$ by continuity. The equality $q(x)=0$ would imply $\lambda q(x, e_a)+\lambda^2=q(x+\lambda e_a) \geqq 0$ for small values of λ, hence $q(x, e_a)=0$ if $x_a>0$. Since the equations $q(x, e_a)=0$ have integral coefficients, there would exist a vector $y \in \mathbf{Q}^n$ such that $y_a>0=q(y, e_a)$ if $x_a>0$ and $y_a=0$ if $x_a=0$; hence $y \in \partial V \cap \mathbf{Q}^n$ and $q(y)=0$, a contradiction. Now let $x \in \mathbf{R}^n \backslash (V \cup \partial V)$ and choose $z \in \mathbf{Z}^n$ such that $z \gg 0$ and $q(z) \leqq 0$. The line $z+\mathbf{R}x$ intersects ∂V in $z+\lambda_0 x \neq 0$ and $z+\lambda_1 x \neq 0$, where $\lambda_0=\max_{x_a>0}\{-z_a/x_a\}<0<\lambda_1=\min_{x_a<0}\{-z_a/x_a\}$. Therefore $q(z+\lambda x)=q(z)+\lambda q(z, x)+\lambda^2 q(x)$ is >0, $\leqq 0$ and >0 for $\lambda=\lambda_0$, 0 and λ_1 respectively. This implies $q(x)>0$.

2) The set $C_q=\{x \in \mathbf{Z}^n \backslash (V \cup \partial V) \mid q(x)=1\}$ is finite. (Indeed, denoting by $\|\_\|$ the Euclidian norm on $\mathbf{R}^n$, the set $C'=\{x \in \mathbf{R}^n \backslash V \mid \|x\|=1\}$ is compact, and with $\xi=\min_{x \in C'}\{q(x)\}>0$ we have $\|x\|^2 \leqq \|x\|^2\, \xi^{-1} q(\|x\|^{-1} x)=\xi^{-1} q(x)=\xi^{-1}$ for each $x \in C_q$.) If T_{ab} is an inflation of q, then $q'=qT_{ab}$ is critical (by 1)), and T_{ab} induces an injection $C_{q'} \to C_q$, which is not surjective since $e_b-e_a \in$

$C_q \backslash T_{ab}(C_{q'})$. Consequently, in a sequence of unit forms $q = q^0, q^1, \ldots, q^s$ where q^r is obtained from q^{r-1} by an inflation, the unit forms are pairwise distinct.

3) Since $n \geqq 3$, we have $0 < q(e_a \pm e_b) = 2 \pm q_{ab}$ for all $a \neq b$. Thus, there are only finitely many critical forms on $\mathbf{Z}^n$, and we can choose a unit form $q' = qT$ where T is an iterated inflation of q involving a maximal number of inflations. We then have $-1 \leqq q'_{ab} \leqq 0$ for all $a \neq b$. Since q' is not positive, the weighted graph $Q_{q'}$ contains an extended Dynkin graph Δ as a full subgraph. But since q' and q_Δ both are critical they coincide.

4) If Δ and Γ are extended Dynkin graphs and $A: \mathbf{Z}^{\Delta_0} \to \mathbf{Z}^{\Gamma_0}$ is an isomorphism such that $q_\Delta = q_\Gamma A$, then Δ and Γ are isomorphic. Indeed, remove from Δ a vertex a and from Γ a vertex b where $y_{\Delta a} = y_{\Gamma b} = 1$ and denote the resulting Dynkin graphs by Δ' and Γ' respectively. Since $A(\operatorname{rad} q_\Delta) = \operatorname{rad} q_\Gamma$, A induces an isomorphism $A': \mathbf{Z}^{\Delta'_0} \to \mathbf{Z}^{\Gamma'_0}$ such that $q_{\Delta'} = q_{\Gamma'} A'$. In particular, the (symmetric) coefficient matrices of $q_{\Delta'}$ and $q_{\Gamma'}$ have the same determinant. It follows that $\Delta' \simeq \Gamma'$ and therefore $\Delta \simeq \Gamma$. qed.

1.3 Theorem 1.2 shows that each critical form q is nonnegative and that $\operatorname{rad} q$ is generated by a positive vector y satisfying $y \leqq y_\Delta$ where $\mathbf{Z} y_\Delta = \operatorname{rad} q_\Delta$. (If T_{ab} is an inflation of q then $\operatorname{rad} qT_{ab}$ is generated by $T_{ab}^{-1} y = y + y_b e_a > y$.) Since the coordinates of y_Δ are bounded by 6 we obtain:

COROLLARY. *A unit form q is weakly positive iff $q(x) > 0$ whenever $x > 0$ and $x_a \leqq 6$ for all indices a.*

Remark 1. The classification of all critical forms on $\mathbf{Z}^n$ is equivalent to the classification of all pairs of a positive unit form on $\mathbf{Z}^{n-1}$ together with a strictly positive root. Namely, if q is a critical form on $\mathbf{Z}^n$, the radical generator $y \gg 0$ of q has some coordinate 1, say $y_n = 1$ (as seen above, $y \leqq y_\Delta$ and y_Δ has some coordinate 1). Now, $q(y - e_n) = q(-e_n) = 1$ and $q(e_a, e_n) = -q(e_a, y - e_n)$ for all $a \neq n$. This shows that the restriction $q' = q^{\{1, \ldots, n-1\}}$ of q has a strictly positive root $y' = y - e_n$ (considered as an element of $\mathbf{Z}^{n-1}$), and that q can be recovered from q' and y' by $q(x' + x_n e_n) = q'(x') + x_n^2 - q'(x', y')x_n$ for all $x' + x_n e_n \in \mathbf{Z}^{n-1} \times \mathbf{Z}$. Conversely, given a positive unit form q' on $\mathbf{Z}^{n-1}$ and a vector $y' \gg 0$ such that $q'(y') = 1$, the same formula for q yields a nonnegative unit form on $\mathbf{Z}^n$ with $\operatorname{rad} q = \mathbf{Z}(y' + e_n)$.

Remark 2. Each indecomposable nonnegative unit form q on $\mathbf{Z}^n$ with $\operatorname{rad} q = \mathbf{Z} y \neq 0$ is $\mathbf{Z}$-equivalent to q_Δ for some extended Dynkin graph Δ. Further, if $\operatorname{rad} q_\Delta = \mathbf{Z} y_\Delta$, then $|y_a| \leqq |y_{\Delta a}|$ for all a and a suitable numeration of the vertices of Δ; in particular, $|y_{a_0}| = 1$ for some a_0.

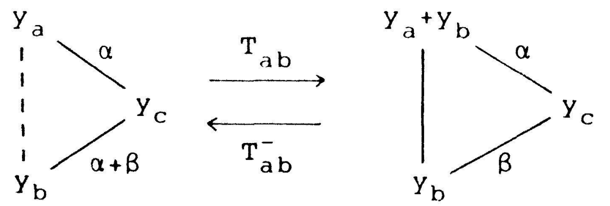
Indeed, we can assume $y > 0$ (otherwise consider qA where $A \in \mathrm{Gl}(n, \mathbf{Z})$ fixes e_a if $y_a \geqq 0$ and maps e_a onto $-e_a$ if $y_a < 0$). Now consider a sequence of unit forms $q = q^0, q^1, q^2, \ldots$ where $q^{r+1} = q^r T_{a_r b_r}$, $q^r_{a_r b_r} = 1$, $\mathrm{rad}\, q^r = \mathbf{Z} y^r$ and $y^r_{b_r} > 0$ for $r = 0, 1, 2, \ldots$; then $y^0 < y^1 < y^2 < \cdots$ and the $q^0, q^1, q^2, \ldots$ are pairwise distinct. Since all coefficients satisfy $-2 \leqq q^r_{ab} \leqq 1$, the sequence must stop, say at q' with $\mathrm{rad}\, q' = \mathbf{Z} y'$, $y' \geqq y^0$. We then have $q'_{ab} \leqq 0$ whenever $y'_a > 0$ and $a \neq b$. But $y'_a > 0$ for each a. (Otherwise, since inflations preserve indecomposability, there are indices b and c such that $y'_b = 0$, $y'_c > 0$ and $q'_{bc} < 0$. We infer $q'(e_b, y') = \sum_{a=1}^n q'_{ba} y'_a = \sum_{y'_a > 0} q'_{ba} y'_a < 0$, contradicting $y' \in \mathrm{rad}\, q'$.) Thus $Q_{q'}$ is an extended Dynkin graph (1.1). To prove the last assertion we can assume $y \gg 0$ (otherwise consider a suitable restriction of q). In this case, the assertion has already been seen.

2. Critical diagrams

2.1 For a further combinatorial analysis of critical forms we need some more notation.

We call (Q, y) a *weighted pair*, if Q is a weighted graph (in the sense of 1.1) and y is an element of $\mathbf{Z}^{Q_0}$. If moreover the components of y are >0, have no common divisor and satisfy $2y_a + \sum_{a \neq b} Q_{ab} y_b = 0$ for each $a \in Q_0$ (i.e. $y \in \mathrm{rad}\, q_Q$), we call (Q, y) a *diagram*. Each critical form q with $\mathrm{rad}\, q = \mathbf{Z} y$ and $y > 0$ gives rise to a diagram (Q_q, y); but not every diagram is of this shape.

(Q, y) $\qquad\qquad$ $(Q, y)T_{ab}$



$c \neq a, b$
arbitrary

Figure 2

Two weighted pairs (Q, y) and (Q', y') are called $\mathbf{Z}$-equivalent, if $q_{Q'} = q_Q T$ and $y' = T^{-1} y$ for some isomorphism $T : \mathbf{Z}^{Q'_0} \to \mathbf{Z}^{Q_0}$; if T maps the natural basis of $\mathbf{Z}^{Q'_0}$ onto that of $\mathbf{Z}^{Q_0}$, we call (Q, y) and (Q', y') isomorphic. Of special interest are the transformations T_{ab} (1.2) and its inverse T^-_{ab}, which fixes e_c for $c \neq b$ and maps e_b onto $e_b + e_a$. If $a, b \in Q_0$ satisfy $Q_{ab} = 1$, we call T_{ab} an *inflation* of (Q, y) and write $(Q, y)T_{ab} = (Q_{q_Q T_{ab}}, T^-_{ab} y)$; if $Q_{ab} = -1$ and $y_a > y_b$, we call T^-_{ab} a *deflation* of (Q, y) and write $(Q, y)T^-_{ab} = (Q_{q_Q T^-_{ab}}, T_{ab} y)$. Under these conditions, $(Q, y)T_{ab}$ and $(Q, y)T^-_{ab}$ are diagrams if so is (Q, y). What changes in (Q, y) under T_{ab} is shown in Figure 2.

We call a diagram (Q, y) *critical*, if the unit form q_Q is critical or, equivalently, nonnegative of corank 1. In this case, (Q, y) is $\mathbf{Z}$-equivalent to some well determined extended Dynkin diagram (Δ, y_Δ) (1.2); we shall say that (Q, y) is of *type* Δ. Clearly, diagrams obtained by inflations and deflations of critical diagrams are critical of the same type.

THEOREM. 1) *A diagram is critical of type* $\tilde{A}_n$ *iff it is isomorphic to* $(\tilde{A}_n, y_{\tilde{A}_n})$.

2) *The critical diagrams of type* $\tilde{D}_n$, $n \geqq 4$, *are those with* $n+1$ *vertices shown in Figure* 3.

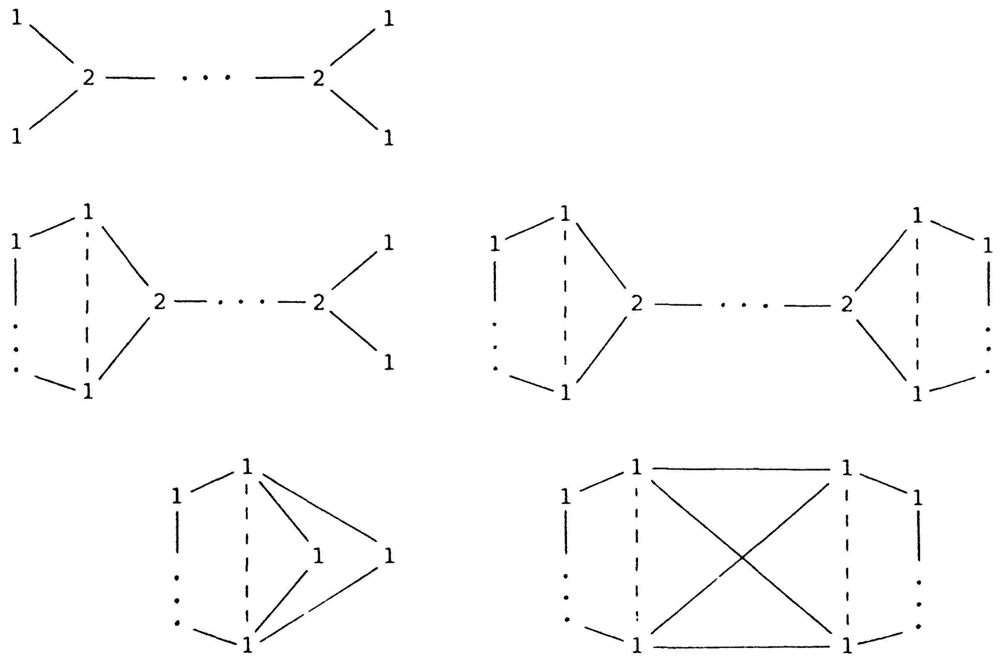


Figure 3

For the proof, we have to determine all diagrams which can be obtained by iterated deflations from (Δ, y_Δ) where $\Delta = \tilde{A}_n$ or $\Delta = \tilde{D}_n$. Clearly there is no deflation of $(\tilde{A}_n, y_{\tilde{A}_n})$. The examination of the case $\Delta = \tilde{D}_n$ is left to the reader as an easy exercise.

2.2 The deflation algorithm used to determine the critical diagrams of type $\tilde{A}_n$ and $\tilde{D}_n$ is not appropriate in the general case, because there are too many critical diagrams of type $\tilde{E}_n$. However, the combinatorial complexity displayed by the diagrams of type $\tilde{E}_8$ is related to the linear subgraph

6——5——4——3——2——1

of $\tilde{E}_8$, and we can attain some simplification if we examine the effect of deflations restricted to this subgraph.

For this, we call a non-empty set A of vertices of a weighted pair (Q, y) a *branch*, if its elements admit a lexicographic indexation $a_1, a_2, \ldots, a_{11}, a_{12}, \ldots, a_{21}, \ldots, a_{111}, \ldots$ such that:

a) $Q_{ac} = Q_{bc}$ for all $c \notin A$ if a, b are *joint-vertices* of A, i.e. have index-length one: $a = a_i$, $b = a_j$
b) $Q_{ab} = -1$ if $a = a_{ij\cdots p}$ and $b = a_{ij\cdots pq}$
c) $Q_{ab} = 1$ if $a = a_{ij\cdots pq}$ and $b = a_{ij\cdots pr}$, $q \neq r$
d) $Q_{ab} = 0$ in all cases $a \in A$, $b \in Q_0$ not mentioned above
e) $y_a - 1$ equals the number of *successors* of a in A, i.e. of vertices of the form $a_{ij\cdots pq_1\cdots q_r}$, $r \geqq 1$, if $a = a_{ij\cdots p}$

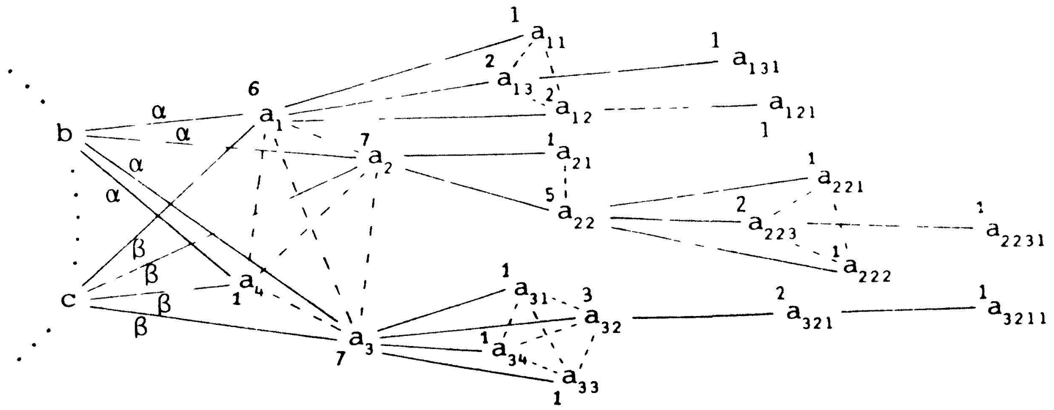


Figure 4

The *link* of A is by definition the family $(Q_{ac})_{c \in Q_0 \setminus A}$ where a is a fixed joint-vertex of A, and the *ramification* is the sequence $m = (m_1, m_2, \ldots)$ where m_s denotes the number of vertices $a_{i_1\cdots i_s}$ with index-length s. For a given cardinality $|A|$ of A, the branch with the smallest ramification (in the lexicographic ordering) is the *twig*:

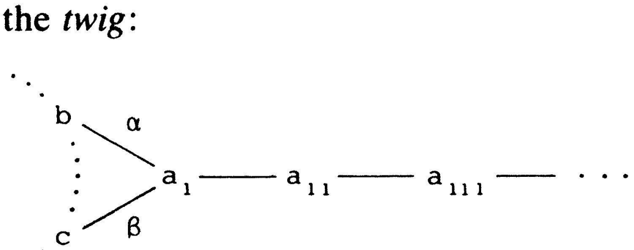


Figure 5

PROPOSITION. *Let a, b belong to a branch A of (Q, y) with ramification m.*
a) *If $Q_{ab} = 1$, A is a branch of $(Q, y)T_{ab}$ with the same link as in (Q, y) and with ramification $<m$.*

b) *If* $Q_{ab} = -1$ *and* $y_a > y_b$, *A is a branch of* $(Q, y)T_{ab}^-$ *with the same link as in* (Q, y) *and with ramification* $>m$.

Proof. Evident on the pictures: In case a), if $a = a_{ij\cdots pq}$ and $b = a_{ij\cdots pr}$, the index-length s of b increases by 1, and the first changing component m_s of m is replaced by $m_s - 1$. In case b), if $a = a_{ij\cdots p}$ and $b = a_{ij\cdots pr}$, the index-length s of b decreases by 1, and the first changing component m_{s-1} of m is replaced by $m_{s-1} + 1$. In each case, the joint-vertices of A different from b coincide in (Q, y), $(Q, y)T_{ab}$ and $(Q, y)T_{ab}^-$ respectively. qed.

The proposition shows that each branch A of (Q, y) can be transformed into a twig by iterated inflations located in A. Of course, the indexation of the vertices of A in the resulting weighted pair will depend on the performed inflations. But the isomorphism class of the new weighted pair depends only on (Q, y) and A (see Figure 4 and Figure 5). We shall say that it is obtained from (Q, y) by *stretching* the branch A. Up to isomorphism, there are 1, 2, 4, 9, 20 and 48 branches which can be stretched into a twig of cardinality 1, 2, 3, 4, 5 and 6 respectively.

2.3. PROPOSITION. *If two branches A and B of* (Q, y) *intersect, then* $A \cup B$ *is a branch of* (Q, y), *unless* $|A|, |B| > 1$, $A \cap B = \{a\}$ *and a is the only joint-vertex of A and of B.*

Proof. Suppose that $a_{i_1\cdots i_s} = b_{j_1\cdots j_t}$ for some lexicographic indexation of the points $a_{hi\cdots}$ of A and $b_{jk\cdots}$ of B. If $s \neq t$, say $s < t$, we have $a_{i_1\cdots i_{s-r}} = b_{j_1\cdots j_{t-r}}$ for all $r \in \{0, \ldots, s-1\}$, a_{i_1} is a successor of $b_{j_1\cdots j_{t-s}}$ in B, and so are all joint-vertices of A by 2.2c) and d); this implies $A \subset B$. If $s = t$, we have $a_{i_1\cdots i_p} = b_{j_1\cdots j_p}$ for all $p \in \{1, \ldots, s\}$; in particular, $a_{i_1} = b_{j_1}$, i.e. the sets $\underline{A}$ and $\underline{B}$ of joint-vertices of A and B intersect. In case $|\underline{A} \cup \underline{B}| > 1$, the set of successors of points of $\underline{A} \cap \underline{B}$ in A and B coincide (for instance, if a' is an immediate successor of $a \in \underline{A} \cap \underline{B}$ in A, we have $Q_{aa'} = -1$ and $Q_{ca'} = 0$ for all $c \in (\underline{A} \cup \underline{B})\backslash\{a\}$, hence $a' \in B$ by 2.2a)); it follows that $A \cup B$ is a branch with set of joint-vertices $\underline{A} \cup \underline{B}$. In case $|\underline{A} \cup \underline{B}| = 1$ but $|A \cap B| > 1$, the unique joint-vertex $a \in \underline{A} \cap \underline{B}$ has some common immediate successor in A and B; therefore all immediate successors of a in A and B coincide (2.2c) and d)) and $A = B$. In the remaining case $|\underline{A} \cup \underline{B}| = 1$ and $A = \underline{A}$ or $B = \underline{B}$, we clearly have $A \subset B$ or $B \subset A$. qed.

The proposition implies that any two distinct maximal branches of (Q, y) are disjoint, or else they intersect in a unique joint-vertex. Consequently, if we stretch all maximal branches of (Q, y), the isomorphism class of the resulting weighted pair $(\bar{Q}, \bar{y})$ does not depend on the order of stretchings (indeed, two inflations T_{ab} and T_{cd} commute if $a \neq d \neq b \neq c$, and a branch A remains unaffected by T_{cd} if $c,\ d \notin A$). Moreover, each branch of $(\bar{Q}, \bar{y})$ is a twig. (Namely, denote by $A_1, \ldots, A_r$ the maximal branches of (Q, y); they become twigs in $(\bar{Q}, \bar{y})$. Assume that B is a branch of $(\bar{Q}, \bar{y})$ having at least two joint-vertices. If $B \cap A_i = \emptyset$ for all $i \in \{1, \ldots, r\}$, B is a branch of (Q, y) not contained in a maximal one, contradiction. If $B \cap A_i \neq \emptyset$ for some i, $B \cup A_i$ is a branch of $(\bar{Q}, \bar{y})$ by the above propositon, hence a branch of (Q, y); but $B \cup A_i \neq A_i$, contradicting the maximality of A_i.) We shall say that $(\bar{Q}, \bar{y})$ is *stretched*, i.e. $(\bar{\bar{Q}}, \bar{\bar{y}}) = (\bar{Q}, \bar{y})$, or the *stretched form* of (Q, y). For instance, up to isomorphism, $(\tilde{A}_n, y_{\tilde{A}_n})$, $(\tilde{D}_4, y_{\tilde{D}_4})$, $(\tilde{D}_{n+1},\ y_{\tilde{D}_{n+1}})$, $(\tilde{E}_6, y_{\tilde{E}_6})$, $(\tilde{E}_7, y_{\tilde{E}_7})$ and $(\tilde{E}_8, y_{\tilde{E}_8})$ are the stretched forms of 1, 2, 3, 10, 30 and 48 diagrams respectively.

2.4 The notion of a branch does not essentially simplify the deflation algorithm, which is based on the edges a—b such that $y_a \neq y_b$; but it reduces the classification of all critical diagrams to that of the stretched critical diagrams. In order to determine these, it is more convenient to proceed by induction on the number of vertices. This is possible with the following 'funnel-construction'.

Consider a *bunch* of a weighted pair (Q, y), i.e. a set U of at least two pairwise disjoint twigs. Each twig $I \in U$ has one joint-vertex, which we denote by a_I. From (Q, y) and U we construct a new weighted pair $(Q, y)^U = (Q^U, y^U)$ as follows: Cut off all twigs $I \in U$ with the exception of the joint-vertices a_I. For the remaining vertices of Q, set $Q^U_{ab} = Q_{ab}$ if a or b is not of the form a_I, $Q^U_{a_I a_J} = Q_{a_I a_J} + 1$ if $I,\ J \in U$ and $y^U_a = y_a$ for all a. Finally, add a twig $\hat{U}$ with cardinality $-1 + \sum_{I \in U} |I|$ and joint-vertex a_U such that $Q^U_{a_U a_I} = -1$ if $I \in U$ and $Q^U_{a_U b} = 0$ if $b \in Q_0 \backslash \bigcup_{I \in U} I$. The pair $(Q, y)^U$ is a diagram iff so is (Q, y), and the numbers of vertices of Q and Q^U are related by $|Q^U_0| = |Q_0| + |U| - 1 > |Q_0|$.

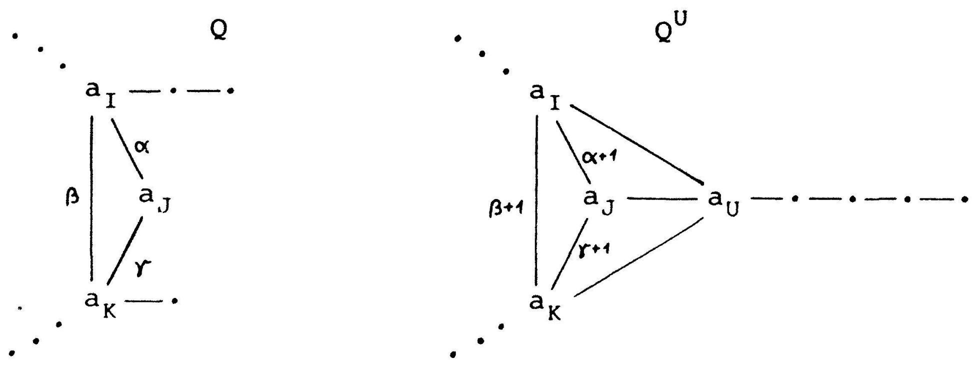


Figure 6

EXAMPLES. The bunches shown in Figure 7 give rise to the diagrams $(\tilde{A}_{n+1},\ y_{\tilde{A}_{n+1}})$, $(\tilde{D}_4,\ y_{\tilde{D}_4})$, $(\tilde{D}_{n+1}, y_{\tilde{D}_{n+1}})$, $(\tilde{E}_6, y_{\tilde{E}_6})$, $(\tilde{E}_7, y_{\tilde{E}_7})$ and $(\tilde{E}_8, y_{\tilde{E}_8})$ respectively.

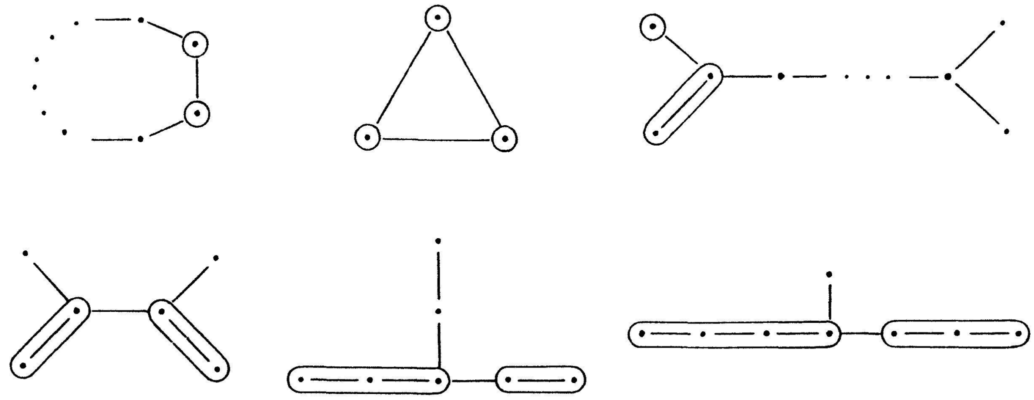


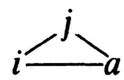
Figure 7

THEOREM. *Each stretched critical diagram not of type $\tilde{A}_1$ is isomorphic to $(Q, y)^U$ for some diagram (Q, y) and some bunch U.*

Proof. It is enough to prove the following assertion: Each critical (Q', y') not of type $\tilde{A}_1$ contains a branch A such that $Q'_{ba} \in \{0, -1\}$ for all $b \in Q'_0 \backslash A$, $a \in A$, and that the set $A_{Q'} = \{b \in Q'_0 \backslash A \mid \exists a \in A,\ Q'_{ba} \neq 0\}$ has at least two elements. (Indeed, if (Q', y') is stretched, A is a twig, and we can construct (Q, y) and U as follows: Delete from Q' the twig A. For the remaining vertices of Q, set $Q_{bc} = Q'_{bc}$ if $b \notin A_{Q'}$ or $c \notin A_{Q'}$, $Q_{bc} = Q'_{bc} - 1$ if $b, c \in A_{Q'}$ and $y_b = y'_b$ for all b. Finally, graft a new twig I_b with joint-vertex b and cardinality y_b for each $b \in A_{Q'}$, and set $U = \{I_b \mid b \in A_{Q'}\}$.)

Clearly, the assertion is true for extended Dynkin diagrams. In general, we proceed by induction on the maximal number of deflations producing (Q', y') out of an extended Dynkin diagram. So assume that the assertion holds for (Q', y') and let $i, j \in Q'_0$ be such that $Q'_{ij} = -1$ and $y'_i > y'_j$. We must show that $(Q'', y'') = (Q', y')T^-_{ij}$ contains a branch A' with the required link.

In case $i, j \in A$, we can choose $A' = A$ (Proposition 2.2).

Next, consider the case $i, j \notin A$. If $i \notin A_{Q'}$, we have $Q'_{ia} = 0$ for all $a \in A$; so A and its link remains unaffected by T^-_{ij}. Now assume that $i \in A_{Q'}$. This implies $j \notin A_{Q'}$; otherwise Q' would contain the critical subgraph 

for some $a \in A$, which is impossible. Thus, for each joint-vertex $a \in A$ and each $b \in A_{Q'} \backslash \{i\}$ we have the situation of Figure 8 in Q' and Q'' respectively. We infer that $A' = A$ is a branch of (Q'', y'') with $A'_{Q''} = A_{Q'} \cup \{j\}$.

Now assume that $j \notin A$ but $i \in A$, i.e. $j \in A_{Q'}$ and i is a joint-vertex of A. Since $0 < y'_j < y'_i$, the set A' of successors of i in A is non-empty and therefore is a branch of (Q', y') with $A'_{Q'} = \{i\}$. As in the preceding case, A' is a branch of (Q'', y'') with $A'_{Q''} = \{i, j\}$.

Finally, we must consider the case $i \notin A$ but $j \in A$, i.e. $i \in A_{Q'}$ and $j \in A$

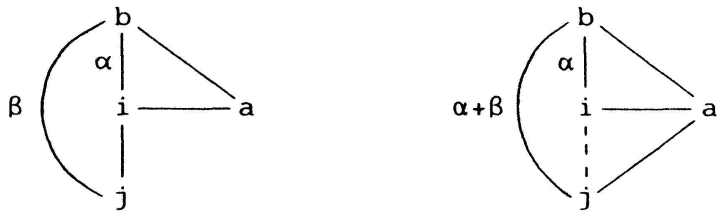


Figure 8

(= set of joint-vertices of A). Then we have $\underline{A} \neq \{j\}$. (Otherwise, since (Q', y') is a diagram and $A_{Q'} \neq \{i\}$, we would have $0 = 2y'_j - (y'_j - 1) - \sum_{b \in A_{Q'}} y'_b = y'_j - y'_i + 1 - \sum_{b \neq i} y'_b \leqq y'_j - y'_i$, contradicting $y'_i > y'_j$.) Thus, for all $b \in A_{Q'} \backslash \{i\}$ and all $a \in \underline{A} \backslash \{j\}$, we have the situation of Figure 9 in Q' and Q'' respectively. We infer, that the subbranch A' of A consisting of $\underline{A} \backslash \{j\}$ and all successors of points of $\underline{A} \backslash \{j\}$ is a branch of (Q'', y'') and that $A'_{Q''} = A_{Q'}$ has the required property. qed.

Figure 9

2.5 Theorem 2.4 raises the question of describing the *critical* bunches of a given diagram (Q, y), i.e. the bunches U such that $(Q, y)^U$ is critical.

Given a bunch U of (Q, y), we say that an inflation T_{bc} of (Q, y) is *U-admissible* if the vertex b belongs to no twig of U. In this case, U is also a bunch of $(Q, y)T_{bc}$, and the diagrams $((Q, y)T_{bc})^U$ and $(Q, y)^U T_{bc}$ are isomorphic. In particular, U is critical in (Q, y) iff it is critical in $(Q, y)T_{bc}$.

The following proposition implies that only critical diagrams admit critical bunches.

PROPOSITION. *Let U be a bunch of a diagram (Q, y). Then U is critical iff there is an iterated U-admissible inflation T of (Q, y) such that $(Q, y)T$ is extended Dynkin and U critical in $(Q, y)T$.*

Proof. Clearly the condition is sufficient. To prove necessity, note first that, if $(Q, y)^U$ is critical, each sequence of composable U-admissible inflations of (Q, y) must stop, since so does the sequence of associated inflations of $(Q, y)^U$. So we can assume further on that (Q, y) admits no U-admissible inflation. We must show that this forces (Q, y) to be extended Dynkin. If Q_{bc} was $\geqq 1$ for some vertices b, c of Q, we would have $Q_{bc} \geqq 2$, or else b and c would be the joint-vertices of two twigs of U; in each case, we would have $Q^U_{bc} \geqq 2$, which is impossible if $(Q, y)^U$ is critical. Thus, $Q_{bc} \leqq 0$ for all $b \neq c$. Since y is a radical-vector of q_Q with strictly positive components, all connected components of Q must be extended Dynkin graphs [BMW]. Now assume that P is one of these components but not the only one, and denote by V the set of twigs of U lying in P. Then P^V or P (in case $|V| = 1$) can be identified with a full weighted subgraph of Q^U supported by a proper subset of Q^U_0. So we have $q_{Q^U}(x) = 0$ for some vector $x \neq 0$ supported by this proper subset. Contradiction. qed.

Remark. If $U \subset V$ are bunches of (Q, y) and V is critical, then so is U. Indeed, we can assume $V = U \cup \{A\}$ for some branch A of (Q, y). Then $W = \{A, \hat{U}\}$ is a bunch of $(Q, y)^U$. If a and b denote the joint-vertices of A and $\hat{U}$, we have $(Q, y)^{UW} T_{ba} \simeq (Q, y)^V$. Hence, if $(Q, y)^V$ is critical, then so is $(Q, y)^{UW}$ and therefore $(Q, y)^U$.

2.6 If we want to use our funnel-construction to classify the critical diagrams with n vertices, we need the critical bunches of *all* critical diagrams with $<n$ vertices. On the other hand, we only list the stretched critical diagrams. So we have to describe the critical bunches of an arbitrary diagram (Q, y) in terms of the stretched form $(\bar{Q}, \bar{y})$.

We first notice that each twig I of (Q, y) is contained in exactly one maximal branch of (Q, y) by Proposition 2.3. This again becomes a twig of $(\bar{Q}, \bar{y})$ and contains exactly one *bud* (= twig of cardinality 1) of $(\bar{Q}, \bar{y})$, which we denote by $\bar{I}$. If U is a bunch of (Q, y), we denote by $\bar{U}$ the bunch of buds of $(\bar{Q}, \bar{y})$ formed by all $\bar{I}$ where $I \in U$.

PROPOSITION. *A bunch U of a diagram (Q, y) is critical iff $\bar{U}$ is critical in $(\bar{Q}, \bar{y})$ and $|\bar{U}| = |U|$.*

Proof. Let a—a'— $\cdots$ be a twig I of U with joint-vertex a and cardinality >1. If we set $I' = I \setminus \{a\}$ and $U' = (U \setminus \{I\}) \cup \{I'\}$, Figure 10 shows that $(Q, y)^U \simeq (Q, y)^{U'} T^-_{aa'} T_{a'a} T_{ad}$, where d denotes the joint-vertex of the branch $\hat{U}'$ of $(Q, y)^{U'}$ (b is the joint-vertex of some other twig of U, and c is some vertex belonging to no twig of U). We infer that U is critical iff so is U'. So we may assume further on that U is a bunch of buds.

Now suppose that $(Q, y)^U$ is critical or that $|U| = |\bar{U}|$. Then $Q_{bc} \leqq 0$ for any two buds $\{b\}, \{c\} \in U$ which lie in a common branch of (Q, y). We infer that we can stretch the various maximal branches of (Q, y) using only U-admissible inflations. In this way, U and $\bar{U}$ coincide, and $(Q, y)^U$ and $(\bar{Q}, \bar{y})^{\bar{U}}$ are $\mathbf{Z}$-equivalent. qed.

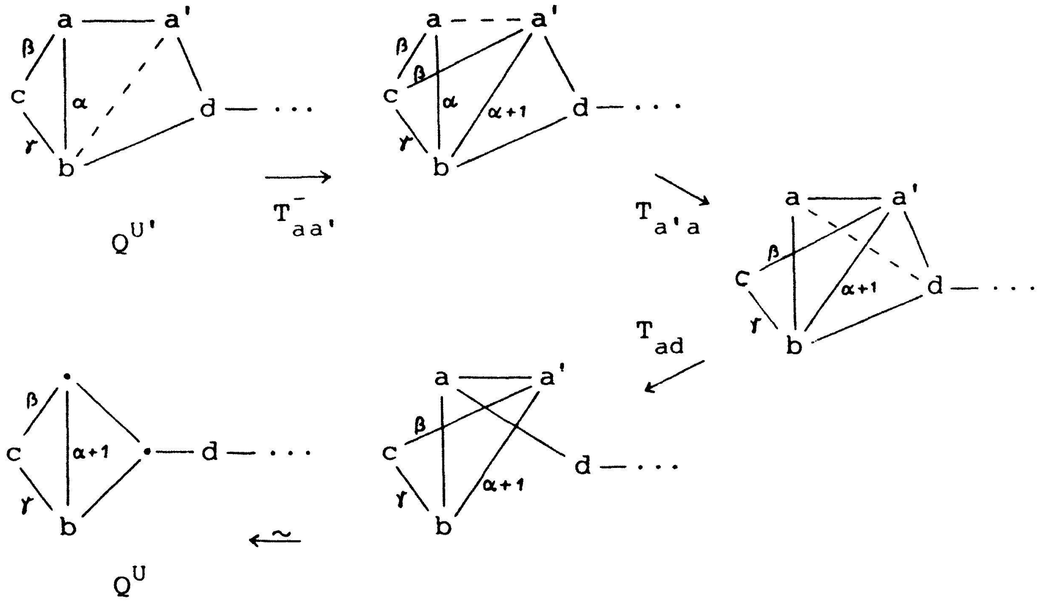


Figure 10

2.7 The Propositions 2.6 and 2.5 reduce the classification of all critical bunches to the description of the critical bunches of buds U of the extended Dynkin diagrams (Δ, y_Δ). These can be easily determined directly and are listed in Figure 11 below. Indeed, our knowledge of the critical diagrams of type $\tilde{A}_n$ and $\tilde{D}_n$ reduces the examination to the case where Δ has $\leqq 8$ vertices. In these few remaining cases, we expect the reader to determine himself when a produced diagram $(\Delta, y_\Delta)^U$ is critical (for instance, use Remark 2.5, replace some buds of U by twigs of cardinality >1 in the way suggested by 2.6, or consider iterated inflations of $(\Delta, y_\Delta)^U$). In the list, each pictured extended Dynkin graph Δ has $n+1$ vertices, the critical bunch of buds U is given by the encircled vertices, and the symbol $\boxed{\tilde{A}_{n+1}}$, $\boxed{\tilde{D}_{n+1}}$, . . . describes the type of $(\Delta, y_\Delta)^U$.

Using Proposition 2.5, we can describe the critical bunches of buds of an arbitrary critical diagram (Q, y) of type $\tilde{D}_n$. We remind that Q has one of the shapes shown in Figure 12, where mixed edges are to be removed if $r=1$ or $s=1$. The critical bunches of buds U of (Q, y) are those satisfying the following conditions: 1) $U \subset \{\{a_i\}, \{a_{i+1}\}, \{b_j\}, \{b_{j+1}\}\}$ for some indices $0 \leqq i < r$, $0 \leqq j < s$, 2) $|U| + n \leqq 9$ if $\{\{a_i\}, \{a_{i+1}\}\} \neq U \neq \{\{b_j\}, \{b_{j+1}\}\}$.

2.8 Now we are ready to construct the stretched critical diagrams of type $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

First note that, if $(Q, y)^U$ is to be stretched, (Q, y) must be *stretched modulo* U, i.e. each branch of (Q, y) which is not a twig must contain a twig of U. In

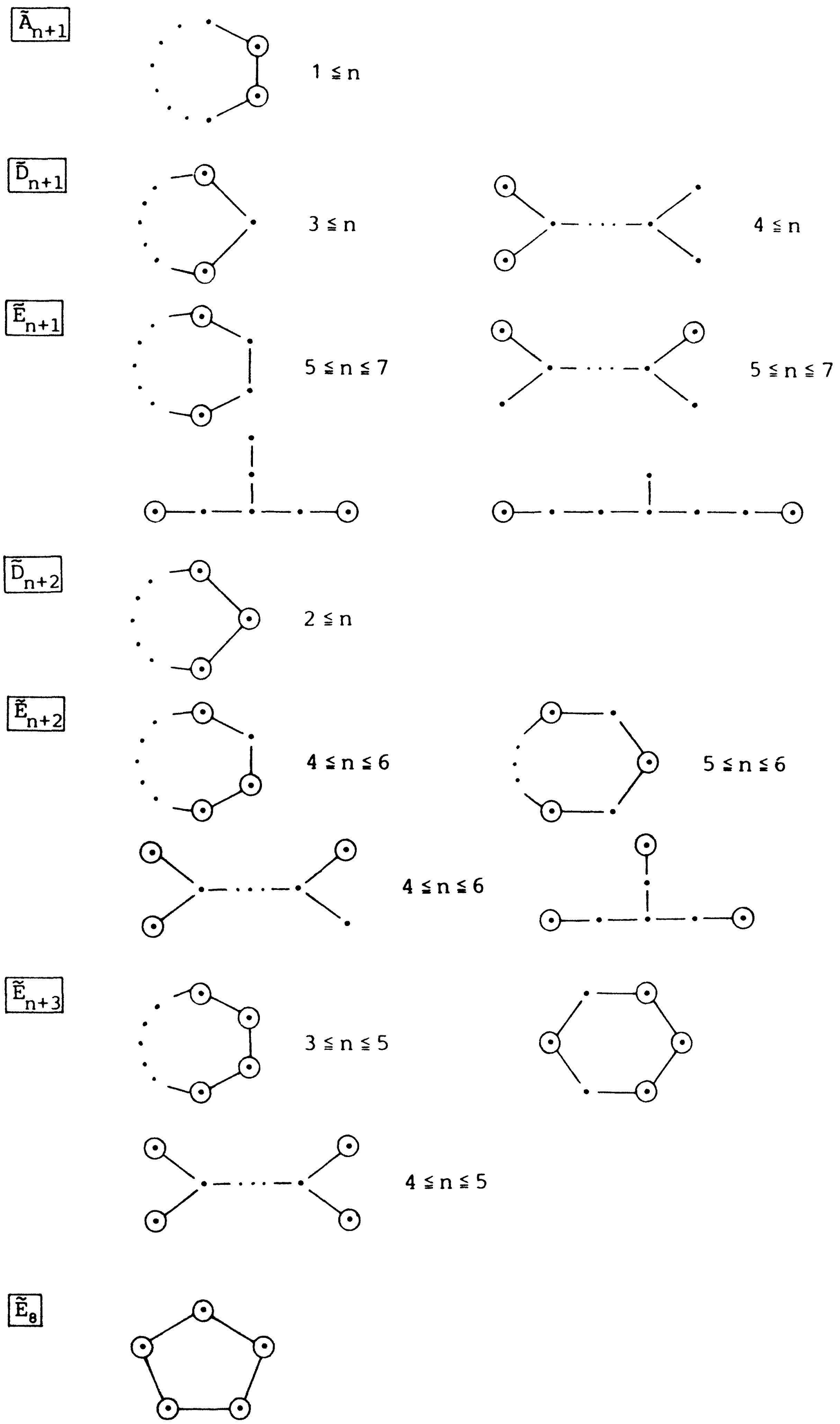

$\tilde{A}_{n+1}$
$1 \leqq n$
$\tilde{D}_{n+1}$
$3 \leqq n$
$4 \leqq n$
$\tilde{E}_{n+1}$
$5 \leqq n \leqq 7$
$5 \leqq n \leqq 7$
$\tilde{D}_{n+2}$
$2 \leqq n$
$\tilde{E}_{n+2}$
$4 \leqq n \leqq 6$
$5 \leqq n \leqq 6$
$4 \leqq n \leqq 6$
$\tilde{E}_{n+3}$
$3 \leqq n \leqq 5$
$4 \leqq n \leqq 5$
$\tilde{E}_8$

Figure 11

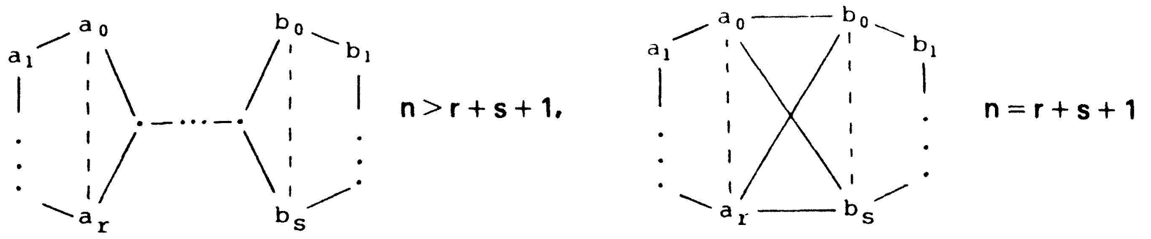


Figure 12

particular, the maximal branches of (Q, y) of cardinality 2, 3 or 4 must have one of the shapes shown in Figure 13; up to symmetry, one of the twigs with underlined joint-vertex must be contained in U.

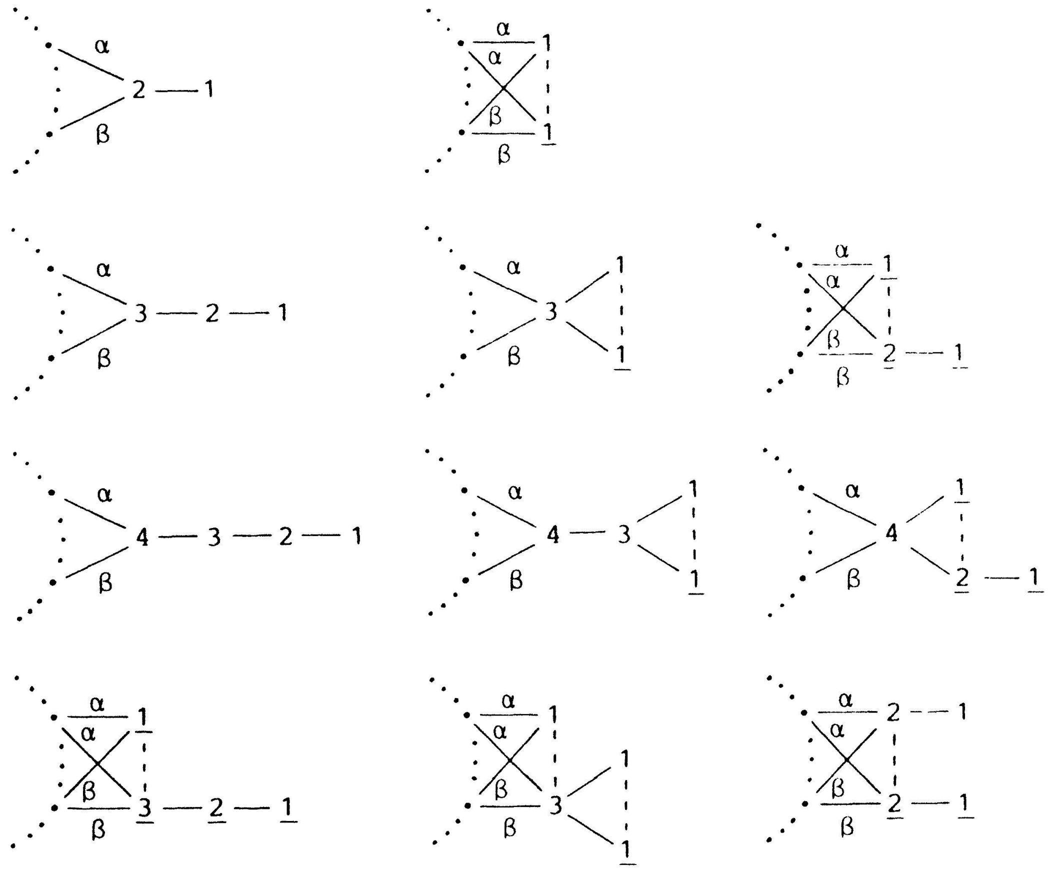


Figure 13

2.9 In the following theorem we present the diagrams in a truncated form; we encircle the joint-vertices of some twigs of cardinality >1 and cut off the rest of these twigs. Further, to each diagram (Q, y) we attach a graph G_Q, which has the buds of (Q, y) as vertices (note that each twig contains exactly one bud) and an edge A—B iff $\{A, B\}$ is a critical bunch of (Q, y). Then each critical bunch of buds of (Q, y) is the vertex set of some complete subgraph of G_Q by Remark 2.5.

THEOREM. 1) *Up to isomorphism, the stretched critical diagrams of type* $\tilde{E}_6$ *are those of Figure* 14.

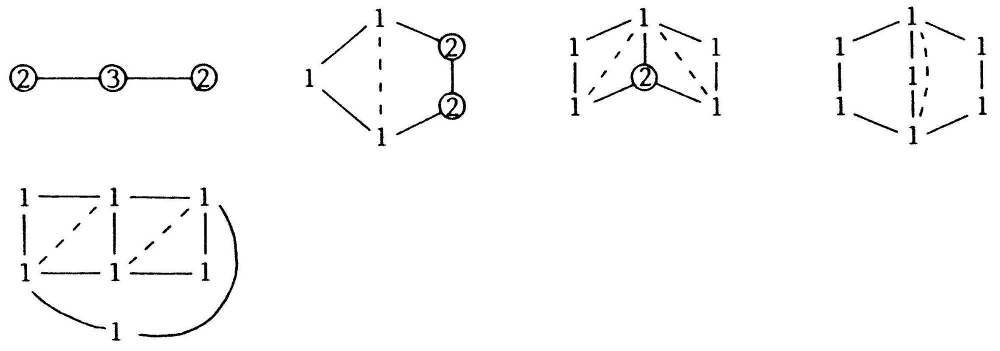


Figure 14

The critical bunches of buds are the vertex sets of triangles $\triangle$ *or of edges* — *in the corresponding graph in Figure* 15.

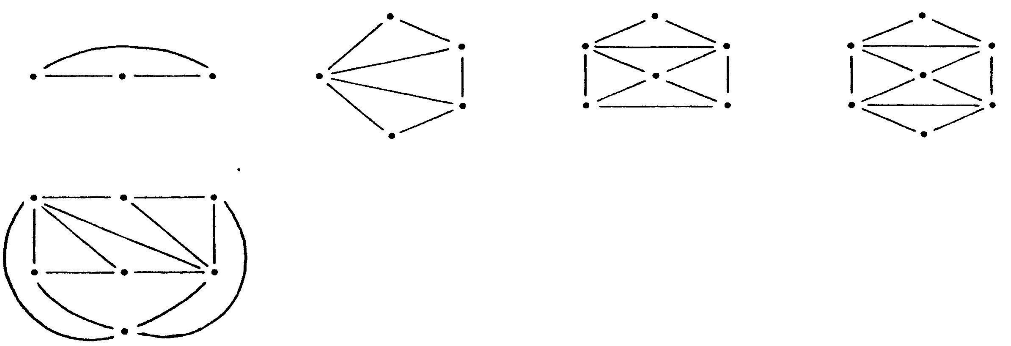


Figure 15

2) *Up to isomorphism, the stretched critical diagrams of type* $\tilde{E}_7$ *are those of Figure* 16. *The critical bunches of buds have cardinality* 2. *The corresponding graphs consist of the full points and the edges of Figure* 17.

3) *Up to isomorphism, the stretched critical diagrams of type* $\tilde{E}_8$ *are those of Figure* 18 *and the diagrams of the shape* $(Q, y)^U$ *where* (Q, y) *is critical of type* $\tilde{E}_7$, U *is a critical bunch and* (Q, y) *is stretched modulo* U.

Proof. The construction of the stretched critical diagrams should be clear: In case $\tilde{E}_6$, we must determine all diagrams $(Q, y)^U$ where (Q, y) is a critical diagram with $\leqq 6$ vertices and U is a critical bunch of (Q, y) of appropriate cardinality and shape (see 2.6, 2.7 and 2.8). In this way we have to examine 14 pairs $((Q, y), U)$; 5 of them yield nonstretched diagrams, and further 4 yield stretched diagrams which are isomorphic to others. Similarly, in the case $\tilde{E}_7$ ($\tilde{E}_8$),

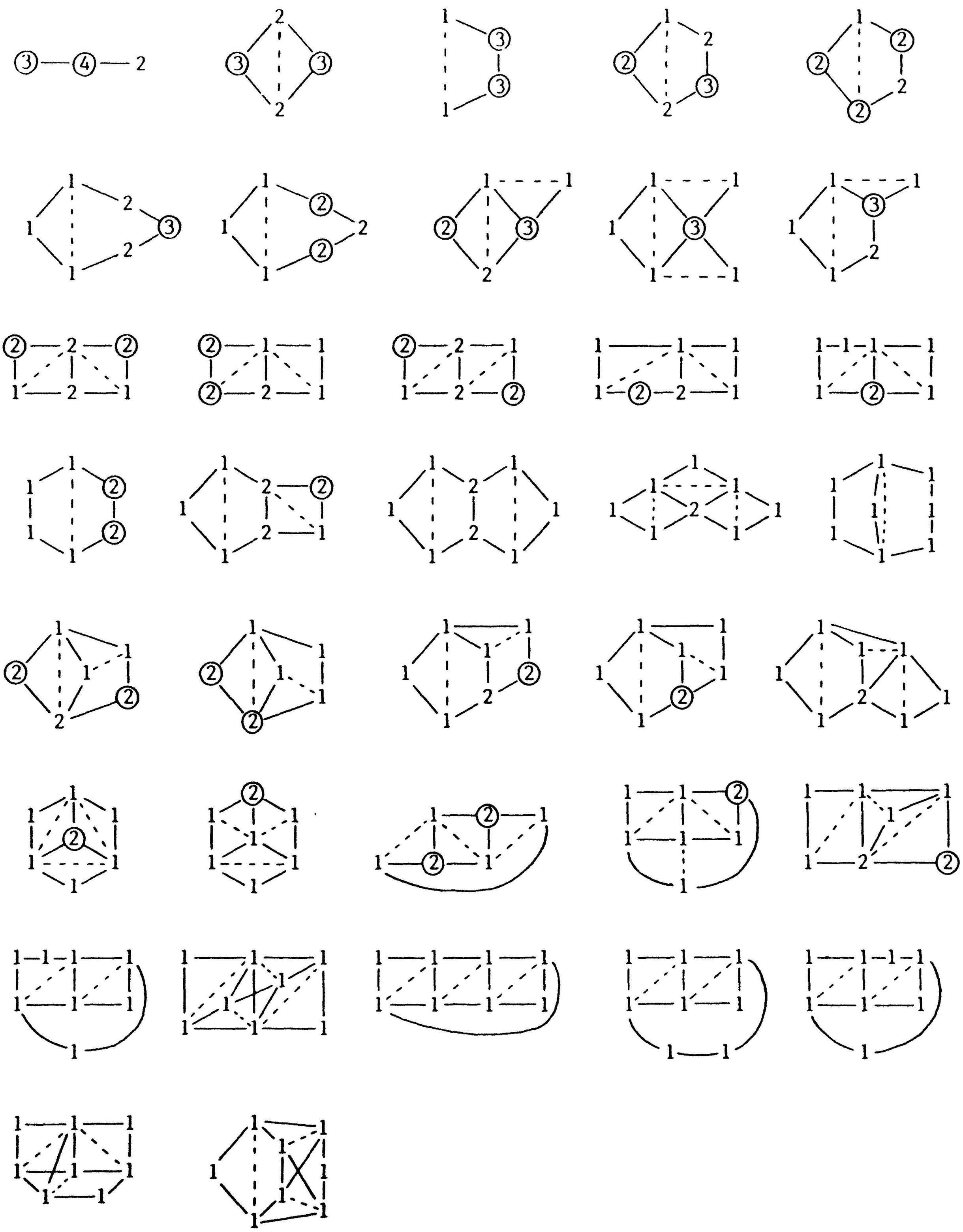


Figure 16

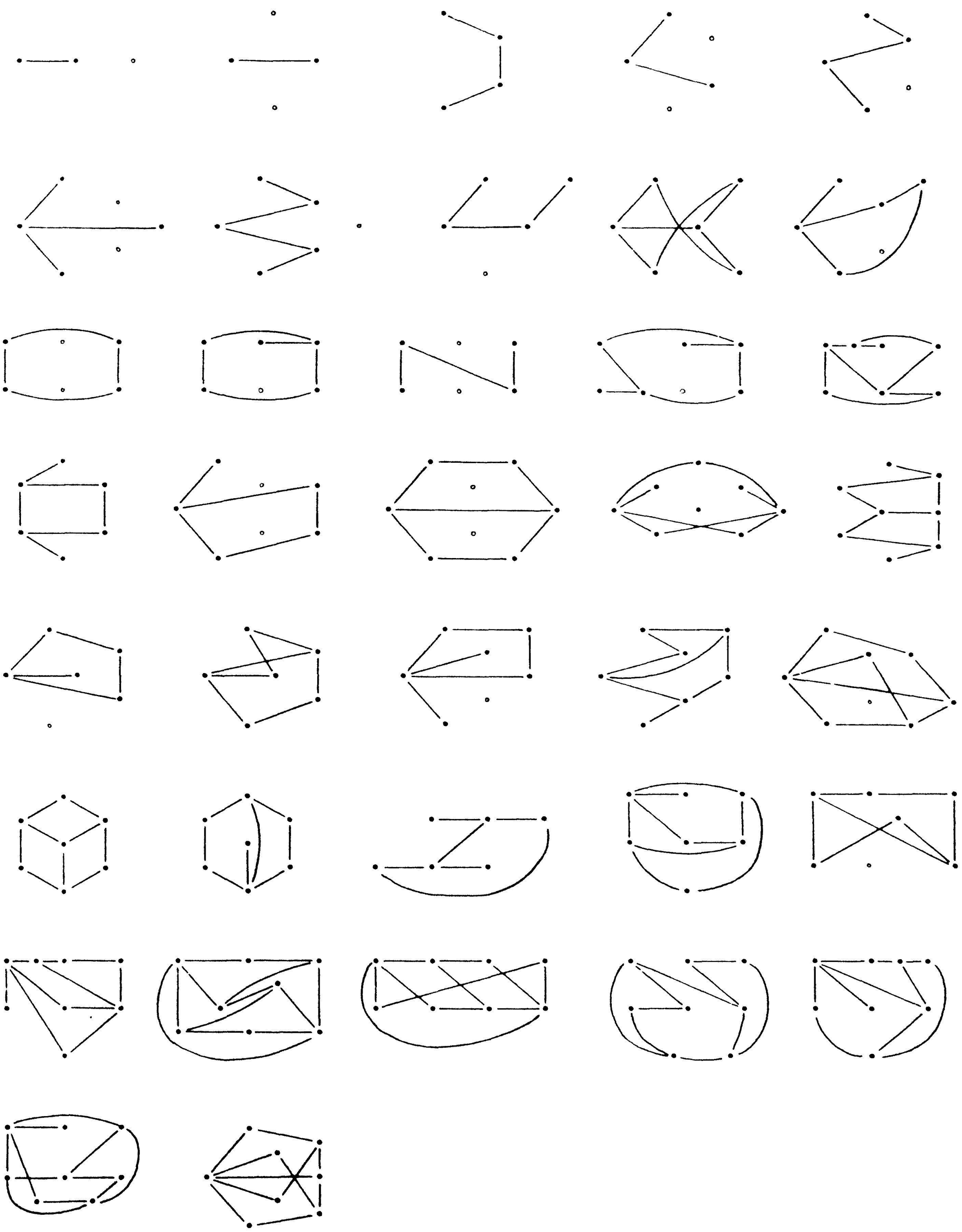


Figure 17

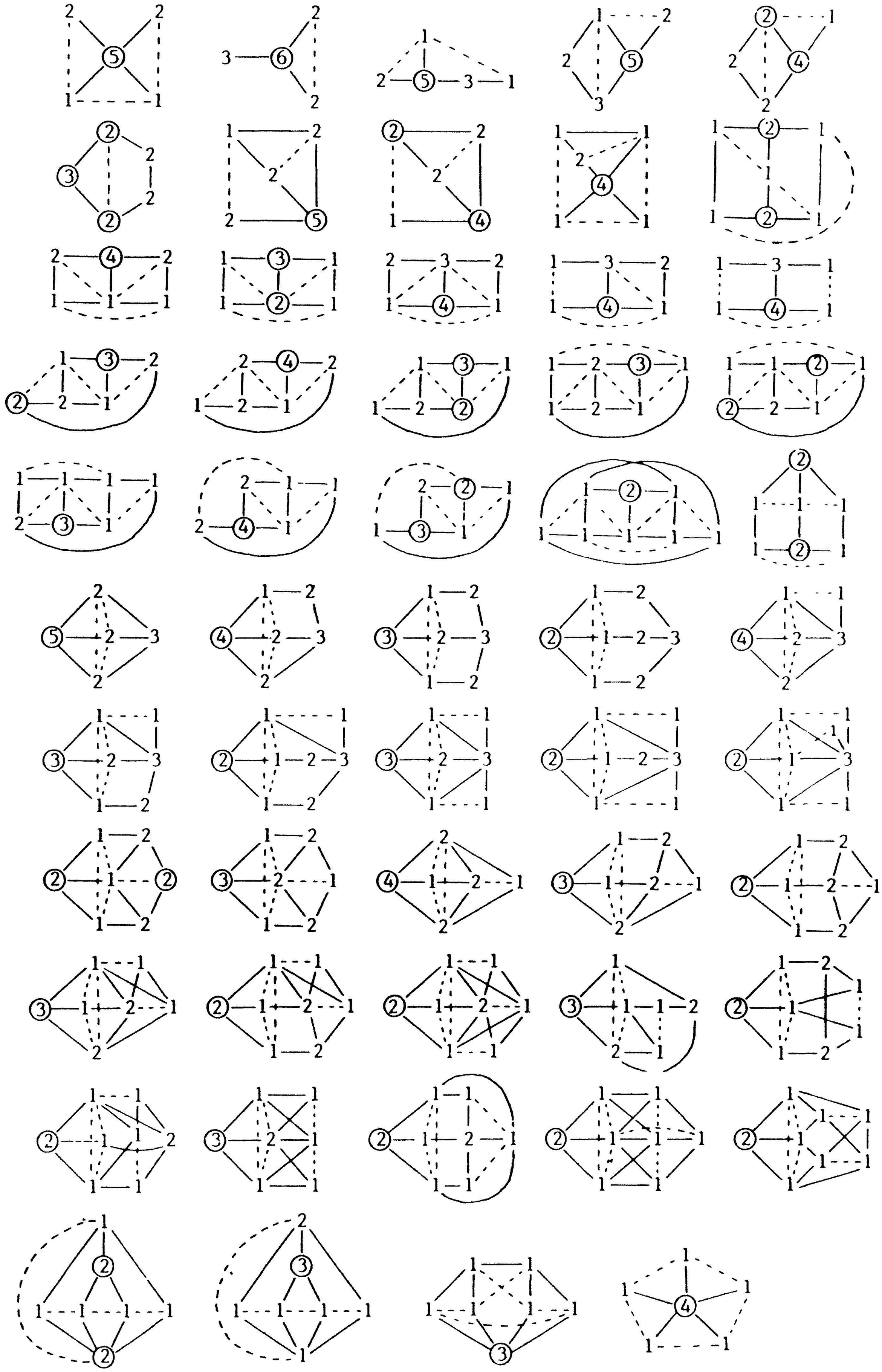


Figure 18

we have to examine 86 (672) pairs, where $11 + 38$ ($13 + 194$) are redundant. Thus, in case $\tilde{E}_8$ we obtain 465 stretched critical diagrams; but we have listed here only those which are not isomorphic to $(Q, y)^U$ for some (Q, y) of type $\tilde{E}_7$.

Finally, we show how to compute the critical bunches of buds of the stretched critical diagrams (Q, y) of a fixed type ($\tilde{E}_6$ or $\tilde{E}_7$). We proceed by decreasing induction on y. The induction basis, i.e. the case where Q is extended Dynkin, has been treated in 2.7. If (Q, y) is not extended Dynkin, we choose vertices a and b of Q such that $Q_{ab} = 1$. If $(y_a, y_b) \neq (1, 1)$, say $y_a \neq 1$, the inflation T_{ab} of (Q, y) is certainly admissible for each bunch of buds of (Q, y) (2.5); so the critical bunches of buds of (Q, y) coincide with those of $(Q, y)T_{ab}$, which are known by induction because the 'radical vector' of the stretched form of $(Q, y)T_{ab}$ is $\geqq T_{ab}^{-}y > y$. In the remaining case $y_a = y_b = 1$, we must consider $(Q, y)T_{ab}$ as well as $(Q, y)T_{ba}$. Since no critical bunch of (Q, y) can contain both $\{a\}$ and $\{b\}$, a bunch of buds is critical in (Q, y) iff it is so in $(Q, y)T_{ab}$ or in $(Q, y)T_{ba}$. qed.

A complete list of the stretched critical diagrams of type $\tilde{E}_8$ can be obtained from the author upon request.

Below, we give the complete numbers of critical diagrams of the respective extended Dynkin types:

$\tilde{A}_n$	1
$\tilde{D}_n$	$\begin{cases} \frac{1}{4}(n-1)^2 & \text{if } n \geqq 5 \text{ odd} \\ \frac{1}{4}(n-2)n & \text{if } n \geqq 4 \text{ even} \end{cases}$
$\tilde{E}_6$	17
$\tilde{E}_7$	142
$\tilde{E}_8$	1717

3. Applications in representation theory

In this section, we show how our list of critical forms involves the well-known lists of Kleiner–Nazarova–Roiter [K, NR] and of Bongartz–Happel–Vossieck [Bo1], [HV]. These play a decisive rôle in the characterization of representation-finite partially ordered sets and finite-dimensional algebras.

3.1 Let $(S, \leqq)$ be a finite partially ordered set. According to Drozd [D], S is representation-finite (in the sense of Nazarova and Roiter) iff the unit form $q_S(x) = x_0^2 + \sum_{a \in S} x_a^2 - \sum_{a \in S} x_0 x_a + \sum_{a<b} x_a x_b$ is weakly positive (x_0 denotes an

additional variable). Now, q_S is weakly positive iff it has no critical restriction (see the introduction). This induces us to pick out all those critical forms which can occur as restrictions of unit forms of the shape q_S, i.e. which themselves have such a shape. This results in confirming the statement of Kleiner, Nazarova and Roiter saying that S is representation-finite iff it does not contain a subset for which the induced order structure is one of the following (compare our approach with Ringel's [Ri]):

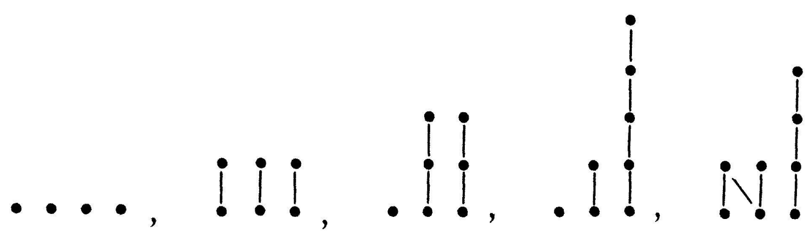


Figure 19

3.2 Now consider a finite-dimensional algebra A over an algebraically closed field k and denote by $\operatorname{mod} A$ the category of finite-dimensional (right) A-modules. Covering theory [BG], [G3], [BGRS] reduces the characterization of representation-finite algebras to the case where there is a bound on the length of sequences of nonzero noninvertible morphisms

$$M_r \xrightarrow{f_r} M_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = P$$

where all $M_i \in \operatorname{mod} A$ are indecomposable and P is projective; in this case we call A *prehereditary*. Prehereditary algebras enjoy great popularity in representation theory, because these algebras and a great deal of their indecomposable modules can easily been constructed from combinatorial data (see [G2]).

Denote by $S_1, \ldots, S_n$ an exhaustive sequence of nonisomorphic simple A-modules and consider the quadratic form $q_A : \mathbf{Z}^n \to \mathbf{Z}$ defined by

$$q_A(x) = \sum_{i,j=1}^{n} \left(\sum_{l=0}^{2} (-1)^l \dim_k \operatorname{Ext}_A^l (S_i, S_j) \right) x_i x_j .$$

LEMMA. *If A is prehereditary, then q_A is a unit form, and for all i, j we have* $\operatorname{Ext}_A^1 (S_i, S_j) = \operatorname{Ext}_A^1 (S_j, S_i) = 0$ *or* $\operatorname{Ext}_A^2 (S_i, S_j) = \operatorname{Ext}_A^2 (S_j, S_i) = 0$.

Proof. Here and in the following, we denote by $<$ the smallest transitive relation on the indecomposable A-modules which satisfies $X < Y$ if there is some noninvertible nonzero morphism $X \to Y$. So, if A is prehereditary, there is no cycle $P_i < X < P_i$ involving the projective cover P_i of some S_i.

Note that $\mathrm{Ext}^l_A(S_i, S_j) \neq 0$ for some $l > 0$ implies $P_j < P_i$. From this it follows, that q_A is a unit form ($\mathrm{End}_A(S_i) \simeq k$!) and that $\mathrm{Ext}^1_A(S_i, S_j) \neq 0$ implies $\mathrm{Ext}^2_A(S_j, S_i) = 0$. It remains to show that $\mathrm{Ext}^1_A(S_i, S_j) \neq 0$ also implies $\mathrm{Ext}^2_A(S_i, S_j) = 0$. Indeed, from $\mathrm{Ext}^1_A(S_i, S_j) \neq 0$ and the Auslander–Reiten formula $\dim_k \mathrm{Ext}^1_A(X, Y) \leqq \dim_k \mathrm{Hom}_A(\mathrm{Tr}\, DY, X)$ [Au], we obtain $\mathrm{Hom}_A(\mathrm{Tr}\, DS_j, S_i) \neq 0$, hence $P_i < \mathrm{Tr}\, DS_j$. We infer that $0 = \dim_k \mathrm{Hom}_A(\mathrm{Tr}\, DS_j, P_i) \geqq \dim_k \mathrm{Hom}_A(\mathrm{Tr}\, DS_j, \mathrm{rad}\, P_i) \geqq \dim_k \mathrm{Ext}^1_A(\mathrm{rad}\, P_i, S_j) = \dim_k \mathrm{Ext}^2_A(S_i, S_j)$, where $\mathrm{rad}\, P_i$ denotes the radical of P_i. qed.

3.3 According to Bongartz [Bo1], a prehereditary algebra A is representation-finite iff q_A is weakly positive. Our purpose is to sift out a minimal set of critical forms in which q_A has a restriction (see the introduction) for each representation-infinite prehereditary algebra A. For this, we consider the weighted graph Q_{q_A} (see 1.1) and the quiver $\vec{Q}_A$ of A, given by the vertices $1, \dots, n$ and $\dim_k \mathrm{Ext}^1_A(S_i, S_j)$ arrows from i to j.

PROPOSITION. *If A is prehereditary, then Q_{q_A} satisfies one of the following conditions*:

a) *Q_{q_A} contains an edge of weight $\leqq -2$ or a full weighted subgraph isomorphic to $\tilde{A}_m$ for some $m \geqq 1$.*

b) *For each full weighted subgraph of Q_{q_A} of the form*

$$\begin{tikzcd}[arrows=dash] & a_1 \arrow[r] & \cdots \arrow[r] & a_{s-1} \arrow[dr] & \\ a_0 \arrow[ur] \arrow[rrrr, dashed] & & & & a_s \end{tikzcd}, \qquad s \geqq 2,$$

the corresponding full subquiver of $\vec{Q}_A$ has the shape $a_0 \to a_1 \to \cdots \to a_s$ or $a_0 \leftarrow a_1 \leftarrow \cdots \leftarrow a_s$.

Proof. Assume that b) fails. Then $\vec{Q}_A$ contains a full subquiver $\vec{W}$ (Figure 20) which satisfies $0 \leqq s_1 < s_2 < \cdots < s_{t-1} \leqq s_t =: s$, $1 \leqq t$ odd, and the following two

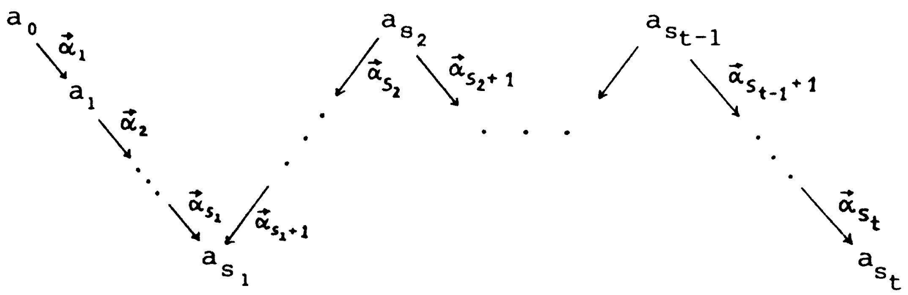


Figure 20

conditions:

1) $(Q_{q_A})_{a_i a_j} \leqq 0$ whenever $a_i \neq a_j$ are connected by a path in $\vec{W}$,
2) there exists a path

$$\omega : a_0 = b_0 \xrightarrow{\vec{\beta}_1} b_1 \xrightarrow{\vec{\beta}_2} \cdots \xrightarrow{\vec{\beta}_r} b_r = a_s, \quad r \geqq 1,$$

of $\vec{Q}_A$ which is not contained in $\vec{W}$. (Note that there exists a path $i \to \cdots \to j$ whenever $\operatorname{Ext}_A^2(S_i, S_j) \neq 0$ (3.2).) Among all subquivers of $\vec{Q}_A$ with these properties we choose one with a minimal number of vertices. We then have $0 < s_1$, $s_{t-1} < s$, and the convex hull $\vec{Q}'$ of $\vec{W}$ in $\vec{Q}_A$ (= full subquiver of $\vec{Q}_A$ formed by the vertices x lying on some path $a_i \to \cdots \to x \to \cdots \to a_j$) is the union of $\vec{W}$ and of a quiver $\vec{Q}''$ with unique source a_0, unique sink a_s and such that $\vec{W} \cap \vec{Q}'' = \{a_0, a_s\}$.

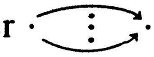
Let $A' = eAe$ where e is the sum of primitive idempotents of A corresponding to the vertices of $\vec{Q}'$. Then $q_{A'}$ can be identified with the restriction of q_A to $\mathbf{Z}^{Q'_0}$ [Bo1]. Further, since A is prehereditary, so is A' (the functor $- \otimes_{A'} eA : \operatorname{mod} A' \to \operatorname{mod} A$ preserves projectivity and is fully faithful). So we can assume further on that $A = A'$, i.e. $\vec{Q}_A = \vec{Q}'$.

Now assume that also a) fails. Since the full weighted subgraph of Q_{q_A} formed by the vertices of $\vec{W}$ and ω is not isomorphic to some $\tilde{A}_m$ there is an edge of weight $\geqq 1$ connecting vertices of ω, i.e. $\operatorname{Ext}_A^2(S_{b_i}, S_{b_j}) \neq 0$ for some $0 \leqq i < j \leqq r$.

If $j < r$, i.e. if there is a path $b_j \to \cdots \to b_{r-1}$, we consider the A-module M given by the following representation of $\vec{Q}_A : M_{a_0} = \cdots = M_{a_s} = M_{b_{r-1}} = k$, $M_{\vec{\alpha}_1} = \cdots = M_{\vec{\alpha}_s} = M_{\vec{\beta}_r} = id_k$ and $M_a = M_{\vec{\alpha}} = 0$ for all other vertices a and arrows $\vec{\alpha}$ of $\vec{Q}_A$. Using the notation of 3.2, we then have $P_{b_i} \preccurlyeq P_{a_0} < M < S_{b_{r-1}} \preccurlyeq S_{b_j}$ (where $\preccurlyeq$ means $<$ or $\simeq$). Further $S_{b_j} < P_{b_i}$ since $0 \neq \operatorname{Ext}_A^2(S_{b_i}, S_{b_j}) \simeq \operatorname{Ext}_A^1(\operatorname{rad} P_{b_i}, S_{b_j})$. So we have a cycle $P_{b_i} < M < P_{b_i}$, which is impossible.

In the case $j = r$, let M' be the A-module given by $M'_{a_0} = \cdots = M'_{a_s} = k$, $M'_{\vec{\alpha}_1} = \cdots = M'_{\vec{\alpha}_s} = id_k$ and $M'_a = M'_{\vec{\alpha}} = 0$ otherwise. Let M'' be the factor-module of M' given by $M''_{a_{s_{t-1}}} = \cdots = M''_{a_{s_t}} = k$, $M''_{\vec{\alpha}_{s_{t-1}+1}} = \cdots = M''_{\vec{\alpha}_{s_t}} = id_k$ and $M''_a = M''_{\vec{\alpha}} = 0$ otherwise. Then clearly $P_{b_i} \preccurlyeq P_{a_0} < M' \preccurlyeq M''$. Further, we have $S_{b_j}\ (= S_{a_s}) \hookrightarrow M''$ and $\operatorname{Ext}_A^1(S_{b_i}, M''/S_{b_j}) = 0$ (since $t > 1$ if $i = 0$ and $\operatorname{Ext}_A^1(S_{b_i}, S_{a_m}) = 0$ for all $s_{t-1} \leqq m \leqq s_t$). We infer that the natural map $\operatorname{Ext}_A^2(S_{b_i}, S_{b_j}) \to \operatorname{Ext}_A^2(S_{b_i}, M'')$ is injective and obtain a cycle $P_{b_i} < M' \preccurlyeq M'' < P_{b_i}$. Contradiction. qed.

3.4 Bongartz and Happel–Vossieck have produced well-known lists of prehereditary algebras, whose quadratic form is critical [Bo2], [HV]. The algebras of the list of Happel–Vossieck [HV], including the algebras of 'frame'

$\tilde{A}_m$ and the path algebra of the quiver  with at least two arrows, will here be called BHV-*algebras*.

THEOREM. *A prehereditary algebra A is representation-finite iff q_A has no restriction of the form $q_{A'}$ where A' is BHV.*

Proof. By the result of Bongartz [Bo1] mentioned in 3.3, the necessity is clear. To prove sufficiency it is enough to show that if q_A has no coefficient $\leqq -2$ and no critical restriction of type $\tilde{A}_m$ then each critical restriction q of q_A has the form $q_{A'}$ where A' is BHV. Indeed, the Proposition 3.3 immediately implies that Q_{q_A} – and therefore Q_q – does not contain a full weighted subgraph of one of the shapes shown in Figure 21 ($\cdot\overset{=}{——}\cdot$ and $\cdot\overset{\cdot}{——}\cdot$ denote subgraphs of the form $a_0—a_1—\cdots—a_m$ where $m \geqq 1$ and $m \geqq 0$ respectively).

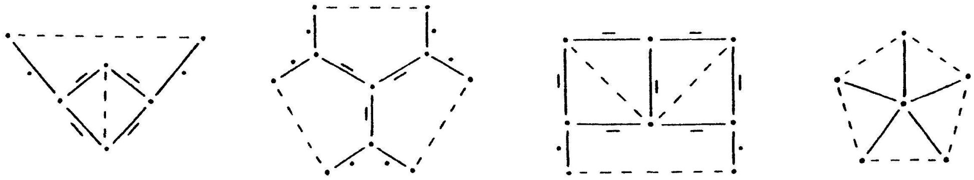
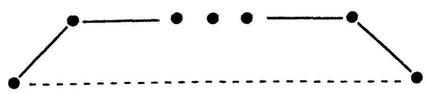


Figure 21

An inspection of the list of critical diagams (2.1 and 2.9) shows that the diagrams (Q, y) where Q contains no such subgraph are isomorphic to $(Q_{q_{A'}}, y')$ for some BHV-algebra A'. For the inspection, the following remarks might be useful:

1) If Q contains no full weighted subgraph of Figure 21 then neither does the weighted graph $\bar{Q}$ of the stretched form $(\bar{Q}, \bar{y})$. (For instance, in Figure 14, Figure 16 and Figure 18, only the first 4, 20 and 6 stretched forms respectively are possible.) Further, if A is a branch of (Q, y) having a joint-vertex c which lies in a full weighted subgraph of the form  , then A is a sub-branch of the branch shown in Figure 22 (compare [HV]).

2) If (Q, y) is critical of type $\tilde{E}_7$ and U is a critical bunch such that Q^U contains no full weighted subgraph as shown in Figure 21, then Q itself does not contain such a subgraph, and the two buds of $(\bar{Q}, \bar{y})$ lying in $\bar{U}$ (see 2.6) are contained in two twigs whose joint-vertices are connected by an edge of weight -1 in $\bar{Q}$. qed.

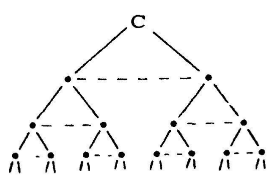


Figure 22

In case A is given by a graded tree [BG], the above theorem is already known by the criterion of Bongartz [Bo2, Bo3]: A is representation-finite iff it contains no BHV-algebra as a *convex* subalgebra eAe (i.e. where $e \in A$ is an idempotent corresponding to the vertices of a convex subquiver of $\vec{Q}_A$). It can be shown, that our theorem also implies this stronger version.

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Mathematisches Institut II
Freie Universität Berlin
Arnimallee 3
D-1000 Berlin 33

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