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On groups of smooth maps into a simple compact Lie group

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Let X be a closed smooth manifold and let G be a connected compact real Lie group with simple Lie algebra $\mathbf{g}$. Let M_0G be the connected component of the group $\mathscr{C}^\infty(X, G)$ of *smooth maps* from X to G, with respect to the $\mathscr{C}^\infty$-compact-open topology. Then M_0G is a nice example of a well behaved infinite dimensional Lie group; see [*M*] and [*PS*]. However, I am interested here in some properties of M_0G as an *abstract group*.

The main result below describes automorphisms of M_0G. Let $M\,\mathrm{Aut}\,(G)$ be the group of smooth maps from X to the group $\mathrm{Aut}\,(G)$ of automorphisms of G, and let $\mathscr{D}(X)$ be the group of smooth diffeomorphisms of X. Consider the natural action of $\mathscr{D}(X)$ on $M\,\mathrm{Aut}\,(G)$ defined by

$$\varphi(\beta) = \beta\varphi^{-1} \quad \text{for} \quad \varphi \in \mathscr{D}(X), \qquad \beta \in M\,\mathrm{Aut}\,(G)$$

and the associated semi-direct product $M\,\mathrm{Aut}\,(G) \rtimes \mathscr{D}(X)$ with multiplication

$$(\alpha, \varphi)(\beta, \psi) = (\alpha\varphi(\beta), \varphi\psi).$$

This acts on M_0G by automorphisms

$$(\alpha, \varphi)(\gamma)(x) = \alpha(x)(\gamma(\varphi^{-1}(x))) \quad \text{for} \quad \begin{cases} \alpha \in M\,\mathrm{Aut}\,(G) & \varphi \in \mathscr{D}(X) \\ \gamma \in M_0G & x \in X \end{cases}.$$

Any automorphism of M_0G happens to be of this form; thus, in particular, it is continuous in any decent sense.

THEOREM I. *The group of all automorphisms of M_0G coincides with $M\,\mathrm{Aut}\,(G) \rtimes \mathscr{D}(X)$.*

The proof of Theorem I has a crucial ingredient, due to E. Cartan (see [C] and [vdW]): any homomorphism between semi-simple compact Lie groups is smooth.

The proof depends also on the classification of *maximal normal subgroups of* M_0G. For $x \in X$, let $\varepsilon_x : M_0G \to G$ be the evaluation at x and let $\check{\varepsilon}_x : M_0G \to G/C(G)$ be the composition of ε_x with the canonical projection of G onto the quotient by its centre. As $G/C(G)$ is simple as an abstract group, the kernel of $\check{\varepsilon}_x$ is clearly a maximal normal subgroup of M_0G. Conversely:

PROPOSITION II. *Any proper normal subgroup of* M_0G *is contained in* Ker $(\check{\varepsilon}_x)$ *for some* $x \in X$.

One may think of Proposition II as a global result, which follows from a related local result. To state the latter, denote by n the dimension of X. Given a smooth manifold Y, denote by $\mathscr{C}^\infty_{n,Y}$ the set of germs at the origin of smooth maps from $\mathbf{R}^n$ to Y, and let $\varepsilon : \mathscr{C}^\infty_{n,Y} \to Y$ be the evaluation map. For $Y = G$ as above, let again $\check{\varepsilon}$ be the composition of ε with the projection from G onto $G/C(G)$. Then $\mathscr{C}^\infty_{n,G}$ is a *local group* in the following sense:

PROPOSITION III. *Any proper normal subgroup of* $\mathscr{C}^\infty_{n,G}$ *is contained in* Ker $(\check{\varepsilon})$.

Everything here works equally well for maps and germs which are of class $\mathscr{C}^k$ for some $k \geqslant 0$. The proof of Proposition III works also in the real analytic setting. Though the real analytic analogues of Theorem I and Proposition II look plausible, they are not covered by our proofs which use partitions of unity. Also, we believe that Proposition III may be proved for a simple Lie group which is not necessarily compact, but the class of groups for which Theorem I holds is not so clear.

Theorem I, with extra smoothness assumptions about automorphisms of M_0G, is due to Pressley and Segal. Indeed, its statement appears in Chapter III of a preliminary version of [*PS*]. But the proof there is not quite explicit, and I thought it worthwhile to write it up as follows. I am grateful to both Andrew Pressley and Grame Segal for their patience in discussing parts of their forthcoming book, as well as to Armand Borel who has encouraged me to write up the full proofs for a general simple compact Lie group (and not just for $SU(2)$).

1. Proof of Proposition III when $G = SU(2)$

Any $g \in G = SU(2)$ has two eigenvalues z_g and $\overline{z_g}$ of modulus 1. Assume that g lies in the set $G_{\text{reg}} = G - \{\pm 1\}$ or regular elements. Agree that notations are such that $\operatorname{Im}(z_g) > 0$, define $t_g \in]0, \pi[$ by $z_g = \exp(it_g)$, let u_g be the orthogonal projection of $\mathbf{C}^2$ onto the eigenspace of g corresponding to z_g, and let $u'_g = 1 - u_g$ be the projection onto the other eigenspace. One has

$$g = z_g u_g + \overline{z_g} u'_g \quad \text{for all} \quad g \in G_{\text{reg}}.$$

We identify the projective complex line $P^1_{\mathbf{C}}$ with the space of orthogonal projections of $\mathbf{C}^2$ onto a line

$$P^1_{\mathbf{C}} = \{u \in M_2(\mathbf{C}) \mid u^* = u = u^2 \text{ and trace}(u) = 1\}$$

where u^* is the adjoint matrix of u. Set $A = \,]0, \pi[$.

LEMMA 1. *The map*

$$\begin{cases} G_{\text{reg}} \to P^1_{\mathbf{C}} \times A \\ g \mapsto (u_g, t_g) \end{cases}$$

is an analytic diffeomorphism.

Proof. Clear. Analyticity follows for example from holomorphic functional calculus (see [*TS*], Chapter I, §4, no 11, Proposition 16).

Observe that this diffeomorphism is G-equivariant, where G acts on G_{reg} by conjugation, on $P^1_{\mathbf{C}}$ as usual, and on A in the trivial way.

Any $u \in P^1_{\mathbf{C}}$ is of the form $u = \begin{pmatrix} a & b \\ \bar{b} & 1-a \end{pmatrix}$ with $a \in \mathbf{R}$, $a \geqslant 0$, $b \in \mathbf{C}$, $a^2 + |b|^2 = a$.

LEMMA 2. *Let $\mathcal{U}$ be the open subset of $P^1_{\mathbf{C}}$ consisting of those $\begin{pmatrix} a & b \\ \bar{b} & 1-a \end{pmatrix}$ for which $a > 0$. There exists a smooth map $\begin{cases} \mathcal{U} \to SU(2) \\ u \mapsto g_u \end{cases}$ such that $g_u \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g_u^{-1} = u$ for all $u \in \mathcal{U}$.*

Proof. Given $u = \begin{pmatrix} a & b \\ \bar{b} & 1-a \end{pmatrix} \in \mathcal{U}$, set $\rho = a^{1/2}$ and $\sigma = -b/\rho$. Then $g_u = \begin{pmatrix} \rho & \sigma \\ -\bar{\sigma} & \rho \end{pmatrix}$ works.

Let $\mathcal{O}$ be a neighbourhood of the origin in $\mathbf{R}^n$ and let $\gamma : \mathcal{O} \to G$ be a smooth map with values in G_{reg}. Then $t_{\gamma(x)}$ depends smoothly on x by Lemma 1, and defines at the origin a germ of smooth map from $\mathbf{R}^n$ to A; this germ depends only on the germ $\underline{\gamma}$ of γ, and will be denoted by $\underline{t}_{\underline{\gamma}}$ below. Similarly for $\underline{u}_{\underline{\gamma}}$. We write also $\underline{t}$ and $\underline{u}$ if the reference to $\underline{\gamma}$ is clear.

LEMMA 3. *Consider two germs $\underline{\gamma}$, $\underline{\delta}$ in $\mathscr{C}^\infty_{n,G}$ with $\underline{\gamma}(0)$, $\underline{\delta}(0) \in G_{\text{reg}}$. Set $Y = P^1_{\mathbf{C}} \times A$ and denote the germs in $\mathscr{C}^\infty_{n,Y}$ associated to $\underline{\gamma}$ and $\underline{\delta}$ by $(\underline{u}, \underline{s})$ and $(\underline{v}, \underline{t})$.*
Then $\underline{\gamma}$ and $\underline{\delta}$ are conjugate in $\mathscr{C}^\infty_{n,G}$ if and only if $\underline{s} = \underline{t}$.

Proof. One may choose representatives γ, u, s, δ, v, t defined in a common neighbourhood $\mathcal{O}$ of the origin of $\mathbf{R}^n$ in such a way that $\gamma(x)$, $\delta(x) \in G_{\text{reg}}$ for all $x \in \mathcal{O}$. To prove the non trivial implication, we assume that $\underline{s} = \underline{t}$, and indeed that $s = t$.

Suppose first that the range of $u(0)$ is not orthogonal to the range of $v(0)$. (If $P^1_{\mathbf{C}}$ is identified with S^2, this means that $u(0)$ and $v(0)$ are not antipodal.) In appropriate coordinates

$$u(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad v(0) = \begin{pmatrix} c & d \\ \bar{d} & 1-c \end{pmatrix} \quad \text{with} \quad c > 0.$$

If

$$u(x) = \begin{pmatrix} a(x) & b(x) \\ \overline{b(x)} & 1-a(x) \end{pmatrix} \quad \text{and} \quad v(x) = \begin{pmatrix} c(x) & d(x) \\ \overline{d(x)} & 1-c(x) \end{pmatrix} \quad \text{for} \quad x \in \mathcal{O},$$

we may assume $\mathcal{O}$ small enough so that $a(x) \neq 0$ and $c(x) \neq 0$ for all $x \in \mathcal{O}$. By Lemma 2, there exists a map $\zeta : \mathcal{O} \to G$ such that $\zeta(x)u(x)\zeta(x)^{-1} = v(x)$ for all $x \in \mathcal{O}$. As $s = t$ it follows that $\zeta\gamma\zeta^{-1} = \delta$.

In case $u(0)$ and $v(0)$ are orthogonal, define a new germ $\underline{\gamma}'$ with $\underline{t}' = \underline{t}$ such that $\underline{u}'(0)$ is orthogonal neither to $\underline{u}(0)$ nor to $\underline{v}(0)$. The argument above shows that $\underline{\gamma}'$ is conjugate both to $\underline{\gamma}$ and $\underline{\delta}$.

LEMMA 4. *Let $s \in A$. For any $t \in A$ with* $\sin(t/2) < \sin s$ *set*

$$\zeta_t = \begin{pmatrix} \left(1 - \dfrac{\sin^2(t/2)}{\sin^2 s}\right)^{1/2} & \dfrac{\sin(t/2)}{\sin s} \\ -\dfrac{\sin(t/2)}{\sin s} & \left(1 - \dfrac{\sin^2(t/2)}{\sin^2 s}\right)^{1/2} \end{pmatrix}$$

$$\eta_t = \begin{pmatrix} \exp(is) & 0 \\ 0 & \exp(-is) \end{pmatrix} \zeta_t \begin{pmatrix} \exp(-is) & 0 \\ 0 & \exp(is) \end{pmatrix} \zeta_t^{-1}$$

$$\delta_t = \begin{pmatrix} \exp(it) & 0 \\ 0 & \exp(-it) \end{pmatrix}$$

Then η_t is in G_{reg} and is conjugate to δ_t.

Proof. Compute

$$\operatorname{trace}(\eta_t) = 2\frac{\sin^2 s - \{1-\cos(2s)\}\sin^2(t/2)}{\sin^2 s}$$
$$= 2\cos t = \operatorname{trace}(\delta_t).$$

LEMMA 5. *Consider two germs $\underline{\gamma}$, $\underline{\delta}$ in $\mathscr{C}^\infty_{n,G}$ with $\underline{\gamma}(0)$, $\underline{\delta}(0) \in G_{\mathrm{reg}}$, and let $\underline{u}$, $\underline{s}$, $\underline{v}$, $\underline{t}$ be as in Lemma* 3.

There exists a neighbourhood of the identity $\mathscr{V} \subset G$ (depending on $\underline{\gamma}$) such that $\underline{\delta}$ is in the normal subgroup $\mathscr{C}^\infty_{n,G}$ generated by $\underline{\gamma}$ as soon as $\underline{\delta}(0) \in \mathscr{V} \cap G_{\mathrm{reg}}$.

Proof. Let $\mathscr{V}$ and γ, u, s, δ, v, t be as in the proof of Lemma 3. By this Lemma 3, we may suppose that

$$\gamma(x) = \begin{pmatrix} \exp(is(x)) & 0 \\ 0 & \exp(-is(x)) \end{pmatrix}$$

$$\delta(x) = \begin{pmatrix} \exp(it(x)) & 0 \\ 0 & \exp(-it(x)) \end{pmatrix}$$

and that $s(x)$ is bounded away from $\{0, \pi\}$ for all $x \in \mathscr{O}$. By Lemma 4, there exists for all $\underline{\delta} \in \mathscr{C}^\infty_{n,G}$ with $\sin(\underline{t}(0)/2) < \sin(\underline{s}(0))$ a germ $\underline{\zeta} \in \mathscr{C}^\infty_{n,G}$ such that the eigenvalue parts of $\underline{\delta}$ and $\underline{\gamma}\underline{\zeta}\underline{\gamma}^{-1}\underline{\zeta}^{-1}$ are equal. One concludes using Lemma 3 again.

For a while, let G be an arbitrary connected Lie group. We introduce a topology on $\mathscr{C}^\infty_{n,G}$ making it a connected (though not Hausdorff) topological group.

For any integer $k \geq 0$, the set $J^k_0(\mathbf{R}^n, G)$ of jets of order k of smooth maps from $\mathbf{R}^n$ to G is naturally a finite dimensional connected Lie group. These constitute a projective system with inverse limit the connected topological group $J^\infty_0(\mathbf{R}^n, G)$, and there is a natural projection π from $\mathscr{C}^\infty_{n,G}$ onto $J^\infty_0(\mathbf{R}^n, G)$. The topology put on $\mathscr{C}^\infty_{n,G}$ is that for which a subset S is open if and only if $\pi(S)$ is open in the infinite jet group.

The only open subgroup of $\mathscr{C}^\infty_{n,G}$ is $\mathscr{C}^\infty_{n,G}$ itself. The point $\{1\}$ has for closure the set of germs which are flat in the appropriate sense.

Assume now that the Lie algebra of G is simple, and let $\mathscr{E} : \mathscr{C}^\infty_{n,G} \to G/C(G)$ be as in the introduction. Let N be a normal subgroup of $\mathscr{C}^\infty_{n,G}$. If $N \not\subset \operatorname{Ker}(\mathscr{E})$ then $\mathscr{E}(N) = G/C(G)$, because $G/C(G)$ is simple as an abstract group. In particular N contains a germ $\underline{\gamma}$ with $\gamma(0)$ regular.

From now on until the end of section 1, we set $G = SU(2)$ again.

PROPOSITION 6. *Proposition III of the introduction holds for $G = SU(2)$.*

Proof. Let N be a normal subgroup of $\mathscr{C}^\infty_{n,G}$ with $N \not\subset \operatorname{Ker}(\check{\varepsilon})$. We have just seen that N contains a germ $\underline{\gamma}$ with $\underline{\gamma}(0) \in G_{\text{reg}}$. Lemma 5 shows that N contains any germ $\underline{\delta}$ with $\underline{\delta}(0)$ regular and near enough 1. As these constitute a non empty open subset of $\mathscr{C}^\infty_{n,G}$, the group N is open, and consequently is all of $\mathscr{C}^\infty_{n,G}$.

We end this section with a digression. Though regular germs can be diagonalized by Lemma 3, the condition of regularity cannot be removed. The following example illustrates this. It comes from [R]; see also [J].

Set $n = 1$ and $\theta(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Define for $x \in \mathbf{R}^*$

$$\theta(x) = \exp(-x^{-2}) \begin{pmatrix} \cos(2/x) & \sin(2/x) \\ \sin(2/x) & -\cos(2/x) \end{pmatrix}.$$

Define also $\gamma(x) = \exp(i\theta(x))$; then $\gamma \in \mathscr{C}^\infty(\mathbf{R}, SU(2))$. Outside the origin, $\gamma(x)$ has eigenvectors $\begin{pmatrix} \cos(1/x) \\ \sin(1/x) \end{pmatrix}$ and $\begin{pmatrix} \sin(1/x) \\ -\cos(1/x) \end{pmatrix}$ with eigenvalues $\exp(i \exp \times (-x^{-2}))$ and $\exp(-i \exp(-x^{-2}))$. But there is no germ $\underline{\delta}$ such that $\underline{\delta}\underline{\gamma}\underline{\delta}^{-1}$ is diagonal. Of course, one may diagonalize γ with a Borel map [A]!

2. Proof of Proposition III in the general case

Let G be a connected compact Lie group with simple Lie algebra $\mathbf{g}$.

It is sufficient to prove proposition III for simply connected groups. Indeed, suppose the proposition holds for the universal covering $\tilde{G}$ of G (which is still compact by Weyl's theorem). The short exact sequence

$$\{1\} \to \pi_1(G) \to \tilde{G} \to G \to \{1\}$$

induces a sequence

$$\{1\} \to \pi_1(G) \to \mathscr{C}^\infty_{n,\tilde{G}} \xrightarrow{p} \mathscr{C}^\infty_{n,G} \to \{1\}$$

which is again exact (here elements of $\pi_1(G)$ are viewed as constant germs). Let

$N \subset \mathscr{C}^\infty_{n,G}$ be a normal subgroup which is not contained in the kernel of $\check{\varepsilon} : \mathscr{C}^\infty_{n,G} \to G/C(G)$. As $\tilde{G}/C(\tilde{G}) \approx G/C(G)$, the normal subgroup $\tilde{N} = p^{-1}(N)$ is not contained in the kernel of $\mathscr{C}^\infty_{n,\tilde{G}} \to \tilde{G}/C(\tilde{G})$. If Proposition III holds for $\tilde{G}$, then $\tilde{N} = \mathscr{C}^\infty_{n,\tilde{G}}$ and thus $N = \mathscr{C}^\infty_{n,G}$, so that the proposition holds also for G.

From now on we assume G to be *simply connected.* Using as much as possible the notations of [Lie], we denote by T a maximal torus in G, by $\mathbf{t}$ its Lie algebra, and by A an alcôve in $\mathbf{t}$ (namely a connected component of the subset of $\mathbf{t}$ consisting of those $x \in \mathbf{t}$ with $\exp(x)$ regular). Lemma 1 above is a particular case of

LEMMA 7. *The map*

$$\varphi_A : \begin{cases} (G/T) \times A \to G_{\mathrm{reg}} \\ (gT, s) \mapsto g(\exp(s))g^{-1} \end{cases}$$

is an analytic diffeomorphism. It is G-equivariant if G acts canonically on G/T, trivially on A, and by conjugation on the set G_{reg} of regular elements in G.

Proof. For the first claim, see Proposition 4b in [Lie], page 51 (where $H_A = \{1\}$ by remark 1 of page 45). The second claim is clear.

LEMMA 8. *There exist an integer $k \geq 1$ and*

open subsets $\mathscr{U}_1, \mathscr{U}_2, \ldots, \mathscr{U}_k$ in G/T
base points $0_1 \in \mathscr{U}_1, 0_2 \in \mathscr{U}_2, \ldots, 0_k \in \mathscr{U}_k$
smooth maps $\mu_j : \mathscr{U}_j \to G$ for $j = 1, 2, \ldots, k$

such that

$\mu_j(u)0_j = u$ for each $u \in \mathscr{U}_j$, for $j = 1, 2, \ldots, k$.

Proof. This is because $G \to G/T$ defines a locally trivial bundle with compact base.

Lemma 3 can in turn be generalized as follows.

LEMMA 9. *Consider two germs $\underline{\gamma}$, $\underline{\delta}$ in $\mathscr{C}^\infty_{n,G}$ with $\gamma(0), \delta(0) \in G_{\mathrm{reg}}$. Set $Y = (G/T) \times A$. Denote by $(\underline{u}, \underline{s})$, $(\underline{v}, \underline{t})$ the germs in $\mathscr{C}^\infty_{n,Y}$ which are composition of $\underline{\gamma}$, $\underline{\delta}$ and φ_A^{-1} (see Lemma 7). Then $\underline{\gamma}$ and $\underline{\delta}$ are conjugate in $\mathscr{C}^\infty_{n,G}$ if and only if $\underline{s} = \underline{t}$.*

Proof. We assume $\underline{s} = \underline{t}$, and we show that $\underline{\gamma}$ is conjugate to $\underline{\delta}$.

Suppose first that there exists $j \in \{1, \ldots, k\}$ with $\underline{u}(0), \underline{v}(0) \in \mathscr{U}_j$ (notations of

Lemma 8). We may choose representatives γ, u, s, δ, v, t of $\underline{\gamma}, \ldots, \underline{t}$ defined in a neighbourhood $\mathcal{O}$ of the origin in $\mathbf{R}^n$ such that $s(x) = t(x)$ and $u(x)$, $v(x) \in \mathcal{U}_j$ for all $x \in \mathcal{O}$. By Lemma 8, there exists a map $\mu : \mathcal{O} \to G$ such that $\mu(x)u(x) = v(x)$ for all $x \in \mathcal{O}$. By Lemma 7 one has $\mu(x)\gamma(x)\mu(x)^{-1} = \delta(x)$ for all $x \in \mathcal{O}$.

In the general case, one may find a sequence $\underline{u}_0 = \underline{u}, \underline{u}_1, \ldots, \underline{u}_m = \underline{v}$ such that, for each $i \in \{1, \ldots, m\}$, there exists some $j \in \{1, \ldots, k\}$ with $\underline{u}_{i-1}(0)$, $\underline{u}_i(0) \in \mathcal{U}_{j(i)}$. By the argument above, there exists $\mu_i : \mathcal{O} \to G$ with $\mu_i(x)u_{i-1}(x) = u_i(x)$ for all x in a neightbourhood $\mathcal{O}$ of the origin in $\mathbf{R}^n$. Then the product $\underline{\mu_m\mu_{m-1}} \cdots \underline{\mu_1}$ conjugates $\underline{\gamma}$ in $\underline{\delta}$.

As we have not been able to generalize Lemma 4, we proceed as follows by reduction to the case of $SU(2)$.

Choose a root $\alpha : T \to U$ of G with respect to T (with U the unit circle of the complex plane). Let $\nu : SU(2) \to G$ be a morphism of Lie groups such that

(a) the image of ν and the kernel of α commute

(b) for $a \in U$ one has $\nu\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \in T$ and $\alpha\nu\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} = a^2$ (see [Lie], page 31).

Let ψ be the inner automorphism of G defined by $\nu\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$\psi(t) = t \qquad \text{for all} \quad t \in \operatorname{Ker}(\alpha)$$

$$\psi(t) = t^{-1} \qquad \text{for all} \quad t \in T \cap \operatorname{Im}(\nu)$$

(see [Lie], page 33). As

$$T = (\operatorname{Ker}(\alpha)) \cdot (T \cap \operatorname{Im}(\nu))$$

the commutator $t\psi(t)^{-1}$ lies in $T \cap \operatorname{Im}(\nu)$ for all $t \in T$. Moreover there exists an open dense subset $T_{\mathrm{reg}+}$ in T (indeed in $T \cap G_{\mathrm{reg}}$) such that $t\psi(t)^{-1}$ is not in the image by ν of the centre of $SU(2)$ for any $t \in T_{\mathrm{reg}+}$. We denote by $\mathbf{t}_{\mathrm{reg}+}$ the inverse image by $\exp_T$ of $T_{\mathrm{reg}+}$ in $\mathbf{t}$. Then

$$G_{\mathrm{reg}+} = \varphi_A((G/T) \times (A \cap \mathbf{t}_{\mathrm{reg}+}))$$

is an open dense subset of G.

For example, let $G = SU(l+1)$ for some $l \geqslant 2$ and let T be the torus of diagonal matrices. If α maps $\operatorname{diag}(z_1, \ldots, z_{l+1})$ to $z_1 z_2^{-1}$, then the image of ν

consists of matrices of the form

$$\begin{pmatrix} a & b & & & \\ -\bar{b} & \bar{a} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad a, b \in \mathbf{C} \qquad |a|^2 + |b|^2 = 1$$

and $T_{\text{reg}+}$ consists of matrices $\operatorname{diag}(z_1, \ldots, z_{l+1})$ where the z_j's are all distinct and where $z_1 \neq -z_2$. We return now to the general case for G.

LEMMA 10. *Let N be a normal subgroup of $\mathscr{C}^\infty_{n,G}$ which contains a germ $\underline{\gamma} \in \mathscr{C}^\infty_{n,G}$ with $\underline{\gamma}(0) \in G_{\text{reg}+}$. Then N contains any constant germ.*

Proof. Let $\gamma : \mathscr{O} \to G$ be a representative of $\underline{\gamma}$. By Lemma 9 we may assume that there exists a smooth map $s : \mathscr{O} \to A \cap \mathbf{t}_{\text{reg}+}$ with $\gamma(x) = \exp(s(x))$ for $x \in \mathscr{O}$. Define $\delta : \mathscr{O} \to G$ by $\delta(x) = \gamma(x)\psi(\gamma(x))^{-1}$. As δ is the commutator of γ with the constant map $\mathscr{O} \to G$ of value $v\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the germ $\underline{\delta}$ lies in N. Moreover

$$\underline{\delta}(0) \notin \left\{ v\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

by definition of $G_{\text{reg}+}$. It follows from Proposition 6 that N contains any constant germ with value in $\operatorname{Im}(v)$. As G is simple (up to its centre), this implies that N contains any constant germ at all.

LEMMA 11. *Let N be as in Lemma 10. Then $N = \mathscr{C}^\infty_{n,G}$.*

Proof. Let c be a Coxeter element of the Weyl group of T. Let m be an element in the normalizer $N_G(T)$ of class c in W, and denote by f the smooth map $\begin{cases} T \to T \\ t \mapsto mtm^{-1}t^{-1} \end{cases}$. The derivative of f at the identity is $L(f) : \begin{cases} \mathbf{t} \to \mathbf{t} \\ x \mapsto c(x) - x \end{cases}$. Now $L(f)$ is an automorphism because Coxeter elements do not have 1 as eigenvalue (we repeat here part of [Lie], page 33). By the implicit function theorem, this implies that there exist a neighbourhood $\mathscr{V}$ of 1 in T and a smooth map $\chi : \mathscr{V} \to T$ such that

$$t = m\chi(t)m^{-1}\chi(t)^{-1} \quad \text{for all} \quad t \in \mathscr{V}.$$

Consider now m as a constant germ in $\mathscr{C}^\infty_{n,G}$; then $m \in N$ by Lemma 10. Let $\underline{\gamma} \in G_{\mathrm{reg}}$ be a germ with

$$\underline{\gamma}(0) \in \bigcup_{g \in G} g\mathscr{V}g^{-1},$$

and let γ be a representative of $\underline{\gamma}$. Let δ be a conjugate of γ with values in T (see Lemma 9). As

$$\delta(x) = m\chi(\delta(x))m^{-1}\chi(\delta(x))^{-1} \qquad x \in \mathcal{O}$$

and as $m \in N$, *one has* $\underline{\delta} \in N$ and also $\underline{\gamma} \in N$. Consequently N contains an open subgroup of $\mathscr{C}^\infty_{n,G}$.

As $\mathscr{C}^\infty_{n,G}$ is connected, this ends the proof.

The argument used for Proposition 6 above proves now Proposition III for any simply connected compact Lie group with simple Lie algebra.

3. Proof of Proposition II.

The support of a smooth map $\gamma: X \to G$ is the closure of $\{x \in X \mid \gamma(x) \neq 1\}$.

LEMMA 12. *Let $(\mathscr{U}_j)_{1 \le j \le n}$ be an open covering of X and let $\gamma \in M_0G$. There exists a finite sequence $(j_1, \ldots, j_N)$ of indices in $\{1, \ldots, n\}$ and there exist smooth maps $\gamma_k \in M_0G$ with* supp $(\gamma_k) \in \mathscr{U}_{j_k}$ *for $k \in \{1, \ldots, N\}$ such that $\gamma = \gamma_1\gamma_2 \cdots \gamma_N$.*

Proof. The group M_0G has a Lie algebra $\mathscr{C}^\infty(X, \mathbf{g})$ made of smooth maps from X to the Lie algebra $\mathbf{g}$ of G, and an exponential map

$$\mathrm{EXP}: \mathscr{C}^\infty(X, \mathbf{g}) \to M_0G$$

defined by $(\mathrm{EXP}\,\zeta)(x) = \exp(\zeta(x))$, where $\exp: \mathbf{g} \to G$ is the exponential of G. As EXP is a local diffeomorphism and as M_0G is connected, there exists $\zeta_1, \ldots, \zeta_m \in \mathscr{C}^\infty(X, \mathbf{g})$ with

$$\gamma = \mathrm{EXP}(\zeta_1)\,\mathrm{EXP}(\zeta_2) \cdots \mathrm{EXP}(\zeta_m).$$

Let $(\lambda_j)_{1 \le j \le n}$ be a smooth partition of unity subordinated to $(\mathscr{U}_j)_{1 \le j \le n}$; as the

$\lambda_j\zeta_1$'s commute, one has

$$\mathrm{EXP}(\zeta_1) = \prod_{i \leq j \leq n} \mathrm{EXP}(\lambda_1\zeta_1).$$

The same holds for each ζ_k and this proves the lemma (with $N = nm$).

In the following lemma, G is identified with the set of constant maps in M_0G. Given $A, B \subset M_0G$, we write (A, B) for the subgroup of M_0G generated by the commutators $\gamma\delta\gamma^{-1}\delta^{-1}$ with $\gamma \in A$ and $\delta \in B$.

LEMMA 13. *The group M_0G is* perfect. More precisely $M_0G = (G, M_0G)$.

Proof. Choose a maximal torus T in G. Let $m \in G$ and $\chi : \mathcal{V} \to T$ be as in the proof of Lemma 11.

Let first $\gamma \in M_0G$ be a map with values in the neighbourhood $\mathcal{V}$ of 1 in T. Define $\delta : X \to T$ by $\delta(x) = \chi(\gamma(x))$; then $\gamma = m\delta m^{-1}\delta^{-1} \in (G, M_0G)$.

As conjugates of T cover G, there exist tori $T_1, \ldots, T_k \subset G$ which are conjugated to closed subtori of T, and such that

$$\begin{cases} T_1, \ldots, T_k \to G \\ (g_1, \ldots, g_k) \mapsto g_1g_2 \cdots g_k \end{cases}$$

is a local diffeomorphism. It follows that any $\gamma \in M_0G$ near enough to the identity is a product $\gamma_1\gamma_2 \cdots \gamma_k$ with $\gamma_j \in M_0T_j \subset (G, M_0G)$ for $j = 1, \ldots, k$.

Now M_0G is generated by elements near the identity, and the lemma follows.

LEMMA 14. *Let N be a maximal normal subgroup in M_0G. There exists $a \in X$ such that N contains the group N_aG of those $\gamma \in M_0G$ which have trivial germ at a.*

Proof. Set $S = M_0G/N$; it is a group which is perfect, by Lemma 13, and simple, by maximality of N. Write $\pi : M_0G \to S$ the canonical projection.

For any open subset $\mathcal{U}$ of X, denote by $M_{\mathcal{U}}$ the normal subgroup of M_0G of those maps which have supports in $\mathcal{U}$. Define

$$\mathcal{W} = \{x \in X \mid x \text{ has a } nbd\ \mathcal{U} \text{ with } \pi(M_{\mathcal{U}}) = 1\};$$

this is an open subset of X.

If there were two distinct points y, z in $X - \mathcal{W}$, one could choose neighbourhoods $\mathcal{U}$, $\mathcal{V}$ of y, z with $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\pi(M_{\mathcal{U}}) = \pi(M_{\mathcal{V}}) = S$. Then

$$S = (S, S) = (\pi(M_{\mathcal{U}}), \pi(M_{\mathcal{V}})) = \pi(M_{\mathcal{U}}, M_{\mathcal{V}}) = \pi(1) = 1$$

which is absurd. By Lemma 12 and by compacity of X (which is crucial here), one cannot have $\mathcal{W} = X$. Hence there exists $a \in X$ with $\mathcal{W} = X - \{a\}$. By Lemma 12 again one has $N_{\mathcal{W}} \subset \operatorname{Ker}(\pi)$. But $M_{\mathcal{W}}$ is precisely the group $N_a G$.

We may now finish the proof of Proposition II. Let N be a maximal normal subgroup in $M_0 G$. Notations being as above, $M_0 G / N_a G$ is isomorphic to $\mathscr{C}^{\infty}_{n,G}$, with n the dimension of X. By Proposition III, the image of N in $M_0 G / N_a G$ is the group of those germs with value at a in the centre of G. In other words N is equal to the kernel of $\check{\varepsilon}_a$.

4. Proof of Theorem I.

The next lemma is an easy illustration "that group properties often force considerably more regularity than is explicitly postulated" (quoted from Chapter V in [MZ]).

LEMMA 15. *Let X be a compact smooth manifold, let G be a connected compact Lie group, and let $\alpha : X \times G \to G$ be a map such that*

$$\left.\begin{array}{l} \alpha(x, g)\alpha(x, h) = \alpha(x, gh) \\ \alpha(x, 1) = 1 \end{array}\right. \qquad \text{for all } x \in X \text{ and } g, h \in G.$$

If α is separately smooth, then α is smooth.

Proof. We check first that α is continuous. Let d be a distance which defines the topology of G. Let $(x, g) \in X \times G$ and let $\varepsilon > 0$. As $y \to \alpha(y, g)$ is continuous, one has $d(\alpha(y, g), \alpha(x, g)) < \varepsilon$ for y near enough to x. As the set of automorphisms of the compact group G is equicontinuous, one has $d(\alpha(z, h), \alpha(z, g)) < \varepsilon$ for h near enough to g and uniformly with respect to $z \in X$. It follows that

$$d(\alpha(y, h), \alpha(x, g)) \leq d(\alpha(y, h), \alpha(y, g)) + d(\alpha(y, g), \alpha(x, g)) < 2\varepsilon$$

for (y, h) near enough to (x, g).

The next step is to translate the problem from G to its Lie algebra $\mathbf{g}$. Let $\mathcal{U}$ be a neighbourhood of the origin in $\mathbf{g}$ and let $\mathcal{V}$ be a neighbourhood of 1 in G such that $\exp : \mathcal{U} \to \mathcal{V}$ and $\log : \mathcal{V} \to \mathcal{U}$ are diffeomorphisms inverse to each other. Let $\mathcal{V}'$ be a neighbourhood of 1 in $\mathcal{V}$ such that $\alpha(X \times \mathcal{V}') \subset \mathcal{V}$ and let $\mathcal{U}' = \log(\mathcal{V}')$. Define $\beta : X \times \mathbf{g} \to \mathbf{g}$ by $\beta(x, \zeta) = \log(\alpha(x, \exp \zeta))$ for $x \in X$ and $\zeta \in \mathcal{V}'$, and extend by linearity for larger ζ. Then β is continuous, separately smooth, and linear in ζ.

Consider linear coordinates $(\zeta_1, \ldots, \zeta_m)$ on $\mathbf{g}$. By linearity, one may write

$$\beta(x, \zeta)_i = \sum_{1 \le j \le m} \beta_{ij}(x)\zeta_j \qquad i = 1, \ldots, m.$$

By assumption, $x \to \beta(x, \zeta)_i$ is smooth for any $(\zeta_1, \ldots, \zeta_m) \in \mathbf{R}^m$ and for $i \in \{1, \ldots, m\}$. It follows that each $\beta_{i,j}$ is smooth, so that β is smooth. Consequently the restriction of α to $X \times \mathcal{V}'$ is smooth.

Let $(x, g_0) \in X \times G$. For g near enough to g_0 one has $g_0^{-1}g \in \mathcal{V}'$ and $\alpha(x, g) = \alpha(x, g_0)\alpha(x, g_0^{-1}g)$ depends smoothly on (x, g). Thus α is smooth.

We need a second smoothness lemma, for which compacity does not help.

LEMMA 16. *Let X be a smooth manifold, let G be a connected Lie group, and let $\varphi: X \to X$ be a map such that* $\begin{cases} X \to G \\ x \to \gamma(\varphi(x)) \end{cases}$ *is smooth for all $\gamma \in M_0G$. Then φ is smooth.*

Proof. Let I be the open unit interval, let I^N be a cube of large dimension, and let $t_j: I^N \to I$ be the jth projection ($j = 1, \ldots, N$). Let $s: X \to I^N$ be a smooth embedding of X in the cube, and let $u: I \to G$ be a smooth embedding. Consider the map $\tau_j = ut_js$, which is in M_0G for $j \in \{1, \ldots, N\}$.

For each j, the map $\tau_j\varphi$ is smooth by hypothesis. As u is an embedding, $t_js\varphi$ is smooth. It follows that $s\varphi$ is smooth. As s is an embedding, φ is smooth.

Proof of Theorem I. Let $\Phi: M_0G \to M_0G$ be an automorphism. For each $x \in X$, consider the endomorphism α_x of G defined by

$$\begin{array}{ccc} M_0G & \xrightarrow{\Phi} & M_0G \\ \uparrow{\scriptstyle \text{constant}} & & \downarrow{\scriptstyle \varepsilon_x} \\ G & \overset{\alpha_x}{\dashrightarrow} & G \end{array}$$

Then α_x is smooth by [C]. If α_x was not injective, it would be trivial (because the Lie algebra of G is simple) and Φ would not be onto by Lemma 13. Hence $\alpha_x \in A = \operatorname{Aut} G$ for all $x \in X$. Moreover $\alpha_x(g) = \Phi(g)(x)$ depends smoothly on x for each $g \in G$. It follows from Lemma 15 that α is smooth. Upon replacing Φ by $(\alpha^{-1}, id_x)\Phi$, we may now prove Theorem I under the additional assumption that $\alpha_x \doteq id_G$ for all $x \in X$.

The automorphism Φ permutes maximal normal subgroups of M_0G. By Proposition II, there exists a bijection $\varphi: X \to X$ such that $\Phi(\operatorname{Ker} \check{\varepsilon}_x) = \operatorname{Ker} \check{\varepsilon}_{\varphi(x)}$ for all $x \in X$. Define a homomorphism $\tilde{\Phi}$ from M_0G to the group of all maps from

X to G by $\tilde{\Phi}(\gamma)=\gamma\varphi^{-1}$. We have to show that $\tilde{\Phi}=\Phi$ and that φ is smooth.

Let $\gamma \in M_0G$. For any $x \in X$ one may write $\gamma = g_x\gamma_x$ with $g_x \in G$ (a constant map) and $\gamma_x \in \operatorname{Ker}(\varepsilon_{\varphi^{-1}(x)})$. As $\Phi(g_x)(x)=g_x$ and $\Phi(\gamma_x)\in \operatorname{Ker}\check{\varepsilon}_x$ one has $\Phi(\gamma)(x)=z_{\gamma,x}g_x$ for some $z_{\gamma,x}\in C(G)$. The map $\gamma \to z_{\gamma,x}$ is a homomorphism; it is constant, because M_0G is perfect and $C(G)$ is abelian. Consequently $\Phi(\gamma)(x)=g_x$. Also $\tilde{\Phi}(\gamma)(x)=\gamma(\varphi^{-1}(x))=g_x$. As this holds for any x one has $\tilde{\Phi}(\gamma)=\Phi(\gamma)$. But γ is arbitrary, so that $\tilde{\Phi}=\Phi$. Finally, φ is smooth by Lemma 16.

Remark. Consider now the topological group $\mathscr{C}^0(X, G)$ of continuous maps from X to G, and its connected component $\mathscr{G}=\mathscr{C}^0(X,G)^0$. Let $\mathscr{X}$ be the set of maximal normal subgroups of $\mathscr{G}$; because of Theorem I, it can also be viewed as the set of homomorphisms of $\mathscr{G}$ onto $G/C(G)$. One may define a topology on $\mathscr{X}$ as follows: given $\chi_0 \in \mathscr{X}$, and given an integer $k \geq 1$, elements $\gamma_1,\ldots,\gamma_k \in \mathscr{G}$, and a real number $\varepsilon>0$, define a basic neighbourhood of χ_0 by

$$(\chi_0; \gamma_1, \ldots, \gamma_k, \varepsilon)=\{\chi \in \mathscr{X} \mid d(\chi(\gamma_j), \chi_0(\gamma_j))<\varepsilon \quad \text{for} \quad j=1,\ldots,k\}$$

where d is a distance defining the topology of $G/C(G)$. Then the "Gelfand map" $X \to \mathscr{X}$ which associates to a point x the homomorphism $\check{\varepsilon}_x$ is a homeomorphism.

It follows that $\mathscr{C}^0(X,G)^0$ and $\mathscr{C}^0(Y,G)^0$ are isomorphic (as topological groups) if and only if the compact topological spaces X and Y are homeomorphic.

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