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On the outradius of the Teichmüller space

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Dedicated to Professor Kôtaro Oikawa on his 60th birthday

Abstract. Let Γ be a fuchsian group which preserves the unit disc Δ and hence also its complement Δ^* in the Riemann sphere $\hat{\mathbb{C}}$. The Bers embedding represents the Teichmüller space $T(\Gamma)$ of Γ in the space $B(\Delta^*, \Gamma)$ of bounded quadratic differentials for Γ in Δ^*. Then, $T(\Gamma)$ is included in the closed ball centred at the origin of radius 6 in $B(\Delta^*, \Gamma)$ with respect to the norm employed in a paper by Nehari [The Schwarzian derivative and Schlicht functions; Bull. Amer. Math. Soc. 55 (1949), 545–551]. In other words the outradius $o(\Gamma)$ of $T(\Gamma)$ is not greater than 6. The purpose of this paper is to give a complete characterization of a fuchsian group Γ for which the outradius $o(\Gamma)$ of $T(\Gamma)$ attains this extremal value 6. The main theorem is: Let Γ be a fuchsian group preserving Δ^*. Then the outradius $o(\Gamma)$ of the Teichmüller space $T(\Gamma)$ equals 6 if and only if for any positive number d, either (i) there exists a hyperbolic disc of radius d precisely invariant under the trivial subgroup, or (ii) there exists the collar of width d about the axis of a hyperbolic element of Γ.

Introduction

Let Γ be a fuchsian group which preserves the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and hence also the complement Δ^* of the closure Δ in the Riemann sphere $\hat{\mathbb{C}}$. In this paper we treat the Teichmüller space $T(\Gamma)$ of Γ represented as a subregion of the Banach space $B(\Delta^*, \Gamma)$ of bounded quadratic differentials for Γ by means of the Bers embedding ([1]). By a classical theory of Nehari [8], in $B(\Delta^*, \Gamma)$ the Teichmüller space $T(\Gamma)$ is included in the closed ball of radius 6 centered at the origin. In other words the outradius $o(\Gamma)$ of $T(\Gamma)$ does not exceed 6. The purpose of this paper is to give a complete characterization of those fuchsian groups Γ for which $o(\Gamma)$ attains the extremal value 6. Our main theorem is:

THEOREM 1.1. *Let Γ be a fuchsian group preserving Δ^*. Then the outradius $o(\Gamma)$ of the Teichmüller space $T(\Gamma)$ equals* 6 *if and only if Γ satisfies one of the following conditions*:

(O_1) *For any positive number d, there exists a hyperbolic disc of radius d which is precisely invariant under the trivial subgroup $\{1\}$ of Γ.*

(O_2) *For any positive number d, there exists the collar of width d about the axis of a hyperbolic element of Γ.*

We shall refer to the notations in the theorem in Section 1. Theorem 1.1 means a geometric condition of the fuchsian group Γ reflects the property that $o(\Gamma)$ equals 6 or not. First we discuss in Section 1 preliminary notions and definitions concerning fuchsian groups and Teichmüller spaces. In Section 2 we give some lemmas needed to prove Theorem 1.1. A proof of Theorem 1.1 is carried out in Sections 3 and 4. The final section, Section 5, is devoted to some examples concerning the conditions (O_1) and (O_2) of Theorem 1.1.

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1. Preliminaries

1.1. Our basic reference for the content of this section is [2]. Let Möb $(\hat{\mathbb{C}})$ be the group of Möbius transformations of the Riemann sphere $\hat{\mathbb{C}}$ onto itself; that is, mappings

$$\gamma(z) = (az + b)/(cz + d), \qquad a, b, c, d \in \mathbb{C}, \qquad ad - bc = 1.$$

Let $\Delta^* = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$. The hyperbolic metric defined on Δ^*,

$$\rho^*(z)\,|dz| = (|z|^2 - 1)^{-1}\,|dz|$$

has constant curvature -4. Geodesics with respect to this metric or *hyperbolic lines* are circles or straight lines which are orthogonal to the unit circle $\{|z| = 1\}$. Denote by Möb (Δ) the subgroup of Möb $(\hat{\mathbb{C}})$ which preserves the unit disc Δ (as well as Δ^*). Then all transformations of Möb (Δ) are of the form:

$$\gamma(z) = (az + \bar{b})/(bz + \bar{a}), \qquad |a|^2 - |b|^2 = 1. \tag{1.1}$$

A Möbius transformation (1.1) is an orientation preserving hyperbolic motion, which means that $\rho^*(z) = \rho^*(\gamma(z))\,|\gamma'(z)|$ for $z \in \Delta^*$. The hyperbolic distance between z and w in Δ^* will be denoted by $d(z, w)$.

A *fuchsian group* Γ is a subgroup of Möb (Δ) which acts discontinuously on Δ and hence also on Δ^*. Let G be a subgroup of Γ which is also a fuschsian group. A subset D of Δ^* is said to be *precisely invariant* under G if $\gamma(D) = D$ for all $\gamma \in G$ and $\gamma(D) \cap D = \varnothing$ for all $\gamma \in \Gamma - G$. Let γ be a hyperbolic element of Γ. The hyperbolic line in Δ^* connecting the fixed points of γ is called the axis of γ and denoted by $A^*(\gamma)$. Let Stab $(\Gamma, A^*(\gamma))$ be the stabilizer of $A^*(\gamma)$ in Γ, that is,

$$\operatorname{Stab}(\Gamma, A^*(\gamma)) = \{\eta \in \Gamma : \eta(A^*(\gamma)) = A^*(\gamma)\}.$$

Then Stab $(\Gamma, A^*(\gamma))$ is either the infinite cyclic group generated by a hyperbolic transformation or a group generated by two elliptic elements of order 2. Let $U^*(d)=U^*(d, A^*(\gamma))=\{z\in\Delta^*: d(z, A^*(\gamma))<d\}$. We say that $U^*(d, A^*(\gamma))$ is the *collar of width d about* $A^*(\gamma)$ if $U^*(d)$ is precisely invariant under Stab $(\Gamma, A^*(\gamma))$.

Let $\gamma(z)=(az+\bar{b})/(bz+\bar{a})$ be a transformation in Möb (Δ). If $b\neq 0$, then ∞ is not fixed by γ. In this case the *isometric circle* $I(\gamma)$ of γ is defined to be $\{z: |bz+\bar{a}|=1\}$. We mean by the exterior of $I(\gamma)$ the complementary region of $I(\gamma)$ which contains ∞. Let Γ be a fuchsian group. Suppose that ∞ is not fixed by any element of $\Gamma-\{1\}$. The set $\mathscr{F}(\Gamma)$ of all points in Δ^* exterior to the isometric circles of all the transformations of $\Gamma-\{1\}$ is referred to here the *Ford region* for Γ in Δ^*. The Ford region is a fundamental region for the action of Γ in Δ^* ([2, Sec. 15]).

1.2. Let Γ be a fuchsian group. A holomorphic function ϕ in Δ^* is a *bounded quadratic differential* for Γ if it satisfies

$$\phi(z)=\phi(\gamma(z))\gamma'(z)^2 \quad \text{for all} \quad \gamma\in\Gamma \quad \text{and} \quad z\in\Delta^*, \tag{1.2}$$

and

$$\|\phi\|=\sup_{z\in\Delta^*}\rho^*(z)^{-2}|\phi(z)|<\infty, \tag{1.3}$$

where $\rho^*(\infty)^{-2}|\phi(\infty)|$ means $\lim_{z\to\infty}\rho^*(z)^{-2}|\phi(z)|$. The Banach space of bounded quadratic differentials for Γ with the norm, $\| \ \|$, defined by (1.3) is denoted by $B(\Delta^*, \Gamma)$.

A quasiconformal automorphism w of $\hat{\mathbb{C}}$ is said to be compatible with a fuchsian group Γ if the correspondence $\gamma\to w\gamma w^{-1}$ defines an isomorphism of Γ into Möb $(\hat{\mathbb{C}})$. Let $Q(\Delta^*, \Gamma)$ be the family of all quasiconformal automorphisms of $\hat{\mathbb{C}}$ compatible with Γ and conformal in Δ^*. The *Teichmüller space* $T(\Gamma)$ of Γ is the set of the Schwarzian derivatives $[w]=(w''/w')'-(1/2)(w''/w')^2$ of $w\in Q(\Delta^*, \Gamma)$ in Δ^*. By the well known properties of the Schwazian derivative, $[w](z)$, $w\in Q(\Delta^*, \Gamma)$, satisfies (1.2). Moreover as the Schwarzian derivative of a univalent function in Δ^*, it holds that ([8]):

$$\|[w\|\leqq 6. \tag{1.4}$$

Thus $T(\Gamma)\subset B(\Delta^*, \Gamma)$. The outradius $o(\Gamma)$ of $T(\Gamma)$ is defined to be $\sup\{\|\phi\|:\phi\in T(\Gamma)\}$. By (1.4) we have $o(\Gamma)\leqq 6$.

2. Some lemmas

LEMMA 2.1. *Let $\{\gamma_n\}$ be a sequence in* Möb (Δ). *If there exists $R>1$ such that $|\gamma_n(\infty)|>R$ for all n, then $\{\gamma_n\}$ contains a subsequence which converges to a transformation in* Möb (Δ).

Proof. Let $\gamma_n(z)=(a_nz+\bar{b}_n)/(b_nz+\bar{a}_n)$, $|a_n|^2-|b_n|^2=1$. By assumption $|\gamma_n(\infty)|=|a_n/b_n|=|a_n|\,(|a_n|^2-1)^{-1/2}>R>1$. It follows that $|a_n|\leqq R(R^2-1)^{-1/2}$ and $|b_n|\leqq(R^2-1)^{-1/2}$. Choose subsequences $|a_{n_j}|$ and $|b_{n_j}|$ in such a way that $\lim_{j\to\infty}a_{n_j}=a_0$ and $\lim_{j\to\infty}b_{n_j}=b_0$, for some a_0 and $b_0\in\mathbb{C}$. Then the subsequence $\{\gamma_{n_j}\}$ converges to $\gamma_0(z)=(a_0z+\bar{b}_0)/(b_0z+\bar{a}_0)$.

LEMMA 2.2. *Let $\{\gamma_n\}$ be a sequence in* Möb (Δ) *converging to the identity transformation* 1. *Then there exist a subsequence $\{\gamma_{n_j}\}$ and a sequence of integers $\{k(j)\}$ such that $\gamma_{n_j}^{k(j)}$ converges to a transformation γ which is neither* 1 *nor an elliptic transformation of order* 2.

Proof. By replacing $\{\gamma_n\}$ with a suitable subsequence, if necessary, we may assume that $\{\gamma_n\}$ contains only elliptic or parabolic or hyperbolic elements. Here we give a proof only for the case where $\{\gamma_n\}$ contains only elliptic elements, but other cases can be treated in a similar manner.

Assume that γ_n is elliptic of order ν_n. Let p_n be the fixed point of γ_n in Δ. Then γ_n^k corresponds to the matrix in $SL(2,\mathbb{C})$

$$\frac{1}{p_n-\bar{p}_n^{-1}}\begin{bmatrix}\lambda_n^{-k}p_n-\lambda_n^k\bar{p}_n^{-1} & (\lambda_n^k-\lambda_n^{-k})p_n\bar{p}_n^{-1}\\ -(\lambda_n^k-\lambda_n^{-k}) & \lambda_n^kp_n-\lambda_n^{-k}\bar{p}_n^{-1}\end{bmatrix},$$

where $\lambda_n=e^{2\pi i/\nu_n}$. Since $\gamma_n\to 1$, we have that $\nu_n\to\infty$ and $(\lambda_n-\lambda_n^{-1})/(p_n-\bar{p}_n^{-1})\to 0$ as $n\to\infty$. Choose a subsequence $\{\gamma_{n_j}\}$ so that p_{n_j} converges to a point p. If $|p|<1$, then let $k(j)$ be a nearest integer to $\nu_n/3$. Then $\gamma_{n_j}^{k(j)}$ converges to the elliptic transformation γ of order 3 represented by the matrix

$$\frac{1}{p-\bar{p}^{-1}}\begin{bmatrix}\lambda^{-1}p-\lambda\bar{p}^{-1} & (\lambda-\lambda^{-1})p\bar{p}^{-1}\\ -(\lambda-\lambda^{-1}) & \lambda p-\lambda^{-1}\bar{p}^{-1}\end{bmatrix},$$

where $\lambda=e^{2\pi i/3}$. If $|p|=1$, then choose $k(j)$ so that $|\lambda_{n_j}^{k(j)}-\lambda_{n_j}^{-k(j)}|/|p_{n_j}-\bar{p}_{n_j}^{-1}|$ $(j=1,2,\ldots)$ are bounded and bounded below by a positive number. Then, by replacing $\{\gamma_{n_j}\}$ with some subsequence, we may assume that $(\lambda_{n_j}^{k(j)}-$

$\lambda_{n_j}^{-k(j)})/(p_{n_j}-\bar{p}_{n_j}^{-1})$ converges to a number $\zeta \in \mathbb{C}-\{0\}$. Then $|\gamma_{n_j}^{k(j)}(\infty)|$ converges to $|\zeta|^{-1}(1+|\zeta|^2)^{1/2}>1$. By Lemma 2.1 $\gamma_{n_j}^{k(j)}$ converges to a Möbius transformation γ. Since $|p_{n_j}-\bar{p}_{n_j}^{-1}|\to 0$, we have that $|\lambda_{n_j}^{k(j)}-\lambda_{n_j}^{-k(j)}|\to 0$. Thus it follows that $|\mathrm{tr}\,\gamma_{n_j}^{k(j)}|^2=|(\lambda_{n_j}^{k(j)}-\lambda_{n_j}^{-k(j)})^2+4|\to 4$ and hence that $|\mathrm{tr}\,\gamma|^2=4$. Since $|\gamma(\infty)|=|\zeta|^{-1}(1+|\zeta|^1)^{1/2}\neq\infty$, γ cannot be the identity transformation. Now we can conclude that γ is a parabolic transformation.

LEMMA 2.3. *Let* $k(z)=z+z^{-1}$ *be defined in* Δ^*. *Then,*

(i) $[k](z)=6(z^2-1)^{-2}$;

(ii) $\rho^*(z)^{-2}\,|[k](z)|\leqq 6$, *where the equality holds if and only if* z *lies on the segment* $A_0^*=\{z\in\Delta^*:\mathrm{Im}\,z=0\}\cup\{\infty\}$;

(iii) *If* γ *is a transformation in* Möb (Δ) *for which*

$$[k](z)=[k](\gamma(z))\gamma'(z)^2, \tag{2.1}$$

then γ *is either the identity transformation, or an elliptic transformation of order* 2 *which preserves* A_0^*, *or a hyperbolic transformation with the axis* A_0^*.

Proof. Conclusions (i) and (ii) are verified by a direct calculation. Now let $\gamma(z)=(az+\bar{b})/(bz+\bar{a})$ be a transformation satisfying (2.1). Then (i) yields

$$(z^2-1)^2=((a^2-b^2)z^2+2(a\bar{b}-\bar{a}b)z-(\bar{a}^2-\bar{b}^2))^2.$$

Comparing the coefficients of the above polynomials in z, it follows that $a\bar{b}=\bar{a}b$ and $a^2+\bar{b}^2=(-a+\bar{b})^2/(-b+\bar{a})^2=1$. Therefore γ preserves the set of two points $\{1,-1\}$. If γ fixes each of 1 and -1, then γ is either the identity transformation or a hyperbolic transformation with the axis A_0^*. If γ interchanges 1 and -1, then γ is elliptic of order 2 and preserves A_0^*.

3. Proof of theorem 1.1.(I)

In this section we shall prove the "only if" part of Theorem 1.1, that is, that a fuchsian group Γ with $o(\Gamma)=6$ satisfies one of the conditions (O_1) and (O_2). The proof is rather lengthy, hence we divide it into several steps (3.1–3.5).

3.1. Suppose that $o(\Gamma)=6$ holds for a fuchsian group Γ. Let $\{\varepsilon_n\}$ be a sequence of positive numbers decreasing to 0. By definition there exists a sequence of bounded quadratic differentials $\{\phi_n\}$ in $T(\Gamma)$ for which $6-\varepsilon_n<\|\phi_n\|\leqq 6$. Choose a point z_n of Δ^* so that $6-\varepsilon_n<\rho^*(z_n)^{-2}\,|\phi_n(z_n)|\leqq 6$. Here by replacing z_n with another point nearby if necessary, we may assume that z_n is not fixed by any element of $\Gamma - \{1\}$. Conjugate Γ by a transformation h_n in Möb (Δ) with $h_n(\infty) = z_n$ and denote $h_n \Gamma h_n^{-1}$ by Γ_n. (Here we remark that, by the choice of z_n, the Ford region for Γ_n can be defined. We use this fact in the next paragraph.) It is easy to see that $\psi_n(z) = \phi_n(h_n(z))h_n'(z)^2$ belongs to $T(\Gamma_n)$, and that $6 - \varepsilon_n < \rho^*(\infty)^{-2} |\psi_n(\infty)| \leqq 6$. Let f_n be the solution of the differential equation $[f_n](z) = \psi_n(z)$, which has the following normalized form

$$f_n(z) = z + b_{n1}z^{-1} + b_{n2}z^{-2} + \cdots .$$

Here we may assume that b_{n1} is real positive, because if otherwise, we need only to replace $h_n(z)$ with $h_n(e^{-i\theta}z)$ and $f_n(z)$ with $e^{i\theta}f_n(e^{-i\theta}z)$, where $\theta = -(1/2) \arg b_{n1}$. Since $\psi_n \in T(\Gamma_n)$, f_n is univalent in Δ^*. Then the area theorem (cf. [9; p. 19]) yields

$$b_{n1}^2 + \sum_{\nu=2}^{\infty} \nu \, |b_{n\nu}|^2 \leqq 1. \tag{3.1}$$

Since $6 - \varepsilon_n < \rho^*(\infty)^{-2} |\psi_n(\infty)| = 6b_{n1} \leqq 6$, b_{n1} converges to 1. Then the inequality (3.1) implies that f_n converges to $k(z) = z + z^{-1}$ locally uniformly in Δ^* with respect to the spherical metric of $\hat{\mathbb{C}}$. Also, ψ_n converges to $[k](z) = 6(z^2 - 1)^{-2}$ locally uniformly in Δ^* with respect to the spherical metric.

3.2. For each n we choose a $\gamma_n \in \Gamma_n - \{1\}$ in such a way that the radius r_n of the isometric circle of γ_n is the largest of those of the elements in $\Gamma_n - \{1\}$. If $\{r_n\}$ contains a subsequence $\{r_{n_j}\}$ which converges to 0, then the Ford region $\mathscr{F}(\Gamma_{n_j})$ converges to Δ^*. This means that Γ satisfies the condition (O_1), since Γ and Γ_{n_j} are conjugate in Möb (Δ) and we can find a disc contained in $\mathscr{F}(\Gamma_{n_j})$ whose hyperbolic radius tends to ∞ as j tends to ∞. Therefore we need only to consider the case that $\{r_n\}$ is bounded below by a positive number r_0. Then, for $\gamma_n(z) = (a_n z + \bar{b}_n)/(b_n z + \bar{a}^n)$, $|a_n|^2 - |b_n|^2 = 1$, we have $|\gamma_n(\infty)| = |a_n/b_n| \geqq (1 + r_0^2)^{1/2}$. By Lemma 2.1 it follows that a subsequence of $\{\gamma_n\}$ converges to a Möbius transformation γ_0. For convenience we denote this subsequence again by $\{\gamma_n\}$.

Since $\gamma_n \in \Gamma_n$, it holds that $\psi_n(z) = \psi_n(\gamma_n(z))\gamma_n'(z)^2$. By letting $n \to \infty$ in this equation, we obtain that $[k](z) = [k](\gamma_0(z))\gamma_0'(z)^2$. By Lemma 2.3, γ_0 is either the identity transformation or an elliptic transformation preserving A_0^* or a hyperbolic transformation with the axis A_0^*.

3.3. The above sequence $\{\gamma_n\}$ contains only finitely many parabolic elements and elliptic elements of order >2. To see this, assume that $\{\gamma_n\}$ contains an infinite subsequence $\{\eta_n\}$, where η_n is either parabolic or elliptic of order >2. As the limit of η_n, γ_0 is the identity transformation. By Lemma 2.2 there exist a subsequence $\{\eta_{n_j}\}$ and a sequence of integers $\{k(j)\}$ such that $\eta_{n_j}^{k(j)}$ converges to a Möbius transformation η which is neither the identity nor elliptic of order 2. As the limit of parabolic or elliptic elements, η is not hyperbolic. By letting $j \to \infty$ in the equation $\psi_{n_j}(z) = \psi_{n_j}(\eta_{n_j}^{k(j)}(z))(\eta_{n_j}^{k(j)})'(z)^2$, we obtain that $[k](z) = [k](\eta(z))\eta'(z)^2$. However this contradicts Lemma 2.3. In a similar manner we can show that for a subsequence $\{\eta_n\}$ of hyperbolic elements in $\{\gamma_n\}$, if any, the axis of η_n converges to A_0^*. Assume that the axis of η_n does not converge to A_0^*. Then γ_0 is again the identity transformation. Choose a subsequence $\{\eta_{n_j}\}$ and a sequence of integers $\{k(j)\}$ so that $\eta_{n_j}^{k(j)}$ converges to a nontrivial transformation η. As the limit of hyperbolic elements η is either parabolic or hyperbolic. If η is hyperbolic, then by the assumption the axis of η is different from A_0^*. Then as above we can deduce a contradiction.

Now, by eliminating a finite number of elements and choosing a subsequence, we can assume that one of the following two cases occurs for $\{\gamma_n\}$:

Case (A). $\{\gamma_n\}$ contains only elliptic elements of order 2, and γ_0 is an elliptic transformation of order 2 and preserves A_0^*.

Case (B). $\{\gamma_n\}$ contains only hyperbolic elements, and γ_0 is either the identity transformation or a hyperbolic transformation. Moreover the axis $A^*(\gamma_n)$ converges to A_0^*.

We shall consider the two cases (A) and (B) separately.

3.4. Case (A). In this case we shall show that Γ satisfies the condition (O_1), or otherwise that we can transfer the argument to Case (B).

We take a $\delta_n \in \Gamma_n$ in such a way that the radius r_n' of the isometric circle of δ_n is the largest of those of elements in $\Gamma_n - \{1, \gamma_n\}$. If $\{r_n'\}$ contains a subsequence converging to 0, then we can see easily that Γ satisfies the condition (O_1). On the other hand if $\{r_n'\}$ is bounded below by a positive number, then by Lemma 2.1 a subsequence of $\{\delta_n\}$ converges to a Möbius transformation δ_0. Choosing a subsequence and relabelling if necessary, we can assume that Case (A) or (B) occurs for $\{\delta_n\}$. If Case (A) occurs, then both γ_n and δ_n are elliptic of order 2. Since $\gamma_n \neq \delta_n$, $\gamma_n\delta_n$ is hyperbolic and converges to $\gamma\delta$. By applying the argument in 3.3 we can show that Case (B) occurs for $\{\gamma_n\delta_n\}$. Therefore Case (B) occurs for $\{\delta_n\}$ or $\{\gamma_n\delta_n\}$.

3.5. Case (B). In this case we shall show that Γ satisfies the condition (O_2).

Let R_n be the region exterior to the isometric circles of γ_n and γ_n^{-1}. Among the elements of $\Gamma_n - \mathrm{Stab}\,(\Gamma_n, A^*(\gamma_n))$ whose isometric circles meet R_n, choose a δ_n which has the isometric circle of the largest radius r_n'. Assume that $\{r_n'\}$ is bounded below by a positive number. By Lemma 2.1 we can find a subsequence, which is denoted again by $\{\delta_n\}$, of $\{\delta_n\}$ converging to a Möbius transformation δ_0. As in 3.2 we can show that δ_0 preserves A_0^*. For $n \geqq 0$ denote by G_n the group generated by γ_n and δ_n. Since $\delta_n \in \Gamma_n - \operatorname{Stab}(\Gamma_n, A^*(\gamma_n))$, G_n is non-elementary for $n \geqq 1$. Let $\chi_n: G_0 \to G_n$ be the mapping which is the canonical extension of the correspondings $\chi_n(\gamma_0) = \gamma_n$ and $\chi_n(\delta_0) = \delta_n$. By [4, Proposition 1 and Theorem 2] G_n is fuchsian and χ_n is homomorphism of G_0 onto G_n for sufficiently large n. Thus G_0 is a non-elementary fuchsian group. However G_0 is elementary because both γ_n and δ_n preserve A_0^*. This is a contradiction. Therefore $\{r_n'\}$ contains a subsequence converging to 0. Since Γ is conjugate to Γ_n in $\text{Möb}(\Delta)$, Γ satisfies the condition (O_2). The first half of the proof is now completed.

4. Proof of Theorem 1.1.(II)

In this section we shall prove the "if" part of Theorem 1.1 and complete the proof.

4.1. We consider also the action of a fuchsian group Γ on the unit disc Δ with the hyperbolic metric $\rho(z)\,|dz| = (1-|z|^2)^{-1}\,|dz|$. Let $R(z) = 1/\bar{z}$ be the reflection with respect to the unit circle. Note that R is an isometry of $(\Delta^*, \rho^*\,|dz|)$ onto $(\Delta, \rho\,|dz|)$ and that R conjugate Γ to itself; $R^{-1}\Gamma R = \Gamma$. Let A_0 be the segment $\{z \in \Delta: \operatorname{Im} z = 0\}$. If $\text{Möb}(A_0)$ denotes the subgroup of $\text{Möb}(\Delta)$ which preserves A_0 and also A_0^*, then each element γ of $\text{Möb}(A_0) - \{1\}$ has the following form:

$$\gamma(z) = \frac{(1+\lambda)z + (1-\lambda)}{(1-\lambda)z + (1+\lambda)}, \qquad \lambda > 0 \quad \text{and} \quad \lambda \neq 1, \tag{4.1a}$$

if γ is hyperbolic, and

$$\gamma(z) = \frac{(1+\lambda)z - (1-\lambda)}{(1-\lambda)z - (1+\lambda)}, \qquad \lambda > 0 \quad \text{and} \quad \lambda \neq 1, \tag{4.1b}$$

if γ is elliptic of order 2. Neither a parabolic transformation nor an elliptic transformation of order $\neq 2$ belongs to $\text{Möb}(A_0)$.

Let V be the segment $\{\alpha: -3/2 < \alpha < 1/2\}$. For each α in V, define a function μ_α *in* $\hat{\mathbb{C}}$:

$$\mu_\alpha(z) = \begin{cases} (\delta(\alpha) - 1)(1-z^2)/(1-\bar{z}^2) & \text{for } z \in \Delta, \\ 0 & \text{for } z \in \bar{\Delta}^*, \end{cases} \tag{4.2}$$

where $\delta(\alpha) = (1-2\alpha)^{1/2}$ (with $\delta(0)=1$). Note that ess. sup $|\mu_\alpha| = |\delta(\alpha)-1| < 1$. A direct calculation using (4.1a) and (4.1b) yields:

$$\mu_\alpha(z) = \mu_\alpha(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) \quad \text{for all} \quad \gamma \in \text{Möb}\,(A_0). \tag{4.3}$$

Let $h(z) = i(1-z)/(1+z)$ be a Möbius transformation sending Δ^* onto the lower half plane $\mathscr{H}^*$. Then the Beltrami equation

$$w_{\bar{z}} = \mu_\alpha w_z$$

has a solution W_α expressed by

$$W_\alpha(z) = \begin{cases} -2i\delta(-i)^\delta/[h(z)\overline{h(z)}^{\delta-1} - (-i)^\delta] & \text{for} \quad z \in \Delta \\ -2i\delta(-i)^\delta/[h(z)^\delta - (-i)^\delta] & \text{for} \quad z \in \bar{\Delta}^*, \end{cases}$$

with $\delta = \delta(\alpha)$ and a fixed branch of z^δ in $\mathscr{H}^*$. The function W_α is a quasiconformal automorphism of $\hat{\mathbb{C}}$, conformal in Δ^* and has the normalized form: $W_\alpha(z) = z + O(|z|^{-1})$ as $z \to \infty$. Moreover, $[W_\alpha](z) = 4\alpha(z^2-1)^{-2}$ in Δ^*.

Remark. We learned the above construction of the function W_α from a paper by Kalme [5].

For the remainder of this section, the notion of convergence of functions defined in $\hat{\mathbb{C}}$ will be considered in the spherical metric.

4.2. First we shall show that $o(\Gamma) = 6$ holds for any fuchsian group Γ satisfying the condition (O_2). Now let Γ satisfy (O_2). Choose a sequence of positive numbers $\{d_n\}$ increasing to ∞. Conjugate Γ by a transformation h_n in Möb (Δ) so that a hyperbolic element γ_n of $\Gamma_n = h_n\Gamma h_n^{-1}$ has A_0 as its axis and so that the collar $U_0(d_n)$ of width d_n about A_0 exists. Let $G_n = \text{Stab}\,(\Gamma_n, A_0)$. By the definition of collar:

$$\begin{aligned} &\gamma(U(d_n)) = U_0(d_n) \quad \text{for} \quad \gamma \in G_n \quad \text{and} \\ &\gamma(U_0(d_n)) \cap U_0(d_n) = \varnothing \quad \text{for} \quad \gamma \in \Gamma_n - G_n. \end{aligned} \tag{4.5}$$

Define functions $\mu_{\alpha,n}$ $(n = 1, 2, \ldots)$ in $\hat{\mathbb{C}}$:

$$\mu_{\alpha,n}(w) = \begin{cases} \mu_\alpha(z)\gamma'(z)/\overline{\gamma'(z)} & \text{if } w = \gamma(z) \text{ for some } z \in U_0(d_n) \\ & \text{and for some } \gamma \in \Gamma_n. \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

Since $G_n \subset \text{Möb}(A_0)$, we can see by (4.3) and (4.5) that $\mu_{\alpha,n}$ is well defined. The support of $\mu_{\alpha,n}$ is in the closure of $\bigcup_{\gamma\in\Gamma_n} \gamma(U_0(d_n))$ and in particular $\mu_{\alpha,n}=0$ in Δ^*. By definition $\mu_{\alpha,n}$ is a Beltrami coefficient for Γ_n, that is,

$$\text{ess. sup}\,|\mu_{\alpha,n}| = |\delta(\alpha)-1| < 1, \tag{4.7}$$

and

$$\mu_{\alpha,n}(z) = \mu_{\alpha,n}(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) \quad \text{for all} \quad \gamma\in\Gamma_n. \tag{4.8}$$

4.3. Let $W_{\alpha,n}$ be the solution of the Beltrami equation

$$(W_{\alpha,n})_{\bar z} = \mu_{\alpha,n}(W_{\alpha,n})_z$$

which satisfies the normalization condition: $W_{\alpha,n}(z) = z + O(|z|^{-1})$ as $z\to\infty$. Let $K(\alpha) = (1+|\delta(\alpha)-1|)/(1-|\delta(\alpha)-1|)$. As a consequence of (4.6), (4.7) and (4.8) $W_{\alpha,n}$ is $K(\alpha)$-quasiconformal automorphism of $\hat{\mathbb{C}}$, conformal in Δ^* and compatible with Γ_n. From the normalization condition $\{W_{\alpha,n}\}_{n=1}^{\infty}$ forms a normal family ([6, Chap. II]). On the other hand the support of $\mu_\alpha - \mu_{\alpha,n}$ is included in $\Delta - \mathrm{U_0(d_n)}$. Since the Lebesgue measure of $\Delta - U_0(d_n)$ decreases to 0 as $n\to\infty$, a subsequence of $\{\mu_{\alpha,n}\}$ converges a.e. to μ_α ([10, Chap. 4, Proposition 17]). by replacing $\{\mu_{\alpha,n}\}$ and $\{W_{\alpha,n}\}$ with suitable subsequences, and denoting them again by $\{\mu_{\alpha,n}\}$ and $\{W_{\alpha,n}\}$, the following situation arises:

(i) $W_{\alpha,n}$ converges uniformly to a $K(\alpha)$-quasiconformal automorphism $W_{\alpha,0}$ of $\hat{\mathbb{C}}$.

(ii) $\mu_{\alpha,n}$ converges a.e. to μ_α.

Then $W_{\alpha,n}$ is a good approximation to $W_{\alpha,0}$ in the sense given in [6, Chap. IV, 5.4]. Therefore, by the normalization condition, $W_{\alpha,0} = W_\alpha$. Let $\psi_{\alpha,n}(z) = [W_{\alpha,n}](z)$ for $z\in\Delta^*$. Then $\psi_{\alpha,n}\in T(\Gamma_n)$ and $\psi_{\alpha,n}(z)$ converges to $[W_\alpha](z) = 4\alpha(z^2-1)^{-2}$ locally uniformly in Δ^*. In particular it follows that $\|\psi_{\alpha,n}\|\to 4|\alpha|$.

Recall that $\Gamma_n = h_n\Gamma h_n^{-1}$ for some Möbius transformation h_n. Thus, $\phi_{\alpha,n}(z) = \psi_{\alpha,n}(h_n(z))h_n'(z)^2$ belongs to $T(\Gamma)$ and $\|\phi_{\alpha,n}\| = \|\psi_{\alpha,n}\|\to 4|\alpha|$. Choose a positive number ε to be sufficiently small so that $\alpha = -3/2+\varepsilon/4\in V$. Then for sufficiently large n we have that $6-2\varepsilon < \|\phi_{\alpha,n}\|$. Since ε can be arbitrarily small and $o(\Gamma)\leqq 6$ always holds, we can now conclude that $o(\Gamma)=6$, whenever Γ satisfies the condition (O_2).

4.4. The proof for the case where Γ satisfies the condition (O_1) proceeds in a way similar to the preceding case. Now let Γ satisfy (O_1). For a choice of a sequence of positive numbers $\{\delta_n\}$ decreasing to 0, let $D_n = \{z\in\Delta\colon |z|<1-\delta_n\}$. Then we can conjugate Γ by a transformation h_n in $\text{Möb}(\Delta)$ in such a way that

$\Gamma_n = h_n \Gamma h_n^{-1}$ has the following property:

$$\gamma(D_n) \cap D_n = \varnothing \quad \text{for all} \quad \gamma \in \Gamma_n - \{1\}.$$

For an $\alpha \in V$, define functions $\mu_{\alpha,n}$ $(n = 1, 2, \ldots)$ in $\hat{\mathbb{C}}$:

$$\mu_{\alpha,n}(w) = \begin{cases} \mu_\alpha(z)\gamma'(z)/\overline{\gamma'(z)} & \text{if } w = \gamma(z) \text{ for some } z \in D_n \\ & \text{and for some } \gamma \in \Gamma_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then each $\mu_{\alpha,n}$ is a Bertrami coefficient for Γ_n with ess. $\sup|\mu_{\alpha,n}| = |\delta(\alpha) - 1| < 1$. Note that the support of $\mu_\alpha - \mu_{\alpha,n}$ is included in $\Delta - D_n$, whose Lebesgue measure decreases to 0. With the normalized solution $W_{\alpha,n}$ of the Beltrami equation $(W_{\alpha,n})_{\bar z} = \mu_{\alpha,n}(W_{\alpha,n})_z$, now we are in the same situation as in 4.3. Thus by the same argument there we can conclude in this case also that $o(\Gamma) = 6$. Now we complete the proof of Theorem 1.1.

5. Examples

We list some examples of fuchsian groups concerning the conditions (O_1) and (O_2).

(a) Any fuchsian group Γ of the second kind satisfies the condition (O_1). Thus for such a group we have $o(\Gamma) = 6$. Theorem 1.1 includes the former result by Sekigawa and Yamamoto [12], [13].

(b) For each integer n, define two circles $A_n = \{z\colon |z - 2n + 1/2| = 1/2\}$ and $B_n = \{z\colon |z - 2n - 1/2| = 1/2\}$. Then $\gamma_n(z) = ((1 + 4n)z - 8n^2)/(2z + 1 - 4n)$ maps the exterior of A_n onto the interior of B_n. The collection $\{\gamma_n\}$ generates a subgroup Γ of the modular group PSL (2, Z). The region in the upper half plane which is exterior to every A_n and B_n is a fundamental region of Γ and it contains $\{z\colon \mathrm{Im}\, z > 1\}$. Since we can find there a hyperbolic disc of an arbitrary radius, Γ is a fuchsian group of the first kind satisfying the condition (O_1).

(c) Let R be the complex plane with infinitely many punctures $\mathbb{C} - \{2^{-n^2}, -2^{-n^2}\}_{n=1}^{\infty}$ endowed with the hyperbolic metric $\rho_R(z)\,|dz|$. Then the annulus $A_n = \{z\colon 2^{-(n+1)^2} < |z| < 2^{-n^2}\}$ is included in R. Let g_n be a simple closed curve in A_n which separates the two boundary components of A_n. There exists a unique simple closed geodesic g_n' freely homotopic to g_n in R. Denote its length by $l(g_n')$. Next let g_n'' be the simple closed geodesic freely homotopic to g_n with respect to the hyperbolic metric of the annulus A_n. Denote by $l(g_n'')$ and $\tilde{l}(g_n'')$ its length with respect to the hyperbolic metric of R and that of A_n, respectively. Since $A_n \subset R$, it holds that $\rho_R(z) < \rho_{A_n}(z)$ for $z \in A_n$, where ρ_{A_n} is the density of the hyperbolic metric of A_n. Therefore we have $l(g_n') \leqq l(g_n'') < \tilde{l}(g_n'')$. Moreover we have that $\tilde{l}(g_n'') \to 0$ as $n \to \infty$ since the module $\log 2^{(n+1)^2}/2^{n^2}$ of A_n tends to ∞. Thus, for any $\varepsilon > 0$, there exists a simple closed geodesic on R whose length is smaller than ε. Then the collar lemma ([3]) ensures that a fuchsian group representing R satisfies the condition (O_2).

(d) A cyclic group Γ generated by a hyperbolic transformation in Möb(Δ) and its extension of index 2 satisfy both (O_1) and (O_2).

(e) A finitely generated fuchsian group Γ of the first kind satisfies neither of the conditions, and hence the outradius $o(\Gamma)$ is strictly less than 6. This result is first proved by Sekigawa [11]. On the other hand, there exist fuchsian groups Γ_n $(n = 1, 2, \ldots)$ quasiconformally equivalent to Γ, for which $o(\Gamma_n) \to 6$. This is proved in a generalized form in [7].

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