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Holomorphic group actions with many compact orbits

CHRISTOPH GELLHAUS and TILMANN WURZBACHER

Introduction

The purpose of this note is to classify manifolds with holomorphic group actions such that on an open set all orbits are compact. The first result in this direction which reflects the spirit of the present article is due to Holmann: Let X be a connected compact complex space and $\mathbb{C} \times X \to X$ a holomorphic action such that all orbits are compact. Then the $\mathbb{C}$-action factorizes through the action of an one-dimensional compact complex torus (see [Ho2]), i.e. there is a torus $T < \mathrm{Aut}_{\mathcal{O}}(X)$ and the following diagram commutes:

$$\begin{tikzcd} \mathbb{C} \times X \arrow[r] \arrow[d] & X \\ T \times X \arrow[ur] & \end{tikzcd} .$$

For theorems of this type the compactness of the ambient manifold X is obviously necessary. Even in the compact case the following example shows that an analogous result is not true in general: Let G be the unipotent complex group of upper (3×3)-matrices and Γ the subgroup with Gaussian integers as entries. Then the two-dimensional abelian subgroup A of G with a zero as the middle entry of the last column acts holomorphically on the Iwasawa-manifold $X = G/\Gamma$ with compact orbits. However a direct computation shows that the A-action does not factorize through a torus-action.

Motivated by earlier work on automorphism groups of Kähler manifolds ([Ma2], [S], [F]) we consider a class $\mathcal{F}$ of compact complex manifolds which contains for example all meromorphic images of compact Kähler manifolds (see §1 for the details). Our main result is the following (see §3):

THEOREM. *Let X be in $\mathcal{F}$ and G a connected complex subgroup of* $\mathrm{Aut}_{\mathcal{O}}(X)$. *Suppose there is an open set Ω in X such that the G-orbits in Ω are closed. Then:*

(1) *G is a product $S \times T$ where S is semi-simple and T a compact complex torus.*

(2) *X is G-equivariant a product of a homogeneous-rational manifold Q and a manifold M in $\mathcal{F}$ such that $G = S \times T$ acts as a product on $Q \times M$.*
(3) *$X/G = M/T$ is a normal complex space and M is a holomorphic Seifert principal fibre bundle over M/T.*

In particular all G-orbits are compact. Furthermore there is a compact subgroup of G which acts transitively on the orbits. In the case of a $\mathbb{C}^k$-action for arbitrary k the action always factorizes through a holomorphic torus-action.

§1. A class of compact complex manifolds

Let X be a compact complex manifold. We will denote the group of biholomorphic automorphisms of X by $\mathrm{Aut}_{\mathcal{O}}(X)$ which is well-known to be a complex Lie group. The compactness of X implies that every holomorphic vector field $V \in H^0(X, TX)$ has a complete holomorphic flow, thus we can identify $H^0(X, TX)$ with the Lie algebra $\mathrm{aut}_{\mathcal{O}}(X)$ of $\mathrm{Aut}_{\mathcal{O}}(X)$.

The Albanese map ψ_X from X to its Albanese torus $\mathrm{Alb}(X)$ is $\mathrm{Aut}_{\mathcal{O}}(X)$-equivariant, because of its universality property (see [Bl]). Thus there is a Lie group homomorphism $\lambda_X : \mathrm{Aut}_{\mathcal{O}}(X) \to \mathrm{Aut}_{\mathcal{O}}(\mathrm{Alb}\, X)$, the Jacobi-homomorphism, and an induced homomorphism of the respective Lie algebras, denoted by ρ_X. The kernel of ρ_X is called $\mathscr{L}(X)$. In the Kählerian case the following characterization is well-known (see e.g. [Ma2])

$$\mathscr{L}(X) = \{V \in \mathrm{aut}(X) \mid \omega(V) = 0\ \forall \omega \in H^0(X, \Omega^1_X)\}.$$

For the study of the orbit structure of a holomorphic group action on X it is natural to consider the following compatibility condition:

DEFINITION 1.1. *A compact complex manifold X is in class $\mathcal{F}$ if and only if for every closed complex submanifold Y in X*

$$\mathrm{rest}_Y(\mathscr{L}(X) \cap \mathrm{stab}_{\mathrm{aut}(X)} Y) \subset \mathscr{L}(Y),$$

where rest_Y denotes the restriction of vector fields to Y.

Before proceeding, it is convenient to summarize some elementary facts on tori:

LEMMA 1.2. *Let Z be a compact complex torus and $\mathbb{C}^k \times Z \to Z$ a holomorphic action on Z. Then*

(1) *All isotropy groups* $\mathrm{Stab}_{\mathbb{C}^k}\{x\}$ *are equal.*
(2) *If one orbit is closed, then all orbits are biholomorphic equivalent to a fixed torus T and the $\mathbb{C}^k$-action factorizes through a T-action on Z.*
(3) *Suppose $V \in \mathrm{aut}(Z)$ has a zero on Z, then V vanishes on Z.*

Recall that a Lie algebra $\mathfrak{g}$ is a semi-direct product $\mathfrak{r} \rtimes \mathfrak{s}$, where the "radical" $\mathfrak{r}$ is defined to be the maximal solvable ideal and $\mathfrak{s}$ is a maximal semi-simple subalgebra. Let Y be a closed submanifold of X, $\mathfrak{g} := \mathscr{L}(X) \cap \mathrm{stab}_{\mathrm{aut}(X)} Y$, and let $\rho_Y : \mathfrak{g} \to \mathrm{aut}(\mathrm{Alb}(Y))$ be the Jacobi-homomorphism. Then $\rho_Y(\mathfrak{s}) = \{0\}$ and $\rho_Y(\mathfrak{r})$ is abelian.

Now, if X is a Hodge manifold, then X admits a $\mathscr{L}(X)$-equivariant embedding in some $\mathbb{P}_N(\mathbb{C})$ (see e.g. [Ma2]). Thus $\mathfrak{g}$ is contained in $\mathrm{stab}_{\mathrm{aut}(\mathbb{P}_N(\mathbb{C}))} Y$. The Borel Fixed Point Theorem implies that the solvable algebra $\mathfrak{r}$ has a common zero in Y. Hence by (3) of Lemma 1.2 and equivariance of ψ_Y, $\mathfrak{r}$ acts trivially on $\mathrm{Alb}(Y)$. Thus *every Hodge manifold is in $\mathscr{F}$.*

The Borel Fixed Point Theorem for Kähler manifolds [S] implies by the same reasoning as above that *Kähler manifolds are in $\mathscr{F}$.*

A compact complex manifold X is said to be in class $\mathscr{C}$ whenever it is the meromorphic image of a compact Kähler space (see [F]). For example, Moišezon manifolds are in class $\mathscr{C}$. For smooth $X \in \mathscr{C}$ one has $H^0(X, \Omega_X^1) = \psi_X^*(H^0(\mathrm{Alb}\, X, \Omega^1_{\mathrm{Alb}\, X}))$ and thus $\mathscr{L}(X)$ coincides with the "linear" vector fields in the sense of [F]. The proof of Proposition 6.9 in [F] shows:

PROPOSITION 1.3. *Let $X \in \mathscr{C}$ be smooth and Y a $\mathfrak{r}$-stable closed analytic set in X, where $\mathfrak{r}$ is a solvable subalgebra of $\mathscr{L}(X)$. Then $\mathfrak{r}$ has a common zero in Y.*

Thus a *smooth manifold in $\mathscr{C}$ is in $\mathscr{F}$.* It should be mentioned that there are manifolds in class $\mathscr{F}$ which are not in class $\mathscr{C}$, in particular manifolds for which Proposition 1.3 is not true. Certain Inoue-surfaces without curves are examples of this [I].

We remark that the Iwasawa-manifold $X = G/\Gamma$ does not fulfill the compatibility condition under consideration. The Albanese map of X is the principal fibration of X induced by the right action of the center Z of G: $\psi_X : G/\Gamma \to G/(\Gamma \cdot Z)$. Let $Y = \psi_X^{-1}(\psi_X(e\Gamma)) = Z/(Z \cap \Gamma)$, then $\mathrm{Alb}(Y)$ equals Y and $\mathscr{L}(Y) = 0$. But the central vector field of the Lie algebra of G is in $\mathscr{L}(X) \cap \mathrm{stab}_{\mathrm{aut}(X)} Y$ and acts non-trivially on Y.

§2. Abelian group actions

To obtain the main result in the case of abelian Lie algebras, we need some preparations.

LEMMA 2.1. *Let X be a connected compact complex manifold and $T <$ $\mathrm{Aut}_{\mathcal{O}}(X)$ be a compact complex torus. Then*:

(1) $\dim T(x) = \dim T$ *for all* $x \in X$.

(2) *There is an open dense subset U of X with* $\mathrm{Stab}_T\{x\} = \{e\}$ *for all* $x \in U$.

(3) *The quotient X/T is a normal complex space and $X \to X/T$ is a holomorphic Seifert principal fibre bundle.*

Proof. Ad (1). Fix an arbitrary point x in X. Let d be a T-invariant metric on X and $\varepsilon > 0$ such that $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ is holomorphic separable. The connected component T_x^0 of the isotropy of T in x has compact connected complex orbits in $B_\varepsilon(x)$. Now the holomorphic separability of $B_\varepsilon(x)$ implies $T_x^0(y) = y$ for all $y \in B_\varepsilon(x)$. Thus T_x^0 acts trivially on $B_\varepsilon(x)$ and therefore on X.

Ad (2). In the theory of smooth actions of compact Lie groups the following theorem on the principal orbit type is well-known (see e.g. [J]): In a connected G-manifold X there is an open dense subset U such that the isotropy subgroups on U are all conjugate to a fixed subgroup $H < G$ and the conjugacy class (H) of H is the absolute minimum in the partially ordered set of conjugacy classes of isotropy subgroups of G.

In the abelian case this means that there is a fixed subgroup $S < T$ with $T_x = S$ for all x in U. It follows $S = \{e\}$.

Ad (3). This result is proved by Holmann (§2, Satz 1 in [Ho1]). □

LEMMA 2.2. *Let $\mathbb{C}^k \times X \to X$ be a holomorphic action on a complex manifold $X \in \mathcal{F}$ and assume that the induced Lie algebra of vector fields on X is k-dimensional. Suppose there is an open set $\Omega \subset X$ where all orbits are closed. Then the orbit dimension equals k throughout Ω.*

Proof. Fix $x \in \Omega$ and denote the isotropy algebra of $\mathbb{C}^k$ in x by $\mathfrak{b}$. Since $\mathfrak{b}$ stabilizes the image of x in $\mathrm{Alb}(X)$, it acts trivially on $\mathrm{Alb}(X)$ by Lemma 1.2. Hence $\mathfrak{b}$ lies in $\mathcal{L}(X) \cap \mathrm{stab}_{\mathrm{aut}(X)} Y$ for all orbits Y in Ω and thus, $X \in \mathcal{F}$ implies that $\mathfrak{b}$ acts trivially on Ω. Therefore $\mathfrak{b}$ acts trivially on X and $\mathfrak{b} = 0$. □

We now show that in the situation of Lemma 2.2 the action factorizes through a torus-action:

THEOREM 2.3. *Let $\mathbb{C}^k \times X \to X$ be a holomorphic action on a complex* *manifold X in class $\mathscr{F}$. Suppose there is an open set Ω in X where all orbits are closed. Then the $\mathbb{C}^k$-action factorizes through the actions of a torus, i.e. there is a torus $T < \mathrm{Aut}_{\mathcal{O}}(X)$ and the following diagram commutes:*

$$\begin{tikzcd} \mathbb{C}^k \times X \arrow[r] \arrow[d] & X \\ T \times X \arrow[ur] & \end{tikzcd}$$

Remark. For applications in the theory of completely integrable dynamical systems, we note that the theorem still remains true, if we only assume $\mu(\Omega) > 0$ for a suitable measure μ on X (for example the Liouville measure induced by a symplectic form).

Proof of Theorem 2.3. Without loss of generality we may assume that the induced Lie algebra of vector fields on X is k-dimensional. We will denote the Lie algebra of the $\mathbb{C}^k$-vector fields by $\mathbf{a} \subset \mathrm{aut}(X)$.

By Lemma 2.2 the orbit Y of a point y in Ω is a k-dimensional torus, therefore the isotropy Γ_y in y is a cocompact lattice in $\mathbb{C}^k$. Since $X \in \mathscr{F}$ one has:

$$(\mathscr{L}(X) \cap \mathbf{a}) \subset (\mathscr{L}(X) \cap \mathrm{stab}_{\mathrm{aut}(X)} Y) \xrightarrow{\mathrm{rest}_Y} \mathscr{L}(Y) = 0,$$

thus $\mathscr{L}(X) \cap \mathbf{a}$ is zero and all $\mathbb{C}^k$-orbits on $\mathrm{Alb}(X)$ are k-dimensional tori with fixed isotropy $\Gamma < \mathbb{C}^k$ by Lemma 1.2. Now we restrict the Albanese map of X to the orbit of an arbitrary point x in Ω:

$$\psi_X|_{\mathbb{C}^k(x)} : \mathbb{C}^k(x) = \mathbb{C}^k/\Gamma_x \xrightarrow{\Gamma/\Gamma_x} \mathbb{C}^k/\Gamma$$

and

$$\Gamma/\Gamma_x = (\psi_X|_{\mathbb{C}^k(x)})^{-1}(\psi_X(x))$$

is finite.

Since the index of Γ_x in Γ is finite for all x in Ω and there are only countable many lattices of finite index in Γ, say $\{\Gamma_n\}_{n \in \mathbb{N}}$, Ω is the union of countable many closed analytic sets: $\Omega = \bigcup_{n \in \mathbb{N}} (\mathrm{Fix}(\Gamma_n) \cap \Omega)$, where $\mathrm{Fix}(\Gamma_n)$ denotes the fixed point set of Γ_n. Thus a connected component of Ω is contained in $\mathrm{Fix}(\Gamma_{n_0})$ for some $n_0 \in \mathbb{N}$. Therefore the cocompact lattice Γ_{n_0} is in the ineffectivity Λ of the $\mathbb{C}^k$-action on X and the action factorizes through the action of the torus $T := \mathbb{C}^k/\Lambda < \mathrm{Aut}_{\mathcal{O}}(X)$. $\square$

§3. General group actions

The goal of this section is to prove the theorem stated in the introduction. As a first step we prove the Borel–Remmert theorem for a single compact orbit.

LEMMA 3.1. *Let Γ be a discrete subgroup in a Lie group G and Δ a finite subset of Γ. Then $Z_G(\Delta)\cdot\Gamma$ is closed in G.*

Proof. Let $\{x_n\}\subset Z_G(\Delta)$, $\{\gamma_n\}\subset\Gamma$ be sequences with $\lim_{n\to\infty} x_n\cdot\gamma_n = z\in G$. For γ in Δ one has

$$z^{-1}\cdot\gamma\cdot z=\lim_{n\to\infty}(\gamma_n^{-1}\cdot x_n^{-1}\cdot\gamma\cdot x_n\cdot\gamma_n)=\lim_{n\to\infty}(\gamma_n^{-1}\cdot\gamma\cdot\gamma_n)$$

and therefore $\gamma_n^{-1}\cdot\gamma\cdot\gamma_n$ converges in G and thus in Γ. Since Γ is discrete this sequence is constant for large $n\geq n_0(\gamma)$. Thus we find a $n_0\in\mathbb{N}$ with:

$$\gamma_{n+1}^{-1}\cdot\gamma\cdot\gamma_{n+1}=\gamma_n^{-1}\cdot\gamma\cdot\gamma_n \quad \text{for all } n\geq n_0 \text{ and for all } \gamma\in\Delta,$$

i.e. $\gamma_{n+1}\cdot\gamma_n^{-1}\in Z_G(\Delta)$ for $n\geq n_0$. This implies $\gamma_n=v_n\cdot\gamma_{n_0}$ for some $v_n\in Z_G(\Delta)$ for all $n\geq n_0$. Now, with $z_n:=x_n\cdot v_n$, it follows:

$$z_n\cdot\gamma_{n_0}=x_n\cdot v_n\cdot\gamma_{n_0}=x_n\cdot\gamma_n\xrightarrow{n\to\infty} z$$

and

$$z_n\xrightarrow{n\to\infty} z\cdot\gamma_{n_0}=:z'\in Z_G(\Delta)$$

since $Z_G(\Delta)$ is closed. Thus $z=z'\cdot\gamma_{n_0}$ is in $Z_G(\Delta)\cdot\Gamma$. □

We apply this more technical lemma to obtain:

LEMMA 3.2. *Let G be a complex Lie group and Γ a discrete cocompact subgroup of G. Then there is a complex subgroup $H<G$ such that $H/(H\cap\Gamma)$ is a positive dimensional compact torus in G/Γ.*

Proof. Without loss of generality we may assume G to be connected. If Γ is central in G, G/Γ is already a complex torus. Thus we can assume that there is an element δ in $\Gamma\backslash Z_G$.

Since the conjugation action of G on G is holomorphic, the sets

$$B_k(G)=\{x\in G \mid \dim_{\mathbb{C}} G(x)<k\}$$

are closed analytic sets in G (see e.g. [H, O]). Taking k equal to the dimension of G we get $B_k(G) = G$ since $\exp(\mathfrak{g})$ is contained in $B_k(G)$. Thus the centralizer of an element in G is always positive dimensional.

Let now G_1 be the connected component of $Z_G(\delta)$ for the above $\delta \in \Gamma \backslash Z_G$. We have: $\dim_{\mathbb{C}} G > \dim_{\mathbb{C}} G_1 > 0$ and by Lemma 3.1. $G_1 \cdot \Gamma$ is closed in G. Therefore $G_1/(G_1 \cap \Gamma)$ is compact in G/Γ and by induction we find $H < G$ such that $H \cdot (e\Gamma) = H/(H \cap \Gamma)$ is a compact complex torus in G/Γ. □

PROPOSITION 3.3. *Let X be in $\mathcal{F}$ and G a complex subgroup of* $\mathrm{Aut}_{\mathcal{O}}(X)$. *Suppose the orbit $G(x_0)$ through a point x_0 in X is connected and closed. Then $G(x_0) \cong T \times Q$, where T is a complex torus and Q a homogeneous-rational manifold.*

Proof. Without loss of generality we may assume G to be connected and the image of x_0 under the Albanese map of X to be the neutral element of $\mathrm{Alb}(X)$. Furthermore let us denote the isotropy of G in x_0 by H.

We consider the Tits fibration of the compact orbit

$$\pi : G(x_0) = G/H \xrightarrow{N/H} G/N =: Q,$$

where N is the normalizer of H^0 in G and the base Q is well-known to be homogeneous-rational. First we want to show that the restriction $\tilde{\psi} = \psi_X|_{N/H}$ is a finite map. Since $\tilde{\psi}$ is N-equivariant, the image $\tilde{\psi}(N/H) = \lambda_X(N) \cdot \psi_X(x_0)$ is a subtorus of $\mathrm{Alb}(X)$. The image of N in the automorphism group of N/H, denoted by M, has the same dimension as N/H. Thus $\tilde{\psi}$ is M-equivariant

$$\tilde{\psi} : N/H = M/\Gamma \xrightarrow{L/\Gamma} M/L,$$

where L is a closed subgroup of M, which contains a discrete subgroup Γ.

If $\tilde{\psi}$ is not finite L/Γ is positive dimensional and by Lemma 3.2 we find a subgroup L' of L such that $Y := L'/(L' \cap \Gamma)$ is a positive dimensional torus in L/Γ. Consider a vector field V in $\mathfrak{g}$ which is tangent to $Y \subset \psi_X^{-1}(\psi_X(x_0))$. Then, since X is in class $\mathcal{F}$, $V \in (\mathcal{L}(X) \cap \mathrm{stab}_{\mathrm{aut}(X)} Y) \xrightarrow{\mathrm{rest}_Y} \mathcal{L}(Y)$. But Y is a torus and thus $V|_Y \equiv 0$ which contradicts the construction of Y. Therefore Y cannot be positive dimensional, thus $\tilde{\psi} : N/H \to \psi_X(N/H) \subset \mathrm{Alb}(X)$ is finite and N/H is a torus.

Since $\pi_1(Q) = 1$, it follows that N is connected and therefore the Tits fibration

$$\pi : G/H \xrightarrow{N/H} G/N = Q$$

is a holomorphic torus principal bundle. In order to show that π is holomorphically trivial, we consider the Albanese map of the orbit:

$$G(x_0) = G/H \xrightarrow{J/H} G/J = \mathrm{Alb}\,(G(x_0)).$$

Let $G = R \cdot S$ be a Levi–Malcev decomposition of G. Now S acts trivially on $\mathrm{Alb}\,(G(x_0))$. Thus $S < J^0$ and consequently J^0 acts transitively on Q. Hence

$$\dim\,(J^0/(H \cap J^0)) \geq \dim\,(J^0/(N \cap J^0)) = \dim Q.$$

By the universality of $\mathrm{Alb}\,(G/H)$ we get

$$\dim \mathrm{Alb}\,(G/H) \geq \dim \psi(G/H) \geq \dim \psi(N/H) = \dim\,(N/H).$$

Therefore

$$\begin{aligned}\dim\,(J^0/(H \cap J^0)) &= \dim\,(J/H) = \dim\,(G/H) - \dim \mathrm{Alb}\,(G/H)\\ &\leq \dim\,(G/H) - \dim\,(N/H) = \dim\,(J^0/(N \cap J^0)).\end{aligned}$$

Hence

$$\hat{\pi} := \pi|_{J^0/(H \cap J^0)} : J^0/(H \cap J^0) \to J^0/(N \cap J^0)$$

has discrete fibers. The homotopy sequence of this bundle yields by $\pi_1(Q) = 1$ that $\hat{\pi}$ is already biholomorphic. Thus the J^0-orbit in G/H is a holomorphic section of the torus principal bundle given by π. □

The above single orbit decomposition is reflected in a striking way by the structure of the whole manifold.

THEOREM 3.4. *Let X be in $\mathcal{F}$ and G a connected complex subgroup of* $\mathrm{Aut}_{\mathcal{O}}(X)$. *Suppose there is an open set Ω in X such that the G-orbits in Ω are closed. Then*:

(1) *G is a product $S \times T$ where S is semi-simple and T a compact complex torus.*

(2) *X is G-equivariant a product of a homogeneous-rational manifold Q and a manifold M in $\mathcal{F}$ such that $G = S \times T$ acts as a product on $Q \times M$.*

(3) *$X/G = M/T$ is a normal complex space and M is a holomorphic Seifert principal fibre bundle over M/T.*

Proof. Ad (1). By Proposition 3.3 one has $G(x) = Q(x) \times T(x)$ for all x in Ω. Let $S \cdot R$ be a Levi–Malcev-decomposition for G. Since on an orbit in Ω, S acts only on the rational part and R only on the torus part, it follows that G is a product $S \times A$, where A is abelian. Now by Theorem 2.3 A is a torus $T < \mathrm{Aut}_{\mathcal{O}}(X)$.

Ad (2). Since each S-orbit in Ω is of the form S/P_x, where P_x is a parabolic subgroup of S, we first look for a "generic" P. Let K be a fixed maximal compact subgroup of S. Then, on Ω, S and K have the same orbits since each S-orbit is simply-connected (see [Mo]). For the K-action we have a principal orbit, i.e. there is an open dense set in X with orbit type (L) for some fixed $L < K$ (see e.g. [J]). Thus there is an open set $\Omega_1 \subset \Omega$ such that $S(x) = K(x) = K/L_x$ with $(L_x) = (L)$ for all x in Ω_1. Since K/L is homogeneous-rational, L contains a maximal compact real torus of K. The smallest complex subgroup of S containing this torus is a maximal torus of S in the sense of linear algebraic groups; we will denote this group by H. Obviously H is in $L^{\mathbb{C}}$.

Every S-orbit in Ω_1 contains a point x such that $L_x = L$. Denoting $\mathrm{Stab}_S\{x\}$ by P_x, $L = K \cap P_x$ implies $H < L^{\mathbb{C}} < P_x$ and therefore $\mathrm{Fix}(P_x) \subset \mathrm{Fix}(L^{\mathbb{C}}) = \mathrm{Fix}(L) \subset \mathrm{Fix}(H)$. Furthermore $0 < |\mathrm{Fix}(H) \cap S(x)| < \infty$ since H is contained only in finitely many Borel groups of S and each Borel group is contained only in finitely many parabolic subgroups of S (for these standard facts on linear algebraic groups see e.g. [Hu]). A fortiori $0 < |\mathrm{Fix}(L) \cap K(x)| < \infty$ holds for all x in Ω_1.

Now, since $L^{\mathbb{C}}$ is reductive, $\mathrm{Fix}(L) = \mathrm{Fix}(L^{\mathbb{C}})$ is the disjoint union of closed connected complex submanifolds of X: $\mathrm{Fix}(L) = \bigcup_{i=1}^{N} M_i$. We want to find a component M_{i_0} of $\mathrm{Fix}(L)$ such that $|M_{i_0} \cap K(x)| > 0$ for all x in a possibly smaller open set $\Omega_2 \subset \Omega_1$. Assume $K(x) \cap M_1 = \emptyset$ for some x in Ω_1. Then we find an open K-stable neighbourhood V of $K(x)$ in Ω_1 such that $V \cap M_1 = \emptyset$. Since $\mathrm{Fix}(L)$ has only finitely many components, inductively we find the desired Ω_2 such that for some component $M := M_{i_0}$ of $\mathrm{Fix}(L)$ we have $0 < |M \cap K(x)| < \infty$ for all x in Ω_2. It follows $\dim M + \dim S(x) = \dim X$ for all x in Ω_2.

The above argument on algebraic groups immediately implies that $L^{\mathbb{C}}$ lies only in finitely many parabolic groups, say $P_1, \ldots, P_k$. Let x be in $M \cap \Omega_2$ then $P_x = P_j$ for some $j \in \{1, \ldots, k\}$ and therefore $M \cap \Omega_2 = \bigcup_{j=1}^{k} (\mathrm{Fix}(P_j) \cap (M \cap \Omega_2))$. Thus there is a $P_{j_0} =: P$ such that $\mathrm{Fix}(P)$ and M have a common component on Ω_2. Hence M is contained in $\mathrm{Fix}(P)$. By the construction of Ω_2, on each S-orbit in Ω_2 there is a point x such that $P \subset P_x$ and thus by the constant orbit dimension on Ω_2 we have $(P_x) = (P)$ for all x in Ω_2.

The map $f: S \times M \to X$ defined by $f(s, x) = s(x)$ factorizes through a proper holomorphic map $\bar{f}: S/P \times M \to X$, since M is in $\mathrm{Fix}(P)$. Since $\Omega_2 \subset \mathrm{Im}(\bar{f})$, it follows that $\bar{f}$ is surjective by the proper mapping theorem of Remmert (see e.g. [CAS]).

Part (2) of the theorem will be proved if we can show that $\bar{f}$ is injective. First we observe that all S-orbits on X are closed, since the surjectivity of $\bar{f}$ implies that each S-orbit contains a point of M. Now K acts transitively on each S-orbit in X. Remember that by the principal orbit theorem there exists an open dense set U in X such that $(L_x) = (L)$ (L as above!) for all x in U. For x in $U \cap M$, $\bar{f}: S/P \times \{x\} \cong K/L \to K/L_x = S/P_x = S(x)$ is bijective. Thus $(P_x) = (P)$ for all x in U.

Now assume there is an x in M with $P_x \neq P$. i.e. $L_x \neq L$. Let d be a L_x-invariant metric on X and $\varepsilon > 0$ such that $B_\varepsilon(x) = \{z \in X \mid d(z, x) < \varepsilon\}$ is holomorphic separable. Since U is dense we find a y in $B_\varepsilon(x) \cap M \cap U$, in particular one has $L_y = L$. The L_x-orbit of y turns out to be complex by the following biholomorphisms:

$$S(y) \xrightarrow[\sim]{\bar{f}^{-1}} S/P \times \{y\} \xrightarrow[\sim]{\tau} S/P \times \{x\},$$

since $\tau \circ \bar{f}^{-1}(L_x(y))$ is the fibre through (eP, x) of $\bar{f}$ restricted to $S/P \times \{x\}$. Thus $L_x(y)$ is finite and in fact equal to $\{y\}$ by the homotopy sequence of the bundle given by $\bar{f}|_{S/P \times \{x\}}$. This shows $L_x = L$ and therefore S has fixed orbit type (P) on X. In particular we have that $\bar{f}: S/P \times \{x\} \to S(x)$ is bijective for all x in M and P has exactly one fixed point on each S-orbit in X.

Assume (sP, x) and $(s'P, x')$ are mapped to the same point of X under $\bar{f}$, then it follows that the S-orbit of x and x' are the same. By the uniqueness of the P-fixed point on an S-orbit x equals x'. But $\bar{f}$ restricted to $S/P \times \{x\}$ is injective and thus (sP, x) equals $(s'P, x')$ which shows the injectivity of $\bar{f}$.

Now $\bar{f}: S/P \times M \to X$ is biholomorphic and induces an T-action on $S/P \times M$. Since $\pi_1: S/P \times M \to S/P$ is T-equivariant, and a complex torus action on the homogeneous-rational S/P is trivial, it follow that T only acts on the second factor M. Thus $\bar{f}$ is G-equivariant with respect to the product action on $S/P \times M$ and the given action on X.

Ad (3). This follows immediately from Theorem 2.3. □

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REFERENCES

[Bl] BLANCHARD, A. *Sur les variétés analytiques complexes*, Ann. Sci. ecole norm. super. *73* (1956), 157–202.

[CAS] GRAUERT, H. and REMMERT, R. Coherent Analytic Sheaves, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1984.

[F] FUJIKI, A. *On automorphism groups of compact Kähler manifolds,* Invent. math. *44* (1978), 225–258.

[Ho1] HOLMANN, H. *Seifertsche Faserräume,* Math. Ann. *157* (1964), 138–166.

[Ho2] HOLMANN, H. *Analytische periodische Strömungen auf kompakten komplexen Räumen,* Comment. Math. Helvetici *52* (1977), 251–257.

[H, O] HUCKLEBERRY, A. T. and OELKEKLAUS, E. *Classification theorems for almost homogeneous spaces,* Janvier 84, 9. Revue de l'Institut Elie Cartan, Nancy, 1984.

[Hu] HUMPHREYS, J. E. Linear algebraic groups, Springer-Verlag, New York–Heidelberg–Berlin, 1981.

[I] INOUE, M. *On surfaces of class* VII_0, Invent. math. *24* (1974), 269–310.

[J] JÄNICH, K. *Differenzierbare G-Mannigfaltigkeiten,* Lecture Notes in Mathematics 59, Springer-Verlag, Berlin–Heidelberg–New York, 1968.

[Ma1] MATSUSHIMA, Y. *Sur les espaces homogènes Kählériens d'un groupe de Lie réductif,* Nagoya Math. J. *11* (1957), 53–60.

[Ma2] MATSUSHIMA, Y. *Holomorphic vector fields on compact Kähler manifolds,* Conf. Board Math. Sci. Regional Conf. Ser. in Math. 7, Amer. Math. Soc. 1971.

[Mo] MONTGOMERY, D. *Simply connected homogeneous spaces,* Proc. Amer. Math. Soc. *1* (1950), 467–469.

[S] SOMMESE, A. J. *Extension Theorems for reductive group actions on compact Kähler manifolds,* Math. Ann. *218* (1975), 107–116.

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