Grunsky coefficient inequalities, Caratheodory metric and extremal quasiconformal mappings.

Autor(en): **Krushkal, S.L.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **64 (1989)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-48968

Comment. Math. Helvetici 64 (1989) 650–660

00100-2571/89/040650-11\$01.50 + 0.20/0

Grunsky coefficient inequalities, Caratheodory metric and extremal quasiconformal mappings

S. L. KRUSHKAL'

1. It is well-known that a meromorphic function $f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$ in $\Delta^* = \{z \in \bar{\mathbf{C}}: |z| > 1\}$ is univalent in Δ^* (and belongs to the class Σ) if and only if for its Grunsky coefficients α_{mn}, which are determined from the expansion

$$\log\frac{f(z)-f(\zeta)}{z-\zeta} = -\sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m}\zeta^{-n}, \tag{1}$$

the inequality

$$\left|\sum_{m,n=1}^{\infty} \sqrt{mn}\,\alpha_{mn}x_m x_n\right| \leq \|x\|^2 \quad \left(\|x\| = \left(\sum_{1}^{\infty} |x_n|^2\right)^{1/2}\right) \tag{2}$$

holds for any $x = (x_1, x_2, \ldots) \in l^2$. There are also other univalence criteria, expressed by the inequalities which are equivalent to (2).

If we are to consider the subclass $\Sigma(k)$, $0 \leq k < 1$, which is formed by functions $f \in \Sigma$, having k'-quasiconformal extensions onto the whole sphere $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ with $k' \leq k$, then we obtain as an immediate corollary of the holomorphic compressibility of the Carathéodory metric that the inequality (2) can be sharpened as follows:

$$\left|\sum_{m,n=1}^{\infty} \sqrt{mn}\,\alpha_{mn}x_m x_n\right| \leq k\,\|x\|^2. \tag{3}$$

(In fact, as it follows from the results of [4], on the right hand of (3) one can change the multiplier k by the possibly smaller value $\tanh c_T(S_f, 0)$, where

$$S_f = (f''/f')' - (f''/f')^2/2$$

is the Schwarzian derivative of a function f in Δ^* and $c_T(\cdot, \cdot)$ is the Carathéodory metric on the universal Teichmuller space T.)

A question of how to characterize a class of functions for which the inequality (3) is both necessary and sufficient to belong to $\Sigma(k)$, was raised by different authors, starting from [7] and was solved in [5]. To formulate the result which was obtained in [5], we need some notations. Let

$$\Delta=\{z\in\mathbb{C}:|z|<1\},$$

$$\kappa(f)=\sup_{\|x\|_{l^2}=1}\left|\sum_{m,n=1}^{\infty}\sqrt{mn}\,\alpha_{mn}x_mx_n\right|,$$

$$A(\Delta)=\{\varphi\in L_1(\Delta):\varphi \text{ is holomorphic in } \Delta\},$$

$$B(\Delta)=\{\mu\in L_\infty(\mathbb{C}):\mu\mid\Delta^*=0,\ \|\mu\|<1\},$$

Let ϕ be the projection $B(\Delta)\to T$ and let $w^\mu(z)$ be the quasiconformal automorphism of the sphere $\bar{\mathbb{C}}$ with complex dilatation (Beltrami coefficient) $\mu_w=w_{\bar z}/w_z\in B(\Delta)$, normalized by $w^\mu(z)=z+0(|z|^{-1})$ as $z\to\infty$; in what follows, when considering the functions $\mu\in L_\infty(\Delta)$, we will suppose them to be extended by zero on Δ^*.

Consider on the sphere

$$C=\{\varphi\in A(\Delta):\|\varphi\|=1\}$$

the set C^0 of elements of following type

$$\varphi(z)=\frac{1}{\pi}\sum_{m,n=1}^{\infty}\sqrt{mn}\,x_mx_nz^{m+n-2}=\frac{1}{\pi}\sum_{m+n=2}^{\infty}\sqrt{mn}\,x_mx_nz^{m+n-2} \tag{4}$$

with $x=(x_n)\in l^2$, $\|x\|=1$.

The above-mentioned result of [5] states:

THEOREM A. *Equality*

$$\kappa(f)=\inf\{\|\mu_w\|_\infty:w^\mu|_{\Delta^*}=f\} \tag{5}$$

holds if and only if f is the restriction to Δ^* of a quasiconformal automorphism w^{μ_0} of the sphere $\bar{\mathbb{C}}$ with the dilatation μ_0, satisfying the condition

$$\sup_{\varphi\in C^0}\left|\iint_\Delta\mu_0(z)\varphi(z)\,dx\,dy\right|=\|\mu_0\|_\infty\quad(z=x+iy). \tag{6}$$

Geometrically this condition means, as it is shown in [5], that the

Carathéodory metric c_T in the holomorphic disk

$$\{\phi(t\mu_0): t \in \Delta\} \subset T$$

coincides with the Teichmuller–Kobayashi metric d_T of the space T by which it is majorized; the latter is connected with extremal quasiconformal mappings for which the sphere C is of crucial role.

2. In principle Theorem A solved completely the above mentioned problem, however, it cån be essentially improved by describing the structure of the set C^0. Along with this I mention here some corollaries of this theorem which are of paramount importance for the theory of univalent functions and the Teichmuller spaces theory.

THEOREM 1. *The set C^0 consists of the functions $\varphi \in C$, which are the squares of functions holomorphic in Δ*:

$$C^0 = \{\varphi \in C : \varphi = \psi^2\}$$

(*so that every $\varphi \in C^0$ can have in Δ zeros of even order only*).

Proof. (a). Let $\varphi = \psi^2$, where

$$\psi(z) = \sum_0^\infty c_n z^n \in L_2(\Delta);$$

then condition $\|\varphi\|_{A(\Delta)} = 1$ yields

$$\|\psi\|_{L_2} = \pi \sum_0^\infty \frac{|c_n|^2}{n+1} = 1.$$

Therefore, setting $x_n = \sqrt{\pi/n}\, c_{n-1}$, $n = 1, 2, \ldots$, and $x = (x_n)$ we shall have

$$\|x\|_{l^2} = \pi \sum_1^\infty |c_{n-1}|^2/n = 1$$

and

$$\begin{aligned}\varphi(z) &= \frac{1}{\pi} \sum_1^\infty \sqrt{m}\, x_m z^{m-1} \sum_1^\infty \sqrt{n}\, x_n z^{n-1} \\ &= \frac{1}{\pi} \sum_{m+n=2}^\infty \sqrt{mn}\, x_m x_n z^{m+n-2},\end{aligned} \tag{7}$$

and thus $\varphi \in C^0$.

(b) On the other hand, let $\varphi \in C^0$, i.e. is of type (4). Then two cases are possible:

Case 1. $\varphi(z) = z^{2p}\omega(z)$ near $z = 0$, where $p \geqslant 0$ is an integer and ω is holomorphic, $\omega(0) \neq 0$. Then one can set

$$\psi(z) = \frac{1}{\sqrt{\pi}} \sum_1^\infty \sqrt{n}\, x_n z^{n-1}. \tag{8}$$

The series on the right hand side of (8) converges absolutely for $|z| < 1$ since by Schwarz inequality we have

$$\left(\sum_1^\infty |x_n \sqrt{n}\, z^{n-1}|\right)^2 \leqslant \|x\|^2 \sum_1^\infty n\, |z|^{2n-2} < \|x\|^2 \sum_1^\infty n\, |z|^{n-1}$$
$$= \|x\|2\,(1 - |z|)^{-2}.$$

Consequently, ψ is holomorphic in Δ, and analogously to (7) we obtain, that $\psi^2 = \varphi$.

Case 2. Near $z = 0$

$$\varphi(z) = z^{2p+1}\omega(z), \quad \text{where} \quad p \geqslant 0, \quad \omega(0) \neq 0. \tag{9}$$

We will show that *such a case can't occur for the functions of type* (4). Indeed, let's denote $\varphi(z) = \sum_0^\infty a_l z^l$, then from (4) it follows that

$$\frac{\pi}{2} a_l = \sqrt{1(l+1)}\, x_1 x_{l+1} + \sqrt{2l}\, x_2 x_l + \sqrt{3(l-1)}\, x_3 x_{l-1} +$$
$$\cdots + \begin{cases} \dfrac{\sqrt{(l+1)(l+3)}}{2} x_{(l+1)/2} x_{(l+3)/2}, & \text{if } \ l \text{ is odd;} \\ \left(\dfrac{1}{2} + 1\right) x^2_{(l/2)+1}, & \text{if } \ l \text{ is even.} \end{cases} \tag{10}$$

Setting here successively $l = 1, 2, \ldots, 2p$, we must have

$$a_0 = 0, \ldots, \qquad a_{2p} = 0,$$

by virtue of (9). Hence the equalities

$$x_1 = 0, \ldots, x_{p+1} = 0$$

follow, but then because of (10) it must be true also that $a_{2p+1}=0$ which is invalid due to (9) – that is a contradiction! This completes the proof of the theorem.

3. COROLLARY 1. *For any function $f \in \Sigma(k)$ having k-quasiconformal extension $\hat{f}^{k(\bar{\varphi}/|\varphi|)}$ on $\bar{\mathbb{C}}$ with*

$$\varphi = \psi^2 \quad (\varphi \in A(\Delta)\setminus\{0\}), \tag{11}$$

the equality $\kappa(f)=k$ holds.

There is a close connection between the Grunsky matrix $A=(\alpha_{mn}(f))$ and the Fredholm eigenvalues λ_j of the curve $L=f(|z|=1)$, which was discovered by Schiffer [12]. Let us recall that in the case of smooth curve L these are the eigenvalues of the integral equation

$$h(z)=\frac{\lambda}{\pi}\int_L h(\zeta)\frac{\partial}{\partial n_\zeta}\log\frac{1}{|\zeta-z|}ds_\zeta, \qquad z\in L.$$

In certain questions the least non-trivial positive eigenvalue λ_1 plays a crucial role. This eigenvalue can be defined for any (oriented) closed Jordan curve $L\subset\bar{\mathbb{C}}$ by the equality

$$\frac{1}{\lambda_1}=\sup\frac{|D_G(h)-D_{G^*}(h)|}{D_G(h)+D_{G^*}(h)}$$

where G and G^* are respectively the interior and the exterior of L, and the supremum is taken over all functions h, continuous in $\bar{\mathbb{C}}$ and harmonic in $\bar{\mathbb{C}}\setminus L$; here

$$1/\lambda_1=\kappa(f) \tag{12}$$

(see, for instance, [1], [8], [13], [14]). However, so far the sharp values of λ_1 were found for a few curves only. Corollary 1 (and (5), (6)) distinguishes a class of functions f, for which by virtue of (12) one can immediately point out the exact value of the least eigenvalue for the curve $f(|z|=1)$.

COROLLARY 2. *If $\varphi\in A(\Delta)\setminus\{0\}$ has zeroes of even order only, then in the holomorphic disk*

$$\{\phi(t\bar{\varphi}/|\varphi|): t\in\Delta\}\subset T \tag{13}$$

the Carathéodory metric and the Teichmuller–Kobayashi metrics coincide.

This statement can be derived from the proof of the theorem A in [5] but we give a direct proof for the completeness of the exposition and in view of the great importance for some applications.

Due to the condition of Corollary 2, one can distinguish in Δ a single-valued branch of function $\sqrt{\varphi}$, therefore having φ normalized, we obtain that $\varphi \in C^0$.

Now we note that the Grunsky coefficients $\alpha_{mn}(f)$ depend holomorphically on $\omega = S_f$ as elements of the Banach space $B_2(\Delta^*)$ consisting of the holomorphic functions in Δ^* with finite norm $\|\omega\| = \sup(|z|^2 - 1)^2 |\omega(z)|$; we will write $\alpha_{mn}(\omega)$ and use standard biholomorphic imbedding of the space T into $B_2(\Delta^*)$.

Consider for $x = (x_n) \in l^2$ with $\|x\| = 1$ the function

$$h(\psi) = \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(\psi) x_m x_n \in \mathrm{Hol}\,(T, \Delta)$$

and take its lifting

$$\hat{h}(\mu) = h \circ \phi : B(\Delta) \to \Delta$$

onto the ball $B(\Delta)$. Then for $\mu = t\bar{\varphi}/|\varphi|$ for any $t \in \Delta$ (by Schwarz lemma) we must have

$$|\hat{h}(t\bar{\varphi}/|\varphi|)| \leq |t|. \tag{14}$$

On the other hand the application of the standard variation formula for the mappings $f^{\nu} \in \Sigma(k)$ yields

$$\alpha_{mn}(f^{\nu}) = -\frac{1}{\pi} \iint_{\Delta} \nu(z) z^{m+n-2}\, dx\, dy + 0(\|\nu\|^2), \qquad \|\nu\| \to 0,$$

whence we obtain that differential of function $\hat{h}$ in zero is given by

$$d\hat{h}(0)\nu = \frac{1}{\pi} \iint_{\Delta} \nu(z) \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m x_n z^{m+n-2}\, dx\, dy.$$

In particular, for $\nu = t\bar{\varphi}/|\varphi|$ taking into account that $\varphi \in C^0$ we will have

$$d\hat{h}(0)\left[t \frac{\bar{\varphi}}{|\varphi|}\right] = \frac{t}{\pi} \iint_{\Delta} |\varphi(z)|\, dx\, dy = t,$$

and hence in (14) we must have equality: $\hat{h}(t\bar{\varphi}/|\varphi|) = t$. But this is equivalent to the coincidence of the metrics c_T and d_T in the disk (13).

The statement of the Corollary 2 is an analogue for the universal space T of the Kra theorem [3] on the coincidence of the Carathéodory and Teichmuller–Kobayashi metrics on Abelian Teichmuller disks in finite-dimensional Teichmuller spaces. It is of interest that the proof of this theorem is essentially based on the intrinsic properties of the closed Riemann surfaces, while our arguments use only function theoretic considerations. Nevertheless it is likely that the similar fact must be valid for *all* Teichmüller spaces.

The importance of these results is due, in particular, to the fact that in the first place these metrics in general don't coincide, and secondly, it is due to recently discovered unexpected applications of the Carathéodory metric to the solution of variational problems of geometric function theory. (See [6]).

4. As the Corollary 1 shows, the condition (11) is sufficient for the validity of equality (5). The question arises when vice versa the equality (5) implies the validity of condition (11), i.e. the question of the necessity of condition (11).

Kühnau [8] proved that if a function $f \in \Sigma(k)$ maps the boundary circle $\{|z| = 1\}$ onto an *analytic* curve (i.e. f is holomorphic in the closure $\overline{\Delta^*}$), then the equality $\kappa(f) = k$ can really hold only when (11) is valid; the proof is based on fine properties of the least nontrivial eigenvalue λ_1 of the curve $f(|z| = 1)$ and the Faber polynomials.

Theorems A and 1 allow us essentially to decrease the required order of smoothness of the boundary curves.

THEOREM 2. *Let $f \in \Sigma(k)$ maps the circle $\{|z| = 1\}$ onto a curve of class $C^{2,\alpha}$, $0 < \alpha < 1$, and $\kappa(f) = k$. Then f admits (a unique) k-quasiconformal extension $\hat{f}$ onto $\bar{\mathbb{C}}$ and its complex dilatation in Δ has the form*:

$$\mu_{\hat{f}} = k\bar{\varphi}/|\varphi| \quad \text{with} \quad \varphi = \psi^2 \in A(\Delta) \setminus \{0\}.$$

The *proof* is based on the so-called frame mapping criterion which was established by Strebel [15].

Namely, let h be an orientation preserving homeomorphism of the unit circle $\partial\Delta$ onto itself, which can be extended up to a quasiconformal automorphism w of the disk $\bar{\Delta}$ and let $Q(h)$ be the class of all quasiconformal automorphisms of $\bar{\Delta}$, which coincide with h on $\partial\Delta$. Let us set

$$K(h) = \inf_{Q(h)} K(w),$$

where, as usual, $K(w) = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$. Exclude the trivial case when h is the restriction of a linear-fractional automorphism of the sphere $\bar{\mathbb{C}}$ onto Δ and consider now quasi-conformal extensions $\tilde{h}$ of the homeomorphism h into the rings $\{1-\delta \leqslant |z| \leqslant 1\}$, $\delta > 0$, which are called (interior) *frame mappings* for h. Let us take the exact lower bound over maximal dilatations of all such mappings:

$$H(h) = \inf_{\tilde{h}|_{\partial\Delta}=h} K(w).$$

It is clear that always $H(h) \leqslant K(h)$.

Strebel's result, which we need, states the following. As is known, the extremal (minimizing $K(w)$) w_0 in $Q(h)$ is characterized by the equality

$$\sup_{\varphi \in C} \left| \iint_\Delta \mu_{w_0} \varphi \, dx \, dy \right| = \|\mu_{w_0}\|_\infty; \tag{15}$$

however here a sequence maximizing the left part of (15) may degenerate, i.e. it may converge to zero on all compact sets in Δ. It is proved in [15] that if

$$H(h) < K(h), \tag{16}$$

then *no* maximizing sequence can be degenerating, and consequently in $Q(h)$ there is (a unique) extremal Teichmuller mapping with complex dilatation $\mu_{w_0} = k_0 \bar{\varphi}_0/|\varphi_0|$, where $k_0 = (K(h)-1)/(K(h)+1)$, $\varphi_0 \in A(\Delta)\setminus\{0\}$; in particular, if homeomorphism h is twice continuously differentiable and besides $h'(t) \neq 0$ and $h''(t)$ is bounded, then $H(h) = 1$ and the condition (16) holds.

Let us return to our situation and note that because of the well-known properties of conformal mappings in the closed domains we have $f|_{\partial\Delta} \in C^{2,\alpha}$ and a conformal homeomorphism $g\colon \operatorname{int} L \xrightarrow{\text{ontu}} \Delta$ is extended to the closure $\overline{\operatorname{int} f(\Delta)}$ to a conformal homeomorphsim of the class $C^{2,\alpha}(\overline{\operatorname{int} f(\Delta)})$ which means that the condition (16) holds for the homeomorphism $g \circ f|_{\partial\Delta}$. Hence, excluding the trivial case $f = \mathrm{id}$ by virtue of the Strebel theorem we obtain that f admits (a unique) k-quasiconformal extension $\hat{f}$ onto $\bar{\mathbb{C}}$. Its dilatation in Δ has the form $\mu_{\hat{f}} = k\bar{\varphi}_0/|\varphi_0|$, where $\varphi_0 \in A(\Delta)\setminus\{0\}$; it is determined up to positive factor. Normalize φ_0 by the condition $\|\varphi_0\| = 1$, i.e. $\varphi_0 \in C$.

Let us show that in fact φ_0 must belong to the set C^0. Indeed, since by assumption, $\kappa(f) = k$ we may apply the Theorem A and hence the equality

$$\sup_{\varphi \in C^0} \left| \iint_\Delta (\bar{\varphi}_0/|\varphi_0|)\varphi \, dx \, dy \right| = 1$$

must hold. Let

$$\varphi_p(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m^{(p)} x_n^{(p)} z^{m+n-2}, \qquad p = 1, 2, \ldots,$$

be a corresponding maximizing sequence in C^0 (with $x^{(p)} = (x_n^{(p)}) \in l^2$, $\|x^{(p)}\| = 1$), which means

$$\lim_{p\to\infty} \left| \iint_{\Delta} (\bar{\varphi}_0/|\varphi_0|)\varphi_p \, dx\, dy \right| = 1.$$

Since the sphere $\{\|x\|_{l^2} = 1\}$ is weakly compact, one can choose a subsequence, from the sequence $\{x^{(p)}\}$, which we denote again by $\{x^{(p)}\}$, weakly converging to a certain $x^0 = (x_n^0)$ with $\|x^0\| \leq 1$. This is equivalent to the convergence of the coordinates:

$$\lim_{p\to\infty} x_n^{(p)} = x_n^0 \quad \text{for all } n \geq 1.$$

If it were so that $x^0 = 0$ then φ_p would converge to zero locally uniformly in Δ, and hence the sequence $\{\varphi_p\} \subset C^0$ would be degenerating, which contradicts the above-mentioned result of [15]. Therefore, $x^0 \neq 0$ and f must have Teichmuller extension $\hat{f}_*$ on $\bar{\Delta}$ with $\mu = k\bar{\varphi}_*/|\varphi_*|$ where

$$\varphi_*(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m^0 x_n^0 z^{m+n-2}. \tag{17}$$

Passing, if necessary, to $x^0/\|x^0\|$ from x^0, one may assume that in the expression (17) $\|x^0\| = 1$. Then also $\|\varphi_*\| = 1$ (since $\varphi_p = \psi_p^2$ and $\varphi_* = \lim_{p\to\infty} \psi_p^2 = \psi_*^2$) and from the unqiueness of Teichmuller extension it follows that $\varphi_0 = \varphi_*$, and hence it belongs to C^0. The theorem is proved.

5. The arguments we have made can be extended to the more general class of curves $f(\partial\Delta)$ containing also non smooth curves for which the induced homeomorphism $h = g \circ f|_L$ of the circle $\{|z| = 1\}$ onto $\{|w| = 1\}$ satisfies the equality

$$\lim_{\tau\to 0} \frac{h(\vartheta + \tau) - h(\vartheta)}{h(\vartheta) - h(\vartheta - \tau)} = 1 \quad (\vartheta = \arg z,\, h(\vartheta) = \arg w)$$

uniformly in ϑ. This equality, as is shown in [15], is just equivalent to the condition $H(h) = 1$.

It is obvious from the proof of Theorem 2 that the main purpose there was to guarantee of the absence of degenerating sequences $\{\varphi_n\} \subset C$. For this we used the Strebel condition for frame mappings, but it is only a sufficient condition.

The situation concerning the presence of degenerating sequences is here the same as in the general theory of extremal quasiconformal mappings, but now it is necessary to take those sequences which are in C^0, so that (5) holds. Examples of such sequences were known and some were pointed out by Reich [10], [11].

Namely, in [10] an example of a quasiconformal automorphism w of the disc Δ with $\mu_w = k\bar{\varphi}_0/|\varphi_0|$ was constructed, where $\varphi_0 = \psi^2 \in C^0$ and for $\bar{\varphi}_0/|\varphi_0|$ there is a degenerating sequence $\{\varphi_n\}$, also belonging to C^0. Having extended μ_w by zero onto Δ^*, we come to $\hat{f} \in \Sigma(k)$ for which (5) and (11) hold, though the Strebel condition isn't valid.

On the other hand, we consider the affine stretching

$$F_K(\zeta) = K\xi + i\eta = \frac{K+1}{2}\zeta + \frac{K-1}{2}\bar{\zeta}, \qquad K > 1 \quad (\zeta = \xi + i\eta)$$

of the half-strip $\Pi_+ = \{\zeta : \operatorname{Re}\zeta > 0, 0 < \operatorname{Im}\zeta < 1\}$; evidently

$$\mu_{F_K}(\zeta) \equiv (K-1)/(K+1) = k,$$

so that $\omega_0(\zeta) \equiv 1$ $(\arg \mu_{F_K} = \arg \omega_0)$ and $\iint_{\Pi_+} |\omega_0|\, d\xi\, d\eta = \infty$. Let us take the sequence

$$\omega_n(\zeta) = \frac{1}{n} e^{-\zeta/n}, \qquad \zeta \in \Pi_+ \quad (n = 1, 2, \ldots);$$

it is evident that $\omega_n \to 0$ locally uniformly in Π_+, and at the same time

$$\iint_{\Pi_+} |\omega_n(\zeta)|\, d\xi\, d\eta = 1, \qquad \left|\iint_{\Pi_+} \omega_n(\zeta)\, d\xi\, d\eta\right| = 1 - 0\left(\frac{1}{n}\right). \tag{18}$$

Having Δ mapped conformally onto Π_+ using a function $\zeta = g(z)$, we construct $f^\mu \in \Sigma(k)$ with μ equal to zero in Δ^* and equal to $k\bar{\varphi}_0/|\varphi_0|$ in Δ, where $\varphi_0 = g'^2$. Then the corresponding sequence

$$\{\varphi_n(z) = (\omega_n \circ g) g'^2\} \subset C^0$$

is degenerating for μ, but by virtue of (18), the equality (6) holds, and hence $\kappa(f^\mu) = k$; however $\varphi_0 \notin A(\Delta)$.

Thus, *in general the condition* (11) contrary to (6) *is not necessary.*

6. Let us raise now the question to what extent one can spread the above obtained results onto the arbitrary quasiconformal disks D (which are different

from the usual disk). As to the extending of the Corollary 2, it is immediately derived from this corollary itself, since the conformal mapping $D \to \Delta$ induces a so-called *admissible bijection* of the space T, for which metrics and property (11) are preserved.

However, the use of univalence criteria by Grunsky [2] and Milin (see [9], ch. 6, §2), which generalize the inequality (2), leads now to the appearence of some additional quantities in the corresponding inequalities, generalizing (3), and the question remains open, when the case of equality might occur.

REFERENCES

[1] AHLFORS, L. V. *Remarks on the Neumann–Poincaré integral equation.* Pacif. J. Math., *2* (1952), 271–280.

[2] GRUNSKY, H. *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen.* Math. Z., *45* (1939), H. 1, 29–61.

[3] KRA, I. *The Carathéodory metric on Abelian Teichmüller disks.* J. d'Analyse Math., *40* (1981), 129–143.

[4] KRUSHKAL', S. L. *Invariant metrics on Teichmüller spaces and quasiconformal extendability of analytic functions.* Ann. Acad. Sci. Fenn., Ser. AI. Math., *10* (1985), 249–253.

[5] KRUSHKAL', S. L. *On the Grunsky coefficient conditions.* Sibirian Math. J., *28* (1987), N1, 138–145.

[6] KRUSHKAL', S. L. *A new approach to the variational problems of the theory of quasiconformal mappings.* Doklady Akad. Nauk SSSR, 292 (1987), N6, 1297–1300.

[7] KÜHNAU, R. *Verzerrungssatze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen.* Math. Nachr., *48* (1971), 77–105.

[8] KÜHNAU, R. *Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q-quasikonforme Fortsetzbarkeit?* Comment. Math. Helv., *61* (1986), 290–307.

[9] MILIN, I. M. *Univalent functions and ortonormal systems.* Nauka: Moscow, 1975.

[10] REICH, E. *On the relation between local and global properties of boundary values for extremal quasiconformal mappings.* – In: Discontinuous Groups and Riemann Surfaces. Princeton: Princeton Univ. Press, 1974, p. 391–407 (Ann. of Math. Stud., 79).

[11] REICH, E. *Quasiconformal mappings of the disk with given boundary values.* – In. Advances in Complex Function Theory, Berlin–Heidelberg–New York: Springer, 1976, p. 101–137 (Lecture Notes in Math., 505).

[12] SCHIFFER, M. *Fredholm eigenvalues and Grunsky matrices.* Ann. Polon. Math., *39* (1981), 149–164.

[13] SCHOBER, G. *Estimates for Fredholm eigenvalues based on quasiconformal mappings.* – In: Numerische, insbesondere approximationstheoretische Behandlung von Funktionalgleichungen, Berlin–Heidelberg–New York: Springer, 1973, p. 211–217.

[14] SPRINGER, G. *Fredholm eigenvalues and quasiconformal mapping.* Acta Math., *111* (1964), 121–142.

[15] STREBEL, K. *On the existence of extremal Teichmüller mapping.* J. d'Analyse Math., *30* (1976), 464–480.

Institute of Mathematics
Sibirian Branch of the USSR Academy of Sciences
Novosibirsk, 630090, USSR

Received February 1, 1988