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The level of real projective spaces

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1. Introduction

In this paper we determine the level of the real projective space $\mathbf{RP}^{2m-1}$ with the $\mathbf{Z}/2$-action induced by multiplication by the complex number i. By definition (see [DL]), the level of a topological space X with a free $\mathbf{Z}/2$-action is the number

$$s(X) = \min \{n : \text{there exists a } \mathbf{Z}/2\text{-equivariant map } f : X \to S^{n-1}\},$$

where the sphere S^{n-1} is equipped with the antipodal $\mathbf{Z}/2$-action. We abbreviate $s(\mathbf{RP}^{2m-1})$ by $s(m)$.

The previously known results about $s(m)$ seem to be the following, P. E. Conner and E. E. Floyd proved $s(1) = 2$, $s(2) = 3$, $s(3) = 5$ [CF] and A. Pfister and the author obtained the estimates $m + 1 \leq s(m) \leq \frac{1}{2}(3m + 1)$ [PS].

The main result of this paper is the computation of $s(m)$.*

THEOREM. *Let $m \geq 2$. Then*

$$s(m) = \begin{cases} m+1 & \text{if } m = 0, 2 \bmod 8 \\ m+2 & \text{if } m = 1, 3, 4, 5, 7 \bmod 8 \\ m+3 & \text{if } m = 6 \bmod 8 \end{cases}$$

Remark. The invariant $s(m)$ is related to the following purely algebraic invariant

$$r(m) = \min \left\{ n : \begin{array}{l} \text{there exists a complex quadratic form } q : \mathbf{C}^m \to \mathbf{C}^n \\ \text{such that } \operatorname{im}(q) : \mathbf{R}^{2m} \to \mathbf{R}^n \text{ is anisotropic} \end{array} \right\}$$

Here $\operatorname{im}(q)$ denotes the imaginary part of q which is a real quadratic form. It is called anisotropic if $\operatorname{im}(q)^{-1}(0) = 0$. By normalizing and restricting $\operatorname{im}(q)$ it

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* This result was also proved by M. C. Crabb using somewhat different arguments in his preprint "Periodicity in $\mathbf{Z}/4$-equivariant stable homotopy theory".

induces a $\mathbf{Z}/4$-equivariant map $S^{2m-1} \to S^{n-1}$ where $\mathbf{Z}/4$ acts by multiplication by i (resp. -1) on the domain (resp. range). Passing to the quotient we get a $\mathbf{Z}/2$-equivariant map $\mathbf{RP}^{2m-1} \to S^{n-1}$. This shows $r(m) \geq s(m)$. The 8-periodicity of $s(m)$ suggests that there might be a way to use Clifford algebras to construct $\mathbf{Z}/2$-equivariant maps $\mathbf{RP}^{2m-1} \to S^{s(m)-1}$ or even quadratic forms $\mathbf{C}^m \to \mathbf{C}^{s(m)}$ with anisotropic imaginary part.

The proof of the theorem uses the following reformulation of the level of X. Let L be the real line bundle $X \times_{\mathbf{Z}/2} \mathbf{R} \to Y$ over the quotient space $Y = X/\mathbf{Z}/2$. If $f: X \to S^{n-1}$ is a $\mathbf{Z}/2$-equivariant map then by passing to the quotient the equivariant map $id \times f: X \to X \times S^{n-1}$ gives a nowhere vanishing section of nL. Conversely a nowhere vanishing section of nL gives rise to an equivariant map f as above. Hence the level of X can equivalently be characterized as the smallest n such that nL has a nowhere vanishing section. An obstruction for the existence of such a section is the cohomotopy Euler class, which we discuss in section 2.

In section 3 we use K-theory methods to show the non-vanishing of the cohomotopy Euler class of nL for certain n's, where L is the non-trivial line bundle over the $\mathbf{Z}/4$-lens space L^{2m-1}, the quotient space of $\mathbf{RP}^{2m-1}$. This implies a lower bound for $s(m)$. It should be emphasized that these K-theory restrictions are stronger than those imposed by the vanishing of the K-theory Euler class. A study of the K-theory Euler class only leads to the lower bound $s(m) \geq m+1$, the same bound as obtained in [PS].

In section 4 we use the Adams spectral sequence and a vanishing result for its E_2-term to show that the cohomotopy Euler class vanishes in certain cases. That leads to an upper bound for $s(m)$ which agrees with the lower bound derived in section 3 except for $m = 4 \bmod 8$.

Finally in section 5 we prove the inequality $s(m+n) \geq s(m) + s(n)$ and use it to compute $s(m)$ for $m = 4 \bmod 8$.

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2. The cohomotopy Euler class

In this section we discuss the cohomotopy Euler class and its properties and recall the definition of the (cohomotopy) Gysin sequence.

Throughout this section let X be a finite CW complex and let α be an n-dimensional vector bundle over X. We choose a metric for α and denote by $S(\alpha)$ (resp. $D(\alpha)$) the sphere bundle (resp. disk bundle) of α. The Thom space $T(\alpha)$ is by definition the quotient space $D(\alpha)/S(\alpha)$. The zero section of α induces a map $i: X \to T(\alpha)$ or, more generally, a map $i: T(\beta) \to T(\alpha \oplus \beta)$ for a vector bundle β over X. If α' is an n'-dimensional inverse bundle of α then a trivialization of $\alpha \oplus \alpha'$ induces a map $t: T(\alpha \oplus \alpha') \to S^{n+n'}$. For n' large the vector bundle α' is unique and we define the cohomotopy Euler class $e(\alpha)$ as the composition $T(\alpha') \to T(\alpha \oplus \alpha') \to S^{n+n'}$ of i and t.

If α has a nowhere vanishing section s then the zero section can be deformed into s and hence i is homotopic to the constant map since we can assume that s is a section of $S(\alpha)$. Thus $e(\alpha)$ is homotopic to the constant map.

At this point it is convenient to use the language of Thom spectra. A general reference for spectra is [S]. With our assumption that X is a finite CW-complex Thom spectra of (virtual) vector bundles over X are easily defined as follows. If α is a n-dimensional vector bundle then its Thom spectrum $M\alpha$ is the n-th desuspension of the suspension spectrum of $T(\alpha)$. Note that with this definition the bottom cell of $M\alpha$ is in dimension 0. The notion of Thom spectrum can be extended to virtual vector bundles. For example $M(-\alpha) = M(\alpha')$, where α' is an inverse to α.

For n' large the set $[T(\alpha'), S^{n+n'}]$ of homotopy classes of maps from $T(\alpha')$ to $S^{n+n'}$ is isomorphic to $\{T(\alpha'), S^{n+n'}\}$, the group of homotopy classes of maps from the suspension spectrum of $T(\alpha')$ to the suspension spectrum of $S^{n+n'}$. Via suspension isomorphism $\{T(\alpha'), S^{n+n'}\}$ can be identified with $\{M(-\alpha), S^n\} = \pi^n(M(-\alpha))$.

Using these identifications the cohomotopy Euler class $e(\alpha)$ is an element of $\pi^n(M(-\alpha))$. We think of $\pi^n(M(-\alpha))$ as a "twisted" cohomotopy group of X and hence we use the notation $\pi^n(X; -\alpha)$. The big advantage of the cohomotopy Euler class is the following.

PROPOSITION 2.1 ([C, Prop. 2.4]). *If α is an n-dimensional vector bundle over a finite* CW-*complex X and* $\dim X < 2(n-1)$ *then α has a nowhere vanishing section if and only if its cohomotopy Euler class vanishes.*

The classical obstruction for finding a non-where vanishing section of an orientable vector bundle α is the usual Euler class of α which is an element of $H^n(X; \mathbf{Z})$ (see e.g. [MS]). If α is a complex vector bundle of dimension k this Euler class is the k-th Chern class $c_k(\alpha) \in H^{2k}(X; \mathbf{Z})$. The usual Euler class and the cohomotopy Euler class are related as follows. Using the notation $H^n(X; -\alpha)$ for $H^n(M\alpha; \mathbf{Z})$ the Hurewicz homomorphism

$$h : \pi^n(X; -\alpha) = \pi^n(M(-\alpha)) \to H^n(M\alpha; \mathbf{Z}) = H^n(X; -\alpha) \tag{2.2}$$

maps $e(\alpha)$ to a (twisted) cohomology class $e_{\mathbf{Z}}(\alpha)$ which we call the cohomology Euler class of α. If α is oriented $e_{\mathbf{Z}}(\alpha)$ corresponds to the usual Euler class under the Thom isomorphism $H^n(X; -\alpha) \cong H(X; \mathbf{Z})$.

Replacing $\mathbf{Z}$-cohomology by $\mathbf{Z}/2$-cohomology there is a corresponding Hurewicz map $h_{\mathbf{Z}/2} : \pi^n(X; -\alpha) \to H^n(X; \mathbf{Z}/2)$ (note that here we don't need α to be oriented) and

$$h_{\mathbf{Z}/2}(e(\alpha)) = w_n(\alpha) \text{ (the } n\text{-th Stiefel Whitney class of } \alpha). \tag{2.3}$$

The Euler class has the following multiplicative property. Assume that α and β are n-dimensional (resp. m-dimensional) vector bundles over X. Then

$$e(\alpha \oplus \beta) = e(\alpha)e(\beta), \tag{2.4}$$

where the product on the right hand side is the cup product for (twisted) cohomotopy

$$\pi^n(X; -\alpha) \otimes \pi^m(X; -\beta) \rightarrow \pi^{n+m}(X; -(\alpha \oplus \beta))$$

defined as follows. Let f, g be elements of $\pi^n(X; -\alpha)$ resp. $\pi^m(X; -\beta)$ which are represented by maps of spectra $f: M(\alpha') \rightarrow S^n$ resp. $g: M(\beta') \rightarrow S^m$, where α' resp. β' are inverse bundles of α resp. β. Then their cup product is given by the composition

$$M(\alpha' \oplus \beta') \xrightarrow{M\Delta} M(\alpha' \times \beta') = M(\alpha') \wedge M(\beta') \xrightarrow{f \wedge g} S^n \wedge S^m = S^{n+m}, \tag{2.5}$$

where $\alpha' \times \beta'$ is the product bundle over $X \times X$ whose Thom spectrum can be identified canonically with the smash product $M(\alpha') \wedge M(\beta')$. The diagonal map $\Delta: X \rightarrow X \times X$ is covered by a bundle map $\alpha' \oplus \beta' \rightarrow \alpha' \times \beta'$ which induces a map $M\Delta$ between the Thom spectra. The multiplicative property (2.4) follows easily from the definitions of the Euler class and the cup product.

Another tool we need is the Gysin sequence. Let α be an n-dimensional vector bundle over X. Then by definition of the Thom space there is a cofibration

$$S(\alpha) \xrightarrow{p} X \xrightarrow{i} T(\alpha) = \Sigma^n M\alpha, \tag{2.6}$$

where p is the projection map and i denotes the inclusion of the zero section. It induces long exact sequences

$$\rightarrow \pi^{i-n}(X; \alpha) \xrightarrow{i^*} \pi^i X \xrightarrow{p^*} \pi^i S(\alpha) \xrightarrow{\partial} \pi^{i-n+1}(X; \alpha) \rightarrow \text{and} \tag{2.7}$$

$$\rightarrow H^{i-n}(X; \alpha) \xrightarrow{i^*} H^i(X; \mathbf{Z}) \xrightarrow{p^*} H^i(S(\alpha); \mathbf{Z}) \xrightarrow{\partial} H^{i-n+1}(X; \alpha) \rightarrow, \tag{2.8}$$

which we refer to as the cohomotopy (resp. cohomology) Gysin sequence for $S(\alpha)$. If α is orientable we can replace the twisted cohomology group $H^{i-n}(X; \alpha) = H^{i-n}(M\alpha; \mathbf{Z})$ by $H^{i-n}(X; \mathbf{Z})$ using the Thom isomorphism and this gives the usual Gysin sequence (see e.g. [MS]). More generally, if β is a vector bundle over X then there is a cofibration

$$T(p^*\beta) \xrightarrow{p} T(\beta) \xrightarrow{i} T(\alpha \oplus \beta) \tag{2.9}$$

inducing long exact sequences

$$\to \pi^{i-n}(X; \alpha \oplus \beta) \xrightarrow{i^*} \pi^i(X; \beta) \xrightarrow{p^*} \pi^i(S(\alpha); p^*\beta) \xrightarrow{\partial} \pi^{i-n+1}(X; \alpha \oplus \beta) \tag{2.10}$$

and

$$\to H^{i-n}(X; \alpha \oplus \beta) \xrightarrow{i^*} H^i(X; \beta) \xrightarrow{p^*} H^i(S(\alpha); p^*\beta) \xrightarrow{\partial} H^{i-n+1}(X; \alpha \oplus \beta), \tag{2.11}$$

which we call the cohomotopy (resp. cohomology) Gysin sequence for $S(\alpha)$ with coefficients in β. It follows from the definition of the cohomotopy Euler class that the map i^* in these sequences is the multiplication by the cohomotopy (resp. cohomology) Euler class.

3. A lower bound for $s(m)$

The goal of this section is the proof of the following.

PROPOSITION 3.1. *Let L be the non-trivial real line bundle over the $\mathbf{Z}/4$-lens space L^{2m-1} with $m \geq 2$. If $m = 2k - 2$ and $k = 0 \bmod 4$ or $m = 2k - 1$ then the cohomotopy Euler class of $2kL$ is non-trivial.*

This implies that $2kL$ does not have a nowhere vanishing section or, equivalently, there is no $\mathbf{Z}/2$-equivariant map $\mathbf{RP}^{2m-1} \to S^{2k-1}$. Hence we obtain the following estimate on $s(m)$.

COROLLARY 3.2. *Let $m \geq 2$. Then*

$$s(m) \geq \begin{cases} m+1 & \text{if } m = 0, 2, 4 \bmod 8 \\ m+2 & \text{if } m = 1, 3, 5, 7 \bmod 8. \\ m+3 & \text{if } m = 6 \bmod 8 \end{cases}$$

Proof of Proposition 3.1. We observe that L^{2m-1} can be identified with the sphere bundle of H^4, the fourth tensor power of the Hopf bundle H over the complex projective space $\mathbf{CP}^{m-1}$. Moreover the pull back of H^2 under the projection map $p: L^{2m-1} = S(H^4) \to \mathbf{CP}^{m-1}$ is $2L$.

This can be seen as follows. The Hopf bundle H can be written as the vector bundle associated to the standard 1-dimensional complex representation of S^1 given by multiplication by $z \in S^1$. Thus H^2 corresponds to the representation given by multiplication by z^2 and $p^*(H^2)$ corresponds to its restriction to the subgroup $\mathbf{Z}/4$ of S^1 generated by $i \in S^1$. This representation of $\mathbf{Z}/4$ is the sum of two copies of the non-trivial 1-dimensional real representation of $\mathbf{Z}/4$ whose assocated vector bundle is L.

The naturality of the Euler class then implies $p^*(e(kH^2)) = e(2kL)$. To analyze $p^*(e(kH^2))$ we use the Gysin sequence for the sphere bundle $S(H^4)$. Writing down the Gysin sequences for cohomotopy (resp. cohomology) with coefficients in $-kH^2$ (see (2.10) resp. (2.11)) and identifying the twisted cohomology groups with untwisted ones using the Thom isomorphism we get the following commutative diagram

$$\begin{tikzcd}
 & \pi^{2k-2}(\mathbf{CP}^{m-1}; H^4\text{-}kH^2) \arrow[r, "i^*"] \arrow[d, "h"] & \pi^{2k}(\mathbf{CP}^{m-1}; \text{-}kH^2) \arrow[r, "p^*"] \arrow[d, "h"] & \pi^{2k}(L^{2m-1}; \text{-}2kL) \arrow[r] \arrow[d, "h"] & {} \\
{} \arrow[r] & H^{2k-2}(\mathbf{CP}^{m-1}; \mathbf{Z}) \arrow[r, "i^*"] & H^{2k}(\mathbf{CP}^{m-1}; \mathbf{Z}) \arrow[r, "p^*"] & H^{2k}(L^{2m-1}; \mathbf{Z}) \arrow[r] & {}
\end{tikzcd}$$

Here the vertical map h is the Hurewicz map. It maps the cohomotopy Euler class of kH^2 to the cohomology Euler class $e_{\mathbf{Z}}(kH^2)$.

Recall that the cohomology of $\mathbf{CP}^{m-1}$ is a truncated polynomial ring $H^*(\mathbf{CP}^{m-1}; \mathbf{Z}) \cong \mathbf{Z}[x]/(x^m)$ whose generator $x \in H^2(\mathbf{CP}^{m-1}; \mathbf{Z})$ is the first Chern class of the Hopf bundle. Hence $e_{\mathbf{Z}}(H^2) = c_1(H^2) = 2x$ and $e_{\mathbf{Z}}(kH^2) = (e_{\mathbf{Z}}(H^2))^k = 2^k x^k$. The induced map i^* in cohomology is multiplication by $e_{\mathbf{Z}}(H^4) = c_1(H^4) = 4x$.

To prove proposition 3.1 assume $e(2kL) = 0$. Then the cohomotopy exact sequence implies that $e(kH^2)$ is of the form $i^*(y)$ for some $y \in \pi^{2k-2}(\mathbf{CP}^{m-1}; H^4 - kH^2)$. The commutativity of the diagram implies $i^*(h(y)) = h(i^*(y)) = h(e(kH^2)) = e_{\mathbf{Z}}(kH^2) = 2^k x^k$ and hence $h(y) = 2^{k-2} x^{k-1}$. But this contradicts the following proposition.

PROPOSITION 3.3. *Let $m \geq 2$. If $m = 2k-2$ and $k = 0 \bmod 4$ or $m = 2k-1$ then the index of the Hurewicz homomorphism $h: \pi^{2k-2}(\mathbf{CP}^{m-1}; H^4 - kH^2) \to H^{2k-2}(\mathbf{CP}^{m-1}; \mathbf{Z}) \cong \mathbf{Z}$ is multiple of 2^{k-1}.*

To prove this proposition we first characterize the index of h as the "codegree" of some vector bundle and then use the K-theory methods of [CK] of obtain estimates for this codegree. If α is an orientable (virtual) vector bundle over a space X then $cd(\alpha)$, the codegree of α, is defined as the index of the Hurewicz map $\pi^0 M \in \to H^0(M\alpha; \mathbf{Z}) \cong \mathbf{Z}$.

LEMMA 3.4. *If α is some (virtual) vector bundle over $\mathbf{CP}^{m-1}$ then the index of the Hurewicz map $h: \pi^{2r}(\mathbf{CP}^{m-1}; \alpha) \to H^{2r}(\mathbf{CP}^{m-1}; \mathbf{Z})$ is the codegree of $\alpha + rH$ over $\mathbf{CP}^{m-r-1}$.*

Proof. Consider the cofibration

$$\mathbf{CP}^{r-1} \to \mathbf{CP}^{m-1} \xrightarrow{pr} \mathbf{CP}^{m-1}/\mathbf{CP}^{r-1}.$$

It is well known that the cofiber $\mathbf{CP}^{m-1}/\mathbf{CP}^{r-1}$ can be identified with the Thom space of the vector bundle rH over $\mathbf{CP}^{m-r-1}$. Moreover there is a corresponding cofibration with "coefficients in α" which induces the following long exact sequence of cohomotopy groups.

$$\pi^{2r-1}(\mathbf{CP}^{r-1}; \alpha) \to \pi^0(\mathbf{CP}^{m-r-1}; \alpha + rH) \xrightarrow{pr^*} \pi^{2r}(\mathbf{CP}^{m-1}; \alpha)$$
$$\to \pi^{2r}(\mathbf{CP}^{r-1}; \alpha)$$

The groups $\pi^{2r-1}(\mathbf{CP}^{r-1}; \alpha)$ and $\pi^{2r}(\mathbf{CP}^{r-1}; \alpha)$ vanish for dimensional reasons and hence pr^* is an isomorphism. The same argument shows that pr induces an isomorphism in cohomology, too. Hence the index of the Hurewicz map

$$h: \pi^{2r}(\mathbf{CP}^{m-1}; \alpha) \to H^{2r}(\mathbf{CP}^{m-1}; \mathbf{Z})$$

is equal to the index of

$$h: \pi^0(\mathbf{CP}^{m-r-1}; \alpha + rH) \to H^0(\mathbf{CP}^{m-r-1}; \mathbf{Z}),$$

which is the codegree of $\alpha + rH$. Q.E.D.

We estimate the codegree of $H^4 - kH^2 + (k-1)H$ using the K-theory method of [CK]. It is based on the fact that the Hurewicz map factors through K-theory. More precisely the Hurewicz map $h: \pi^0 M\alpha \to H^0(M\alpha; \mathbf{Z})$ composed with the inclusion $i: H^0(M\alpha; \mathbf{Z}) \to H^*(M\alpha; \mathbf{Q})$ is the composition of the K-theory Hurewicz map $h_K: \pi^0 M\alpha \to K^0 M\alpha$ and the Chern character $ch: K^0 M\alpha \to H^*(M\alpha; \mathbf{Q})$.

The codegree of α is by definition the index of $\operatorname{im}(h)$ in $H^0(M\alpha; \mathbf{Z})$ or, alternatively, the index of $\operatorname{im}(i \circ h)$ in $\operatorname{im}(i)$. It is hence a multiple of the index of $\operatorname{im}(i) \cap \operatorname{im}(ch)$ in $\operatorname{im}(i)$ which is called the K-theory codegree of α and denoted by $cd^K(\alpha)$.

For computations the following characterization of $cd^K(\alpha)$ is useful.

LEMMA 3.5 ([CK], Prop. 3.2). *Let α be a complex vector bundle over a finite* CW *complex X with torsion free homology. Then*

$$cd^K(\alpha) = \min \{m \in N \mid m \cdot ch^{-1} \operatorname{Todd}(-\alpha) \in K^0X \otimes \mathbf{Q} \text{ is integral}\}$$

Here $\operatorname{Todd}(\alpha) \in H^*(X; \mathbf{Q})$ *is the* Todd *genus of α. It is multiplicative, i.e.*

$$\operatorname{Todd}(\alpha + \beta) = \operatorname{Todd}(\alpha) \cdot \operatorname{Todd}(\beta),$$

and if L is a complex line bundle then

$$\operatorname{Todd}(L) = (\exp(c_1(L)) - 1)/c_1(L).$$

LEMMA 3.6 ([CK], p. 16). *Let L be a complex line bundle. Then* $ch^{-1} \operatorname{Todd}(-L) = \log(\lambda + 1)/\lambda \in K^0X \otimes \mathbf{Q}$, *where* $\lambda = L - 1 \in K^0X$ *and* $\log(\lambda + 1)$ *is the standard power series of the natural logarithm.*

Proof. $ch(\log(\lambda + 1)/\lambda) = \log(ch(\lambda + 1)/ch(\lambda)) = \log(ch(L)/(ch(L) - 1)) = c_1(L)/(\exp(c_1(L)) - 1) = \operatorname{Todd}(L)^{-1} = \operatorname{Todd}(-L)$. Q.E.D.

LEMMA 3.7. *The K-theory codegree of $H^4 - kH^2 + (k-1)H$ over $\mathbf{CP}^{k-1}$ is a multiple of 2^{k-1}.*

Proof. Recall that $K^0\mathbf{CP}^{k-1}$ is the truncated polynomial ring $\mathbf{Z}[\eta]/(\eta^k)$ where $\eta = H - 1$. To compute the highest power of 2 in the denominator of $ch^{-1} \operatorname{Todd}(-(H^4 - kH^2 + (k-1)H))$ it is convenient to rewrite everything in terms of the new variable $y = \eta/2$. A look at the power series

$$\left(\frac{\log(\eta + 1)}{\eta}\right) = 1 - \frac{\eta}{2} + \frac{\eta^2}{3} - \frac{\eta^3}{4} + \cdots$$

shows that it represents an element in $\mathbf{Z}_{(2)}[y]$, where $\mathbf{Z}_{(2)}$ denotes the integers localized at 2, i.e. all rational numbers whose denominator is prime to 2. Moreover computing modulo the ideal $2\mathbf{Z}_{(2)}[y]$ we have $\log(\eta + 1)/\eta = 1 - y$. More generally, if λ is an element of $\mathbf{Z}[\eta]$ with vanishing constant term then

$$\left(\frac{\log(\lambda + 1)}{\lambda}\right) = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{3} - \frac{\lambda^3}{4} + \cdots = 1 - \frac{\lambda}{2} \bmod 2\mathbf{Z}_{(2)}[y].$$

In particular we get

$$ch^{-1}\,\mathrm{Todd}\,(-H^4)=\frac{\log(\eta+1)^4}{(\eta+1)^4-1}=1-\frac{4\eta+6\eta^2+4\eta^3+\eta^4}{2}=1 \bmod 2\mathbf{Z}_{(2)}[y]$$

and

$$ch^{-1}\,\mathrm{Todd}\,(-H^2)=\frac{\log((\eta+1)^2)}{(\eta+1)^2-1}=1-\frac{2\eta+\eta^2}{2}=1 \bmod 2\mathbf{Z}_{(2)}[y].$$

Using the multiplicativity of the Todd genus and the fact that the Chern character is a ring homomorphism we obtain

$$ch^{-1}\,\mathrm{Todd}\,(-H^4-kH^2+(k-1)H))=(1-y)^{k-1} \bmod 2\mathbf{Z}_{(2)}[y].$$

Expressing $(1-y)^{k-1}$ as a power series in η we see that $m=2^{k-1}$ is the smallest power of 2 such that $m(1-y)^{k-1}\in\mathbf{Z}_{(4)}[\eta]/(\eta^k)$. Since $2^{k-2}(2\mathbf{Z}_{(2)}[y])$ is contained in $\mathbf{Z}_{(2)}[\eta]/(\eta^k)$ the same conclusion holds for $ch^{-1}\,\mathrm{Todd}\,(-(H^4-kH^2+(k-1)H))$. It follows from (3.5) that the codegree of $H^4-kH^2+(k-1)H$ is a multiple of 2^{k-1}. Q.E.D.

Together the lemmata 3.4 and 3.7 provide the proof of proposition 3.3 except if $k=0 \bmod 4$. In that case we have to show that the codegree of $H^4-kH^2+(k-1)H$ over $\mathbf{CP}^{k-2}$ is a multiple of 2^{k-1}. This sharper estimate can be obtained by considering the KO-theory codegree which is defined analogous to the K-theory codegree by replacing the Chern character $ch: K^0M\alpha\to H^*(M\alpha;\mathbf{Q})$ by the Pontrjagin character $ph: KO^0M\alpha\to H^*(M\alpha;\mathbf{Q})$ which is the composition of the complexification map $KO^0M\alpha\to K^0M\alpha$ and the Chern character. The same arguments as before show that the codegree is a multiple of the KO-theory codegree which in turn is a multiple of the K-theory codegree. Hence the proof of proposition 3.3 is completed with the proof of the following lemma.

LEMMA 3.8. *Let $k=0 \bmod 4$. Then the KO-theory codegree of $H^4-kH^2+(k-1)H$ over $\mathbf{CP}^{k-2}$ is a multiple of 2^{k-1}.*

Proof. Consider the cofibration $\mathbf{CP}^{k-2}\to\mathbf{CP}^{k-1}\to\mathbf{CP}^{k-1}/\mathbf{CP}^{k-2}=S^{2k-2}$ and its induced long exact sequence in KO-theory

$$\to KO^{-1}S^{2k-2}\to KO^0\mathbf{CP}^{k-1}\to KO^0\mathbf{CP}^{k-2}\to KO^0S^{2k-2}\to.$$

It follows that $KO^0\mathbf{CP}^{k-1}\to KO^0\mathbf{CP}^{k-2}$ is an isomorphism since the other two terms vanish by Bott periodicity. Hence the KO-codegree of $H^4 - kH^2 + (k-1)H$ as a bundle over $\mathbf{CP}^{k-2}$ is the same as its codegree as a bundle over $\mathbf{CP}^{k-1}$ which is a multiple of 2^{k-1} by (3.7). Q.E.D.

4. An upper bound for $s(m)$

The main result of this section is the following.

PROPOSITION 4.1. *Assume $m = 2k$ and $k = 0, 1 \bmod 4$ or $m = 2k - 1$. Then the cohomotopy Euler class of $(2k+1)L$ over L^{2m-1} vanishes.*

By proposition 2.1 this implies that $(2k+1)L$ has a nowhere vanishing section or, equivalently, that there is a $\mathbf{Z}/2$-equivariant map $\mathbf{RP}^{2m-1} \to S^{2k}$. Hence we obtain the following upper estimate for $s(m)$.

COROLLARY 4.2.

$$s(m) \leq \begin{cases} m+1 & \text{if } m = 0, 2 \bmod 8 \\ m+2 & \text{if } m = 1, 3, 5, 7 \bmod 8 \\ m+3 & \text{if } m = 4, 6 \bmod 8 \end{cases}$$

Proposition 4.1 is proved using the Adams spectral sequence, notably a "vanishing line" for its E_2-term (see 4.4). We begin by describing the properties of the Adams spectral sequence which are relevant to us. General references are the books of Adams [A] and Switzer [S].

Let X, Y be finite spectra and let p be a fixed prime. We say that a map $X \to Y$ has $\mathbf{Z}/p$-Adams filtration $\geq s$ if it can be written as a composition

$$X \to Z_1 \to \cdots \to Z_{s-1} \to Y$$

of s maps which are all trivial in Z/p-cohomology. This defines a filtration on the abelian group $[X, Y]$ of homotopy classes of maps $X \to Y$ or, more generally, on $[X, Y]_n = [\Sigma^n X, Y]$. We denote by $F_s[X, Y]_n$ the subgroup of elements of filtration $\geq s$ in $[X, Y]_n$. Note that in the case where X (resp. Y) is the sphere spectrum S^0 this defines a filtration of the homotopy (resp. cohomotopy) groups of spectra.

This filtration is compatible with the smash product, i.e. if $f \in F_s[X, Y]_n$ and $f' \in F_{s'}[X', Y']_{n'}$ then $f \wedge f' \in F_{s+s'}[X \wedge X', Y \wedge Y']_{n+n'}$. This follows directly from the definition since if f factors as $X \to Z_1 \to \cdots \to Z_{s-1} \to Y$ and f' factors as $X' \to Z'_1 \to \cdots \to Z'_{s-1} \to Y'$ then there is the following factorization for $f \wedge f'$.

$$X \wedge X' \to Z_1 \wedge X' \to \cdots \to Z_{s-1} \wedge X' \to Y \wedge X' \to Y \wedge Z'_1$$
$$\to \cdots \to Y \wedge Z'_{s'-1} \to Y \wedge Y'$$

The compatibility of the Adams filtration with the smash product implies its compatibility with the cup product (see 2.5), which we state as a lemma for further reference.

LEMMA 4.3. *If α and α' are vector bundles over a space X and f, f' are elements of $\pi^n(X; \alpha)$ (res. $\pi^{n'}(X; \alpha')$) of Adams filtration $\geq s$ (resp. $\geq s'$) then their cup product has filtration $\geq s + s'$.*

Associated to the Adams filtration on $[X, Y]_n$ there is a corresponding spectral sequence $E_r^{s,t}(X, Y)$, the Adams spectral sequence. It converges to the p-primary part of $[X, Y]_n$, i.e.

$$E_\infty^{s,t}(X, Y) \cong F_s[X, Y]_{t-s}/F_{s+1}[X, Y]_{t-s},$$

where $F_s[X, Y]_{t-s}$ denotes the elements of filtration s in $[X, Y]_{t-s}$. Moreover the intersection of all $F_s[X, Y]_{t-s}$ consists of the torsion elements of $[X, Y]_{t-s}$ whose order is prime to p. Its E_2-term is

$$E_2^{s,t}(X, Y) = \mathrm{Ext}_A^{s,t}(H^*Y, H^*X),$$

where H^*X (resp. H^*Y) denotes the cohomology of X (resp. Y) with coefficients in $\mathbf{Z}/p$, which is a module over the mod p Steenrod algebra A. The differentials have the form

$$d_r : E_r^{s,t}(X, Y) \to E_r^{s+r, t+r-1}(X, Y).$$

For $p = 2$ let A_0 be the subalgebra of A which is generated by $Sq^1 \in A$. This is an exterior algebra since $Sq^1Sq^1 = 0$. J. F. Adams proved the following homological vanishing theorem.

PROPOSITION 4.4 ([A], Thm. 3, p. 62]). *Let M be a graded A-module which is free over A_0 and $(l-1)$-connected, i.e. trivial in dimensions $<l$. Then $\mathrm{Ext}_A^{s,t}(M, \mathbf{Z}/2)$ is zero if $t - s < l + F(s)$, where $F(s)$ is the numerical function defined by $F(4r) = 8r$, $F(4r+1) = 8r+1$, $F(4r+2) = 8r+2$ and $F(4r+3) = 8r+4$.*

COROLLARY 4.5. *Let X be a finite spectrum whose $\mathbf{Z}/p$-cohomology vanishes for p odd and whose $\mathbf{Z}/2$-cohomology is free as an A_0-module and trivial above dimension d. Let $\alpha \in \pi^n X$ be an element of Adams filtration s. Then $\alpha = 0$ provided $d - n < F(s)$.*

Proof of the corollary. Consider the Adams spectral sequence $E_r^{s,t}(X, S^0)$ converging to $[X, S^0]_{-n} = \pi^n X$. For p odd all terms are zero and hence the cohomotopy groups of X are torsion groups whose orders are powers of 2.

From now on let $p = 2$. $E_2^{s,t}(X, S^0)$ is equal to $\mathrm{Ext}_A^{s,t}(\mathbf{Z}/2, H^*X) = \mathrm{Ext}_A^{s,t}(DH^*X, \mathbf{Z}/2)$, where DH^*X is the dual of the graded A-module H^*X which is defined as follows. If M is a graded A-module and M_i denotes the elements of degree i in M then $(DM_i) = \mathrm{Hom}\,(M_{-i}, \mathbf{Z}/2)$. The left A-module structure on M induces a right A-module structure on $DM = \mathrm{Hom}\,(M, \mathbf{Z}/2)$ which is then converted into a left A-module structure using the canonical anti-automorphism χ of the Steenrod algebra.

Our assumption that H^*X vanishes in dimensions bigger than d implies that DH^*X is $(-d-1)$-connected. Moreover, DH^*X is free as A_0-module since H^*X is A_0-free and $\chi(Sq^1) = Sq^1$. It follows from proposition 4.4 that $E_2^{s,t}(X, S^0)$ and hence $E_\infty^{s,t}(X, S^0)$ vanishes for $t - s + d < F(s)$. This means that the filtration quotient $F_s\pi^n X/F_{s+1}\pi^n X = E_\infty^{s,t}(X, S^0)$ is zero for $d - n = d + t - s < F(s)$, which implies that the element $\alpha \in \pi^n X$ is in the intersection of all filtration groups and hence a torsion element of odd order. Thus $\alpha = 0$. Q.E.D.

After these preparations we now prove proposition 4.1. The idea is to use corollary 4.5 to prove the vanishing of the cohomotopy Euler class $e((2k+1)L) \in \pi^n M(-(2k+1)L)$. We first show that $M(-(2k+1)L)$ satisfies the assumptions of (4.5), i.e. that

i) $H^*(M(-(2k+1)L); \mathbf{Z}/2)$ is free as A_0-module

ii) $H^*(M(-(2k+1)L); \mathbf{Z}/p) = 0$ for p odd

Ad i) The $\mathbf{Z}/2$-cohomology ring of L^{2m-1} is $\mathbf{Z}[x]/(x^m) \otimes E(y)$, where x is a 2-dimensional cohomology class, $y = w_1(L)$ is the first Stiefel Whitney class of L and $E(y)$ is the exterior algebra generated by y. As abelian group the $\mathbf{Z}/2$-cohomology of the Thom spectrum $M(-(2k+1)L)$ is isomorphic to the $\mathbf{Z}/2$-cohomology of L^{2m-1} via Thom isomorphism. It is given by multiplication with the Thom class $U \in H^0(M(-(2k+1)L); \mathbf{Z}/2)$. The computation $Sq^1U = w_1(-(2k+1)L)U = yU$, $Sq^1(x^sU) = x^syU$ for $s < m$ shows that the $\mathbf{Z}/2$-cohomology of the Thom spectrum is a free A_0-module.

Ad ii) Note that $-(2k+1)L$ is non-orientable since its first Stiefel–Whitney class is non-trivial and hence there is no Thom isomorphism for $\mathbf{Z}/p$-cohomology. Instead we use the Gysin sequence for $S(L)$ with coefficients in $-(2k+2)L$ (see

(2.11))

$$\to H^{i-1}(L^{2m-1}; -(2k+1)L) \to H^i(L^{2m-1}; -(2k+2)L)$$
$$\xrightarrow{p^*} H^i(S(L); -(2k+2)p = L) \to .$$

Here $H^i(\ )$ is the cohomology with $\mathbf{Z}/p$-coefficients. The bundle $-(2k+2)L$ is orientable and hence p^* can be identified with the map induced by p in (untwisted) $\mathbf{Z}/p$-cohomology whch is an isomorphism since L^{2m-1} and $S(L) = \mathbf{RP}^{2m-1}$ have the $\mathbf{Z}/p$-cohomology of a point. Thus $H^*(M(-(2k+1)L); \mathbf{Z}/p) = H^*(L^{2m-1}; -(2k+1)L)$ vanishes.

Next we estimate the Adams filtration of the cohomotopy Euler class of $(2k+1)L$ using the general properties of the Euler class stated in section 2. Note that $w_2(2L) = w_1(L)^2 = y^2 = 0$. This implies that $e(2L)$ has at least Adams filtration 1, since $w_2(2L)$ is the image of $e(2L)$ under the Hurewicz map. Hence $e(2kL) = e(2L)^k$ has at least filtration k by (2.4) and (4.3).

Finally we apply (4.5) to the Euler class $e((2k+1)L) \in \pi^{2k+1}M(-(2k+1)L)$. In this case $d = 2m-1$ (the dimension of $M(-(2k+1)L)$), $n = 2k+1$ and $s = k$ (the filtration of $(2k+1)L$). Thus the inequality $d - n < F(s)$ reduces to $2k-2 < F(k)$ (in the case $m = 2k$, $k = 0, 1 \bmod 4$) respectively to $2k-4 < F(k)$ (in the case $m = 2k-1$). Inspection of the numerical function $F(k)$ (see 4.4) shows that these inequalities hold. Corollary (4.5) then implies $e((2k+1)L) = 0$. Q.E.D.

5. Determination of $s(m)$

An inspection of the lower and upper estimates for $s(m)$ obtained in the last two sections show that they agree except for $m = 4 \bmod 8$ where we have the inequalities $m+1 \le s(m) \le m+3$.

PROPOSITION 5.1. *$s(m) = m+2$ for $m = 4 \bmod 8$.*

The main ingredients of the proof are the knowledges of $s(m)$ for other values of m and the following lemma.

LEMMA 5.2. *$s(m+n) \le s(m) + s(n)$*

Proof of the lemma. Let $f: \mathbf{RP}^{2m-1} \to S^{s(m)-1}$ and $g: \mathbf{RP}^{2n-1} \to S^{s(n)-1}$ be $\mathbf{Z}/2$-equivariant maps. Denote by $\bar{f}: S^{2m-1} \to S^{s(m)-1}$ resp. $\bar{g}: S^{2n-1} \to S^{s(n)-1}$ the composition of f resp. g with the projection map from the sphere to projective space. These maps are $\mathbf{Z}/4$-equivariant with respect to the $\mathbf{Z}/4$-action given by multiplication by $i \in \mathbf{C}$ on the domain and multiplication by -1 on the range. Then also their join

$$\tilde{f} * \tilde{g} : S^{2(m+n)-1} = S^{2m-1} * S^{2n-1} \to S^{s(m)-1} * S^{s(n)-1} = S^{s(m)+s(n)-1}$$

is a $\mathbf{Z}/4$-equivariant map. Passing to the quotient we obtain a $\mathbf{Z}/2$-equivariant map $\mathbf{RP}^{2(m+n)-1} \to S^{s(m)+s(n)-1}$ showing that $s(m+n) \leq s(m) + s(n)$. Q.E.D.

Proof of the proposition. Let $m = 4 \bmod 8$. Then using the lemma and our computations of $s(m)$ we obtain the inequalities $s(m) \leq s(m-2) + s(2) = (m-1) + 3 = m + 2$ and $m + 5 = s(m+2) \leq s(m) + s(2) = s(m) + 3$. Thus $s(m) = m + 2$. Q.E.D.

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