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Mean curvature functions of codimension-one foliations

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0. Introduction

Let F be a transversely oriented codimension-one foliation of a closed connected manifold M. If we choose a Riemannian metric g on M, then we have a smooth function $H(x)$ on M. Here $H(x)$ is the mean curvature function at x of the leaf L_x of F through x with respect to the unit vector field N orthogonal to F and its direction coincides with the given transverse orientation. We call $H(x)$ the mean curvature function of F with respect to g. Recently, Walczak [8], [9] studied the following problem for some special foliations:

[Q] *Which smooth function on M can be written as a mean curvature function with respect to some Riemannian metric on M?*

In this paper, we consider this problem from the view point given by Sullivan [6], and give an answer to this question. As a corollary, we can give another proof of the results in Walczak [8], [9]. These are done in Sections 1 and 2. In Section 3, we apply it for the case of Reeb foliations of S^3.

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1. Preliminary and result

In this paper, we work in the C^∞-category. In what follows, we always assume that foliations are of codimension-one and transversely oriented, and that the ambient manifolds are closed, connected, oriented, and of dimension $n+1 \geqq 3$, unless otherwise stated.

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Let g be a Riemannian metric on M. Then there is a unique vector field orthogonal to F whose direction coincides with the given transverse orientation. We denote this vector field by N. We given an orientation to F as follows: Let $\{E_1, \ldots, E_n\}$ be an oriented local orthonormal frame for TF. Then the orientation of M given by $\{N, E_1, \ldots, E_n\}$ coincides with the given one of M.

We denote the mean curvature of a leaf L at x with respect to N by $H(x)$, that is,

$$H = \sum_{i=1}^{n} \langle \nabla_{E_i} E_i, N \rangle$$

where $\langle , \rangle$ means $g(,)$ and ∇ is the Riemannian connection of (M, g) and $\{E_i\}$ is a local orthonormal frame for TF with $\dim F = n$. We also define an n-form X_F on M by

$$X_F(V_1, \ldots, V_n) = \det\left((\langle E_i, V_j \rangle)_{i,j=1,\ldots,n} \right) \quad \text{for } V_j \in TM,$$

where $\{E_1, \ldots, E_n\}$ is an oriented local orthonormal frame for TF. The restriction $X_F \mid L$ is the volume element of $(L, g \mid L)$ for $L \in F$. First we have the following.

PROPOSITION 1 (Rummler [5]). *$dX_F = -HdV(M, g) = \mathrm{div}_g(N)dV(M, g)$, where $dV(M, g)$ is the volume element of (M, g) and $\mathrm{div}_g(N)$ is the divergence of N with respect to g, i.e.,*

$$\mathrm{div}_g(N) = \sum_{i=1}^{n} \langle \nabla_{E_i} N, E_i \rangle$$

Let f be a smooth function on M. We call f *admissible* if there is a Riemannian metric g on M so that $-f$ coincides with the mean curvature function of F with respect to g, and set

$$C_{\mathrm{Ad}} = \{f : \text{admissible}\}.$$

Note that if $H \equiv 0$, then F is a minimal foliation. If we can find such a g, then we call F *taut*. We also set

$$C_{\pm} = \{f : f(x) > 0 > f(y) \text{ for some } x, y \in M\}.$$

Then, by Proposition 1, we have $C_{\mathrm{Ad}} \subset C_{\pm} \cup \{0\}$. From this view point, we have:

THEOREM (Sullivan [7]). $0 \in C_{\mathrm{Ad}}$, *i.e., F is taut if and only if each compact leaf of F is cut out by a closed transversal.*

THEOREM (Walczak [9]). *F is taut if and only if $C_{\mathrm{Ad}} = C_{\pm} \cup \{0\}$.*

Now recall the set-up of Sullivan [6]. Let $\mathbf{D}_p$ be the space of smooth p-forms on M and $\mathbf{D}_p^*$ be the dual space of $\mathbf{D}_p$, i.e., the space of p-currents. Then we have:

THEOREM (Schwartz [5]). $(\mathbf{D}_p^*)^* = \mathbf{D}_p$.

Let $x \in M$ and $B = \{e_1, \ldots, e_n\}$ be an oriented basis of T_xF. We define a Dirac current $\delta_{x,B}$ by

$$\delta_{x,B}(\phi) = \phi_x(e_1 \wedge \cdots \wedge e_n) \quad \text{for } \phi \in \mathbf{D}_n.$$

And set

$$C_F = \text{the closed convex cone in } \mathbf{D}_n^* \text{ spanned by Dirac currents } \delta_{x,B} \text{ for } x \in M.$$

PROPOSITION 2 (Sullivan [6]). *C_F is a compact convex cone cone in $\mathbf{D}_n^*$. Here "compact" means that there is a continuous linear functional $L : \mathbf{D}_n^* \to \mathbb{R}$ so that the set $L^{-1}(1) \cap C_F$ is compact.*

We shall call a compact set $L^{-1}(1) \cap C_F$ of the cone C_F the *base* of C_F and denote it by $\mathbf{C}$. Let $d : \mathbf{D}_p \to \mathbf{D}_{p+1}$ be the exterior differentiation and $\partial : \mathbf{D}_{p+1}^* \to \mathbf{D}_p^*$ be the dual of d, i.e., $\langle d\phi, c \rangle = \langle \phi, \partial c \rangle$ for $\phi \in \mathbf{D}_p$, $c \in \mathbf{D}_{p+1}^*$, and $\langle , \rangle$ means the natural coupling $\mathbf{D}_p \times \mathbf{D}_p^* \to \mathbb{R}$. Set $B = \partial(\mathbf{D}_{n+1}^*)$ and $Z = \operatorname{Ker} \partial : \mathbf{D}_n^* \to \mathbf{D}_{n-1}^*$.

MAIN THEOREM. *For $f \in C^\infty(M)$, the following three conditions are equivalent.*

(1) $f \in C_{\mathrm{Ad}}$.

(2) *There are an n-form ω and an oriented volume form dV on M so that $d\omega = f\,dV$ and ω is positive on F. Here "positive" means that $\omega_x(e_1 \wedge \cdots \wedge e_n) > 0$ for all oriented basis $\{e_1, \ldots, e_n\}$ of T_xF and $x \in M$.*

(3) *There is an oriented volume form dV on M so that*
   (i) $\int_M f\,dV = 0$, *and*
   (ii) $\int_c f\,dV > 0$ *for all $c \in \partial^{-1}(\mathbf{C} \cap B)$.*

In case F is taut, Sullivan [7] showed that $\mathbf{C} \cap B = \varnothing$. And it is easy to show the existence of dV so that $\int_M f\, dV = 0$ if $f \in C_{\pm}$. Thus we have another proof of the results of Walczak [8], [9].

2. Proof of main theorem

We follow the proofs given in Sullivan [6], [7]. To do this we need a Hahn–Banach theorem of the following type (cf. [1]):

THEOREM OF HAHN–BANACH. *Let V be a Frechet space, W be a closed subspace of V, and C be a compact convex cone at the origin $0 \in V$. And let $\rho : W \to \mathbb{R}$ be a continuous linear functional of W with $\rho(v) > 0$ for $v \in C \cap W - \{0\}$. Then there is an extension $\eta : V \to \mathbb{R}$ of ρ so that $\eta(v) > 0$ for $v \in C - \{0\}$.*

Proof of Main Theorem. First note that (2) *implies* (3). Because the condition (ii) of (3) reads as $\langle f\, dV, c\rangle = \langle d\omega, c\rangle = \langle \omega, \partial c\rangle$, and $\int_c f\, dV > 0$, for $\partial c \in \mathbf{C}$ and ω is positive on F.

(1) *implies* (2). Now fix a Riemannian metric g on M with $-f = H$, and set $dV = dV(M, g)$ and $\omega = X_F$ (see §1). Then, by Proposition 1, $dX_F = -H\, dV(M, g) = f\, dV$, that is, (1) implies (2).

(3) *implies* (1). Here we use the Hahn–Banach Theorem quoted above under the following situation: $V = D_n^*$, $W = B$, $C = C_F$, and ϕ is given as follows: As $\int_M f\, dV = 0$, there is an n-form ω so that $d\omega = f\, dV$. By the theorem of Schwartz, we can regard ω as a continuous linear functional $k : \mathbf{D}_n^* \to \mathbb{R}$. If we restrict k on B, then this map $k \mid B$ is independent of the choice of ω. And, by condition (ii), $\phi = k \mid B$ satisfies the hypotheses. Thus, we have a continuous linear functional $L : \mathbf{D}_n^* \to \mathbb{R}$ which satisfies $L(C_F - \{0\}) > 0$, and by the theorem of Schwartz, we have an n-form X on M which is positive on F. Further, the condition $d\omega = f\, dV$ implies $dX = f\, dV$, because $\omega \mid B = X \mid B$.

Now choose a Riemannian metric g as follows: On each leaf L, $X \mid L$ is the volume form of $(L, g \mid L)$, Ker X is orthogonal to F, and on Ker X the metric is determined by requiring $dV(M, g) = dV$, where dV is the n-form in the condition (3). By the relation $f\, dV = dX = -H\, dV(M, g)$, we have $f \in C_{\mathrm{Ad}}$. Q.E.D.

3. Examples and a concluding remark

In this section, we study, as an example, C_{Ad} for Reeb foliations of S^3 and close this paper with a conjecture.

EXAMPLE 1. *A Reeb foliation F_R of S^3*. A Reeb component F_{RC} is a codimension-one foliation of $S^1 \times D^2$ with the boundary $\partial(S^1 \times D^2)$ as a unique compact leaf (cf. Lawson [2]). To get a Reeb foliation F_R of S^3, decompose S^3 as $S^1 \times D^2 \cup D^2 \times S^1$ by identifying $(x, y) \in S^1 \times \partial D^2$ with $(x, y) \in \partial D^2 \times S^1$, and consider a Reeb component F_{RC} on each $S^1 \times D^2$. We give a transverse orientation to F_R and denote by R_+ (resp. R_-) the compact saturated set $S^1 \times D^2$ on that boundary the given transverse orientation is outward (resp. inward).

In this case, by our theorem, $C_{\mathrm{Ad}} = \{f \in C^\infty(S^3) : f(x) > 0$ and $f(y) < 0$ for some $x \in R_+$ and $y \in R_-\}$.

Note that $0 \notin C_{\mathrm{Ad}}$. This fact also shows that we cannot find any Riemannian metric of S^3 which makes each leaf of F_R a hypersurface of constant mean curvature (cf. Ex. 2).

EXAMPLE 2. *A generalized Reeb foliation F_{GR} of S^3*. Here we also use the same notations as in Example 1. We decompose S^3 as $S^1 \times D^2 \cup T^2 \times [0, 1] \cup D^2 \times S^1$ with the canonical identification given in Example 1. A generalized Reeb foliation F_{GR} is given by $F_{RC} \cup \{T^2 \times (t)\}_{t \in [0, 1]} \cup F_{RC}$. Then we also have $C_{\mathrm{Ad}} = \{f : f(x) > 0$ and $f(y) < 0$ for some $x \in R_+$ and for some $y \in R_-\}$.

We choose a transverse orientation so that R_+ is the one of $S^1 \times D^2$ whose boundary is identified with $T^2 \times \{0\}$, and set $R_+(a) = R_+ \cup \{T^2 \times (t); 0 \leqq t \leqq a\}$. The above condition is equivalent to the following:

For each $f \in C_{\mathrm{Ad}}$ there is a volume element dV of S^3 so that $\int_{S^3} f\, dV = 0$ and $\int_{R_+(a)} f\, dV > 0$ for all $a \in [0, 1]$.

Note that $0 \notin C_{\mathrm{Ad}}$. But we can find a Riemannian metric of S^3 so that each leaf of F_{GR} is a hypersurface of constant mean curvature. To see this, we simply choose a smooth function h of $[0, 1]$ satisfying $h(t) \equiv 1$ near 0, and $h(t) \equiv -1$ near 1, and lift this function onto $T^2 \times [0, 1]$ naturally, and set f on S^3 to be 1 on R_+, -1 on R_-, and h on $T^2 \times [0, 1]$. It is clear that $f \in C_{\mathrm{Ad}}$ (cf. Oshikiri [3]).

Condition (3) in our theorem is not easy to apply to arbitrary foliations. By considering the above examples, it seems to be plausible to conjecture that condition (3) is equivalent to the following:

(3*) There is an oriented volume form dV on M so that
- (i) $\int_M f\, dV = 0$, and
- (ii) $\int_D f\, dV > 0$ for any foliated compact domain D with the transverse orientation of F being outward everywhere on ∂D.

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