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Introduction

Let Γ be a lattice in $\mathbb{R}^2$ and q a real valued function in $L^2(\mathbb{R}^2/\Gamma)$. For each k in $\mathbb{R}^2$ the self-adjoint boundary value problem

$$(-\Delta + q)\psi = \lambda\psi$$

$$\psi(x + \gamma) = e^{i\langle k, \gamma\rangle}\psi(x), \quad \text{for all } \gamma \text{ in } \Gamma,$$

has a discrete spectrum customarily denoted by

$$E_1(k) \le E_2(k) \le E_3(k) \le \cdots.$$

The eigenvalue $E_n(k)$, $n \ge 1$, defines a function of k called the nth band function. It is continuous and periodic with respect to the lattice

$$\Gamma^* := \{b \in \mathbb{R}^2 | \langle \gamma, b\rangle \in 2\pi Z \quad \text{for all } \gamma \text{ in } \Gamma\}$$

dual to Γ.

To study analytic properties of band functions, we introduce the set

$$B(q) := \{(k, \lambda) \in \mathbb{C}^2 \times \mathbb{C} | \text{ there is a nontrivial } \psi \text{ in } H^2_{\text{loc}}(\mathbb{R}^2) \text{ satisfying}$$
$$(-\Delta + q)\psi = \lambda\psi, \text{ and } \psi(x + \gamma) = e^{i\langle k, \gamma\rangle}\psi(x) \text{ for all } \gamma \text{ in } \Gamma\}$$

which is referred to as the complex *Bloch variety* for q. For all $n \ge 1$, the graph of $E_n(k)$ in $\mathbb{R}^2 \times \mathbb{R}$ is a subset of $B(q)$. Conversely, the intersection $B(q) \cap (\mathbb{R}^2 \times \mathbb{R})$ coincides with these graphs. The Bloch variety contains all points that can possibly be reached by analytic continuation of any band function.

In section 1 it is shown that for any q in $L^2(\mathbb{R}^2/\Gamma)$, the Bloch variety $B(q)$ is a transcendental complex analytic hypersurface in $\mathbb{C}^2 \times \mathbb{C}$. Thus, when q is real, the union of the graphs of $E_n(k)$, $n \ge 1$, is a real analytic hypersurface in $\mathbb{R}^2 \times \mathbb{R}$.

The dual lattice Γ^* acts on $\mathbb{C}^2 \times \mathbb{C}$ by $\Gamma^* \ni b : (k, \lambda) \to (k + b, \lambda)$. Clearly, this actions maps $B(q)$ to itself. In section 3 we shall prove that for all q in $L^2(\mathbb{R}^2/\Gamma)$ the irreducible components of $B(q)$ are translates of each other by elements of Γ^*, in other words $B(q)/\Gamma^*$ is an irreducible complex analytic variety. Therefore, when q is real, the analytic continuation of any given band function is $B(q)$, from which all the other band functions can be found. That is, one band function determines all other band functions.

By the remark above, each of the functions $E_n(k), n \geq 1$, is piecewise real analytic on $\mathbb{R}^2$. For $q = 0$ and all $n \geq 1$, every real analytic piece of $E_n(k)$ continues to an entire function on $\mathbb{C}^2$, namely a translate of $k_1^2 + k_2^2$ by Γ^*. It will be shown in section 3 that when q is real and $B(q)$ contains a component that is the graph of an entire function, then q must be constant. Thus, q is constant, if a single real analytic piece of any one of the band functions extends to an entire function on $\mathbb{C}^2$.

To gain some perspective, let us recall the one-dimensional situation. For p in $L^2(\mathbb{R}/Z)$,

$$B(p) = \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} | \text{there is } \psi \in H^2_{\text{loc}}(\mathbb{R}) \text{ satisfying}$$
$$-\psi'' + p(x)\psi = \lambda\psi, x \in \mathbb{R}, \text{ and } \psi(x + n) = e^{ikn}\psi(x) \text{ for all } n \in Z\}$$

is a transcendental curve [6]. It is immediate from the equation for $B(p)$ given at the end of section 1 that $B(p)/Z^*$ is irreducible. Here, the statement that for real valued p, the function p is constant if and only if $B(p)$ contains a component that is the graph of an entire function is equivalent to a well known theorem of Borg [2]. The connection is made in detail in Section 1.

The most important part of this paper is section 2, where we directionally compactify $B(q)$. The idea is to construct a cradle by inserting $\mathbb{C}^2 \times \mathbb{C}$ in $\mathbb{P}^3$ and blowing up a set of points on $\mathbb{P}^3$ determined by the lattice Γ. The points to be adjoined to $B(q)$ lie in the exceptional planes of the blown up space. However, $B(q)$ is transcendental, so it is only possible to approach the exceptional planes along $B(q)$ "directionally" through a special family of cones. Once the cradle and cones have been specified the directional compactification $\widetilde{B(q)}$ can be computed. The divisor $\widetilde{B(q)} - B(q)$ "at infinity" is the union of curves; one curve for each primitive lattice vector γ in Γ (a lattice vector γ is called primitive if there is no $\delta \in \Gamma$ such that $\gamma = m \cdot \delta$ for some $m \in Z, |m| \geq 2$). The component corresponding to γ is isomorphic to the one-dimensional Bloch variety for the one-dimensional potential $\int_0^1 q(x + \tau\gamma) d\tau$ obtained from q by averaging in direction of γ. These curves lie in different exceptional planes and are attached to $B(q)$ in an essentially C^1 way. The results discussed above are proved by exploiting the structure of this compactification.

Section 4 complements the calculation of the directional compactification of $B(q)$ by describing how the two-dimensional spectral problem degenerates via a geometric optics limit to a one-dimensional problem on the exceptional planes.

When the continuum Laplace operator is replaced by the difference Laplacian, many properties of the corresponding algebraic Bloch variety, including its compactification have been worked out in [5], which, together with [4], served as a motivation for the constructions presented here.

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1. The complex analytic structure of the Bloch spectrum

Fix a lattice Γ in $\mathbb{R}^2$. Without loss of generality we may assume that the torus $\mathbb{R}^2/\Gamma$ has area one. Let

$$\Gamma^{*} = \{b \in \mathbb{R}^2 | \langle b, \gamma \rangle \in 2\pi Z \text{ for all } \gamma \text{ in } \Gamma\}$$

be the dual lattice. We want to show that for every q in $L^2 = L^2(\mathbb{R}^2/\Gamma)$, the Hilbert space of complex valued, square integrable functions on the torus $\mathbb{R}^2/\Gamma$, the corresponding Block variety

$$B(q) = \{(k, \lambda) \in \mathbb{C}^2 \times \mathbb{C} | \text{ there is a function } \psi \text{ in } H^2_{\text{loc}}(\mathbb{R}^2) \text{ satisfying}$$
$$(-\Delta + q)\psi = \lambda\psi \text{ and } \psi(x + \gamma) = e^{i\langle k, \gamma \rangle}\psi(x) \text{ for all } \gamma \text{ in } \Gamma\}$$

is a complex analytic hypersurface in $\mathbb{C}^2 \times \mathbb{C}$.

To do this, it is convenient to put the boundary conditions into the operator. For each k in $\mathbb{C}^2$ set

$$\Delta_k := \Delta + 2ik \cdot \nabla - k^2,$$

where $k^2 = k_1^2 + k_2^2$. If $\varphi \in H^2(\mathbb{R}/\Gamma)$ then the function $\psi(x) := e^{i\langle k, x \rangle}\varphi(x)$ satisfies

$$\Delta\psi(x) = e^{i\langle k, x \rangle}\Delta_k\varphi(x)$$

and

$$\psi(x + \gamma) = e^{i\langle k, \gamma \rangle}\psi(x)$$

for all γ in Γ. Clearly,

$$B(q) = \{(k, \lambda) \in \mathbb{C}^2 \times \mathbb{C} \mid \text{there is a function } \varphi$$
$$\text{in } H^2(\mathbb{R}^2/\Gamma) \text{ with } (-\Delta_k + q)\varphi = \lambda\varphi\}.$$

For all k in $\mathbb{C}^2$, the exponentials $e^{i\langle b, x\rangle}$, $b \in \Gamma^*$, are a complete set of eigenfunctions for $-\Delta_k$ in $L^2(\mathbb{R}^2/\Gamma)$. The associated eigenvalues are $(k + b)^2$, $b \in \Gamma^*$. Hence, the Bloch variety $B(0)$ for $q = 0$ is the union $\bigcup_{b \in \Gamma^*} B_b$ of the quadrics

$$B_b := \{(k, \lambda) \in \mathbb{C}^2 \times \mathbb{C} \mid (k + b)^2 - \lambda = 0\}, \qquad b \in \Gamma^*.$$

Observe that only a finite number of these quadrics can intersect any bounded subset of $\mathbb{C}^2 \times \mathbb{C}$. Therefore, $B(0)$ is a locally finite union of algebraic hypersurfaces and consequently a complex analytic hypersurface in $\mathbb{C}^2 \times \mathbb{C}$.

For general q in L^2 we first obtain an analytic equation for the part of $B(q)$ lying in

$$\mathbb{C}^2 \times \mathbb{C} \backslash B(0) = \mathbb{C}^2 \times \mathbb{C} \backslash \bigcup_{b \in \Gamma^*} B_b.$$

On this set, the product

$$(-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1} = 1 + q \cdot (-\Delta_k - \lambda)^{-1}$$

is defined. Furthermore, the operator $q \cdot (-\Delta_k - \lambda)^{-1}$ is Hilbert–Schmidt since

$$\sum_{b, c \in \Gamma^*} |\langle e^{i\langle b, x\rangle}, q \cdot (-\Delta_k - \lambda)^{-1} e^{i\langle c, x\rangle}\rangle|^2 = \sum_{b, c} \left| \frac{\hat{q}(b - c)}{(k + c)^2 - \lambda} \right|^2$$

$$= \sum_b |\hat{q}(b)|^2 \sum_c \frac{1}{|(k + c)^2 - \lambda|^2}$$

$$= \|q\|_2^2 \sum_c \frac{1}{|(k + c)^2 - \lambda|^2} < \infty$$

where

$$\hat{f}(k) = \int_{\mathbb{R}^2/\Gamma} f(x) e^{i\langle k, x\rangle} dx, \quad \text{for } f \in L^2(\mathbb{R}^2/\Gamma).$$

The determinant of $1 + A$ where A is a trace class operator on a Hilbert space is given by the convergent sum

$$\det (\mathbb{1} + A) = \sum_{n \geq 0} \operatorname{tr} \Lambda^n(A).$$

Here, $\Lambda^n(A), n \geq 0$, is the nth exterior power of A. If A is Hilbert–Schmidt $(\mathbb{1} + A) e^{-A} - \mathbb{1}$ is trace class and one defines the regularized determinant

$$\det_2 (\mathbb{1} + A) := \det((\mathbb{1} + A) \cdot e^{-A}).$$

It has the property that $\det_2 (\mathbb{1} + A) \neq 0$ if and only if $\mathbb{1} + A$ is invertible (see, [9]). So,

$$B(q) \cap (\mathbb{C}^2 \times \mathbb{C} \backslash B(0))$$

$$= \{(k, \lambda) \in \mathbb{C}^2 \times \mathbb{C} \backslash B(0) | \det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1}) = 0\}.$$

To analyze $\det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1})$ it is useful to present $(-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1}$ as a matrix with respect to the Fourier basis $e^{i\langle b, x\rangle}, b \in \Gamma^{\#}$, of $L^2(\mathbb{R}^2/\Gamma)$. It has entries

$$[(-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1}]_{bc} = \delta_{bc} + \frac{\hat{q}(b - c)}{(k + c)^2 - \lambda}$$

and acts on $l^2(\Gamma^{\#})$. We enumerate the elements of $\Gamma^{\#}$ so that c appears later in the list than b if $|c| > |b|$.

Let P_r be the orthogonal projection onto the subspace of $L^2(\mathbb{R}^2/\Gamma)$ spanned by $e^{i\langle b, x\rangle}, |b| \leq r$. The truncated operator $\mathbb{1} + P_r \cdot q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r$ has matrix entries

$$[\mathbb{1} + P_r \cdot q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r]_{bc} = \begin{cases} \delta_{bc} + \dfrac{\hat{q}(b - c)}{(k + c)^2 - \lambda}, & |b|, |c| \leq r \\ \delta_{bc}, & |b| \text{ or } |c| > r \end{cases}$$

It is the direct sum of a principal minor and an identity matrix.

Let L_0^2 be the linear subspace of all functions q in L^2 with mean zero, i.e., $\int_{\mathbb{R}^2/\Gamma} q(x) \, dx = \hat{q}(0) = 0$. Since the Bloch varieties of q and $q - \hat{q}(0)$ only differ by the translation $\lambda \rightarrow \lambda + \hat{q}(0)$ there is no loss of generality in discussing only those potentials with mean zero.

THEOREM 1. *The finite determinants*

$$\det \left(\delta_{bc} + \frac{\hat{q}(b - c)}{(k + c)^2 - \lambda} ; |b|, |c| \leq r \right),$$

converge uniformly on closed bounded subsets of $(\mathbb{C}^2 \times \mathbb{C} \backslash B(0)) \times L_0^2$ *to* $\det_2((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1})$.

Proof. Observe that

$$\det_2(\mathbb{1} + P_r \cdot q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r)$$

$$= \det(\mathbb{1} + P_r \cdot q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r) \exp\left(-\hat{q}(0) \sum_{|c| \le r} \frac{1}{(k+c)^2 - \lambda}\right)$$

$$= \det\left(\delta_{bc} + \frac{\hat{q}(b-c)}{(k+c)^2 - \lambda}; |b|, |c| \le r\right)$$

since $\det_2(\mathbb{1} + A) = \det(\mathbb{1} + A) e^{-\operatorname{tr} A}$ for trace class operators.
Next we claim that

$$\det_2(\mathbb{1} + q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r) = \det\left(\delta_{bc} + \frac{\hat{q}(b-c)}{(k+b)^2 - \lambda}; |b|, |c| \le r\right).$$

The operator $\mathbb{1} + q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r$ has matrix entries

$$[\mathbb{1} + q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r]_{bc} = \begin{cases} \delta_{bc} + \dfrac{\hat{q}(b+c)}{(k+c)^2 - \lambda}, & |c| \le r \\ \delta_{bc}, & |c| > r \end{cases}$$

The columns for $|c| > r$ have a single nonzero entry and that is $\delta_{cc} = 1$. By adding multiples of these columns to those with $|c| < r$, this matrix can be reduced to that of $\mathbb{1} + P_r \cdot q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r$. If we were in a finite number of dimensions their determinants would be the same and the claim would be obvious. Here, first notice that for $s > r$

$$\det_2(\mathbb{1} + P_s \cdot q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r)$$

$$= \det\left(\delta_{bc} + \frac{\hat{q}(b-c)}{(k+c)^2 - \lambda}; |b| \le s, |c| \le r : \delta_{bc}; |b| < s, r < |c| \le s\right)$$

$$= \det\left(\delta_{bc} + \frac{\hat{q}(b-c)}{(k+c)^2 - \lambda}; |b|, |c| \le r\right)$$

by column reduction for finite matrices. Next,

$$\|q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r - P_s \cdot q \cdot (\Delta_k - \lambda)^{-1} \cdot P_r\|_{\mathrm{HS}}^2 = \sum_{\substack{|b| > s \\ |c| \le r}} \frac{|\hat{q}(b-c)|^2}{|(k+c)^2 - \lambda|^2}$$

tends to zero as s goes to ∞. But $\det_2(\mathbb{1} + A)$ is Lipschitz in A with respect to the Hilbert-Schmidt norm; in fact,

$$|\det_2(\mathbb{1} + A) - \det_2(\mathbb{1} + B)| \le \|A - B\|_{HS} \exp[\alpha(\|A\|_{HS} + \|B\|_{HS} + 1)^2]$$

where $\alpha > 0$ is a universal constant. This proves the claim.

Finally,

$$\left| \det_2((-\Delta_k + q - \lambda)(-\Delta_k - \lambda)^{-1}) - \det\left(\delta_{bc} + \frac{\hat{q}(b - c)}{(k + c)^2 - \lambda}; |b|, |c| \le r\right) \right|^2$$

$$= |\det_2((-\Delta_k + q - \lambda)(-\Delta_k - \lambda)^{-1}) - \det_2(\mathbb{1} + q \cdot (-\Delta_k - \lambda)^{-1} \cdot P_r)|^2$$

$$\le \|q\|_2^2 \left(\sum_{|c| > r} \frac{1}{|(k + c)^2 - \lambda|^2} \right) \exp 2\alpha(2\|q \cdot (-\Delta_k - \lambda)^{-1}\|_{HS} + 1)^2$$

as above. Now $\|q\|_2^2 \sum_{|c| > r} 1/|(k + c)^2 - \lambda|^2$ tends uniformly to zero on closed bounded subsets of $(\mathbb{C}^2 \times \mathbb{C} \backslash B(0)) \times L_0^2$ while $\|q \cdot (-\Delta_k - \lambda)^{-1}\|_{HS}$ is uniformly bounded on them. This finishes the proof.

COROLLARY 1. $\det_2((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1})$ *is a holomorphic function on* $(\mathbb{C}^2 \times \mathbb{C} \backslash B(0)) \times L_0^2$. *For each* q *in* L_0^2 *its zero-set equals* $B(q) \cap (\mathbb{C}^2 \times \mathbb{C} \backslash B(0))$.

Proof

$$\det\left(\frac{\delta_{bc} + \hat{q}(b - c)}{(k + c)^2 - \lambda}; |b|, |c| \le r\right)$$

is a holomorphic function on $(\mathbb{C}^2 \times \mathbb{C} \backslash B(0)) \times L_0^2$.

For each *finite* subset G of Γ^* set

$$U_G := \mathbb{C}^2 \times \mathbb{C} \backslash \bigcup_{b \in \Gamma^* \backslash G} B_b.$$

In particular, $U_\varnothing = \mathbb{C}^2 \times \mathbb{C} \backslash B(0)$. These sets form an open cover of $\mathbb{C}^2 \times \mathbb{C}$. Let P_G be the projection onto the subspace spanned by $e^{i\langle b, x\rangle}$, $b \in G$, and define a partial resolvent by

$$(-\Delta_k - \lambda)_G^{-1} := P_G + (-\Delta_k - \lambda)^{-1}(1 - P_G)$$

for (k, λ) in U_G. Its matrix representation is

$$[(-\Delta_k - \lambda)_G^{-1}]_{bc} = \begin{cases} \delta_{bc} & b \in G \\ \dfrac{\delta_{bc}}{(k+b)^2 - \lambda}, & b \notin G \end{cases}$$

and

$$(-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1} = 1 + q \cdot (-\Delta_k - \lambda)_G^{-1} + (-\Delta_k - \lambda - 1)P_G$$

The operator $(-\Delta_k - \lambda)_G^{-1}$ is defined so that $(k, \lambda) \in B(q) \cap U_G$ if and only if $(-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1}$ is not invertible, that is

$$B(q) \cap U_G = \{(k, \lambda) \in U_G \mid \det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1} = 0\}$$

THEOREM 1′. *Let* G *be a finite subset of* Γ^*. *Then*

$$e^{|G|} \prod_{b \in G} ((k+b)^2 - \lambda) \exp (\lambda - (k+b)^2) \det \left(\delta_{bc} + \frac{\hat{q}(b-c)}{(k+c)^2 - \lambda}; |b|, |c| \le r \right),$$

$r > \max_{b \in G} |b|$, *converges uniformly on closed bounded subsets of* $U_G \times L_0^2$ *to* $\det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1})$.

Proof. As above.

COROLLARY 1′

$$\det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1})$$

$$= e^{|G|} \prod_{b \in G} ((k+b)^2 - \lambda) \exp (\lambda - (k+b)^2) \det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1})$$

is a holomorphic function on $U_G \times L_0^2$. *For each* q *in* L_0^2 *its zero set equals* $B(q) \cap U_G$.

The determinants

$$\det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1})$$

for finite subsets G of Γ^* give local equations for $B(q)$ in the cover of open sets U_G. Therefore, $B(q)$ is a complex analytic hypersurface of $\mathbb{C}^2 \times \mathbb{C}$.

We now obtain a global equation for $B(q)$. For $b \in \Gamma^*$ set

$$R_b(k, \lambda) = \exp \left(-\left(\frac{2\langle k, b \rangle + k^2 - \lambda - 1}{b^2 + 1} \right) + \frac{1}{2} \left(\frac{2\langle k, b \rangle + k^2 - \lambda - 1}{b^2 + 1} \right)^2 \right).$$

Then the infinite product

$$\prod_{b \in \Gamma^*} \frac{((k + b)^2 - \lambda)}{b^2 + 1} R_b(k, \lambda)$$

converges to an entire function of finite order on $\mathbb{C}^2 \times \mathbb{C}$ whose zero set equals $B(0)$.

THEOREM 2. *For each* q *in* L_0^2,

$$F(k, \lambda, q) = \prod_{b \in \Gamma^*} \frac{((k + b)^2 - \lambda)}{b^2 + 1} R_b(k, \lambda) \det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1})$$

is an entire function of finite order on $\mathbb{C}^2 \times \mathbb{C}$ *whose zero set is* $B(q)$.

Proof. By Corollary 1′, for any finite subset G of Γ^*

$$F(k, \lambda, q) = \left(\prod_{b \in \Gamma^* \backslash G} \frac{((k + b)^2 - \lambda)}{b^2 + 1} R_b(k, \lambda) \right) \left(e^{-|G|} \prod_{b \in G} \frac{\exp ((k + b)^2 - \lambda)}{b^2 + 1} R_b(k, \lambda) \right)$$

$$\cdot \det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1}).$$

It follows that F is holomorphic on U_G and vanishes precisely on $B(q) \cap U_G$. Consequently, F is an entire function on $\mathbb{C}^2 \times \mathbb{C}$ whose zero set is $B(q)$.

We apply Hadamard's inequality to obtain

$$|\det_2 ((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1})|^2 \le \prod_{b \in \Gamma^*} \left(1 + \frac{\|q\|_2^2}{|(k + b)^2 - \lambda|^2} \right).$$

Thus

$$|F(k, \lambda)|^2 \le \prod_{b \in \Gamma^*} \frac{|(k + b)^2 - \lambda|^2}{(b^2 + 1)^2} |R_b|^2 \prod_{b \in \Gamma^*} \left(1 + \frac{\|q\|_2^2}{|(k + b)^2 - \lambda|^2} \right)$$

$$= \prod_{b \in \Gamma^*} \frac{|(k + b)^2 - \lambda|^2 + \|q\|_2^2}{(b^2 + 1)^2} |R_b|^2,$$

from which one concludes by elementary estimation that F is of finite order.

We have actually shown that $F(k, \lambda, q)$ is an entire function of finite order on $(\mathbb{C}^2 \times \mathbb{C}) \times L_0^2$.

We say that $X \subset \mathbb{C}^n$ is a complex analytic hypersurface of finite order if it is the zero set of an entire function of finite order on $\mathbb{C}^n$. Suppose Y is a component of X.

Since X is the zero set of a function of finite order its indicator of growth is of finite order (cf. [7] 3.6). Therefore, the indicator of growth of Y is also of finite order and hence by the solution of the "Cousin problem with finite order" ([7], 3.30) Y is the zero set of an entire function of finite order. That is, the components of a hypersurface of finite order are also hypersurfaces of finite order.

Suppose $X \subset \mathbb{C}^n \times \mathbb{C}$ is a hypersurface of finite order defined by $f(z, w) = 0$. Let $\mu_X(z)$ be the number (possibly infinite) of w's (counted with multiplicity in $f(z, \cdot)$) for which $(z, w) \in X$. Geometrically, $\mu_x(z)$ is the number of sheets lying over z when X is projected onto $\mathbb{C}^n$. If no fiber of the projection is all of $\mathbb{C}$ then ([7], 3.44) for each $m \geq 0$ $X_m := \{z \in \mathbb{C}^n | \mu_x(z) \leq m\}$ is either $\mathbb{C}^n$ itself or contained in an analytic subvariety of positive codimension.

THEOREM 3. *Let q be a real valued function in $L^2(\mathbb{R}^2/\Gamma)$. Then, the "real Bloch variety"*

$$B_{\mathbb{R}}(q) := \{(k, \lambda) \in \mathbb{R}^2 \times \mathbb{R} | \text{there is a } \varphi \text{ in } H^2(\mathbb{R}^2/\Gamma) \text{ with } (-\Delta_k + q)\varphi = \lambda\varphi\}$$

determines the complex Bloch variety $B(q)$.

Proof. By analytic continuation $B_{\mathbb{R}}(q)$ determines all the components of $B(q)$ that meet $B_{\mathbb{R}}(q)$.

It follows from Theorem 2 (we may assume that the average of q is zero) that every component Y of $B(q)$ is a hypersurface in $\mathbb{C}^2 \times \mathbb{C}$ of finite order. Moreover, for fixed $k \in \mathbb{C}^2$, $(\{k\} \times \mathbb{C}) \cap Y$ is discrete since the corresponding values of λ belong to the spectrum of $-\Delta_k + q$. Therefore, (with the notation introduced above) Y_0 is either $\mathbb{C}^2$ itself – in which case Y is empty and there is nothing to prove – or is contained in an analytic subvariety of $\mathbb{C}^2$ of positive codimension.

For real k, $-\Delta_k + q$ is self adjoint so that $B_{\mathbb{R}}(q) = B(q) \cap (\mathbb{R}^2 \times \mathbb{C})$ and $Y \cap B_{\mathbb{R}}(q) = Y \cap (\mathbb{R}^2 \times \mathbb{C})$. If $Y \cap (\mathbb{R}^2 \times \mathbb{C}) = \varnothing$, then $\mathbb{R}^2 \subset Y_0$. By the last paragraph $\mathbb{C}^2 \subset Y_0$ which implies once again that $Y = \varnothing$. Thus, every nontrivial component of $B(q)$ intersects $B_{\mathbb{R}}(q)$.

Later on we will need some information about one dimensional Bloch varieties. So, let $q \in L^2(\mathbb{R}^1/Z)$. The associated one dimensional Bloch variety is

$$B(q) = \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} | \text{ there is a function } \psi \text{ in } H^2_{loc}(\mathbb{R}^1)$$
$$\text{satisfying } -\psi'' + q(x)\psi = \lambda\psi \text{ and } \psi(x + n) = e^{ikn}\psi(x) \text{ for all } n\}$$

$$= \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} | \text{ there is a function } \varphi \text{ in } H^2(\mathbb{R}^1/Z)$$
$$\text{with } (-D_k^2 + q)\varphi = \lambda\varphi\}$$

where $D_k = (d/dk) + ik$. Once again

$$B(0) = \bigcup_{b \in Z^*} B_b,$$

$$B_b = \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} | (k + b)^2 - \lambda = 0\}, \quad b \in \mathbb{Z}^* = 2\pi\mathbb{Z}$$

Now, in contrast to the two dimensional case, there is a simple equation for $B(0)$; namely,

$$B(0) = \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} | \cos k - \cos \sqrt{\lambda} = 0\}.$$

Notice that the order of $\cos k - \cos \sqrt{\lambda}$ is smaller than what we would obtain had we simply imitated the estimates used in the two dimensional case.

In general, let $y_1(x, \lambda, q)$ and $y_2(x, \lambda, q)$ be the solutions of

$$-y'' + q(x)y = \lambda y \tag{1}$$

satisfying the initial conditions

$$y_1(0, \lambda) = y_2'(0, \lambda) = 1$$

$$y_1'(0, \lambda) = y_2(0, \lambda) = 0.$$

Then, any solution can be written

$$\psi = \psi(0)y_1 + \psi'(0)y_2.$$

It follows that $\psi(x + 1) = e^{ik}\psi(x)$, $x \in \mathbb{R}$, if and only if

$$F(\lambda)\begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix} = e^{ik}\begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix}$$

where $F(\lambda)$ is the Floquet matrix

$$F(\lambda) := \begin{pmatrix} y_1(1, \lambda) & y_2(1, \lambda) \\ y_1'(1, \lambda) & y_2'(1, \lambda) \end{pmatrix}$$

Set

$$\Delta(\lambda, q) = \operatorname{tr} F(\lambda)$$

$$= y_1(1, \lambda) + y_2'(1, \lambda).$$

The characteristic polynomial of $F(\lambda)$ is

$$\det\,(F(\lambda) - \xi I) = \xi^2 - \Delta(\lambda)\xi + |F(\lambda)|$$
$$= \xi^2 - \Delta(\lambda)\xi + 1$$

Therefore,

$$B(q) = \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} | e^{ik} \text{ is a root of } \xi^2 - \Delta(\lambda)\xi + 1 = 0\}$$
$$= \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} | 2 \cos k - \Delta(\lambda) = 0\}.$$

The final task is to connect this equation for $B(q)$ with a determinant.

THEOREM 4. *The holomorphic functions*

$$(2 \cos k - 2 \cos \sqrt{\lambda}) \det\left(\delta_{bc} + \frac{\hat{q}(b - c)}{(k + c)^2 - \lambda} \,;\, |b|, |c| < r\right), \qquad r > 0,[1]$$

converge uniformly on closed bounded subsets of

$$(\mathbb{C} \times \mathbb{C}) \times L_0^2(\mathbb{R}^1/Z) \text{ to } (2 \cos k - 2 \cos \sqrt{\lambda}) \det_2 ((-D_k^2 + q - \lambda)(-D_k^2 - \lambda)^{-1})$$

Furthermore,

$$(2 \cos k - 2 \cos \sqrt{\lambda}) \det_2 ((-D_k^2 + q - \lambda) \cdot (D_k^2 - \lambda)^{-1}) = 2 \cos k - \Delta(\lambda)$$

It is easy to see that $q \cdot (-D_k^2 - \lambda)^{-1}$ is trace class so that the determinant need not be regularized in the statement of the theorem.

Proof. The first part goes just as before. For the second, notice, by examining the matrix, that

$$(2 \cos k - 2 \cos \sqrt{\lambda}) \det_2 ((-D_k^2 + q - \lambda) \cdot (-D_k^2 - \lambda)^{-1}) \qquad (2)$$

is an even, periodic (period 2π) function of k. For each λ, $2 \cos k - \Delta(\lambda)$ has simple roots in k unless $k = n\pi, n = 0, \pm 1, \ldots$, in which case they are exactly double. By

---

[1] $\hat{q}(b) = \displaystyle\int_0^1 q(x)\, e^{-ibx}\, dx, \qquad b \in 2\pi\mathbb{Z}$

evenness and periodicity (2) has at least double roots at these points. Therefore,

$$\frac{2 \cos k - 2 \cos \sqrt{\lambda}}{2 \cos k - \Delta(\lambda)} \det_2 \left((-D_k^2 + q - \lambda) \cdot (-D_k^2 - \lambda)^{-1}\right) \tag{3}$$

is an entire function (bearing in mind that the zero set of (2) is exactly $B(q)$). For each λ let $k = k' + ik''$ go to $+i^\infty$ with $0 \le k' \le 1$. The determinant goes to 1, the factor $(2 \cos k - 2 \cos \sqrt{\lambda})/(2 \cos k - \Delta(\lambda))$ also goes to 1. Therefore, (3) is bounded everywhere in k by periodicity. The theorem follows from Liouvilles theorem.

It is well known ([8] Theorem 1.4) that

$$\Delta(\lambda) = 2 \cos \sqrt{\lambda} + 0\left(\frac{e^{|\mathrm{Im} \sqrt{\lambda}|}}{|\lambda|^{1/2}}\right).$$

So,

$$\det_2 \left((-D_k^2 + q - \lambda) \cdot (-D_k^2 - \lambda)^{-1}\right) = 1 + \frac{e^{|\mathrm{Im} \sqrt{\lambda}|}}{(2 \cos k - 2 \cos \sqrt{\lambda})} 0\left(\frac{1}{|\lambda|^{1/2}}\right)$$

and we obtain a sharp estimate on the order of the holomorphic function defining $B(q)$.

Suppose q in $L^2(\mathbb{R}^1/Z)$ is real valued. Then the spectrum of $-(d^2/dx^2) + q(x)$ acting on $L^2(\mathbb{R}^1)$ is absolutely continuous and is the union of bands determined by $|\Delta(\lambda)| \le 2$. For generic functions q there are infinitely many complementary gaps. It is a classical theorem of Borg [2] that q is constant when there are no gaps at all.

The obvious generalization of Borg's result, that is, a real valued q in $L^2(\mathbb{R}^2/\Gamma)$ is constant if the spectrum of $-\Delta + q$ is a ray, is false. In fact, the spectrum of any q in $L^\infty(\mathbb{R}^2/\Gamma)$ with sufficiently small norm is a ray. Furthermore, no matter how large q is there are never more than a finite number of gaps, [3]. Nevertheless, there is a geometric reformulation of Borg's theorem that can be generalized to higher dimensions.

It is easy to see that the absence of gaps in one dimension is equivalent to the condition that all the roots of $\Delta^2(\lambda) - 4 = 0$ are real and double except the smallest which is simple. If this condition is satisfied, the projection $B(q) \to \mathbb{C}$, $(k, \lambda) \to k$ has rank one at each of the local branches of $B(q)$ passing through any of its points. Therefore, each local section of the projection can be analytically continued to an entire function $\lambda = \lambda(k)$ parametrizing a component of $B(q)$. Conversely it is easy to see that whenever $B(q)$ contains such a graph, there are no gaps. So, we may reformulate Borg's result geometrically as the statement that $B(q)$ contains a component that is the graph of an entire function $\lambda = \lambda(k)$ if and only if q is constant.

2. Directional compactification

In order to "complete" the Bloch variety $B(q)$, $q \in L^2(\mathbb{R}^2/\Gamma)$, by adding points "at infinity" in a meaningful way we will construct a compactification X of the ambient space $\mathbb{C}^2 \times \mathbb{C}$, together with cone-like subsets $\Sigma(\Theta)$, $0 < \Theta < \pi$ of $\mathbb{C}^2 \times \mathbb{C}$. We shall then say that the *directional compactification* of $B(q)$ is the space obtained from $B(q)$ by adding the point-set closure of $B(q) \cap \Sigma(\Theta)$ in X. The pair $X_\Theta := (X, \Sigma(\Theta))$ is referred to as a *cradle* for directional compactification.

It is helpful to motivate the construction by describing its analogue in one dimension. In that case the space X is the two-dimensional complex projective space $\mathbb{P}^2$, and $\mathbb{C} \times \mathbb{C}$ is embedded in $X = \mathbb{P}^2$ by

$$\mathbb{C} \times \mathbb{C} \to \mathbb{P}^2, \qquad (k, \lambda) \to (k, \lambda - k^2, 1).$$

We use the homogeneous coordinates (k, l, μ) on $\mathbb{P}^2$. The embedding of $\mathbb{C} \times \mathbb{C}$ has been such that the closures of the components B_b of the free Bloch variety are the lines with the equation

$$2kb + \mu b^2 - l = 0, \qquad b \in Z^\# = 2\pi\mathbb{Z}$$

They intersect the line "at infinity" $L_\infty := \{(k, l, \mu) \in \mathbb{P}^2 / \mu = 0\}$ at the points $(1, 2b, 0)$.

For a general potential $q \in L^2(\mathbb{R}^1/Z)$ the equation

$$2 \cos k - \Delta(\lambda; q) = 0$$

of the Bloch variety given in the previous section has essential singularities along L_∞. However, suitably modified it extends continuously over most of L_∞, provided one permits k to go to infinity only along rays in the complex plane that have a non-zero angle with the real axis. More precisely, let us define for $0 < \Theta < \pi$

$$\Sigma(\Theta) := \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} / \text{the argument of } k^2 \text{ does not lie between } -\Theta \text{ and } \Theta\}.$$

Recall that for a finite subset G of $2\pi Z$

$$\prod_{b \in G} ((k + b)^2 - \lambda) \det_2 ((-D_k^2 + q - \lambda) \cdot (-D_k^2 - \lambda)^{-1}) = 0$$

is a local equation for $B(q)$ on the open subset U_G of $\mathbb{C} \times \mathbb{C}$.

LEMMA. *Let* $q \in L^2(\mathbb{R}^1/Z)$ *with* $\int_0^1 q(x)\, dx = 0$. *The restriction of*

$$\prod_{b \in G} \left(\frac{(k+b)^2 - \lambda}{k} \right) \cdot \det_2 \left((-D_k^2 + q - \lambda) \cdot (-D_k^2 - \lambda)^{-1} \right)$$

to $\Sigma(\Theta)$ *extends continuously to*

$$L_\infty \backslash \{(0, 1, 0)\} \cup \{(1, 2b, 0)/b \notin G\}$$

and vanishes there only at the points $(1, 2b, 0)$, $b \in G$.

Proof. Consider the affine chart $\mathbb{C}^2 \ni (u, v) \to (1, u, v)$ on $\mathbb{P}^2$. In this chart $L_\infty \backslash \{(0, 1, 0)\}$ is the line given by $v = 0$. The coordinate change between the (k, λ) and the (u, v) coordinates is

$$\lambda = \frac{1}{v^2}(1 + uv), \qquad k = \frac{1}{v}.$$

By Theorem 1.4

$$\det_2 \left((-D_k^2 + q - \lambda) \cdot (-D_k^2 - \lambda)^{-1} \right) = \frac{2\cos k - \Delta(\lambda)}{2\cos k - 2\cos\sqrt{\lambda}}$$

Since

$$\Delta(\lambda) = 2\cos\sqrt{\lambda} + 0\left(\frac{\exp|\mathrm{Im}\sqrt{\lambda}|}{|\lambda|^{1/2}} \right)$$

our function equals

$$\prod_{b \in G} \left(\frac{(k+b)^2 - \lambda}{k} \right) \cdot \left(1 + 0\left(\frac{1}{|\lambda|^{1/2}} \right) \left(\frac{\exp|\mathrm{Im}\sqrt{\lambda}|}{\sin\frac{1}{2}(k + \sqrt{\lambda}) \cdot \sin\frac{1}{2}(k - \sqrt{\lambda})} \right) \right)$$

In the (u, v)-coordinates this becomes

$$\prod_{b \in G} (2b + u + b^2 v)$$

$$\cdot \left(1 + 0\left(\frac{|v|}{|1 + uv|^{1/2}} \right) \frac{\exp|\mathrm{Im}\, 1/v\sqrt{1 + uv}|}{\sin\dfrac{1}{2v}(1 + \sqrt{1 + uv}) \cdot \sin\dfrac{1}{2v}(1 - \sqrt{1 + uv})} \right)$$

It now follows that everything extends continuously to points with $v = 0$ and $u/2 \notin 2\pi\mathbb{Z}\backslash G$ provided that one restricts v to the region $\arg v^2 \notin (-\Theta, \Theta)$, which is $\Sigma(\Theta)$.

By the lemma the closure of $B(q) \cap \Sigma(\Theta)$ clearly intersects $L_\infty \backslash \{(0, 1, 0)\}$ at most in the points $(1, 2b, 0)$, $b \in 2\pi\mathbb{Z}$. With a little more work one sees that all these points occur and moreover that in a neighborhood of each of them $\overline{B(q) \cap \Sigma(\Theta)}$ looks like the cone $\{z \in \mathbb{C} | \, |z| < 1 \text{ and } \arg z^2 \in (-\Theta, \Theta)\}$. The proofs of the last two statements are similar to those we give in the two-dimensional case, so we won't carry them out here.

We now return to the construction of the two-dimensional cradle.

The first step in constructing the total space X is — as in one dimension — to embed $\mathbb{C}^2 \times \mathbb{C}$ in the projective space $\mathbb{P}^3$ by

$$\mathbb{C}^2 \times \mathbb{C} \to \mathbb{P}^3, \qquad (k, \lambda) \to (k; \lambda - k^2, 1) = (k_1, k_2; \lambda - k_1^2 - k_2^2, 1)$$

On $\mathbb{P}^3$ we introduce homogenuous coordinates $(k; l, \mu)$. For $b \in \Gamma^*$ the closure B_b' of B_b in $\mathbb{P}^3$ is given by the equation $2\langle k, b \rangle + b^2\mu - l = 0$. It intersects the "plane at infinity" $H_\infty := \{(k; l, \mu) \in \mathbb{P} | \mu = 0\}$ along the line

$$B_b' \cap H_\infty = \{(k; l, 0) \in H_\infty / 2\langle k, b \rangle - l = 0\}.$$

These lines in H_∞ are not in general position: The points where two or more lines $B_b' \cap H_\infty$ intersect form the set

$$I := \{(\gamma; 4\pi m, 0) / \gamma \text{ is a primitive vector in the lattice } \Gamma \text{ and } m \in \mathbb{Z}\}.$$

A point $(\gamma; 4\pi m, 0)$ of I lies on infinitely many of the components B_b', namely all such components with $b \in \Gamma^*$ and $\langle b, \gamma \rangle = 2\pi m$.

The intersection pattern of the B_b' will be improved if the points of I are blown up. So, for each finite subset S of I let $X(S)$ be the manifold obtained from $\mathbb{P}^3$ by blowing up the points of S. The exceptional set $P_{\gamma, m}$ over a point $(\gamma; 4\pi m, 0)$ of S is a projective plane whose points are in a one-to-one correspondence with the tangent directions in $\mathbb{P}^3$ at the point $(\gamma; 4\pi m, 0)$. The inclusion of $\mathbb{C}^2 \times \mathbb{C}$ in $\mathbb{P}^3$ used above defines an embedding of $\mathbb{C}^2 \times \mathbb{C}$ in $X(S)$. The closure $\bar{B}_b$ of B_b in $X(S)$ is the strict transform of $B_b' \subset \mathbb{P}^3$. (The bar $^-$ will always denote closure in some space $X(S)$, where the set S will be determined by the context).

Whenever B_b' passes through $(\gamma; 4\pi m)$ the strict transform $\bar{B}_b$ meets $P_{\gamma, m}$ in the line that corresponds to the tangent plane of $\bar{B}_b$ at $(\gamma; 4\pi m, 0)$. One easily checks that all these tangent planes are in general position, and so the same holds for all these lines. Precisely, if b_1, b_2, b_3 are different elements of Γ^* with $\langle b_i, \gamma \rangle = 2\pi m$

then $\bar{B}_{b_1} \cap \bar{B}_{b_2} \cap \bar{B}_{b_3} \cap P_{\gamma, m} = \emptyset$. This shows that in $X(S)$ the closures $\bar{B}_b$, $b \in \Gamma^{\#}$ with $\langle b, \gamma \rangle = 2\pi m$ for some $(\gamma; 4\pi m, 0) \in S$ are effectively separated. Therefore, it is natural to introduce as the compactification X of $\mathbb{C}^2 \times \mathbb{C}$ the inverse limit of all the spaces $X(S)$ defined by the natural maps $X(S_1) \to X(S_2)$ for $S_2 \subset S_1$. However, we are not going to use this inverse limit, but work directly with the manifolds $X(S)$ throughout. This completes the definition of the ambient space X for the cradle.

The cones $\Sigma(\Theta)$ are defined in almost the same way as in one dimension. We set

$$\Sigma(\Theta) := \{(k_1, k_2, \lambda) \in \mathbb{C}^2 \times \mathbb{C} \mid \arg(k_1^2 + k_2^2) \notin (-\Theta, \Theta)\}.$$

For each finite subset S of I and $0 < \Theta < \pi$ we define the cradle $X_\Theta(S)$ as the pair $(X(s), \Sigma(\Theta))$, and call

$$\widetilde{B(q)} := B(q) \cup \overline{B(q) \cap \Sigma(\Theta))}$$

the *directional compactification of $B(q)$* in the cradle $X_\Theta(S)$.

For each $(\gamma, 4\pi m, 0) \in S$ let $E_{\gamma, m}$ be the affine piece of the exceptional plane $P_{\gamma, m}$ corresponding to all tangent directions of $\mathbb{P}^3$ at this point that are not tangent to H_∞. We will primarily be interested in the intersection of $\widetilde{B(q)}$ with $E_{\gamma, m}$. To reduce the analysis of this intersection to one calculation in local coordinates we use the action of $\Gamma^{\#}$ on $\mathbb{C}^2 \times \mathbb{C}$ defined by

$$\Gamma^{\#} \ni b : (k, \lambda) \to (k + b, \lambda).$$

Notice that the $\Gamma^{\#}$-action maps $B(q)$ to itself and that all the definitions given in this section are compatible with this action. Precisely,

(1) For each $0 < \Theta < \pi$ and each $b \in \Gamma^{\#}$ there is a compact subset K of $\mathbb{C}^2 \times \mathbb{C}$ and $0 < \Theta_1, \Theta_2 < \pi$ such that $\Sigma(\Theta_1) - K \subset b \cdot (\Sigma(\Theta) - K) \subset \Sigma(\Theta_2)$.

(2) The action extends to $\mathbb{P}^3$ by

$$\Gamma^{\#} \ni b : (k; l, \mu) \to (k + \mu b; l - 2\langle k, b \rangle - \mu b^2, \mu).$$

The extended action preserves the set I.

Hence, each $b \in \Gamma^{\#}$ induces an isomorphism $b^* : X(S) \to X(b \cdot S)$. One checks that

(3) for $(\gamma; 4\pi m, 0) \in S$ this isomorphism b^* maps $E_{\gamma, m}$ to $E_{\gamma, m - (1/2\pi)\langle \gamma, b \rangle}$.

We can now use the $\Gamma^{\#}$-action to reduce considerations about $E_{\gamma, m}$ and its intersection with $B(q)$ to the case $m = 0$. Furthermore the original lattice Γ can be

rotated so that the first component of γ becomes zero, i.e. $\gamma = (0, \gamma_2)$. Then local coordinates around the point $(\gamma; 0, 0) = (0, 1, 0, 0)$ of $\mathbb{P}^3$ are $k_1/k_2, 1/k_2, \mu/k_2$. Blowing up produces coordinates (κ, u, v) around $E_{\gamma, 0}$ in $X(S)$ such that $E_{\gamma, 0} = \{v = 0\}$ and

$$\frac{k_1}{k_2} = \kappa v, \qquad \frac{1}{k_2} = uv, \qquad \frac{u}{k_2} = v.$$

So the coordinate change between the new coordinates and the original coordinates on $\mathbb{C}^2 \times \mathbb{C}$ is

$$k_1 = \kappa, \qquad k_2 = \frac{1}{v}, \qquad \lambda = u + \frac{1}{v^2}.$$

One sees that for $b = (b_1, b_2) \in \Gamma^\#$ the intersection $\bar{B}_b \cap E_{\gamma, 0}$ is empty if $b_2 \neq 0$, and for $b_2 = 0$ this intersection is given by the equation $(\kappa + b_1)^2 - u = 0$. Observe that these are the equations of the free Bloch variety in dimension one.

This concludes our discussion of the cradle $X_\Theta(S)$.

Let us now determine the boundary points of $\widetilde{B(q)}$ lying in $E_{\gamma, m}$. In order to describe the local structure of $\widetilde{B(q)}$ at these points we introduce the following notion: A triple of topological spaces is called *a locally cone-like space of dimension* n if it is homeomorphic to the triple (Z, Z_1, Z_2) where

$$Z := \{z \in \mathbb{C}^{n+1}/|z_i| < 1 \text{ for } i = 1, \ldots, n+1\}$$

$$Z_1 := \left\{z \in Z/\arg z_{n+1}^2 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$$

$$Z_2 := \{z \in Z_1/z_1 = 0\}$$

(see Figure 1).

If $q \in L^2$ and γ is a primitive vector of Γ the average of q in direction of γ is the function on $\{x \in \mathbb{R}^2/\langle x, \gamma \rangle\} = 0$ given by

$$x \to \int_0^1 q(x + \tau \gamma) \, d\tau = \sum_{\substack{b \in \Gamma^\# \\ \langle b, \gamma \rangle = 0}} \hat{q}(b) \, e^{i\langle b, x \rangle}$$

Now project the lattice Γ to the line $\langle x, \gamma \rangle = 0$ and let α be the projected lattice vector of minimal length such that the pair (α, γ) gives the standard orientation of $\mathbb{R}^2$. Observe that $|\alpha| \cdot |\gamma| = |\mathbb{R}^2/\Gamma| = 1$, and that $2\pi(\alpha/|\alpha|^2)$ is a generator of
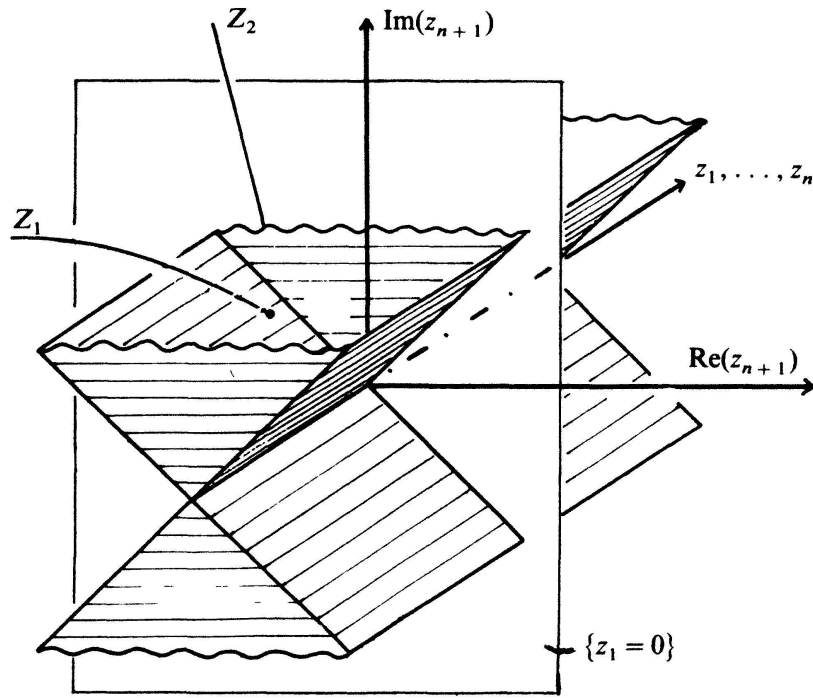
Figure 1

$\{b \in \Gamma^{\#} | \langle b, \gamma \rangle = 0\}$ (see Figure 2). The map $t \to t(\alpha/|\alpha|)$ is a length-preserving parametrization of the line $\langle x, \gamma \rangle 0$. In the t-variable the averaged function

$$\sum_{m = -\infty}^{+\infty} \hat{q}\left(2\pi m \frac{\alpha}{|\alpha|^2}\right) \cdot e^{2\pi i(mt/|\alpha|)}$$

is periodic with periodic $|\alpha|$. Consider the one-dimensional operator

$$-\frac{d^2}{dt^2} + \sum_{m = -\infty}^{+\infty} \hat{q}\left(2\pi m \frac{\alpha}{|\alpha|^2}\right) e^{2\pi i(mt/|\alpha|)}.$$

Setting $s = (t/|\alpha|)$ this operator becomes

$$\frac{1}{|\alpha|^2}\left(-\frac{d^2}{ds^2} + q_\gamma(s)\right)$$

where

$$q_\gamma(s) := \sum_{m = -\infty}^{+\infty} |\alpha|^2 \hat{q}\left(2\pi m \frac{\alpha}{|\alpha|^2}\right) e^{2\pi i s}$$
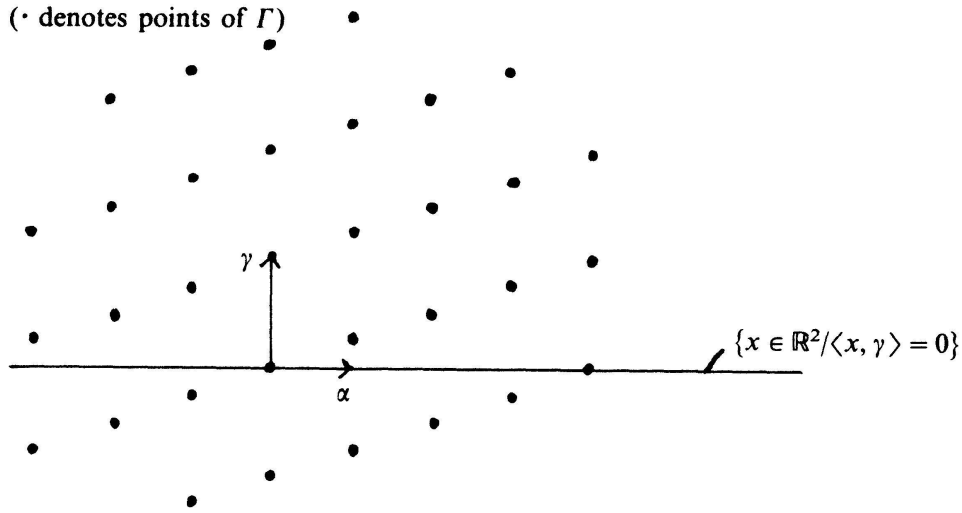
is periodic with period 1.

(· denotes points of Γ)

Figure 2

THEOREM. *Let $q \in L_0^2$, γ a primitive vector of Γ and $m \in \mathbb{Z}$, and let S be a finite subset of I containing $(\gamma, 4\pi m, 0)$. Then for any $0 < \Theta < \pi$*

$$\widetilde{B(q)} \cap E_{\gamma, m}$$

is a complex curve isomorphic to the Bloch-variety $B(q_\gamma)$ of the one-dimensional potential q_γ.

Every smooth point of $\widetilde{B(q)} \cap E_{\gamma, m}$ has a neighborhood U such that the triple $(U, U \cap \Sigma(\Theta), U \cap \Sigma(\Theta) \cap \widetilde{B(q)})$ is a locally cone-like space of dimension 2.

Proof. We may suppose by rotating the lattice Γ and using the Γ^* action that $\gamma = (0, \gamma_2)$ and $m = 0$. Recall that the coordinates (κ, u, v) with

$$k_1 = \kappa, \qquad k_2 = \frac{1}{v}, \qquad \lambda = u + \frac{1}{v^2}$$

have been introduced above in a neighborhood of $E_{\gamma, 0}$ in $X(S)$.

Let p be a point of $E_{\gamma, 0}$ and U a small neighborhood of p in $X(S)$. Then there is a finite set $G \subset \{b \in \Gamma^* | b_2 = 0\}$ such that for all $b \in \Gamma^* \backslash G$ the componente $\bar{B}_b$ of the free Bloch variety does not meet $E_{\gamma, 0} \cap U$. For $(\kappa, u, v) \in U$ the matrix of $(-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1}$ may be organized in the following way

$$[(-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)_G^{-1}]_{bc} = \begin{cases} \delta_{bc} + \dfrac{v\hat{q}(b - c)}{2c_2 + v((\kappa + c_1)^2 + c_2^2 - u)} & \text{if } c_2 \neq 0 \\[3mm] \delta_{bc} + \dfrac{\hat{q}(b - c)}{(\kappa + c_1)^2 - u} & \text{if } c_2 = 0, c \notin G \\[3mm] ((k + c_1)^2 - u)\delta_{bc} + \hat{q}(b - c) & \text{if } c \in G \end{cases}$$

We split it up into blocks

$$
\begin{array}{c}
\xrightarrow{\qquad} c \\[4pt]
\overbrace{c_2 = 0}^{} \quad \overbrace{c_2 \neq 0}^{}
\end{array}
$$

$$
\begin{array}{c}
\Big\downarrow \\[-2pt]
b
\end{array}
\begin{array}{l}
b_2 = 0 \\[6pt]
b_2 \neq 0
\end{array}
\left\{
\begin{pmatrix}
A(\kappa, u, v) & \vdots & B(\kappa, u, v) \\
\cdots\cdots\cdots & & \cdots\cdots\cdots \\
C(\kappa, u, v) & \vdots & D(\kappa, u, v)
\end{pmatrix}
\right.
$$

Observe that $B(\kappa, u, 0) = 0$, $D(\kappa, u, 0) = 1$, so that for $v = 0$ the matrix above has the form

$$
\begin{pmatrix}
A(\kappa, u, 0) & 0 \\
* & 1
\end{pmatrix}
$$

where $A(\kappa, u, 0)$ has entries

$$
A(\kappa, u, 0)_{b_1, c_1} =
\begin{cases}
\delta_{bc} + \dfrac{\hat{q}(b_1 - c_1; 0)}{(\kappa + c_1)^2 - u} & \text{if } (c_1, 0) \notin G \\[12pt]
((\kappa + c_1)^2 - u)\delta_{bc} + \hat{q}(b_1 - c_1, 0) & \text{if } (c_1, 0) \in G.
\end{cases}
$$

Apart from scaling this is the matrix for $(-D_\kappa^2 + q_\gamma - u) \cdot (-D_\kappa^2 - u)_G^{-1}$. By the same kind of argument as used in the proof of Theorem 1.1 one sees that

$$
\det_2 \begin{pmatrix} A & 0 \\ * & 1 \end{pmatrix} = \det_2 A
$$

We set

$$
f(\kappa, u, v) := \det_2 \begin{pmatrix}
A(\kappa, u, v) & B(\kappa, u, v) \\
C(\kappa, u, v) & D(\kappa, u, v)
\end{pmatrix}
$$

Then

$$
f(\kappa, u, 0) = \det_2 A(\kappa, u, 0)
$$

is a meromorphic function on $E_{\gamma, 0}$ and up to scaling of coordinates it coincides with one of the local equations of $B(q_\gamma)$ as discussed in section 1.

For $v \neq 0$ the function $f(\kappa, u, v)$ is an equation for $B(q)$. Therefore we want to show that it is continuous at $E_{\gamma, 0}$ if we approach $E_{\gamma, 0}$ through the cone $\Sigma(\Theta)$. Furthermore after reparametrization it is continuously differentiable in the cone.

Once such regularity has been established it will be possible to show that its zero-set is a locally cone-like space.

The square of the Hilbert–Schmidt norm of the difference

$$\begin{pmatrix} A(\kappa, u, v) & B(\kappa, u, v) \\ C(\kappa, u, v) & D(\kappa, u, v) \end{pmatrix} - \begin{pmatrix} A(\kappa, u, 0) & B(\kappa, u, 0) \\ C(\kappa, u, 0) & D(\kappa, u, 0) \end{pmatrix}$$

is

$$\|q\|_2^2 \cdot \sum_{\substack{c \in \Gamma^* \\ c_2 \neq 0}} \frac{|v|^2}{|2c_2 + v((\kappa + c_1)^2 + c_2^2 - u)|^2}$$

Write v in polar coordinates: $v = t\, e^{i\alpha}$. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sum_{\substack{c \in \Gamma^* \\ c_2 \neq 0}} \frac{|v|^2}{|2c_2 + v((\kappa + c_1)^2 + c_2^2 - u)|^2} = O(|v|) \qquad (*)$$

uniformly on points (κ, u, v) of U satisfying $\varepsilon \leq \alpha \leq \pi - \varepsilon$ and $|t| \leq \delta$. We shall verify this estimate for $(\kappa, u) = (0, 0)$; the general case is a straightforward elaboration.

The sum

$$\sum_{\substack{c \in \Gamma^* \\ c_2 \neq 0}} \frac{|v|^2}{|2c_2 + v|c|^2|^2}$$

is comparable to the integral

$$\int_{|x| \geq 1} \frac{dx}{\left| 2\frac{x_2}{v} + |x|^2 \right|^2}$$

Let $t = 2/s$, i.e. $v = (2/s)(\cos\alpha + i\sin\alpha)$. Then the integral becomes

$$\int_{|x| \geq 1} \frac{dx}{|x|^4 + 2sx_2|x|^2 \cos\alpha + s^2x_2^2}$$

$$= \int_1^\infty \frac{dr}{r} \int_0^{2\pi} \frac{d\varphi}{r^2 + s^2 \sin^2\varphi + 2rs \cos\alpha \sin\varphi}$$

$$= \int_1^\infty \frac{dr}{r} \int_0^{2\pi} \frac{d\varphi}{(r^2 + s^2 \sin^2\varphi)\left(1 + \dfrac{2rs \cos\alpha \sin\varphi}{r^2 + s^2 \sin^2\varphi}\right)}$$

But

$$\frac{2r|s|\,|\sin\varphi|}{r^2+s^2\sin^2\varphi}\le 1,$$

so that

$$1-\frac{2rs\cos\alpha\sin\varphi}{r^2+s^2\sin^2\varphi}\ge 1-|\cos\alpha|=0(\varepsilon^2)$$

Therefore the last integral is bounded above by

$$0(\varepsilon^{-2})\int_1^\infty\frac{dr}{r}\left(\int_0^{2\pi}\frac{d\varphi}{r^2+s^2\sin^2\varphi}\right)=0(\varepsilon^{-2})\int_1^\infty\frac{dr}{r}\left(\frac{4\pi}{r(r^2+s^2)^{1/2}}\right)$$

$$=0(\varepsilon^{-2})\frac{1}{s}\int_1^\infty\frac{dr}{r^2}=0(\varepsilon^{-2})\frac{1}{s}=0(\varepsilon^{-2})\cdot|v|.$$

The estimate

$$\sum_{\substack{c\in\Gamma^\#\\c_2\ne 0}}\frac{|v|^2}{|2c_2+v|c|^2|^2}=0(\varepsilon^{-2})|v|$$

holds uniformly for $\varepsilon<\alpha<\pi-\varepsilon$ and $|t|\le\delta$ for any fixed δ. It is clear from the form of the sum that the real axis must be avoided because the singularities $v=2c_2/|c|^2$ all lie on the real axis and cluster at 0. When $(\kappa,u)\ne 0$ the singularities move off the real axis but lie underneath a parabola. So once ε has been chosen δ must be made so small that the truncated cone $\{te^{i\alpha}|\,|t|<\delta\varepsilon<\alpha<\pi-\varepsilon\}$ is bounded away from the singular set.

Notice that

$$U\cap\Sigma(\Theta)=\left\{(\kappa,u,v)\in U|\arg\left(\kappa^2+\frac{1}{v^2}\right)\notin(-\Theta,\Theta)\right\}$$

is comparable to the cones of the form $\{(\kappa,u,t\,e^{i\alpha})|\varepsilon<\alpha<\pi-\varepsilon\ \underline{\text{and}\ |t|<\delta}\}$. Therefore our estimate (∗) shows that $f(\kappa,u,v)$ is continuous in $\overline{U\cap\Sigma(\Theta)}$. In particular

$$\overline{B(q)\cap\Sigma(\Theta)}\cap(U\cap E_{\gamma,0})$$

is contained in the zero set of $f(\kappa,u,0)=\det_2 A(\kappa,u,0)$, that is the piece of the one-dimensional Bloch variety $B(q_\gamma)$ that intersects U.

We now prove the opposite inclusion. For this purpose let us first assume that $p = (\kappa_0, u_0, 0)$ is a smooth point of the complex analytic curve $\{(\kappa, u, 0) \in E_{y, 0} | f(\kappa, u, 0) = 0\}$. Then the partial derivatives of f with respect to κ and u do not vanish simultaneously at p. The estimate (*) also shows that the function $(\kappa, u, \tau, \alpha) \to f(\kappa, u, \tau^2 e^{i\alpha})$ is C^1 in the variable τ as long as $\varepsilon < \alpha < \pi - \varepsilon$ or $\pi + \varepsilon < \alpha < 2\pi - \alpha$. By similar estimates one can show that this function is actually continuously differentiable in all variables if α is restricted to the intervals above. Now, regard $f(\kappa, u, \tau^2 e^{i\alpha})$ as a map from $\mathbb{C}^2 \times \mathbb{R}^2 = \mathbb{R}^4 \times \mathbb{R}^2$ to $\mathbb{C} = \mathbb{R}^2$. At any point $(\kappa_0, u_0, 0, \alpha)$ with $\varepsilon < \alpha < \pi - \varepsilon$ or $\pi + \varepsilon < \alpha < 2\pi - \varepsilon$ its Jacobian has maximal rank, and the kernel of the Jacobian is transversal to the hyperplane $\tau = 0$. By the implicit function theorem at all such points the zero set of $f(\kappa, u, \tau^2 e^{i\alpha})$ is a real 4-dimensional manifold transversal to the hyperplane $\tau = 0$ in $\mathbb{R}^4 \times \mathbb{R}^2$. The image of this manifold under the map $\mathbb{R}^4 \times \mathbb{R}^2 \ni (\kappa, u, \tau^2 e^{i\alpha}) \to (\kappa, u, \tau^2 e^{i\alpha})$ is a locally cone-like space because this map contracts the α-direction for $\tau = 0$.

To summarize, we have shown that $\overline{B(q) \cap \Sigma(\Theta)}$ is a locally cone-like space at each smooth point of the zero-set of $f(\kappa, u, 0)$ in $E_{y, 0}$. In particular all these smooth points lie in the closure of $B(q) \cap \Sigma(\Theta)$. It is easy to see from the discussion in section 1 that the equation defining the one-dimensional Bloch variety $B(q_y)$ is reduced. Therefore the smooth points are dense in the zero set of $f(\kappa, u, 0)$ and the proof is complete.

3. Applications

It follows immediately from the theorem of section 2 that the Bloch variety $B(q)$ determines the Bloch varieties $B(q_y)$ of all the averaged potentials q_y. This result has been proved in [4] using analytic methods. In the directional compactification the curves $B(q_y)$ are attached to $B(q)$ in a geometric way. We use this to deduce properties of the Bloch variety $B(q)$ itself.

THEOREM 1. *Let* $q \in L^2(\mathbb{R}^2/\Gamma)$. *Then for any two irreducible components* C_1, C_2 *of* $B(q)$ *there is* $b \in \Gamma^*$ *such that* $b \cdot C_1 = C_2$.

This result — which will be proven in a moment — can be expressed more briefly by saying that $B(q)/\Gamma^*$ *is an irreducible complex analytic variety*. Thus, $B(q)$ is determined by any of its irreducible components. If q is real valued then the germ of $B_\mathbb{R}(q)$ at any of its points determines — by analytic continuation — at least one component of $B(q)$. So,

COROLLARY. *Let* q_1, q_2 *be real valued potentials, p a point of* $B_\mathbb{R}(q_1)$ *and* U *a neighborhood of p in* $\mathbb{R}^2 \times \mathbb{R}$. *If* $B_\mathbb{R}(q_1) \cap U = B_\mathbb{R}(q_2) \cap U$ *then* $B(q_1) = B(q_2)$.

This corollary may be reexpressed by saying that the analytic continuation of an arbitrarily small piece of any single band function yields the entire complex Bloch variety and in particular all the other band functions. In this sense it is much stronger than Theorem 3 of Section 1.

The generalization of Borg's theorem suggested at the end of section 1 is

THEOREM 2. *Let $q \in L^2$ be a real valued potential. Assume that there is an entire function $\Lambda : \mathbb{C}^2 \to \mathbb{C}$ such that the Bloch variety $B(q)$ is the union of the graph of Λ and its translates under Γ^*, that is*

$$B(q) = \bigcup_{b \in \Gamma^*} \{(k, \lambda) \in \mathbb{C}^2 \times \mathbb{C}/\lambda = \Lambda(k + b)\}$$

Then q is a constant function.

Both theorems are consequences of the

PROPOSITION. *Let $q \in L_0^2$, γ a primitive vector of Γ and $0 < \Theta < \pi$. Then, for each component C of $B(q)$ there is $m \in \mathbb{Z}$ such that the point set closure of $C \cap \Sigma(\Theta)$ in $X(\{\gamma, 4\pi m, 0\})$ contains at least one irreducible component of $\widetilde{B(q)} \cap E_{q, m}$.*

In particular the directional compactification of every component of $B(q)$ meets some exceptional plane $E_{\gamma, m}$.

We now show that *Theorem* 1 follows from this proposition. Let C_1, C_2 be two components of $B(q)$. As usual, we may assume that $\int q = 0$. By the proposition there are integers m_1, m_2 such that $\overline{C_i \cap \Sigma(\Theta)} \cap E_{\gamma, m_i}$ contains a component C_i' of $\widetilde{B(q)} \cap E_{\gamma, m_i}$. As we observed in the previous section there is $b_1 \in \Gamma^*$ such that $b_1 \cdot E_{\gamma, m_1} = E_{\gamma, m_2}$. Then $b_1 \cdot C_1'$ and C_2' are both components of the curve $\widetilde{B(q)} \cap E_{\gamma, m_2}$, which — by the theorem of the last section — is isomorphic to $B(q_\gamma)$. The analogue of Theorem 1 holds in one dimension, and therefore there is $b_2 \in \Gamma^*$ with $\langle b_2, \gamma \rangle = 0$ such that $b_2 \cdot (b_1 \cdot C_1') = C_2'$. Put $b := b_2 \cdot b_1$. Then the closure of $b \cdot C_1 \cap \Sigma(\Theta)$ also contains C_2'. Since $\widetilde{B(q)} \cap \Sigma(\Theta)$ is a locally cone like space (in the sense of section 2) at each smooth point of C_2' the two components $b \cdot C_1$ and C_2 of $B(q)$ have an open subset in common. Hence, they are equal, proving Theorem 1.

The proof of *Theorem* 2 is similar in spirit. Again we assume that $\int q = 0$. The graph of Λ is a component C of $B(q)$. Below we will show that for each primitive γ in Γ there is $m \in Z$ such that $C \cap \Sigma(\Theta) \cap E_{\gamma, m}$ contains the graph of an entire function. This implies that $B(q_\gamma)$ contains a component which is the graph of an entire function. Then by Borg's theorem q_γ is identically zero. As this holds for all primitive vectors γ the potential q is identically zero.

To prove that $\overline{C \cap \Sigma(\Theta)} \cap E_{\gamma, m}$ contains the graph of an entire function use the Γ^*-action and rotation of the lattice Γ to reduce everything to the case $\gamma = (0, \gamma_2)$ and that $\overline{C \cap \Sigma(\Theta)} \cap E_{\gamma, 0}$ contains a component C' of $\widetilde{B(q)} \cap E_{\gamma, 0}$. In the coordinates κ, u, v with $k_1 = \kappa$, $k_2 = 1/v$, $\lambda = u + 1/v^2$ introduced in Section 2

$$C = \left\{ (\kappa, u, v) / u = \Lambda\left(\kappa, \frac{1}{v}\right) - \frac{1}{v^2}, v \neq 0 \right\}.$$

Since $\widetilde{B(q)}$ is a locally cone like space at all smooth points of C' this implies that

$$\lim_{\substack{\arg v^2 \notin (-\Theta, \Theta) \\ v \to 0}} \Lambda\left(\kappa, \frac{1}{v}\right) - \frac{1}{v^2}.$$

exists for all κ. This expression defines a continuous function $\kappa \to U(\kappa)$ whose graph is contained in $\widetilde{B(q)} \cap E_{\gamma, 0}$. Since this set is a complex analytic curve the function is in fact analytic and parametrizes a component of $\widetilde{B(q)} \cap E_{\gamma, 0}$, completing the proof of Theorem 2.

Now, for the proof of the proposition assume that $\gamma = (0, \gamma_2)$ with $\gamma_2 > 0$. First we want to show that near real points of the planes $E_{\gamma, m}$ the Bloch variety $B(q)$ is contained in a tubular neighborhood of the free Bloch variety $B(0)$. More precisely, let $\pi : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ be the projection $(k, \lambda) \to k = (k_1, k_2)$ and embed $\mathbb{C} \times \mathbb{C}$ into $\mathbb{C} \times P^1$ by $(k_1, k_2) \to (k_1, (1, k_2))$. Observe that, π extends continuously to a map $\bar\pi : \mathbb{C}^2 \times \mathbb{C} \cup E_{\gamma, m} \to \mathbb{C} \times P^1$. For $m = 0$ this map is described in the coordinates (κ, u, v) by $(\kappa, u, v) \to (\kappa, (v, 1))$.

LEMMA. *Fix a compact interval $K_1 \subset \mathbb{R}$, and let $q \in L_0^2$. Then there is $R > 0$ and there are closed subsets $T_{m, n}$ of $\{(k_1, k_2, \lambda) \in \Sigma(\Theta) / k_1 \in K_1, |k_2| \geq R\}$ with the following properties*:

(i) $B(q) \cap \{(k, \lambda) \in \Sigma(\Theta) / k_1 \in \dot{K}_1, |k_2| > R\} \subset \bigcup_{m, n} \dot{T}_{m, n}$

(ii) $\bigcup_{b \in \Gamma^*, \langle b, \gamma \rangle = 2\pi m} (B_b \cap \{(k, \lambda) \in \Sigma(\Theta) / k_1 \in K_1, |k_2| \geq R\} \subset \bigcup_{n=0}^{\infty} T_{mn}$ *for all* $m \in Z$.

(iii) $T_{m, n} \cap T_{m', n'} = \emptyset$ *if* $(m, n) \neq (m', n')$.

(iv) *The closure $\bar T_{m, n}$ of $T_{m, n}$ in $X(\{(\gamma, 4\pi m, 0)\})$ is compact, and the restriction of $\bar\pi$ to $\bar T_{mn}$ is a trivial fibre bundle over $\{(k_1, (v, 1)) \in K_1 \times P^1 | |v| \leq 1/R$ and $\arg k_1^2 + (1/v^2) \notin (-\Theta, \Theta)\}$.*

Intuitively speaking the picture — as represented in Figure 3 — is that near each plane $E_{\gamma, m}$ the directional compactification $\widetilde{B(q)}$ is the union of an infinite number of distinct branches, each lying over the v-plane and each close to a branch of $B(0)$. We emphasize that there is a domain *uniform in n* over which all these branches lie.
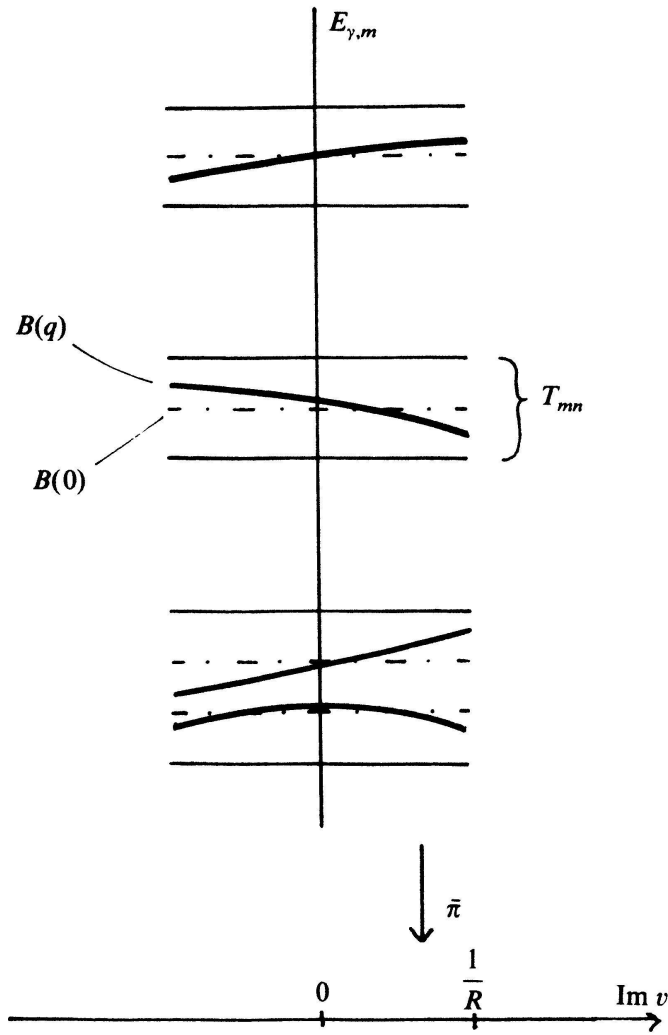
Figure 3

Proof of the Lemma. The square of the Hilbert–Schmidt norm of the operator $q \cdot (-\Delta_k - \lambda)^{-1}$ is

$$\|q\|_2^2 \cdot \sum_{b \in \Gamma^\#} \frac{1}{|(k+b)^2 - \lambda|^2}.$$

Since $\det_2$ is continuous in the Hilbert–Schmidt norm there is an $\varepsilon > 0$ such that

$$\det_2 \left((-\Delta_k + q - \lambda) \cdot (-\Delta_k - \lambda)^{-1} \right) \neq 0$$

whenever

$$\sum_{b \in \Gamma^\#} \frac{1}{|(k+b)^2 - \lambda|^2} \leq \varepsilon.$$

Clearly, in this case $(k, \lambda) \notin B(q)$.

If k_1 is real and $b \in \Gamma^*$ then $(k + b)^2$ has

$$(k_1 + b_1)^2 + (\operatorname{Re} k_2)^2 - (\operatorname{Im} k_2)^2 + 2b_2 \cdot \operatorname{Re} k_2 + b_2^2$$

as real part, and $2b_2 \operatorname{Im} k_2$ as imaginary part. As $b_2 \gamma_2 \in 2\pi \mathbb{Z}$ for $b \in \Gamma^*$ the "polar" set of $\Sigma \, 1/|(k + b)^2 - \lambda|^2$ as a function of λ is a countable union of sequences $(c_{mn})_{n \in Z}$ where

$$\operatorname{Im} c_{mn} = \frac{4\pi}{\gamma_2} (\operatorname{Im} k_2) \cdot m$$

and for each $m \in Z$ the sequence $|c_{mn}|$ grows like $4\pi^2 \gamma_2^2 \cdot n^2$. (see Figure 4)

As in Section 2 one sees that there is $\rho > 0$ such that for real k_1 and $\operatorname{Im} k_2 > \rho \gamma_2 / \pi$

$$\sum_{\substack{b \in \Gamma^* \\ |\operatorname{Im} \lambda - 2b_2 \operatorname{Im} k_2| \geq \rho}} \frac{1}{|(k + b)^2 - \lambda|^2} < \frac{\varepsilon}{2}$$
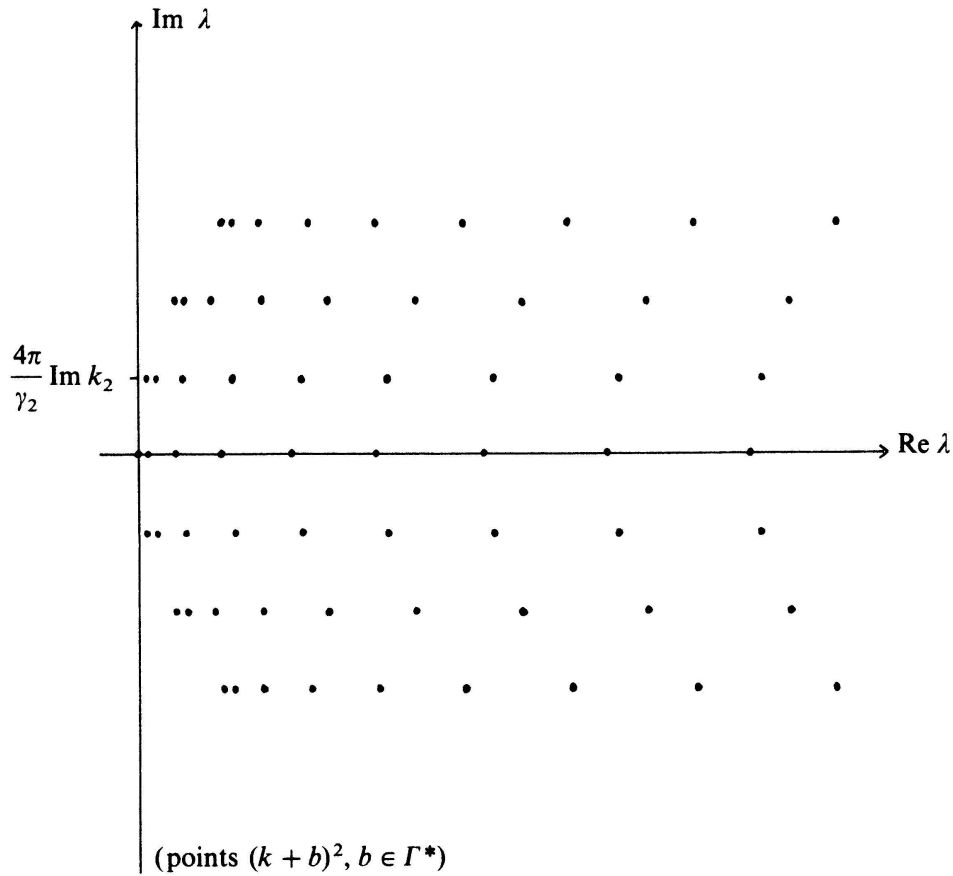

(points $(k + b)^2, b \in \Gamma^*$)

Figure 4

In other words one can ignore all contributions from horizontal strips having distance more than ρ from λ. Choose R so big that $\operatorname{Im} k_2 > \rho \gamma_2 / \pi$ for all $k = (k_1, k_2) \in K_1 \times \mathbb{C}$ with $|k_2| \geq R$ and $\arg (k_1^2 + k_2^2) \notin (-\Theta, \Theta)$.

For $m \in \mathbb{Z}$, set

$$T_m := \left\{ (k, \lambda) \in \Sigma(\Theta) / k_1 \in K_1, |k_2| \geq R \quad \text{and} \quad \left| \operatorname{Im} \lambda - \frac{4\pi}{\gamma_2} (\operatorname{Im} k_2) \cdot m \right| \leq \rho \right\}$$

Clearly

$$B(q) \cap \{ (k, \lambda) \in \Sigma(\Theta) / k_1 \in K_1, |k_2| \geq R \} \subset \bigcup_{m \in \mathbb{Z}} T_m, \quad T_m \cap T_{m'} = \varnothing \quad \text{if} \quad m \neq m',$$

and

$$\bigcup_{\substack{b \in \Gamma^\# \\ \langle b, \gamma \rangle = 2\pi m}} B_b \cap \{ (k, \lambda) \in \Sigma(\Theta) | k_1 \in K_1, |k_2| \geq R \} \subset T_m.$$

Furthermore a point (k, λ) of T_m does not lie on $B(q)$ unless

$$\sum_{\substack{b_1 \in \mathbb{R} \\ (b_1, (2\pi/\gamma_2)m) \in \Gamma^\#}} \frac{1}{|(k + b)^2 - \lambda|^2} > \frac{\varepsilon}{2}$$

For $(k, \lambda) \in T_m$ this sum is less or equal to

$$\sum_{\substack{b_1 \in \mathbb{R} \\ (b_1, (2\pi/\gamma_2)m) \in \Gamma^\#}}$$

$$\times \frac{1}{\left| (k_1 + b_1)^2 - \left[\operatorname{Re} \lambda \cdot \left((\operatorname{Im} k_2)^2 - (\operatorname{Re} k_2)^2 - \frac{4\pi}{\gamma_2} (\operatorname{Re} k_2) m - \frac{4\pi^2}{\gamma_2^2} m^2 \right) \right] \right|^2}$$

Therefore we consider the function

$$t \to \sum_{\substack{b_1 \in \mathbb{R} \\ (b_1, (2\pi/\gamma_2)m) \in \Gamma^\#}} \frac{1}{|(k_1 + b_1)^2 - t|^2}$$

It has infinitely many "poles" on the real axis, but the gaps between them are growing linearly. Since K_1 is compact we may choose

$$\alpha_{m0} < \beta_{m0} < \alpha_{m1} < \beta_{m1} < \cdots$$

such that for all $k \in K_1$

$$\sum_{\substack{b_1 \in \mathbb{R} \\ (b_1, \, 2\pi m/\gamma_2) \in \Gamma^{\#}}} \frac{1}{\left|(k_1 + b_2)^2 - t\right|^2} < \frac{\varepsilon}{2} \quad \text{if} \quad t \notin \bigcup_{n \geq 0} [\alpha_{mn}, \beta_{mn}].$$

Therefore, put

$$T_{m,n} := \left\{ (k, \lambda) \in T_m / \operatorname{Re} \lambda + \right.$$

$$\left. + \left((\operatorname{Im} k_2)^2 - (\operatorname{Re} k_2)^2 - \frac{4\pi}{\gamma_2} (\operatorname{Re} k_2) \cdot m - \frac{4\pi^2}{\gamma_2^2} m^2 \right) \in [\alpha_{mn}, \beta_{mn}] \right\}$$

(see Figure 5)

With this definition the conclusions (i)–(iii) of the lemma are obviously fulfilled. We check conclusion (iv) in the case $m = 0$. In the coordinates κ, u, v the closure $\bar{T}_{0,\,n}$ is described by the conditions

$$|v| \leq \frac{1}{R}, \qquad \arg\left(k_1^2 + \frac{1}{v^2}\right) \notin (-\Theta, \Theta), \qquad k_1 \in K_1,$$

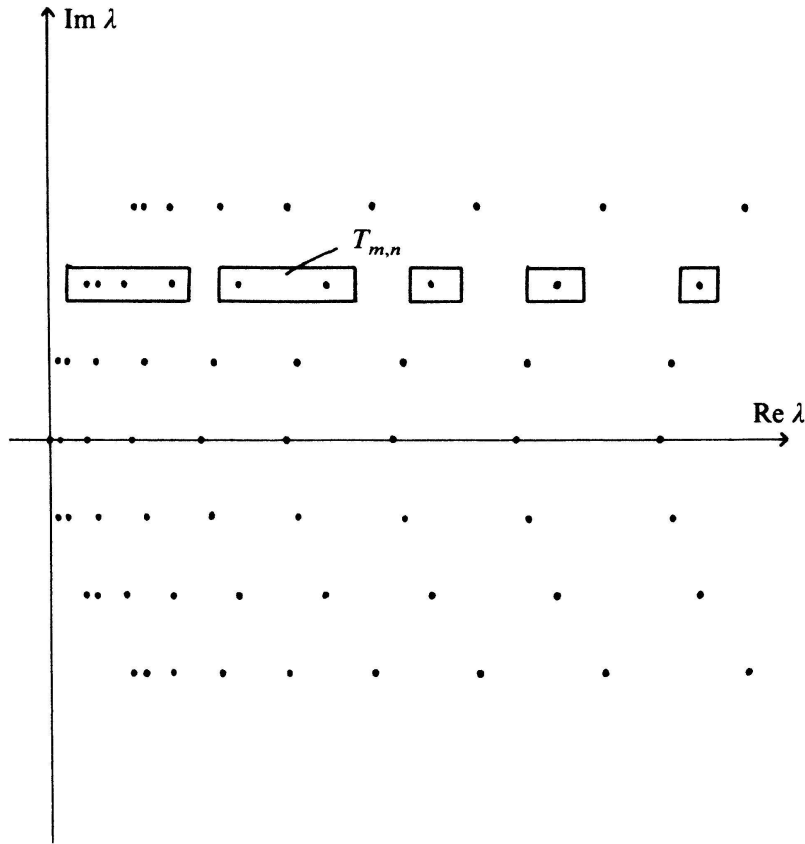$$-\rho \leq \operatorname{Im} u \leq \rho, \qquad \alpha_{mn} \leq \operatorname{Re} u \leq \beta_{mn}.$$

Figure 5

This set is clearly compact and fibers trivially over

$$\left\{(k, (v, 1)) \in \mathbb{P}^1 \Big| |v| \le \frac{1}{R} \text{ and } \arg\left(k_1^2 + \frac{1}{v^2}\right) \notin (-\Theta, \Theta)\right\}$$

Let us now complete the *proof of the proposition*. Suppose C is a component of $B(q)$, and choose a non-empty compact interval $K_1 \subset \mathbb{R}$. In Section 1 we showed that C is the zero set of an entire function of finite order, and that therefore $\pi(C)$ is dense in $\mathbb{C} \times \mathbb{C}$. In particular by the lemma above

$$C \cap \bigcup_{\substack{m \in \mathbb{Z} \\ n \ge 0}} T_{mn} \ne \varnothing.$$

Choose m, n such that $C \cap T_{mn} \ne \varnothing$. Since

$$B(q) \cap \{(k, \lambda) \in \Sigma(\Theta)/k_1 \in \mathring{K}_1, |k_2| > R\} \subset \bigcup_{m, n} \mathring{T}_{mn}$$

the intersection $C \cap \mathring{T}_{mn}$ is a complex subvariety of $\mathring{T}_{mn}$. The restriction of π to $C \cap \mathring{T}_{mn}$ is proper since $\bar{T}_{mn}$ is compact. Therefore $\pi(C \cap \mathring{T}_{mn}) = \{(k_1, k_2) \in \mathring{K}_1 \times \mathbb{C}/|k_2| > R, \arg(k_1^2 + k_2^2) \notin [-\Theta, \Theta]\}$. By part (iv) of the lemma this implies that $C \cap T_{mn}$ meets $E_{\gamma, m}$ and that $K_1 \times \{(0, 1)\} \subset \bar{\pi}(C \cap T_{mn})$. Therefore $\overline{C \cap \Sigma(\Theta)}$ contains smooth points of $\widetilde{B(q)} \cap E_{\gamma, m}$. Since $\widetilde{B(q)}$ is a locally cone-like space of all smooth points of $\widetilde{B(q)} \cap E_{\gamma, m}$ the intersection $\overline{C \cap \Sigma(\Theta)}$ contains a component of $\widetilde{B(q)} \cap E_{\gamma, m}$ whenever it contains one of its smooth points. This concludes the proof of the proposition.

Remark. The constructions and results presented in Sections 1 and 2 carry over with only minor modifications to the case of three dimensions. To apply the implicit function theorem in the proof of the theorem of Section 2 one needs the fact that the derivative of the equations of $B(q_\gamma)$ does not vanish at a generic point of $B(q_\gamma)$ – a fact which for two-dimensional averaged potentials follows directly from Theorem 1. In contrast the estimates in the proof of the lemma above, on which the results of this section are based, use the fact that we are in dimension 2.

4. Degeneration of the spectral problem at infinity

In this section, for the sake of simplicity, we shall assume that q in L^2 is real and (if necessary by adding a constant) that the operator $-\Delta + q$ is positive.

The definition of the Bloch variety may be rephrased in a slightly different form. For $\gamma \in \Gamma$ let S_γ be the shift-operator acting on functions $\varphi \in H^2_{\text{loc}}(\mathbb{R}^2)$ by

$S_\gamma \varphi(x) = \varphi(x + \gamma)$. These operators all commute with each other and with $-\varDelta + q$. For each $(k, \lambda) \in \mathbb{C}^2 \times \mathbb{C}$ denote by $\mathscr{A}_{(k, \lambda)}$ the commutative C^*-algebra of operators generated by $S_\gamma - e^{i\langle k, \gamma \rangle}$, $\gamma \in \Gamma$ and $-\varDelta + q - \lambda$. Then a point (k, λ) lies on $B(q)$ if and only if there is $\Psi \in H^2_{loc}(\mathbb{R}^2)$ such that $A \cdot \Psi = 0$ for all $A \in \mathscr{A}_{(k, \lambda)}$.

Our purpose here is to extend this description of $B(q)$ continuously over $\Sigma(\Theta)$ to the exceptional planes $E_{\gamma, m}$. To do this assume without loss of generality that $\gamma = (0, 1)$ and $m = 0$. Choose a vector $\delta = (\delta_1, \delta_2) \in \Gamma$ such that γ and δ form a basis of Γ. Then $\mathscr{A}_{(k, \lambda)}$ is generated by the three operators

$$S_\gamma - e^{i\langle k, \gamma \rangle}, \qquad S_\delta - e^{i\langle k, \delta \rangle}, \qquad -\varDelta + q - \lambda$$

The coordinates κ, u, v on a neighborhood U of $E_{\gamma, 0}$ with

$$k_1 = \kappa, \qquad k_2 = \frac{1}{v}, \qquad \lambda = u + \frac{1}{v^2}$$

have been introduced in Section 2. For points in the intersection $\mathbb{C}^2 \times \mathbb{C} \cap U$ the algebra $\mathscr{A}_{(k, \lambda)}$ is also generated by

$$A_1 := e^{-i\langle k, \delta \rangle} \cdot S_\delta - 1$$

$$A_2 := v \cdot (-\varDelta + q - \lambda)$$

$$A_3 := (-\varDelta + q)^{1/2} \cdot (1 - e^{i(-\varDelta + q)^{1/2}} \cdot S_\gamma^{-1}) - \frac{1}{v} \sqrt{1 - uv^2}(1 - e^{i(1/v)(\sqrt{1 - uv^2} - 1)})$$

because $v \neq 0$ there. On $\mathbb{C}^2 \times \mathbb{C} \cap U$ we conjugate the algebra $\mathscr{A}_{(k, \lambda)}$ by the multiplication operator $e^{x_2/v}$. If $\Psi \in H^2_{loc}(\mathbb{R}^2)$ with $A \cdot \psi = 0$ for all $A \in \mathscr{A}_{(k, \lambda)}$ then $e^{x_1/v} \cdot \Psi$ lies in the kernel of all the operators of the conjugated algebra $e^{x_2/v} \cdot \mathscr{A}_{(k, \lambda)} \cdot e^{-x_2/v}$. Clearly (k, λ) is in $B(q)$ if and only if the kernel of the conjugated algebra is non-zero.

On $E_{\gamma, 0}$, that is for $v = 0$, the one-dimensional Bloch variety $B(q) \cap E_{\gamma, 0} \cong B(q_\gamma)$ is described by algebras $\mathscr{A}_{\kappa, u}$ generated by the operators

$$-\frac{\partial^2}{dx_1^2} + \int_0^1 q(x_1, t) \, dt - u,$$

$$\frac{\partial}{dx_2} \quad \text{and} \quad e^{-i\kappa\delta_1} \cdot S_\delta^{(1)} - 1$$

where $S_\delta^{(1)}$ is the shift operator $S_\delta^{(1)}\varphi(x_1, x_2) = \varphi(x_1 + \delta_1, x_2)$.

THEOREM. *The conjugated algebra* $e^{x_2/v} \cdot \mathscr{A}_{(k, \lambda)} \cdot e^{-x_2/v}$ *degenerates to* $\mathscr{A}_{\kappa, u}$ *when v goes to zero along the imaginary axis. Precisely, for any Schwartz class function φ, and any $(\kappa, u) \in \mathbb{C} \times \mathbb{C}$*

$$\lim_{\varepsilon \to 0} (e^{-i(x_2/\varepsilon)} \cdot A_1(\kappa, u, i\varepsilon) \cdot e^{i(x_2/\varepsilon)})\varphi(x) = (e^{-i\kappa\delta_1} \cdot S_\delta^{(1)} - 1)\varphi(x)$$

$$\lim_{\varepsilon \to 0} (e^{-i(x_2/\varepsilon)} \cdot A_2(\kappa, u, i\varepsilon) \cdot e^{1(x_2/\varepsilon)})\varphi(x) = -2i \frac{\partial}{\partial x_2} \varphi(x)$$

$$\lim_{\varepsilon \to 0} (e^{-i(x_2/\varepsilon)} \cdot A_3(\kappa, u, i\varepsilon) \cdot e^{1(x_2/\varepsilon)})\varphi(x) = \frac{1}{2i} \left(-\frac{\partial^2}{\partial x_1^2} + \int_0^1 q(x_1, t) \, dt - u \right)\varphi(x)$$

Proof. The first two limits are trivial to compute, so we only verify the last statement. Denote by $\hat{}$ the Fourier transformation

$$\hat{\varphi}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \varphi(x) e^{-i\langle \xi, x \rangle} \, dx$$

Then

$$(e^{-i(x_2/\varepsilon)} \cdot (-\Delta)^{1/2} \cdot (1 - e^{i(-\Delta)^{1/2}} \cdot S_\gamma^{-1}) \cdot e^{i(x_2/\varepsilon)}\varphi) \hat{}(\xi)$$

$$= \left(\xi_1^2 + \left(\xi_2 + \frac{1}{\varepsilon} \right)^2 \right)^{1/2} (1 - e^{i((\xi_1^2 + (\xi_2 + (1/\varepsilon))^2)^{1/2} - (\xi_2 + (1/\varepsilon)))}) \hat{\varphi}(\xi)$$

Expanding the square root

$$\varepsilon \left(\xi_1^2 + \left(\xi_2 + \frac{1}{\varepsilon} \right)^2 \right)^{1/2} = 1 + \xi_2 \varepsilon + \tfrac{1}{2}\xi_1^2\varepsilon^2 + O(|\varepsilon|^3)$$

we get for fixed ξ in R^2

$$\varepsilon \left(\xi_1^2 + \left(\xi_2 + \frac{1}{\varepsilon} \right)^2 \right)^{1/2} (1 - e^{i((\xi_1^2 + (\xi_2 + (1/\varepsilon))^2)^{1/2} - \xi_2 + (1/\varepsilon))}) = \frac{1}{2\pi i} \xi_1^2 \varepsilon + O(\varepsilon^2).$$

Therefore, by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon \to 0} 2i \, e^{i(x_2/\varepsilon)} \cdot (-\Delta)^{1/2} \cdot (1 - e^{i(-\Delta)^{1/2}} \cdot S_\gamma^{-1}) \cdot e^{i(x_2/\varepsilon)}\varphi(x)$$

$$= 2\pi \int_{\mathbb{R}^2} \xi_1^2 \hat{\varphi}(\xi) e^{i\langle \xi, x \rangle} \, dx = -\frac{\partial^2}{dx_1^2} \varphi(x)$$

Since $-\varDelta + q$ is positive its square root is well defined so that $(-\varDelta + q)^{1/2}$ is a classical pseudodifferential operator of order one. Let

$$r_0 := (-\varDelta + q)^{1/2} - (-\varDelta)^{1/2}$$

Then

$$-\varDelta + q = -\varDelta + 2(-\varDelta)^{1/2} \cdot r_0 + [r_0, (-\varDelta)^{1/2}] + r_0^2.$$

or

$$r_0 = \tfrac{1}{2}(-\varDelta)^{-1/2} \cdot q - \tfrac{1}{2}(-\varDelta)^{-1/2} \cdot [r_0, (-\varDelta)^{1/2}] - \tfrac{1}{2}(-\varDelta)^{-1/2} \cdot r_0^2$$

It follows that r_0 is a pseudodifferential operator of order -1 with principal symbol $q(x)/2|\xi|$.

Set

$$R(t) := e^{it(-\varDelta + q)^{1/2}} \cdot e^{-it(-\varDelta)^{1/2}} - \mathbb{1}$$

$$r(t) := e^{it(-\varDelta)^{1/2}} \cdot r_0 \cdot e^{-it(-\varDelta)^{1/2}}$$

Differentiating, one obtains

$$\frac{d}{dt} R = (\mathbb{1} + R) \cdot ir$$

so that

$$R(t) = \int_0^t ir(s) \, ds + \int_0^t R(s) \cdot ir(s) \, ds$$

By Egorov's theorem, $ir(t)$ is a pseudodifferential operator of order -1 for all t and has principal symbol

$$\frac{iq\left(x - \dfrac{\xi t}{|\xi|}\right)}{2|\xi|}$$

Now

$$2i\, e^{-i(x_2/\varepsilon)} \cdot (-\varDelta + q)^{1/2} \cdot (1 - e^{i(-\varDelta + q)^{1/2}} \cdot S_\gamma^{-1}) \cdot e^{i(x_2/\varepsilon)}\varphi(x)$$

$$= 2i\, e^{-i(x_2/\varepsilon)} \cdot (-\varDelta)^{1/2} \cdot (1 - e^{i(-\varDelta)^{1/2}} \cdot S_\gamma^{-1}) \cdot e^{i(x_2/\varepsilon)}\varphi(x)$$

$$- 2i\, e^{-i(x_2/\varepsilon)} \cdot (-\varDelta)^{1/2} \cdot R(1) \cdot e^{i(x_2/\varepsilon)} \cdot (e^{-i(x_2/\varepsilon)} \cdot e^{i(-\varDelta)^{1/2}} \cdot S_\gamma^{-1} \cdot e^{i(x_2/\varepsilon)})\varphi(x)$$

$$+ 2i\, e^{-i(x_2/\varepsilon)} \cdot r_0 \cdot (1 - (1 + R(1))) \cdot e^{i(x_2/\varepsilon)}$$

$$\cdot (e^{-i(x_2/\varepsilon)} \cdot e^{i(-\varDelta)^{1/2}} \cdot S_\gamma^{-1} \cdot e^{i(x_2/\varepsilon)})\varphi(x)$$

The first term tends as above to $-\partial^2/dx_1^2$ as v goes to 0. In the same way one sees that $e^{-i(x_2/\varepsilon)} \cdot e^{i(-\varDelta)^{1/2}} \cdot S_\gamma^{-1} \cdot e^{i(x_2/\varepsilon)}$ goes to the identity operator. The factor $2i\, e^{-i(x_2/\varepsilon)} \cdot r_0 \cdot (1 + R(1))) \cdot e^{i(x_2/\varepsilon)}$ goes to zero. To see this, use the identity

$$\sigma(e^{i(1/\varepsilon)f(x)} \cdot A \cdot e^{-i(1/\varepsilon)f(x)}) = \sigma(A)\left(x, \xi + \frac{1}{\varepsilon}\nabla_x f\right)$$

valid for any pseudodifferential operator A and any function f and the fact that R is generated by solving a Volterra equation. Finally the factor $-2i\, e^{-i(x_2/\varepsilon)} \cdot (-\varDelta)^{1/2 \cdot R(1)} \cdot e^{i(x_2/\varepsilon)}$ converges to $\int_0^1 q(x_1, t)\, dt$. This follows from the fact that the leading order term of $R(1)$ (in powers of r_0) is the pseudodifferential operator $\int_0^1 ir(t)\, dt$ whose principal symbol is

$$\frac{i}{2|\xi|}\int_0^1 q\left(x - \frac{\xi t}{|\xi|}\right) dt.$$

Remark. The theorem may be reformulated and extended by introducing a vector bundle $\mathscr{F}$ over $\Sigma(\Theta) \cup E_{y,0}$ with fibre $H_{\mathrm{loc}}^2(\mathbb{R}^2)$ that is trivial on $\Sigma(\Theta)$ and on $U \cap \Sigma(\Theta)$ and has the multiplication operator $e^{x_2/v}$ as transition function. The algebras $\mathscr{A}_{(\kappa, \gamma)}$ and $\mathscr{A}_{(\kappa, u)}$ glue together to a subbundle $\mathscr{A}$ of the endomorphism bundle End $(\mathscr{F})$ of this vectorbundle. The kernel of $\mathscr{A}$ defines a sheaf on $\Sigma(\Theta)$ whose support is $B(q) \cap \Sigma(\Theta)$ and whose stalk over a point of $B(q) \cap \Sigma(\Theta)$ is generated by the Bloch solutions at this point. It is natural to ask how the Bloch solution behaves as one approaches the exceptional plane. This behavior is given by a geometric optics Ansatz which we have codified in the theorem. For the discrete Laplace operator this has been worked out in detail in [1].

REFERENCES

[1] D. BÄTTIG. *A toroidal compactification of the two dimensional Bloch variety.* Thesis, ETH-Zürich 1988.

[2] G. BORG. *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe.* Acta math. **78**, 1–96 (1946).

[3] B. DAHLBERG and E. TRUBOWITZ. *A remark on two dimensional periodic potentials.* Comment. Math. Helvetici *57*, 130–134 (1982).

[4] G. ESKIN, J. RALSTON and E. TRUBOWITZ. *On isospectral potentials in* $\mathbb{R}^n$. Comm. Pure Appl. Math. *37*, 647–676 (1984).

[5] D. GIESEKER, H. KNÖRRER and E. TRUBOWITZ. *An overview of the geometry of algebraic Fermi curves.* To appear in Proc. Conference on Algebraic Geometry, Salt Lake City 1989.

[6] H. MCKEAN and E. TRUBOWITZ. *Hills operator and hyperelliptic function theory in the presence of infinitely many branch points.* Comm. Pure Appl. Math. *29*, 143–226 (1976).

[7] P. LELONG and L. GRUMAN. *Entire Functions of several complex variables.* Springer Verlag 1986.

[8] J. PÖSCHEL and E. TRUBOWITZ. *Inverse spectral theory.* Academic Press 1987.

[9] B. SIMON. *Trace ideals.* Cambridge University Press 1979.

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