Multiple fibres of a morphism.

Autor(en): **Serrano, Fernando**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **65 (1990)**

PDF erstellt am: **28.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-49726

Comment. Math. Helvetici 65 (1990) 287–298

0010-2571/90/020287-12\$1.50 + 0.20/0

Multiple fibres of a morphism

FERNANDO SERRANO

§0. Introduction

Let us be given a proper, surjective, holomorphic map $\varphi : X \to C$ with connected fibres from a complex manifold onto a smooth quasiprojective curve C. Let $\{m_1, \ldots, m_t\}$ be the (global) multiplicities of the multiple fibres of φ, and denote by F a general fibre. The aim of this paper is to compute the homology of the natural complex of abelian groups

$$H_1(F, \mathbb{Z}) \to H_1(S, \mathbb{Z}) \xrightarrow{\varphi_*} H_1(C, \mathbb{Z}) \to 0$$

in terms of the multiplicities $\{m_1, \ldots, m_t\}$. Namely, a suitable exact sequence

$$H_1(F, \mathbb{Z}) \to H_1(S, \mathbb{Z}) \to H_1(C, \mathbb{Z}) \times G(\varphi) \to 0$$

is constructed, where $G(\varphi) := \mathrm{Coker}\,(f : \mathbb{Z} \to \oplus_i \mathbb{Z}/m_i\mathbb{Z})$ and $f(1) = (\bar{1}, \ldots, \bar{1})$.

Next we will address the question of the variation of $G(\varphi)$ and $\oplus_{i=1}^t \mathbb{Z}/m_i\mathbb{Z}$ under smooth deformations of φ (with the extra assumption that X and C are compact). It will be shown in §2 that both groups are actually invariant under deformation. The proof for $G(\varphi)$ relies on the above exact sequence plus the fact that a smooth analytic map is differentiably locally trivial. Then a base change trick will give the invariance of $\oplus_i \mathbb{Z}/m_i\mathbb{Z}$.

All this generalizes the already known situation for elliptic surfaces: when X is a compact surface and F is a curve of genus 1, the above exact sequence on homology groups can be deduced from the explicit description of the fundamental group of the surface ([8]). For a larger fibre genus such a description is lacking in general. As to the behaviour under deformation, the picture is neater for these two-dimensional elliptic fibrations: Iitaka has proved in [7] that the set of multiplicities of the fibres is a deformation invariant in this case.

Finally, I want to express my thanks to J. Kollar for a helpful remark.

§1. Homology groups

We shall be working over the field of complex numbers. Our complex manifolds are by definition connected, non-singular analytic varieties. A curve C is a quasiprojective complex manifold of dimension one. Equivalently, the smooth compactification of C differs from C at finitely many points only. In this paper a fibration is defined to be a proper, surjective holomorphic map from a complex manifold onto a smooth curve, all of whose fibres are connected. We will also use the following notation:

- $\mathbb{Z}_m :=$ integers $\mathbb{Z}$ modulo $(m)\mathbb{Z}$.
- $\operatorname{tor} H :=$ torsion of an abelian group H.
- $\pi_1(X) :=$ fundamental group of X.
- $h^i\mathcal{O}_X := \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the structure sheaf of X.

Let $\varphi : X \to C$ be a fibration, and $F = \Sigma\, n_i B_i$ a fibre of φ where the B_i's are the irreducible reduced components of F and the n_i's are their multiplicities. Let m be the greatest common divisor of the n_i's. We say that m is the multiplicity of F and write $F = mD$, where $D = \Sigma\,(n_i/m)B_i$. Whenever we say "let mD be a multiple fibre" we shall always mean that m is the multiplicity of mD and $m \geq 2$.

Let $\varphi : X \to C$ be a fibration and let $m_1 D_1, \ldots, m_t D_t$ be all its multiple fibres.

DEFINITION 1.1.

$$G(\varphi) := \operatorname{Coker}\left(\mathbb{Z} \to \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i}\right) \qquad 1 \mapsto (1, \ldots, 1)$$

$$L(\varphi) := \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i}.$$

If μ is the least common multiple of $m_1, \ldots, m_t$, by dualizing the sequence

$$0 \to \mathbb{Z}_\mu \to \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \to G(\varphi) \to 0$$

we obtain an alternative description of $G(\varphi)$ as

$$G(\varphi) = \operatorname{Ker}\left(\bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \to \mathbb{Z}_\mu\right) \qquad (a_1, \ldots, a_t) \mapsto \sum a_i(\mu/m_i).$$

The third characterization that follows will be used later:

LEMMA 1.2. *Write* $\oplus_{i=1}^{t} \mathbb{Z}_{m_i} \simeq \oplus_{j=1}^{k} \mathbb{Z}_{d_j}$ *where each* d_j *divides* d_{j+1}. *Then*

$$G(\varphi) \simeq \bigoplus_{j=1}^{k-1} \mathbb{Z}_{d_j}.$$

Proof. Since $\mu/m_1, \ldots, \mu/m_t$ are relatively prime, we can find integers $\lambda_1, \ldots, \lambda_t$ such that $\Sigma_{i=1}^{t} (\lambda_i \mu/m_i) = 1$. The homomorphism

$$\bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \to \mathbb{Z}_{\mu} \qquad (a_1, \ldots, a_t) \mapsto \sum_{i=1}^{t} a_i(\lambda_i \mu/m_i)$$

is a retraction of $0 \to \mathbb{Z}_{\mu} \to \oplus_{i=1}^{t} \mathbb{Z}_{m_i} \to G(\varphi) \to 0$, and this sequence splits. If we put $G(\varphi) = \oplus_{j=1}^{r} \mathbb{Z}_{e_j}$ with e_j dividing e_{j+1} for all j, then all e_j's divide μ and

$$\bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} = G(\varphi) \oplus \mathbb{Z}\mu = \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{e_j} \right) \oplus \mathbb{Z}_{\mu}.$$

Since the d_j's are uniquely determined, it follows that $(d_1, \ldots, d_{k-1}, d_k) = (e_1, \ldots, e_r, \mu)$. □

Now it comes the main result of this paper. Our proof has been inspired in that of Prop. 1.41 of [2].

THEOREM 1.3. *Let* $\varphi : X \to C$ *be a fibration from the complex manifold* X *onto a smooth curve* C. *Denote by* $m_1 D_1, \ldots, m_t D_t$ *all multiple fibres of* φ, *and let* F *be any smooth fibre, and* $G := G(\varphi)$. *Then there exists an exact sequence*

$$H_1(F, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \to H_1(C, \mathbb{Z}) \times G \to 0$$

induced by φ *and the inclusion of* F *into* X.

Proof. Let

$$\Omega = \{p \in C \mid \varphi^{-1}(p) \text{ is singular}\}, \qquad \tilde{C} = C - \Omega, \quad \tilde{X} = X - (\cup_{p \in \Omega} \varphi^{-1}(p)).$$

Consider the following commutative diagram with exact rows and columns, whose homomorphisms come from the obvious inclusions and restrictions:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & M \arrow[r, "\varepsilon"] & H_1(X, \mathbb{Z}) \arrow[r, "\varphi_*"] \arrow[u] & H_1(C, \mathbb{Z}) \arrow[r] \arrow[u] & 0 \\
 & H_1(F, \mathbb{Z}) \arrow[r] \arrow[u, "f"] & H_1(\tilde{X}, \mathbb{Z}) \arrow[r, "\sigma"] \arrow[u, "g"] & H_1(\tilde{C}, \mathbb{Z}) \arrow[r] \arrow[u, "h"] & 0 \\
 & & N_1 \arrow[r, "\tau"] \arrow[u] & N_2 \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

M, N_1 and N_2 are defined to be the kernels of the corresponding homomorphisms. The second row is exact because $\tilde{X} \to \tilde{C}$ is a C^∞-fibre bundle.

CLAIM 1. *The cokernel of $\tau : N_1 \to N_2$ is a quotient of G.*

Proof of Claim 1. Given $p \in \Omega$, denote by γ_p a simple loop around p in $\tilde{C}$. The group N_2 is generated by all the γ_p, $p \in \Omega$, with the single relation $\Pi_{p \in \Omega} \gamma_p = 0$.

If B is a component of multiplicity n of a fibre $\varphi^{-1}(p)$, $p \in \Omega$, then there is a loop α in $\tilde{X}$ around B such that $\alpha \in N_1$ and $\tau(\alpha) = n\gamma_p$. Consequently, if m is the total multiplicity of $\varphi^{-1}(p)$ then $m\gamma_p \in \operatorname{Im}(\tau)$, and the claim follows.

CLAIM 2. *There exists an exact sequence*:

$$H_1(F, \mathbb{Z}) \xrightarrow{f} M \xrightarrow{\rho} \operatorname{Coker}(\tau) \longrightarrow 0.$$

Proof of Claim 2. Define the map $\rho : M \to \operatorname{Coker}(\tau)$ as follows. Given $x \in M$, there is $y \in H_1(\tilde{X}, \mathbb{Z})$ such that $g(y) = \varepsilon(x)$. Thus $\sigma(y) \in N_2$, and we write $\rho(x)$ as the class of $\sigma(y)$ in $N_2/(\operatorname{Im}(\tau))$. An easy diagram-checking shows that the above sequence is exact. This is nothing else than the so-called Snake Lemma, but later we are going to use the explicit description of the map ρ.

CLAIM 3. *There exists a commutative diagram with exact rows and columns as follows*:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 & \\
H_1(F, \mathbb{Z}) \arrow[r, "f"] \arrow[dr, "j"] & M \arrow[r, "\rho"] \arrow[d, "\varepsilon"] & \operatorname{Coker}(\tau) \arrow[r] \arrow[u] & 0 \\
 & H_1(X, \mathbb{Z}) \arrow[r, "\lambda"] \arrow[d, "\varphi_*"] & G \arrow[u, "\theta"] & \\
 & H_1(C, \mathbb{Z}) \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

Proof of Claim 3. $\theta : G \to \mathrm{Coker}\,(\tau)$ is the epimorphism of Claim 1, and $j = \varepsilon \circ f$ by definition. We must define λ and prove $\rho = \theta \circ \lambda \circ \varepsilon$. The fundamental group $\pi_1(\tilde{C})$ is generated by elements $\alpha_i, \beta_i, \gamma_p, \delta_j$ (for i from 1 up to genus of $\tilde{C}$, $p \in \Omega$, and δ_j corresponding to the "holes" of C) with the unique relation $(\Pi_i\, \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1})(\Pi_j\, \delta_j)(\Pi_{p\in\Omega}\, \gamma_p) = 1$. Given $p \in \Omega$ and $m(p) = \text{multiplicity of } \varphi^{-1}(p)$, there corresponds to $\varphi^{-1}(p)$ a direct summand $\mathbb{Z}_{m(p)}$ in $\oplus_{i=1}^{\ell} \mathbb{Z}_{m_i}$, with $\mathbb{Z}_{m(p)} = 0$ in case $m(p) = 1$. Define an epimorphism $\pi_1(\tilde{C}) \to G$ by mapping γ_p to the image of $\bar{1} \in \mathbb{Z}_{m(p)} \subseteq \oplus_i\, \mathbb{Z}_{m_i}$ in G, and all $\alpha_i, \beta_i, \delta_j$ to 0. We get in this fashion a ramified covering $B \to C$, unramified outside Ω and such that the ramification index on points over $p \in \Omega$ divides $m(p)$. If Y denotes the normalization of $X \times_C B$ then $Y \to X$ is unramified with group G (see the proof of [1], III 9.1, valid in any dimension), and thus it is determined by an epimorphism $\pi_1(X) \to G$ which descends to an epimorphism $\lambda : H_1(X, \mathbb{Z}) \to G$. The preimage of F by $Y \to X$ splits into as many components as the order of G, so that the induced map $\pi_1(F) \to G$ is 0. It follows that $\lambda \circ j = 0$. Finally, the commutativity of the diagram of Claim 3 stems from the description of ρ given in Claim 2 combined with the commutativity of the following diagram:

$$(*)\qquad \begin{tikzcd} H_1(X, \mathbb{Z}) \arrow[r, "\lambda"] & G \arrow[dr, "\theta"] & \\ H_1(\tilde{X}, \mathbb{Z}) \arrow[u, "g"] \arrow[r, "\sigma"] & H_1(\tilde{C}, \mathbb{Z}) \arrow[u, "q"] & \mathrm{Coker}\,(\tau) \\ & N_2 \arrow[u] \arrow[ur] & \end{tikzcd}$$

CLAIM 4. *θ is an isomorphism.*

Proof of Claim 4. Since $\lambda \circ j = 0$, one has a commutative diagram

$$\begin{tikzcd} M/\mathrm{Im}\,(f) \arrow[r, "\sim"] \arrow[dr, "\lambda \circ \bar{\varepsilon}"'] & \mathrm{Coker}\,(\tau) \\ & G \arrow[u, "\theta"'] \end{tikzcd}$$

In particular, Coker (τ) is a direct summand of G. Now it suffices to show that $\lambda \circ \bar{\varepsilon}$ is surjective. The class of the loop γ_p in $H_1(\tilde{C}, \mathbb{Z})$ maps by $q : H_1(\tilde{C}, \mathbb{Z}) \to G$ to the image of $\bar{1} \in \mathbb{Z}_{m(p)} \subseteq \oplus_{i=1}^{\ell} \mathbb{Z}_{m_i}$ in G. By the commutativity of the diagram $(*)$ above, one gets that if $\sigma(x) = \gamma_p$ then $g(x) \in \mathrm{Im}\,(\varepsilon)$, and $(\lambda \circ g)(x)$ is also the image of $\bar{1} \in \mathbb{Z}_{m(p)}$ in G. Consequently $\lambda \circ \bar{\varepsilon}$ is surjective, as we wanted.

CLAIM 5. *The following sequence is exact*:

$$H_1(F, \mathbb{Z}) \xrightarrow{\ j\ } H_1(X, \mathbb{Z}) \xrightarrow{(\lambda, \varphi_*)} G \times H_1(C, \mathbb{Z}) \to 0.$$

Proof of Claim 5. Clearly $\mathrm{Im}(j) \subseteq \mathrm{Ker}(\lambda, \varphi_*)$. Conversely if $x \in \mathrm{Ker}(\lambda, \varphi_*)$ then $x \in M$ and $\rho(x) = 0$, so that $x \in \mathrm{Im}(j)$. Let us finally prove the surjectivity of (λ, φ_*). Let $(y, z) \in G \times H_1(C, \mathbb{Z})$. There exists an element $x \in H_1(H, \mathbb{Z})$ such that $\varphi_*(x) = z$. Since $\lambda \circ \varepsilon$ is surjective, one can find $t \in M$ such that $\lambda(\varepsilon(t)) = y - \lambda(x)$. Then $\lambda(x + \varepsilon(t)) = y$ and $\varphi_*(x + \varepsilon(t)) = z$. This ends the proof of Theorem 1.3. □

For the remainder of this section we will assume all complex manifolds to be projective algebraic.

REMARK 1.4. When X is a compact surface and F is a curve of genus 1 (i.e. when $\varphi : X \to C$ is an elliptic fibration) one has a more accurate information. If φ has a singular fibre other than a multiple of a smooth curve, then the homomorphism $H_1(F, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is the zero map ([2], 1.39). In particular $h^1\mathcal{O}_X = h^1\mathcal{O}_C$ in this case. For the other cases see [11]. In general, the fundamental group of an elliptic surface can be almost completely described ([8]).

A fibration $\varphi : X \to C$ induces a surjective morphism $\mathrm{Alb}(X) \to \mathrm{Alb}(C)$ between the corresponding Albanese varieties, so that one always has the inequality $h^1\mathcal{O}_X \geq h^1\mathcal{O}_C$. Furthermore, one gets the equality $h^1\mathcal{O}_X = h^1\mathcal{O}_C$ if and only if either $h^1\mathcal{O}_X = 0$ or φ coincides with the map from X onto its image by $X \to \mathrm{Alb}(X)$. This is a consequence of the universal property of the Albanese variety and uses in a crucial way the connectedness of the fibre of φ.

Denote by $\mathrm{tor}(H)$ the torsion of an abelian group H. From Theorem 1.3 one immediately gets.

COROLLARY 1.5. *Let J denote the image of $H_1(F, \mathbb{Z})$ in $H_1(X, \mathbb{Z})$. Then there is an exact sequence*

$$0 \to \mathrm{tor}\, J \to \mathrm{tor}\, H_1(X, \mathbb{Z}) \to G.$$

Furthermore, $\mathrm{tor}\, H_1(X, \mathbb{Z}) \to G$ *is surjective provided that* $h^1\mathcal{O}_X = h^1\mathcal{O}_C$. □

We recall that $\mathrm{tor}\, H_1(X, \mathbb{Z}) \simeq \mathrm{tor}\, H^2(X, \mathbb{Z})$ (non-canonically). The following Proposition describes explicitly some of the elements of $\mathrm{tor}\, H^2(X, \mathbb{Z})$ in case $h^1\mathcal{O}_X = h^1\mathcal{O}_C$. Let $m_1D_1, \ldots, m_tD_t$ be the multiple fibres of a fibration $\varphi: X \to C$, and denote μ the least common multiple of $m_1, \ldots, m_t$. Since $\mu/m_1, \ldots, \mu/m_t$ are relatively prime, there exist integers $\lambda_1, \ldots, \lambda_t$ such that $\Sigma_{i=1}^t (\lambda_i\mu/m_i) = 1$ Let $D = \Sigma_{i=1}^t \lambda_iD_i$. Denote by $[E]$ the class in $H^2(X, \mathbb{Z})$ of a divisor E, and $G := G(\varphi)$.

PROPOSITION 1.6. *If* $h^1\mathcal{O}_X = h^1\mathcal{O}_C$, *then the classes* $\{[D_i - (\mu/m_i)D] \mid i = 1, \ldots, t\}$ *generate a subgroup of* tor $H^2(X, \mathbb{Z})$ *isomorphic to* G.

Proof. First we remark that the subgroup generated by these classes is precisely $\{\Sigma_{i=1}^t \alpha_i[D_i] \mid \alpha_i \in \mathbb{Z}, \Sigma_{i=1}^t (\alpha_i/m_i) = 0\}$.

In order to avoid technical difficulties we will reduce the proof to the case $\dim X = 2$. Take successive general hyperplane sections of X so as to get a smooth surface S. We have $h^1\mathcal{O}_S = h^1\mathcal{O}_X$ and $H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ one-to-one ([5], §1). By Lemma 1.8, the multiple fibres of the restriction $\varphi|_S : S \to C$ come as linear sections of the multiple fibres of φ, and have the same multiplicities. Therefore the Proposition is true for X as long as it holds for S. From now onwards we will assume $\dim X = 2$.

If F is a general fibre of φ then

$$\begin{aligned} m_i[D_i - (\mu/m_i)D] &= [m_iD_i] - [\mu D] \\ &= [F] - [F] = 0. \end{aligned}$$

Thus $[D_i - (\mu/m_i)D] \in$ tor $H^2(X, \mathbb{Z})$. Define the homomorphisms:

$$\sigma : \mathbb{Z} \to \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i}, \qquad \rho : \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \to \text{tor } H^2(X, \mathbb{Z})$$

as $\sigma(1) = \Sigma_{i=1}^t \lambda_i e_i$, $\rho(e_i) = [D_i - (\mu/m_i)D]$, where $e_i = (0, \ldots, 0, \bar{1}, 0, \ldots, 0)$, ($\bar{1}$ in the ith-position).

CLAIM 1. *The sequence*

$$\mathbb{Z} \xrightarrow{\sigma} \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \xrightarrow{\rho} \text{tor } H^2(X, \mathbb{Z})$$

is exact.

Proof of Claim 1. First note that

$$\begin{aligned} \rho\left(\sum_{i=1}^{t} \lambda e_i\right) &= \left[\left(\sum_i \lambda_i D_i\right) - \sum_i (\lambda_i \mu/m_i) D\right] \\ &= [D - D] = 0 \end{aligned}$$

Hence $\mathrm{Im}(\sigma) \subseteq \mathrm{Ker}(\rho)$. Now assume $\rho(\Sigma_{i=1}^t \gamma_i e_i) = 0$, and put $\delta := \Sigma_i (\gamma_i \mu/m_i)$. From $[(\Sigma_i \gamma_i D_i) - \delta D] = 0$ it follows that $(\Sigma_i \gamma_i D_i) - \delta D$ belongs to the Picard variety of X, denoted $\mathrm{Pic}^\circ(X)$. As indicated before, the fact that $h^1\mathcal{O}_X = h^1\mathcal{O}_C$

implies that the Albanese varieties of X and C are isomorphic, hence also their Picard varieties are isomorphic. The symbol $\sim$ is going to denote linear equivalence of divisors. Obviously the restriction $\mathrm{Pic}^\circ(C) \to \mathrm{Pic}^\circ(D_k)$ is the zero map, and it follows that $(\Sigma_{i=1}^t \gamma_i D_i - \delta D)_{|D_k} \sim 0$. We know that $(D_i)_{|D_k} \sim 0$ if $i \neq k$, and $(D_k)_{|D_k}$ is torsion of order m_k in $\mathrm{Pic}(D_k)$ ([1]; III 8.3). Combining with $D_{|D_k} \sim \lambda_k (D_k)_{|D_k}$ one gets $(\gamma_k - \delta\lambda_k)(D_k)_{|D_k} \sim 0$, which implies that $\gamma_k - \delta\lambda_k$ is a multiple of m_k. Thus $\Sigma_i \gamma_i e_i = \delta\, \Sigma_i \lambda_i e_i \in \mathrm{Im}(\sigma)$, as we wanted.

CLAIM 2. $\mathrm{Ker}(\sigma) = (\mu)\mathbb{Z}$

Proof of Claim 2. Let $(\nu)\mathbb{Z} := \mathrm{Ker}(\sigma)$. Multiplying the equation $\Sigma_{i=1}^t (\lambda_i \mu/m_i) = 1$ by m_k we obtain that $\lambda_k \mu$ is a multiple of m_k. Hence $\sigma(\mu) = 0$ and one can write $\mu = \nu \cdot d$ for some $d \in \mathbb{Z}$. Since m_i divides $\lambda_i \nu$ we have $\Sigma_i (\lambda_i \nu/m_i) \in \mathbb{Z}$. On the other hand $1 = \Sigma_i (\lambda_i \mu/m_i) = d\, \Sigma_i (\lambda_i \nu/m_i)$, so that $d = 1$ and Claim 2 follows.

The exact sequence

$$0 \to \mathbb{Z}_\mu \xrightarrow{\bar{\sigma}} \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \longrightarrow \mathrm{Im}(\rho) \longrightarrow 0$$

splits because $\bar{\sigma}$ admits a retraction τ defined by $\tau(e_i) = \mu/m_i$. Let $\mathrm{Im}(\rho) \simeq \oplus_{j=1}^r \mathbb{Z}_{b_j}$ with b_j dividing b_{j+1} for all j. Since $\mathrm{Im}(\rho)$ is a quotient of $\oplus_{i=1}^t \mathbb{Z}_{m_i}$ we see that b_r divides μ. Hence

$$\bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \simeq \mathbb{Z}_{b_1} \oplus \cdots \oplus \mathbb{Z}_{b_r} \oplus \mathbb{Z}_\mu$$

The uniqueness of this decomposition together with Lemma 1.2 imply that $\mathrm{Im}(\rho) \simeq G$. □

Finally we will prove some results used before.

LEMMA 1.7. *Let $V \subseteq \mathbb{P}^n$ be a reduced variety of dimension ≥ 2, and denote by $(\mathbb{P}^n)^V$ the variety of hyperplanes. Then* $\dim \{L \in (\mathbb{P}^n)^V \mid L \cap V$ *is non-reduced*$\} \leq n - 2$.

Proof. Let $\Gamma = \{(P, L) \in V \times (\mathbb{P}^n)^V \mid L \cap V$ is non-reduced at $P\}$, and $\Omega = \{(P, L) \in V \times (\mathbb{P}^n)^V \mid L \cap V$ is singular at $P\}$. One has $\dim \Omega = n - 1$ ([6], II 8.18) and $\Gamma \subseteq \Omega$, so that $\dim \Gamma \leq n - 1$. On the other hand, if $\pi : \Gamma \to (\mathbb{P}^n)^V$ denotes the projection and $L \in \mathrm{Im}\, \pi$ then $\dim \pi^{-1}(L) \geq 1$. We conclude $\dim \mathrm{Im}\, \pi \leq n - 2$. □

LEMMA 1.8. *Let $\varphi : X \to C$ be a fibration from the smooth projective variety X of dimension ≥ 3 onto a curve. Let Y be a general hyperplane section of X. Then the* *multiple fibres of the restriction of* φ *to* Y *are exactly the hyperplane sections of the multiple fibres of* φ*, and have their same multiplicities.*

Proof. Let $X \subseteq \mathbb{P}^n$, and set $\Gamma = \{(t, L) \in C \times (\mathbb{P}^n)^V \mid$ multiplicity of $(\varphi^{-1}(t) \cap L)$ is strictly greater than the multiplicity of $\varphi^{-1}(t)\}$. Denote by $\alpha : \Gamma \to C$, $\beta : \Gamma \to (\mathbb{P}^n)^V$ the two projections. For any $t \in C$, the preceding Lemma applied to all the irreducible components of $(\varphi^{-1}(t))_{\mathrm{red}}$ yields $\dim \alpha^{-1}(t) \leq n-2$. Therefore $\dim \operatorname{Im} \beta \leq \dim \Gamma \leq n-1$. □

§2. Families of fibrations

We will consider the following situation. Let X, Y, M be connected complex manifolds (not necessarily compact), and let $f : X \to Y$, $g : Y \to M$ be surjective, proper, flat holomorphic maps with connected fibres. Write $h := g \circ f$, and suppose that all fibres of g are smooth compact curves, and the fibres of h are all compact manifolds. If X_t, Y_t denote the fibres of h and g over $t \in M$, then the induced map $f_t : X_t \to Y_t$ is a fibration as defined at the beginning of §1.

DEFINITION 2.1. With the hypothesis just stated, we will say that $\{f_t : X_t \to Y_t\}_{t \in M}$ is a family of fibrations. For any $0, t \in M$, f_t is called a smooth deformation of f_0.

Now we ask ourselves how do the groups $L(f_t)$ of Definition 1.1 vary for a family of fibrations $\{f_t\}_{t \in M}$. As a matter of fact, we will see that they are all isomorphic. To begin with, the following Proposition shows the invariance of $G(f_t)$ under smooth deformations. The proof relies on the fact that a smooth holomorphic map is differentiably locally trivial. Then we will recall that $G(f_t)$ is a direct summand of $L(f_t)$ and will do a base change in order to obtain the invariance of $L(f_t)$.

PROPOSITION 2.2. *If* $\{f_t : X_t \to Y_t\}_{t \in M}$ *is a family of fibrations, then the groups* $G(f_t)$ *are all isomorphic.*

Proof. Let (X, Y, M, f, g) be the quintuplet which determines the family $\{f_t : X_t \to Y_t\}$, as defined before. In order to fix ideas, we will choose an element $0 \in M$ and will write $R := X_0$, $C := Y_0$, $\varphi := f_0$. The maps f_t are smooth deformations of $\varphi : R \to C$. A theorem of Ehresmann ([3]; compare with [10], page 19, and [12]) states that g and $h := g \circ f$ are differentiably locally trivial. In particular, there exists an analytic open neighbourhood U of $0 \in M$ and a commutative diagram

$$\begin{tikzcd}
h^{-1}(U) \arrow[r, "f"] \arrow[d, "p", "\sim"'] & g^{-1}(U) \arrow[d, "q", "\sim"'] \arrow[ddr, "g"] & \\
R \times U \arrow[r] & C \times U \arrow[dr, "\text{(projection)}"] & \\
(x, t) \arrow[r, mapsto] & (\Psi_t(x), t) \arrow[r] & U
\end{tikzcd}$$

where the vertical arrows p, q are diffeomorphisms, and $\Psi_t : R \to C$ a differentiable map. Choose a point $\xi \in C$ such that $F := \varphi^{-1}(\xi)$ is smooth. The map $f : X \to Y$ is also differentiably trivial in a neighbourhood $V \subseteq g^{-1}(U)$ of $q^{-1}(\xi, 0)$, that is, there exists a diffeomorphism $f^{-1}(V) \simeq F \times V$ making commutative the following diagram

$$\begin{tikzcd}
f^{-1}(V) \arrow[r, "\sim"] \arrow[dr, "f"'] & F \times V \arrow[d, "\text{(projection)}"] \\
& V
\end{tikzcd}$$

Put $W := q(V)$. We have a commutative diagram

$$\begin{tikzcd}
F \times W \arrow[r, "\text{(projection)}"] \arrow[d] & W \arrow[d] \\
R \times U \arrow[r] & C \times U
\end{tikzcd}$$

working as

$$\begin{array}{ccc}
(z; (y, t)) & \mapsto & (y, t) \\
\downarrow & & \updownarrow \\
(\lambda(z, y, t),); t) & \mapsto & ((\Psi_t \circ \lambda)(z, y, t,); t) = (y, t)
\end{array}$$

The left vertical arrow is a differentiable immersion, and $\lambda : F \times W \to R$ is a differentiable map. Let us define $\sigma_t : F \to R(t \in M)$ by $\sigma_t(z) = \lambda(z, \xi, t)$. Notice that $\sigma_t(F)$ is the fibre of Ψ_t over the point $\xi \in C$. Furthermore the maps σ_t, σ_0 are homotopic to each other for t close enough to 0, and thus they induce the same map in homology. With our identifications and Theorem 1.3 we immediately see that the cokernel of $(\sigma_t)_* : H_1(F, \mathbb{Z}) \to H_1(R, \mathbb{Z})$ is isomorphic to $H_1(C, \mathbb{Z}) \times G(f_t)$, whose torsion part is $G(f_t)$. Since $(\sigma_t)_* = (\sigma_0)_*$, it follows that $G(f_t) \simeq G(f_0)$ for t near 0. As a matter of fact, we have just proved that the set of $t \in M$ such that $G(f_t) \simeq G(f_0)$ is open. But similar arguments show that it is also closed, and the connectedness of M finishes our proof. $\square$

THEOREM 2.3. *Let* $\{f_t : X_t \to Y_t\}_{t \in M}$ *be a family of fibrations. Then the groups* $L(f_t)$ *are all isomorphic.*

Proof. Let the family be determined by the maps $f: X \to Y$, $g : Y \to M$ as described at the beginning of this section. Write $h := g \circ f$, and choose a point $0 \in M$. First we will assume that Y_0 is not rational. Let $\sigma : B \to Y_0$ be any étale morphism of degree 2. Since g is differentiably locally trivial, there is a neighbourhood U of $0 \in M$ such that $U \times Y_0$ and $g^{-1}(U)$ are diffeomorphic over U. The composite $(\mathrm{id}, \sigma) : U \times B \to U \times Y_0 \approx g^{-1}(U)$ makes $U \times B$ into a topological covering space of $g^{-1}(U)$. Let V denote the space $U \times B$ endowed with the complex structure induced by $g^{-1}(U)$, and set $W := h^{-1}(U) \times_{g^{-1}(U)} V$. The natural projection $\lambda : W \to V$ defines a family of fibrations parametrized by U. Furthermore, each fibre of multiplicity m of $f_t : X_t \to Y_t$, $t \in U$, lifts to a pair of fibres of $\lambda_t : W_t \to V_t$, both with multiplicity m. Thus $L(\lambda_t) \simeq L(f_t) \oplus L(f_t)$. Combining the invariance of $G(\lambda_t)$ asserted in Theorem 2.2 with Lemma 1.2 yields the invariance of $L(f_t)$ for $t \in U$. Now use the connectedness of M to get that $L(f_t)$ is the same for all $t \in M$.

Next let us suppose that Y_0 is rational. Then $Y_t \simeq \mathbb{P}^1$ for all $t \in M$. It follows from [4] that $g : Y \to M$ is analytically locally trivial, so that $g^{-1}(U)$ is analytically isomorphic to $U \times Y_0$ over U, for some neighbourhood U of $0 \in M$. Let $B \to Y_0$ be any double cover which is unramified over the points of Y_0 where $f_0 : X_0 \to Y_0$ fails to be smooth. Making U smaller if necessary one may assume that the composite $f: h^{-1}(U) \to g^{-1}(U) \approx U \times Y_0$ is a smooth map over all points (t, x) where x is a branch point of $B \to Y_0$. Set $V := U \times B$ and $W := h^{-1}(U) \times_{g^{-1}(U)} V$. Then W is smooth and the projection $\lambda : W \to V$ defines a family of fibrations. One checks that $\lambda_t : W_t \to V_t$. has no other multiple fibres than the ones coming from $f_t : X_t \to Y_t$. Hence also $L(\lambda_t) \simeq L(f_t)^{\oplus 2}$ for all t, and one finishes as before. □

REMARK 2.4. For elliptic fibrations on a compact surface something stronger than Theorem 2.3 holds, namely, that the set of multiplicities of the fibres is invariant under smooth deformations. This was proved by Iitaka in [7].

REFERENCES

[1] W. BARTH, C. PETERS and A. VAN DE VEN, *Compact Complex Surfaces*, Springer, Berlin, Heidelberg, New York 1984.

[2] D. A. COX and S. ZUCKER, *Intersection numbers of sections of elliptic surfaces*. Invent. Math. *53* 1–44 (1979).

[3] C. EHRESMANN, *Sur les espaces fibrés différentiables*. C. R. Acad. Sci. Paris, *224*, 1611–1612 (1947).

[4] W. FISCHER and H. GRAUERT, *Lokal triviale Familien kompakter komplexer Mannigfaltigkeiten*. Nach. Akad. Wiss. Göttingen, II. Math. Phys. Kl. pp. 89–94 (1965).

[5] T. Fujita, *On the hyperplane section principle of Lefschetz*. J. Math. Soc. Japan *32*, 153–169 (1980).
[6] R. Hartshorne, *Algebraic Geometry*, Springer, Berlin, Heidelberg, New York 1977.
[7] S. Iitaka, *Deformations of compact complex surfaces II*. J. Math. Soc. Japan *22*, 247–261 (1970).
[8] S. Iitaka, *Deformations of compact complex surfaces III*. J. Math. Soc. Japan *23*, 692–705 (1971).
[9] M. Levine, *Pluricanonical divisors on Kähler manifolds*. Invent. Math. *74*, 293–303 (1983).
[10] J. Morrow and K. Kodaira, *Complex Manifolds*, Holt, Rinehart and Winston, New York 1971.
[11] F. Serrano, *The Picard group of a quasi-bundle* (preprint).
[12] J. A. Wolf, *Differentiable fibre spaces and mappings compatible with Riemannian metrics*. Mich. Math. J. *11*, 65–70 (1964).

Departament d'Àlgebra i Geometria
Universitat de Barcelona
Gran Vla 585
08007 Barcelona, Spain

Received May 19, 1989/October 30, 1989