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About a problem of Ulam concerning flat sections of manifolds

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§0. Introduction

The problem 68 of the *Scottish Book* [13], stated by S. M. Ulam, says the following:

"*There is a given n-dimensional manifold R with the property that every section of its boundary by a hyperplane of $(n-1)$-dimensions gives an $(n-2)$-dimensional, closed surface (a set of homeomorphic to a surface of the sphere of this dimension). Prove that R is convex.*"

The problem is also included in the list of problems about finite dimensional manifolds of R. J. Daverman [7]. It is listed as the problem M.16 and it is interpreted as follows:

"If M is a compact, $(n+1)$-dimensional manifold with boundary in $\mathbb{R}^{n+1}$ for which every n-dimensional hyperplane H that meets M in more than a point has $H \cap \partial M$ an $(n-1)$-sphere, is M convex?"

Schreier [14] showed that a two-dimensional surface in $\mathbb{R}^3$, each of whose nondegenerate planar sections is a Jordan curve, is the boundary of a convex body. Our first theorem, which is a generalization of Schreier's Theorem, solves this first interpretation of Ulam's Problem.

THEOREM 1. *Let N be a closed, connected n-manifold topologically embedded in $\mathbb{R}^{n+1}$. Suppose that for every n-dimensional hyperplane H that meets N in more than a point, $H - N$ has exactly two components. Then N is the boundary of a convex $(n+1)$-body.*

Let X be a compact subset of $\mathbb{R}^{n+1}$. An n-dimensional hyperplane H of $\mathbb{R}^{n+1}$ is called a *supporting hyperplane* of X if $X \cap H \neq \varnothing$ and X is contained in one of the closed halfspaces of $\mathbb{R}^{n+1}$ determined by H.

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In Theorem 1 we are not using the whole strength of the hypothesis of Ulam's Problem, because the phrase "every n-dimensional hyperplane that meets N in more than one point" almost implies that N is the boundary of a convex body. It will be clear from the proof of Theorem 1 that every supporting hyperplane H of N has $H - N$ connected and, consequently, under this weak hypothesis, $H \cap N$ is a single point. Therefore, a better interpretation of Ulam's Problem is the following one in which no assumption is made on the intersection of M with its supporting hyperplanes. Furthermore, as it is suggested by the commentary of R. D. Mauldin [13] to this problem, it would be interesting to consider intersections with hyperplanes of lower dimensions.

ULAM'S PROBLEM. *Let $1 \leq k \leq n$. If M is a compact $(n+1)$-manifold with boundary in $\mathbb{R}^{n+1}$ for which every k-dimensional hyperplane H that meets the interior of M has $H \cap \partial M$ a $(k-1)$-sphere, is M convex?*

If N is a closed, connected n-manifold topologically embedded in $\mathbb{R}^{n+1}$, let $\mathrm{In}(N)$ denote the bounded component of $\mathbb{R}^{n+1} - N$. Our next theorem solves Ulam's Problem.

THEOREM 2. *Let $1 \leq k \leq n$ and let N be a closed, connected n-manifold topologically embedded in $\mathbb{R}^{n+1}$. Suppose that for every k-dimensional hyperplane H that meets* $\mathrm{In}(N)$, *$H \cap N$ has the Čech-cohomology of a $(k-1)$-sphere. Then N is the boundary of a convex $(n+1)$-body.*

In order to prove Theorem 2 it will be essential to characterize convex bodies in terms of the cohomology of its sections. A compact set X will be called *acyclic* if for every $\lambda \geq 0$, the reduced Čech-cohomology group $\check{H}^{\lambda}(X, \mathbb{Z})$ is zero. For example, if Σ is an n-sphere topologically embedded in $\mathbb{R}^{n+1}$, then $\Sigma \cup \mathrm{In}(\Sigma)$ is acyclic. Our next theorem characterizes convex sets in terms of acyclic sections.

THEOREM 3. *Let $1 \leq k \leq n$ and let K be a compact subset of $\mathbb{R}^{n+1}$. Suppose that for every k-dimensional hyperplane H that meets K, $H \cap K$ is acyclic. Then K is convex.*

Note that Theorem 3 generalizes Aumann's Theorems [1] and [2] (see commentary of R. D. Mauldin to the problem 68 of the Scottish Book [13]) because for a compact subset X of $\mathbb{R}^2$, X is acyclic if and only if X and $\mathbb{R}^2 - X$ are connected (Aumann's definition of simple connectedness).

We may be also interested just in sections with horizontal hyperplanes. In this direction we can obtain, using deep decomposition theorems of R. J. Daverman [4], [6], the following theorem:

THEOREM 4. *Let N be a closed, connected n-manifold topologically embedded in $\mathbb{R}^{n+1}$. Let $\mathbb{R}_t = \mathbb{R}^n \times \{t\}$ and $N_t = \mathbb{R}_t \cap N$. Suppose that N satisfies one of the following two properties:*

(a) *For every $\mathbb{R}_t$ which meets N in more than a point, N_t is a closed, connected $(n-1)$-manifold with the property that $\pi_1(N_t)$ is abelian, $n \geq 5$.*

(b) *For every $\mathbb{R}_t$ which meets N in more than a point, N_t is a closed, connected $(n-1)$-manifold with the property that N_t is locally flat in N.*

Then N is an n-sphere.

If in addition to (b), *N_t is locally flat in $\mathbb{R}_t$, $n \neq 4$, then N is a locally flat n-sphere.*

For 3-dimensional versions of this theorem see [3] and [9].

We let $\mathbb{R}^n$ denote Euclidean n-space and we will identify $\mathbb{R}^k$ with $\{(x_1, \ldots, x_n) \in \mathbb{R}^n / x_{k+1} = \cdots = x_n = 0\}$. Also, we let $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ denote $\{(x_1, \ldots, x_n) \in \mathbb{R}^n / x_n \geq 0\}$ and $\{(x_1, \ldots, x_n) \in \mathbb{R}^n / x_n \leq 0\}$, respectively. Furthermore, if $x \in \mathbb{R}^n$, $B_\epsilon(x) = \{y \in \mathbb{R}^n / \|x - y\| < \epsilon\}$ denotes the open ϵ-ball centered at x. For any space X and $A \subset X$, we use $\mathrm{Int}_X A$, $\mathrm{Cl}_X A$ and $\mathrm{Bd}_X A$ to denote, respectively, the topological interior, closure and boundary of A in X. The subscript will be omitted when the meaning is clear and in that case $\bar{A}$ will also denote the closure of A in X.

In this paper we will use reduced Čech-homology [cohomology] and all our homomorphisms, unless otherwise stated, are induced by inclusions.

§1. The proof of Theorem 1

In this section we will always use reduced Čech-homology [cohomology] with $\mathbb{Z}_2$-coefficients.

Our first task is to prove the following theorem:

THEOREM 1.1. *Let U be a bounded, open subset of $\mathbb{R}^{n+1}$. Suppose that:*

(1) *$H_{n-1}(U) \to H_{n-1}(\bar{U})$ is an epimorphism.*
$[H^{n-1}(\bar{U}) \to H^{n-1}(U)$ is a monomorphism.]

(2) *For every n-dimensional hyperplane H of $\mathbb{R}^{n+1}$ that meets $\bar{U}$ in more than a point, $H^{n-1}(H \cap \mathrm{Bd}\, \bar{U}) = \mathbb{Z}_2$ (i.e. $H - \mathrm{Bd}\, \bar{U}$ has exactly two components).*

Then $\bar{U}$ is convex.

The following two technical lemmas will be needed in the proof of Theorem 1.1.

LEMMA 1.2. *Let U be as in Theorem 1.1 and let H be an n-dimensional hyperplane that meets U. Then $H - \bar{U}$ is the unbounded component of $H - \mathrm{Bd}\, \bar{U}$ and consequently $H - \bar{U}$ is connected.*

Proof. Since $H^{n-1}(H \cap \operatorname{Bd} \bar{U}) = \mathbb{Z}_2$ then $H - \operatorname{Bd} \bar{U}$ has exactly two components, W_1 and W_2. Let W_1 be the bounded component of $H - \operatorname{Bd} \bar{U}$. If $x \in H \cap U$ then $x \in W_1$, otherwise there would be an arc from x to a point in $H - \bar{U}$ which does not intersect $\operatorname{Bd} \bar{U}$. By exactly the same reason and since $H \cap U \neq \emptyset$, every element of W_1 belongs to $\bar{U}$ and also $W_2 \subset H - \bar{U}$. Then $W_2 = H - \bar{U}$. □

LEMMA 1.3. *Let U be as in Theorem* 1.1 *and let H be a supporting hyperplane of $\bar{U}$. Then $H \cap \bar{U}$ consists of a single point.*

Proof. We will give the proof of the homology version. The cohomology version has just the dual proof.

Suppose $H \cap \bar{U} = H \cap \operatorname{Bd} \bar{U} = X$ is not a single point. Hence, $H^{n-1}(X) = \mathbb{Z}_2$ and consequently $H - X$ has exactly two components. Therefore, $H_{n-1}(X) = \mathbb{Z}_2$. Let W be the bounded component of $H - X$ and let $\hat{X} = X \cup W$. It is not difficult to prove that $H_n(\hat{X}) = H_{n-1}(\hat{X}) = 0$ and that if $w \in W$ then $H_{n-1}(X) \to H_{n-1}(\hat{X} - \{w\})$ is an isomorphism. Without loss of generality we may assume that $H = \mathbb{R}^n$, W contains the origin of $\mathbb{R}^{n+1}$ and $\bar{U} \subset \mathbb{R}^{n+1}_+$. For $t \in (0, \infty)$, let $Y_t = \bar{U} \cup (\mathbb{R}^n \times [t, \infty))$.

CLAIM. There is $t_0 \in (0, \infty)$ such that $H_{n-1}(X) \to H_{n-1}(Y_{t_0})$ is zero.

If $H_{n-1}(X) \xrightarrow{i_*} H_{n-1}(\bar{U})$ is zero, then there is nothing to prove. Therefore, let γ be the generator of $H_{n-1}(X) = \mathbb{Z}_2$ and suppose $0 \neq i_*(\gamma) \in H_{n-1}(\bar{U})$. By hypothesis, there is $\beta \in H_{n-1}(U)$ such that $j_*(\beta) = i_*(\gamma)$, where $j : U \hookrightarrow \bar{U}$ is the inclusion. For every $t \in (0, \infty)$, let $U_t = U \cap (\mathbb{R}^n \times (t, \infty))$. Hence, there is $t_0 \in (0, \infty)$ and $\beta_{t_0} \in H_{n-1}(U_{t_0})$ such that $k_*(\beta_{t_0}) = \beta$, where $k : U_{t_0} \to U$ is the inclusion.

Let us consider the following commutative diagram of homomorphisms induced by inclusions:

$$\begin{tikzcd}
\mathbb{Z}_2 = H_{n-1}(X) \arrow[r, "i_*"] \arrow[d] & H_{n-1}(\bar{U}) \arrow[dl, dash] \\
H_{n-1}(Y_{t_0}) & H_{n-1}(U_{t_0}) \arrow[l, "\lambda_*"] \arrow[u, "(jk)_*"']
\end{tikzcd}$$

Since λ_* is zero and $(jk)_*(\beta_{t_0}) = i_*(\gamma)$, we have that $H_{n-1}(X) \to H_{n-1}(Y_{t_0})$ is zero. This concludes the proof of the claim.

Let us return to the proof of Lemma 1.3. Let $Y = \hat{X} \cup Y_{t_0}$. Our next purpose is to prove that $H_n(Y_{t_0}) \to H_n(Y)$ is an epimorphism.

Since the origin of $\mathbb{R}^{n+1}$ is not in $\bar{U}$, then there is $0 < \epsilon < t_0$ such that $B_\epsilon(0) \subset \mathbb{R}^{n+1} - \bar{U}$. Let $0 < t_1 < \epsilon$ be such that $(\mathbb{R}^n \times \{t_1\}) \cap U \neq \emptyset$. Hence, by

Lemma 1.2, there is a proper embedding $\alpha : [0, \infty) \to \mathbb{R}^n \times \{t_1\}$ such that $\alpha(0) = (0, t_1) \in \mathbb{R}^n \times (-\infty, \infty) = \mathbb{R}^{n+1}$, $\alpha([0, \infty)) \subset \mathbb{R}^{n+1} - \bar{U}$ and $\mathbb{R}^{n+1} - \alpha([0, \infty))$ is contractible.

Let S be the n-sphere of radius $\sqrt{\epsilon^2 + t_1^2}$ centered at $(0, t_1)$ and let $V = B_\epsilon(0) \cap \mathbb{R}^n$. Hence, $S \cap \mathbb{R}^n$ is the boundary of V.

Let $f : \mathbb{R}^{n+1} - \{(0, t_1)\} \to S$ be given by

$$f(x) = \left(\frac{x - (0, t_1)}{\|x - (0, t_1)\|}\right)\sqrt{\epsilon^2 + t_1^2} + (0, t_1)$$

for every $x \in \mathbb{R}^{n+1} - \{(0, t_1)\}$. Let us consider the following commutative diagram:

$$\begin{array}{ccc} H_n(Y) & \to & H_n(Y, Y - V) \\ \downarrow f_* & & \cong \uparrow f_* \\ H_n(S) & \to & H_n(S, S \cap \mathbb{R}^{n+1}_+) \end{array}$$

Since $f : (Y, Y - V) \to (S, S \cap \mathbb{R}^{n+1}_+)$ is a homeomorphism of pairs and $f : Y \to S$ is nullhomotopic because it factorizes through $\mathbb{R}^{n+1} - \alpha([0, \infty))$, we have that $H_n(Y) \to H_n(Y, Y - V)$ is zero and consequently that $H_n(Y - V) \to H_n(Y)$ is an epimorphism.

On the other hand, since the inclusion $\hat{X} - V \hookrightarrow \hat{X} - \{0\}$ is a homotopy equivalence and $H_{n-1}(X) \to H_{n-1}(\hat{X} - \{0\})$ is an isomorphism, the long exact sequence of the pair $(\hat{X} - V, X)$ shows that $H_n(\hat{X} - V, X) = 0$ and, by excision, that $H_n(Y - V, Y_{t_0}) = 0$. Thus, $H_n(Y_{t_0}) \to H_n(Y - V)$ is an epimorphism and hence so is $H_n(Y_{t_0}) \to H_n(Y)$.

Let us consider the Mayer-Vietoris long exact sequence corresponding to the decomposition $Y = Y_{t_0} \cup \hat{X}$:

$$\to H_n(Y_{t_0}) \oplus H_n(\hat{X}) \xrightarrow{i} H_n(Y) \xrightarrow{\partial} H_{n-1}(X) \xrightarrow{j} H_{n-1}(Y_{t_0}) \oplus H_{n-1}(\hat{X}) \to$$

Since i is an epimorphism and j is zero, we have that $H_{n-1}(X) = 0$, which is a contradiction. This concludes the proof of Lemma 1.3. □

Proof of Theorem 1.1. Let $cc(\bar{U})$ be the convex hull of $\bar{U}$. We will start by proving that $\mathrm{Bd}\,(cc(\bar{U})) \subset \mathrm{Bd}\,\bar{U}$. Let $x \in \mathrm{Bd}\,(cc(\bar{U}))$ and let H be a supporting hyperplane of $cc(\bar{U})$ through x. By Caratheodory's Theorem (see [10]), there is a finite set $F \subset H \cap \bar{U}$ such that x belongs to the convex hull of F. Therefore, H is a supporting hyperplane of $\bar{U}$ and hence, by Lemma 1.3, $H \cap \bar{U} = \{x\}$.

Let us now prove that $cc(\bar{U}) \subset \bar{U}$. Suppose it is not. Let $y_1 \in cc(\bar{U}) - \bar{U}$, $y_2 \in U \subset \mathrm{Int}\,(cc(\bar{U}))$ and H_1 an n-dimensional hyperplane through y_1 and y_2. Let

$y_3 \in H_1 - cc(\bar{U})$. Note that y_1 and y_2 are in different components of $H_1 - \text{Bd}\,\bar{U}$. Furthermore, since $\text{Bd}\,(cc(\bar{U})) \subset \text{Bd}\,(\bar{U})$, y_3 and y_i, $i = 1, 2$, are in different components of $H_1 - \text{Bd}\,\bar{U}$. Consequently, $H_1 - \text{Bd}\,\bar{U}$ has at least three components, which is a contradiction. Then $cc(\bar{U}) = \bar{U}$ which implies that $\bar{U}$ is convex. □

Let X be a locally compact, metric space. We say that X is *uniformly, homologically n-connected, ulc*^{n}, if given $0 \le i \le n$, any abelian group G and any open cover α of X, there is an open cover β of X with the following property: For every $V \in \beta$ there is $W \in \alpha$ such that $V \subset W$ and $\tilde{H}_i(V, G) \to \tilde{H}_i(W, G)$ is zero. By Theorems X.3.2 and X.6.10 of [15], we know that a bounded, open subset of $\mathbb{R}^{n+1}$ is *ulc*^{n} if and only if each component of its boundary is a generalized manifold. Thus, if N is a closed, connected n-dimensional manifold topologically embedded in $\mathbb{R}^{n+1}$, In (N) is *ulc*^{n}.

As an immediate corollary of Theorem 1.1 and Theorems X.5.12 and X.6.3 of [15], we have the following theorem.

THEOREM 1.4. *Let U be a bounded, ulc^n, open subset of $\mathbb{R}^{n+1}$. Suppose that for every n-dimensional hyperplane H that meets $\bar{U}$ in more than a point, $H - \text{Bd}\,U$ has exactly two components. Then U is convex.*

Theorem 1 follows immediately from Theorem 1.4. Note that Theorem 1 also holds when N is a generalized manifold.

Proof of Theorem 4. Let us suppose that $N_t \ne \emptyset$ if and only if $t \in [0, 1]$. By the proof of Lemma 1.3, Theorem X.5.12 and Theorem X.3.2 of [15], we have that N_0 and N_1 consist each one of a single point. By Corollary 8.9 of [6] in case (a) and Lemma 4.1 of [6] in case (b), we have that M is the suspension of M_t. Consequently, M is an n-sphere and M_t is a homotopy $(n-1)$-sphere. Finally, the last assertion follows from Corollary 5.6 of [4] and Theorem 1 of [5].

§2. Alexander Duality and acyclic sections

The main purpose of this section is to prove Theorem 3. In our proof we will use Alexander Duality. Thus, for completeness, we summarize what we need in the following:

For every compact subset X of $\mathbb{R}^n$, any abelian group G and $0 \le \lambda \le n-1$, there is an isomorphism

$$D^n_\lambda : \check{H}^\lambda(X, G) \to \tilde{H}_{n-\lambda-1}(\mathbb{R}^n - X, G).$$

From this isomorphism we will use the following three facts (see [8]):

(2.1) Let $X, Y \subset \mathbb{R}^n$ be such that X is compact, Y is a compact polyhedron and $X \cap Y = \emptyset$. Let $i : X \to \mathbb{R}^n - Y$ and $j : Y \to \mathbb{R}^n - X$ be the inclusions. Suppose that

$$D^n_\lambda(\alpha) \in \text{Image}\, j_* : \check{H}_{n-\lambda-1}(Y, G) \to \check{H}_{n-\lambda-1}(\mathbb{R}^n - X, G),$$

then

$$\alpha \in \text{Image}\, i^* : \check{H}^\lambda(\mathbb{R}^n - Y, G) \to \check{H}^\lambda(X, G).$$

(2.2) Let X be a compact subset of $\mathbb{R}^{n+1}$. Then the following diagram is commutative (up to sign):

$$\begin{array}{ccc} \check{H}^\lambda(X \cap \mathbb{R}^n, G) & \xleftarrow{i^*} & \check{H}^\lambda(X, G) \\ \downarrow D^n_\lambda & & \downarrow D^{n+1}_\lambda \\ \tilde{H}_{n-\lambda-1}(\mathbb{R}^n - X, G) & \xleftarrow{\partial} & \tilde{H}_{n-\lambda}(\mathbb{R}^{n+1} - X, G) \end{array}$$

where ∂ is the boundary homomorphism which arrives from the Mayer–Vietoris long exact sequence of the decomposition of $\mathbb{R}^{n+1} - X$ induced by $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ and i^* is the homomorphism induced by the inclusion $i : X \cap \mathbb{R}^n \hookrightarrow X$.

(2.3) In particular, if $X \subset \mathbb{R}^n$, since $\partial : \tilde{H}_{n-\lambda}(\mathbb{R}^{n+1} - X, G) \to \tilde{H}_{n-\lambda-1}(\mathbb{R}^n - X, G)$ is an isomorphism, we have that if $D^n_\lambda(\gamma) = \partial\beta$, for $\gamma \in \check{H}^\lambda(X, G)$ and $\beta \in \tilde{H}_{n-\lambda}(\mathbb{R}^{n+1} - X, G)$, then $D^{n+1}_\lambda(\gamma) = \pm\beta$.

For the rest of this paper, unless otherwise stated, we will use reduced Čech-homology [cohomology] with $\mathbb{Z}$-coefficients.

Proof of Theorem 3. It is not difficult to see that it is enough to prove the theorem for $k = n$. The proof is by induction on n. If $n = 1$, compact, acyclic subsets of 1-dimensional hyperplanes in $\mathbb{R}^2$ are closed intervals, so K is convex. Suppose the theorem is true for $n - 1$. We will prove it for n.

We will start by proving that for every supporting hyperplane H of K, $H \cap K$ is convex. For that purpose it will be enough to prove that for every $(n-1)$-dimensional hyperplane Γ of H which meets K, $\Gamma \cap K$ is acyclic.

Let us assume without loss of generality that $\Gamma = \mathbb{R}^{n-1}$, $H = \mathbb{R}^n$ and $K \subset \mathbb{R}^{n+1}_+$. Let $X = \{(x_1, \ldots, x_{n+1}) \in K / x_n \geq 0\}$. Then $X \cap \mathbb{R}^{n-1} = K \cap \mathbb{R}^{n-1}$, $X \cap \mathbb{R}^n = K \cap \mathbb{R}^n_+$ and if $H_t = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} / x_{n+1} = tx_n\}$ is an n-dimensional hyperplane through $\mathbb{R}^{n-1}$, then $X \cap H_t = K \cap H_t$ for every $t > 0$.

We will first prove that for every $i \geq 0$, $\check{H}^i(K \cap \mathbb{R}^n_+) \to \check{H}^i(K \cap \mathbb{R}^{n-1})$ is zero. That is, we will prove that

$$\check{H}^i(X \cap \mathbb{R}^n) \xrightarrow{i^*} \check{H}^i(X \cap \mathbb{R}^{n-1})$$

is zero.

Let α be a nonzero element of $\check{H}^i(X \cap \mathbb{R}^n)$. Let us look at the following commutative diagram (up to sign):

$$\begin{tikzcd}
\check{H}^i(X \cap \mathbb{R}^{n-1}) \arrow[d, equal] & \check{H}^i(X \cap \mathbb{R}^n) \arrow[l, "i^*"'] \arrow[dd, "D_i^{n+1}"] \\
\check{H}^i(X \cap \mathbb{R}^n \cap H_t) \arrow[d, "D_i^n"] & \\
\tilde{H}_{n-i-1}(H_t - (X \cap \mathbb{R}^n)) & \tilde{H}_{n-i}(\mathbb{R}^{n+1} - (X \cap \mathbb{R}^n)) \arrow[l, "\partial"'] \\
\check{H}_{n-i-1}(Y\alpha \cap H_t) \arrow[u, "j_*"] & \tilde{H}_{n-i}(Y_\alpha) \arrow[l, "\partial"'] \arrow[u, "k_*"]
\end{tikzcd}$$

where the first block is as in 2.2, where H_t is playing the role of $\mathbb{R}^n$ and $X \cap \mathbb{R}^n$ the role of X and, where $Y_\alpha \subset \mathbb{R}^{n+1} - (X \cap \mathbb{R}^n)$ is a compact polyhedron with the property that there is $\beta \in \tilde{H}_{n-i}(Y_\alpha)$ such that $k_*(\beta) = D_i^{n+1}(\alpha)$. Furthermore, let $\partial : \check{H}_{n-i}(Y_\alpha) \to \check{H}_{n-i-1}(Y_\alpha \cap H_t)$ be the boundary homomorphism which arises in the Mayer–Vietoris long exact sequence from the decomposition of Y_α induced by $H_t \subset \mathbb{R}^{n+1}$.

Hence, $j_*(\partial\beta) = \pm D_i^n(i^*(\alpha))$ and consequently by 2.1,

$$i^*(\alpha) \in \text{Image } \lambda^* : \check{H}^i(H_t - Y_\alpha) \to \check{H}^i(X \cap \mathbb{R}^{n-1}).$$

Since $Y_\alpha \cap (X \cap \mathbb{R}^n) = \varnothing$ then, for $t > 0$ sufficiently small, $Y_\alpha \cap (X \cap H_t) = \varnothing$ and hence, $X \cap \mathbb{R}^{n-1} \subset X \cap H_t \subset H_t - Y_\alpha$. Therefore, λ^* factorizes through $\check{H}^i(X \cap H_t) = \check{H}^i(K \cap H_t) = 0$, and hence $i^*(\alpha) = 0$.

This proves that $\check{H}^i(K \cap \mathbb{R}^n_+) \to \check{H}^i(K \cap \mathbb{R}^{n-1})$ is zero. Similarly, $\check{H}^i(K \cap \mathbb{R}^n_-) \to \check{H}^i(K \cap \mathbb{R}^{n-1})$ is also zero. Therefore, the Mayer–Vietoris long exact sequence of the decomposition of $K \cap \mathbb{R}^n$ induced by $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, together with the fact that $K \cap \mathbb{R}^n$ is acyclic, imply that $\check{H}^i(K \cap \mathbb{R}^{n-1}) = 0$.

We have proved that $H \cap K$ has acyclic $(n-1)$-dimensional sections. Hence, by induction, $H \cap K$ is convex for every supporting hyperplane H of K. By

Caratheodory's Theorem, we have that $\operatorname{Bd}(cc(K)) \subset \operatorname{Bd} K$ and arguing as in the proof of Theorem 1.1 we conclude that K is convex. □

The next theorem, whose proof uses Alexander Duality, will be used in the proof of Theorem 2.

THEOREM 2.4. *Let U be a bounded, connected, ulc^n, open subset of $\mathbb{R}^{n+1}$. Suppose that for every n-dimensional hyperplane H that meets U, $H \cap \bar{U}$ is acyclic. Then U is convex.*

Proof. Let Γ be a supporting hyperplane of $\bar{U}$. By Theorem 3 and Theorem X.6.3 of [14], it will be enough to prove that $\Gamma \cap \bar{U}$ is acyclic. Without loss of generality we may assume that $\Gamma = \mathbb{R}^n$ and $\bar{U} \subset \mathbb{R}^n_+$. For every $t > 0$, let $Y_t = \bar{U} \cup (\mathbb{R}^n \times [t, \infty))$.

Suppose $\check{H}^i(\bar{U} \cap \mathbb{R}^n)$ is not zero. Then, following the proof of the claim in Lemma 1.3, but using Theorem X.5.9 of [15] instead of (1), it is possible to check, by the naturality of the Universal Coefficient Theorem, that there is an abelian Group G, $t_0 \in (0, \infty)$ and an element $\gamma \in \check{H}^i(\bar{U} \cap \mathbb{R}^n, G)$ which is not in the image of $\check{H}^i(Y_{t_0}, G) \to \check{H}^i(\bar{U} \cap \mathbb{R}^n, G)$.

Clearly, $\tilde{H}_{n-i-1}(\mathbb{R}^n - \bar{U}, G) \to \tilde{H}_{n-i-1}(\mathbb{R}^{n+1}_- - \bar{U}, G)$ sends $D^n_i(\gamma)$ to zero. Note now that for every n-dimensional hyperplane H that meets U, $H^i(H \cap \bar{U}, G) = 0$ (see Section 3 of [12]). Hence, $\tilde{H}_{n-i-1}(\mathbb{R}^n - \bar{U}, G) \to \tilde{H}_{n-i-1}(\mathbb{R}^{n+1}_+ - \bar{U}, G)$ sends $D^n_i(\gamma)$ to zero because there are sufficiently small numbers $t > 0$ with the property that $(\mathbb{R}^n \times \{t\}) \cap U \neq \varnothing$ and consequently with the property that

$$\tilde{H}_{n-i-1}((\mathbb{R}^n \times \{t\}) - \bar{U}, G) \cong \check{H}^i((\mathbb{R}^n \times \{t\}) \cap \bar{U}, G) = 0.$$

Let us consider the following commutative diagram, where the vertical homomorphisms correspond, respectively, to the Mayer–Vietoris long exact sequence of the decomposition of $\mathbb{R}^{n+1} - Y_{t_0}$ and $\mathbb{R}^{n+1} - (\bar{U} \cap \mathbb{R}^n)$ induced by $\mathbb{R}^n \subset \mathbb{R}^{n+1}$:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] \\
\tilde{H}_{n-i}(Y, G) \arrow[r, "j_*"] & \tilde{H}_{n-i}(\mathbb{R}^{n+1} - Y_{t_0}, G) \arrow[r, "k_*"] \arrow[d, "\partial"] & \tilde{H}_{n-i}(\mathbb{R}^{n+1} - (\bar{U} \cap \mathbb{R}^n), G) \arrow[d, "\partial", "\cong"'] \\
 & \tilde{H}_{n-i-1}(\mathbb{R}^n - Y_{t_0}, G) \arrow[r, leftrightarrow, "id", "\cong"'] \arrow[d, "\lambda"] & \tilde{H}_{n-i-1}(\mathbb{R}^n - \bar{U}, G) \arrow[d] \\
 & \begin{array}{c} \tilde{H}_{n-i-1}(\mathbb{R}^{n+1}_+ - Y_{t_0}, G) \\ \oplus \\ \tilde{H}_{n-i-1}(\mathbb{R}^{n+1}_- - Y_{t_0}, G) \end{array} \arrow[d] & {} \\
 & {} &
\end{tikzcd}$$

By the above, $\lambda(D_i^n(\gamma)) = 0$. Thus, let $Y \subset \mathbb{R}^{n+1} - Y_{t_0}$ be a compact polyhedron such that $\partial j_*(\beta) = D_i^n(\gamma)$ for some $\beta \in \tilde{H}_{n-i}(Y, G)$. Hence, $\partial k_* j_*(\beta) = D_i^n(\gamma)$ and then, by 2.3, $k_* j_*(\beta) = \pm D_i^{n+1}(\gamma)$. Therefore, by 2.1, γ is in the image of $\check{H}^i(\mathbb{R}^{n+1} - Y, G) \to \check{H}^i(\bar{U} \cap \mathbb{R}^n, G)$. This is a contradiction because $\bar{U} \cap \mathbb{R}^n \subset Y_{t_0} \subset \mathbb{R}^{n+1} - Y$ and γ is not in the image of $\check{H}^i(Y_{t_0}, G) \to \check{H}^i(\bar{U} \cap \mathbb{R}^n, G)$. Consequently, $\check{H}^i(\bar{U} \cap \mathbb{R}^n) = 0$ and hence, $\bar{U} \cap \mathbb{R}^n$ is acyclic. This concludes the proof of Theorem 2.4. □

REMARK. If in Theorem 2.4, $\bar{U}$ is also an ENR, then the following is a simpler proof. Let $\Gamma = \mathbb{R}^n$ and $\bar{U} \subset \mathbb{R}^n_+$ as above. We will prove that $\bar{U} \cap \Gamma$ is acyclic. Let $\Pi : \bar{U} \to \mathbb{R}$ be the projection on the last coordinate. Then, by 2.2 of [11] or Corollary 3.3 of [12], $\Pi| : \Pi^{-1}(\Pi(U)) \to \Pi(U)$ is an uv^∞ map. Consequently, by Theorems X.5.9 and X.6.3 of [15] and the Vietoris–Begle Theorem (see 3.4 of [11]), $\bar{U} \cap \Gamma$ is uv^∞ and hence, again by 2.2 of [11], acyclic.

§3. The proof of Theorem 2

We will derive Theorem 2 from the following theorem:

THEOREM 3.1. *Let $1 \le k \le n$ and let U be a bounded, ulc^n, open subset of $\mathbb{R}^{n+1}$ such that $\bar{U}$ is an* ENR. *Suppose that for every k-dimensional hyperplane H that meets U, $H \cap U$ is connected and $H \cap \bar{U}$ is acyclic. Then U is convex.*

Proof. Let Γ be a $(k+1)$-dimensional hyperplane that meets U. Then, it is easy to see that $\Gamma \cap U$ is connected. Consequently, by Theorem 2.4, it will be enough to prove that $\Gamma \cap \bar{U}$ is acyclic. Without loss of generality we may assume that $\Gamma = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 = x_2 = \cdots = x_{n-k} = 0\}$.

Let $\rho : \bar{U} \to \mathbb{R}^{n-k}$ be the projection on the first $(n-k)$-coordinates. We would like to show that $\check{H}^i(\rho^{-1}(0)) = 0$, for every $i \ge 0$. Since $\bar{U}$ is an ENR, by 2.2 of [11] or Corollary 3.3 of [12], it will be enough to prove that given $\epsilon > 0$ there is $\delta > 0$ such that

$$\tilde{H}_i(\rho^{-1}(B_\delta(0))) \to \tilde{H}_i(\rho^{-1}(B_\epsilon(0)))$$

is zero, for every $i \ge 0$. By Theorem X.5.9 of [15], it will be enough to prove that for every $i \ge 0$ and $\epsilon > 0$ sufficiently small,

$$\tilde{H}_i(\rho^{-1}(B_\epsilon(0)) \cap U) \to \tilde{H}_i(\rho^{-1}(B_\epsilon(0)))$$

is zero.

Let $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^{n-k+1}$ be the projection of the first $(n-k+1)$-coordinates. Note that $\Pi^{-1}(y)$ is a k-dimensional hyperplane and $\Pi(\Gamma)$ is the line

$$L = \{(x_1, \ldots, x_{n-k+1}) \in \mathbb{R}^{n-k+1}/x_1 = \cdots = x_{n-k} = 0\}.$$

We will first prove that $\Pi(U)$ is convex. Let $x, y \in \Pi(U)$ be any two points and let $x_0, y_0 \in U$ be such that $\Pi(x_0) = x$ and $\Pi(y_0) = y$. Let Γ_1 be the $(k+1)$-dimensional hyperplane which contains $\{x_0\} \cup \Pi^{-1}(y)$. Hence, $\Pi(\Gamma_1)$ is the line in $\mathbb{R}^{n-k+1}$ which contains x and y. Furthermore, since $\Gamma \cap U$ is connected, the closed interval with endpoints x and y is contained in $\Pi(U)$. This proves that $\Pi(U)$ is convex.

Since $\overline{\Pi(U)} = \Pi(\bar{U})$, then $\Pi(\bar{U})$ is convex and $L \cap \Pi(\bar{U})$ is a closed interval whose relative interior is $L \cap \Pi(U)$. Let $\pi : \mathbb{R}^{n-k+1} \to \mathbb{R}^{n-k}$ be the projection on the first $(n-k)$-coordinates. Note that $\pi^{-1}(0) = L$. Hence, for $\epsilon > 0$ sufficiently small, $W = \pi^{-1}(B_\epsilon(0)) \cap \Pi(U)$ is contractible.

Let $\Pi| : \Pi^{-1}(W) \cap \bar{U} \to W$ be the restriction of Π to $\Pi^{-1}(W) \cap \bar{U}$. Clearly, $\Pi^{-1}(W) \cap \bar{U}$ is an ENR and $\Pi|$ is a proper map whose point inverses are acyclic. Then, by 2.1 and 2.2 of [11] or Corollary 3.3 of [12], $\Pi|$ is a uv^∞ map and consequently, by the Vietoris–Begle Theorem (see 3.4 of [11]), $\tilde{H}_i(\Pi^{-1}(W) \cap \bar{U}) = 0$, for every $i \geq 0$.

Since $\rho^{-1}(B_\epsilon(0)) \cap U \subset \Pi^{-1}(W) \cap \bar{U} \subset \rho^{-1}(B_\epsilon(0))$, we have that

$$\tilde{H}_i(\rho^{-1}(B_\epsilon(0)) \cap U) \to \tilde{H}_i(\rho^{-1}(B_\epsilon(0)))$$

is zero. This concludes the proof of Theorem 3.1. □

Proof of Theorem 2. By theorem X.3.2 of [15], $\mathrm{In}(N)$ is a bounded, *ulc*^{n}, open subset of $\mathbb{R}^{n+1}$. Furthermore, $\overline{\mathrm{In}(N)} = N \cup \mathrm{In}(N)$ is an ENR. Let H be a k-dimensional hyperplane that meets $\mathrm{In}(N)$. Therefore, $\check{H}^{k-1}(H \cap N, \mathbb{Z}_2) = \mathbb{Z}_2$, which implies that $H - N$ has exactly two components. Let W be the bounded component of $H - N$. Since $H \cap \mathrm{In}(N) \neq \varnothing$, it is easy to see, as in the proof of Lemma 1.2, that $H \cap \mathrm{In}(N) = W$ and consequently that $H \cap \mathrm{In}(N)$ is connected. Furthermore, using Mayer–Vietoris, it is easy to check that $H \cap \overline{\mathrm{In}(N)} = H \cap (N \cup \mathrm{In}(N)) = (H \cap N) \cup W$ is acyclic. Then, Theorem 3.1 implies that $\mathrm{In}(N)$ is convex and consequently that N is the boundary of a convex $(n+1)$-body. □

REMARK. Note that Theorem 2, $k = n$, holds when N is a generalized manifold and Theorem 2, $1 \leq k < n$, holds when N is a generalized manifold which is also an ENR.

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