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Rational category of the space of sections of a nilpotent bundle

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Abstract. Denote by $\zeta : F \to E \xrightarrow{p} B$ a nilpotent fibration where F is a 1-connected space of finite category and B a finite c.w. complex with non trivial rational cohomology. In this note we compute the rational category of the space Γ_* of continuous pointed sections of ζ.

§1. Introduction

In 1956, R. Thom studied the homotopy type of the space F_f^X of continuous maps of X into F homotopic to a given map f. He computed explicitly the cohomology of F_f^X when F is a product of Eilenberg–Mac Lane spaces [12].

Later on, following ideas of Sullivan, A. Haefliger gave the rational minimal model of the space of sections of a nilpotent bundle [7]. This model has been extensively studied by K. Shibata and M. Vigué-Poirrier [14]. In particular, M. Vigué-Poirrier noted that, if the dimension of X is less than the connectivity of Y, then the rational homotopy Lie algebra of Y^X is isomorphic as a Lie algebra to $H^*(X; \mathbb{Q}) \otimes (\pi_*(\Omega Y) \otimes \mathbb{Q})$.

The aim of this paper is to show that the category of the space of continuous maps from X into Y, and more generally of pointed sections of a fibration, is often infinite.

To be more precise, we prove in fact the following two theorems.

THEOREM 1. *Let Γ_* be the space of continuous pointed sections of a nilpotent fibration $F \to E \to B$. Suppose that*

(1) $\Gamma_* \neq \phi$

(2) *F is a nilpotent space of finite category*

(3) $H^+(B; \mathbb{Q}) \neq 0$ *and* $\dim H^*(B; \mathbb{Q}) < \infty$

(4) $\dim (\pi_*(F) \otimes \mathbb{Q})$ *is infinite.*

Then, $\operatorname{cat}(\Gamma_*) = \infty$.

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In the case where the fibration is trivial, the result is more precise:

THEOREM 2. *Let X be a finite nilpotent c.w. complex and Y be a nilpotent space. We suppose that the rational cohomologies of X and Y are not trivial and one of the two following conditions is satisfied*:

(1) $\dim \pi_*(Y) \otimes \mathbb{Q} = \infty$

(2) $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$ *and there are odd integers p and q such that $H^p(X; \mathbb{Q}) \neq 0$, $\pi_q(Y) \otimes \mathbb{Q} \neq 0$ and $q - p \geq 2$.*

Then the functional space Y^X has infinite category.

This result clearly yields the following corollary previously proved by E. Fadell and S. Husseini.

COROLLARY [3]. *If Y is a 1-connected space of finite category, such that $\tilde{H}^*(Y; \mathbb{Q}) \neq 0$; then the free loop space Y^{S^1} has infinite category.*

The organization of the paper is as follows. We first recall the construction of the Sullivan–Haefliger model for the space of continuous (resp. pointed) sections Γ (resp. Γ_*). We then show how to deduce the two theorems from the model. We also deduce a way to compute explicitly the rational homotopy groups of Γ.

In fact, if X is a 1-connected space and X_0 its rationalization, we have the inequality $\operatorname{cat} X_0 \leq \operatorname{cat} X$ [13]. The integer $\operatorname{cat}(X_0)$ is called the rational category of X and is denoted $\operatorname{cat}_0 X$. Its relevance comes from the fact that $\operatorname{cat}_0(X)$ can be obtained from the minimal model of the space [5].

§2. The Sullivan–Haefliger model

Let $\zeta : F \to E \xrightarrow{p} B$ be a fibration. We suppose that B is a finite nilpotent c.w. complex and F is a nilpotent space with finite Betti numbers. We suppose that $\Gamma_* \neq \phi$.

Let $(A, d_A) \to (A \otimes \Lambda V, D) \to (\Lambda V, d)$ be a minimal K.S. model of ζ [9]. As $\Gamma_* \neq \phi$, we can also suppose that the differential D satisfies:

$$D(V) \subset A \otimes \Lambda^+ V.$$

B is a finite nilpotent c.w. complex. Therefore we can average A is a finite dimensional graded $\mathbb{Q}$-vector space. Denote by S a graded supplementary subspace of the graded vector space formed by the cocycles in A. This gives a direct sum decomposition of $A : A = S \oplus d(S) \oplus T$. We then choose a homogeneous basis

$(a_i)_{i\in I}$ of A by taking the union of homogeneous bases of S, $d(S)$ and T. The graded dual vector space of A will be denoted by $A^\vee$:

$$(A^\vee)_n = \mathrm{Hom}\,(A^n, \mathbb{Q}).$$

$A^\vee$ is naturally equipped with the dual basis a_i^*:

$$\langle a_i^*; a_j\rangle = \delta_{ij}.$$

We now look at the map of algebras defined by:

$$\varepsilon : A\otimes \Lambda V \to A\otimes \Lambda(A^\vee\otimes V) : \varepsilon(v) = \sum_{i\in I} a_i\otimes(a_i^*\otimes v);\quad \varepsilon(a) = a,\quad a\in A.$$

In [7], A. Haefliger shows how to put a uniquely defined differential $d_A\otimes\delta$ on $A\otimes\Lambda(A^\vee\otimes V)$ in such a way that ε becomes a morphism of commutative differential graded algebras. Let W be the quotient of $A^\vee\otimes V$ by the subspace of elements of degree <0, and by the subspace formed by the δ-cocycles in degree 0.

A short computation shows that $\delta(1\otimes v) = 1\otimes dv$, so that the injection $V\cong\mathbb{Q}\otimes V \hookrightarrow A^\vee\otimes V$ induces a K.S. extension:

$$\theta : (\Lambda V, d)\to(\Lambda W, \delta)\to(\Lambda(W/V), \bar{\delta}).$$

THEOREM A (Haefliger, [7]). *θ is a model for the canonical fibration $\Gamma_*\to\Gamma\xrightarrow{p}F$ where p denotes the evaluation on the basis point of B.*

With this model, we can for instance give a rational analogue of the Cohen–Taylor theorem [2].

PROPOSITION. *Let X be a finite wedge of spheres of dimension less than m ($X = \vee_{i=1}^r S^{n_i}$) and Y be a $(m+2)$-connected space, then we have a rational homotopy equivalence*

$$(Y^X)_*\cong\prod_{i=1}^r (Y^{S^{n_i}})_*.$$

Proof. The Haefliger model for $(Y^X)_*$ is $(\Lambda(H_*^+(X;\mathbb{Q})\otimes(\pi_*(Y))^\vee), 0)$. □

§3. The rational homotopy Lie algebra of a space

If S is a nilpotent space with finite Betti numbers, the minimal model of S is a free commutative differential graded algebra $(\Lambda Z, d)$. The graded vector spaces Z^* and $\mathrm{Hom}(\pi_*(S), \mathbb{Q})$ are then isomorphic [11].

The differential d always decomposes in the form $d = d_2 + d_3 + \cdots$, where $d_i(Z) \subset \Lambda^i Z$. This gives on $s^{-1}\,\mathrm{Hom}(Z, \mathbb{Q})$ a structure of Lie algebra by putting:

$$\langle d_2 z; f, g\rangle = (-1)^{\deg(f)+1}\langle sz; [s^{-1}f, s^{-1}g]\rangle$$

$z \in Z; f, g \in \mathrm{Hom}(Z, \mathbb{Q})$.

It is a result of Andrews and Arkowitz [1] that this Lie algebra is isomorphic to the Lie algebra $L_S = \pi_*(\Omega S) \otimes \mathbb{Q}$ obtained on the rational homotopy groups by means of the Whitehead product. An extensive study of L_S has been made these last years with for instance the following result:

THEOREM B ([6], [4]). *If S is a space of finite category, then every nilpotent ideal I of L_S is finite dimensional.*

We now want to compute L_Γ for a given fibration. With the notations of §2, we decompose the differentials D and δ in the form

$$D = D_1 + D_2 + \cdots \qquad D_i(V) \subset A \otimes \Lambda^i(V)$$

$$\delta = \delta_1 + \delta_2 + \cdots \qquad \delta_i(A^\vee \otimes V) \subset \Lambda^i(A^\vee \otimes V)$$

δ_1 is completely defined by d_A and D_1. In fact, put

$$D_1 v_r = \sum_s \alpha_{rs} v_s, \quad \alpha_{rs} \in A^+.$$

We then have:

$$(*)\ \sum_i (-1)^{\deg(a_i)} a_i \otimes \delta_1(a_i^* \otimes v_r) = -\sum_i d_A(a_i) \otimes (a_i^* \otimes v_r) + \sum_i \sum_s \alpha_{rs} \cdot a_i \otimes (a_i^* \otimes v_s)$$

The homology of $(A^\vee \otimes V, \delta_1)$ and $(A^\vee_+ \otimes V, \delta_1)$ are respectively isomorphic to the vector spaces of indecomposable elements of the minimal models of Γ and Γ_*:

We thus have isomorphisms:

(1) $H^*(A^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma) \otimes \mathbb{Q})^\vee$

and

(2) $H^*(A^\vee_+ \otimes V, \delta_1) \cong (\pi_*(\Gamma_*) \otimes \mathbb{Q})^\vee$.

Moreover, the short exact sequence of complexes

$$0 \to (V, 0) \to (A^\vee \otimes V, \delta_1) \to (A^\vee_+ \otimes V, \delta_1) \to 0$$

induces in homology a long exact sequence isomorphic to the dual of the homotopy long exact sequence of the fibration $\Gamma_* \to \Gamma \to F$ [9]:

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(A^\vee_+ \otimes V, \delta_1) \arrow[r, "\Delta"] \arrow[d] & V^{q+1} \arrow[r] \arrow[d] & H^{q+1}(A^\vee \otimes V, \delta_1) \arrow[r] \arrow[d] & H^{q+1}(A^\vee_+ \otimes V, \delta_1) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & (\pi_q(\Gamma_*))^\vee \arrow[r] & (\pi_{q+1}(F))^\vee \arrow[r] & (\pi_{q+1}(\Gamma))^\vee \arrow[r] & (\pi_{q+1}(\Gamma_*))^\vee \arrow[r] & \cdots
\end{tikzcd}$$

REMARK. "D_1 is differential" can be expressed by the fact that the matrix $\mathbf{a}$ consisting of the α_{rs} satisfies $\mathbf{a}^2 + d\mathbf{a} = 0$.

§4. Proof of Theorem 1

We use the notations of §§2 and 3, $(\Lambda V, d) \to (\Lambda W, \delta) \to (\Lambda(W/V, \bar{\delta})$ is a model of the fibration $\Gamma_* \to \Gamma \xrightarrow{p} F$. We consider the linear map

$$D_1 : V \to A \otimes V.$$

There are two cases: Either there exists an infinite sequence of homogeneous linearly independent elements $v_1, v_2, \ldots$ belonging to V such that $D_1(v_i)$ doesn't belong to $D_1(A^+ \otimes V)$, or we can suppose that there exists an integer N such that for every v in V of degree larger than N, $D(v)$ belongs to $A \otimes \Lambda^{\geq 2} V$.

We take a K.S. basis $(v_i)_{i \geq 1}$ of V: $D(v_i) \in A \otimes (v_j)_{j<i}$. Write

$$D_1(v_n) = \sum_{r=1}^{s} \alpha_r \cdot v_r.$$

We obtain $d_A(\alpha_s) = 0$. If $[\alpha_s] = 0$, then $\alpha_s = d_A(b)$ and $D_1(v_n - b \cdot v_s) \in A \otimes (v_j)_{j<s}$.

We then replace v_n by $v'_n = v_n - b \cdot v_s$. If $D_1(v)$ does not belong to $D_1(A^+ \otimes V)$, we can suppose $[a_s] \neq 0$. In this case, formula $(*)$ gives the equality

$$\delta_1(\alpha_s^* \otimes v_s) - (-1)^{\deg(a_s)} 1 \otimes v_n.$$

This means that the element $1 \otimes v_n$ belongs to the image of Δ in the dual homotopy long exact sequence. Recall that the elements in the image of Δ are called the Gottlieb elements of the fibration and let us come back to our dichotomy:

In the first case, the v_i are Gottlieb elements of Γ_* [5]. By [5], we know that the category of a space is greater or equal to the number of its linearly independent Gottlieb elements, so that $\operatorname{cat}(\Gamma_*) = \infty$.

In the second case, put $n = \max\{p \text{ such that } A^p \neq 0\}$. For $q > n + N$, formula $(*)$ yields the isomorphisms

$$H^q((A^\vee \otimes V), \delta_1) = (H(A, d_A)^\vee \otimes V)^q.$$

The injections $\delta(A_p^\vee \otimes V) \subset \Lambda(A^\vee_{<p} \otimes V)$, valid for $p > 0$, imply that the Lie algebra $L = (H(A, d_A)^\vee_+ \otimes V)^{>n+N}$ is a nilpotent Lie algebra of infinite dimension, which is impossible by Theorem B. □

§5. Proof of Theorem 2

In this case, the fibration $(Y^X)_* \xrightarrow{i} (Y^X) \to Y$ admits a section and so $\pi_*(i) \otimes \mathbb{Q}$ is injective. It then results from [5] that $\operatorname{cat}_0((Y^X)_*) \leq \operatorname{cat}_0(Y^X)$. If (1) is satisfied, we deduce from Theorem 1 that $\operatorname{cat}_0(Y^X)$ is infinite.

If (2) is satisfied, choose a non homologically trivial cycle α in A^p and a nonzero element v in V^q. We now have $D_1 = 0$. The formula $(*)$ shows that $a^* \otimes v$ is a δ_1-cycle which is not a δ_1-boundary. $\alpha^* \otimes v$ defines thus a nonzero indecomposable element in the minimal model of Y^X. The definition of δ, as given in §2 implies the following formulas $(**)$ and $(***)$:

$$(**) \quad \sum_i (-)^{\deg(a_i)} a_i \otimes \delta(a_i^* \otimes t) = -\sum_i d_A(a_i) \otimes (a_i^* \otimes t) + \varepsilon(D(t)).$$

$$(***) \quad \varepsilon(v_1 \cdot v_2 \cdots v_r) = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r} \otimes (a_{i_1}^* \otimes v_1)(a_{i_2}^* \otimes v_2) \cdots (a_{i_r}^* \otimes v_r).$$

As $\alpha^2 = 0$ formula $(***)$ shows that $(\alpha^* \otimes v)^n$ can never appear in the decomposition of $\varepsilon(v_q \cdot v_2 \cdots v_r)$, and so by $(**)$, in the differential of an element $\beta^* \otimes t$. It then results from ([8] Proposition 1) that $\operatorname{cat}_0(Y^X) = \infty$. □

If the dimension of X is less than the connectivity of Y, the result we obtain is better.

THEOREM 3. *Let X be a nilpotent space such that there exists an integer $k \geq 1$ with $H^p(X; \mathbb{Q}) = 0$, $p > k$ and $H^k(X; \mathbb{Q}) \neq 0$ and let Y be a $(m-1)$-connected space, non contractible over $\mathbb{Q}$, with $m \geq k+2$, then the functional space Y^X has finite rational category iff the three following conditions are satisfied*:

(1) $\pi_*(Y) \otimes \mathbb{Q} = \pi_{\text{odd}}(Y) \otimes \mathbb{Q}$.

(2) $H^*(X; \mathbb{Q}) = H^{\text{even}}(X; \mathbb{Q})$.

(3) $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$.

Proof. By Theorem 2, Condition 3 is necessary. In this case, we have $\dim \pi_{\text{even}}(Y) \otimes \mathbb{Q} \leq \dim \pi_{\text{odd}}(Y) \otimes \mathbb{Q}$ ([8], Proposition 1), so that $\pi_{\text{odd}}(Y) \otimes \mathbb{Q} \neq 0$. By Theorem 2, the second condition is thus also necessary.

Suppose thus that $H^*(X; \mathbb{Q}) = H^{\text{even}}(X; \mathbb{Q})$. We can suppose that $A^{>k} = 0$. If $\pi_{\text{even}}(Y) \otimes \mathbb{Q} \neq 0$, let's choose a cycle α of A^k defining a nonzero element of $H^k(X; \mathbb{Q})$ and let's choose a nonzero element v in V^{even}. Then, formula (***) shows that no power of $a^* \otimes v$ can appear in the decomposition of $\varepsilon(v_1 \cdot v_2 \cdots v_r)$ for any choice of $v_1, v_2, \ldots, v_r$. Now by (**) no power of $a^* \otimes v$ appear in the expression of a boundary, so that, by ([8]) the category of Y^X has to be infinite.

On the other hand, if the three conditions are satisfied,

$$\pi_*(Y^X) \otimes \mathbb{Q} \cong \pi_{\text{odd}}(Y^S) \otimes \mathbb{Q}$$

is finite dimensional and concentrated in odd degrees. The minimal model of Y^X is thus finite dimensional. This implies that Y^X has the rational homotopy type of a finite c.w. complex. □

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