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The Gauss map of a spacelike constant mean curvature hypersurface of Minkowski space

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The Bernstein problem for maximal (mean curvature zero) spacelike hypersurface of $n+1$ dimensional Minkowski space $\mathbb{E}^{n+1}$ was introduced by Calabi [C] in 1968. He showed that for $n \leq 4$ the only entire maximal graph was a linear supspace. For $n > 4$ the same conclusion was reached by Cheng and Yau in [CY]. Nonlinear entire graphs of constant non-zero mean curvature were found by Treibergs [T].

For hypersurfaces of Euclidean space $\mathbb{E}^{n+1}$, one possible generalization of the Bernstein problem is to study the distribution of normals to a complete constant mean curvature hypersurface. This problem was suggested by Chern in [Ch]. The best result in this direction is that of Hoffman, Osserman, and Schoen [HOS] who showed that the normals to a complete constant mean curvature surface in $\mathbb{E}^3$ cannot lie in a closed hemisphere of S^2, unless the surface is a plane or right circular cylinder.

Here we study the analogous problem for spacelike constant mean curvature hypersurfaces $M \subset \mathbb{E}_1^{n+1}$. The case $n = 2$ was previously discussed by the author in [P]. In order to state the main result we let η be the timelike unit normal field to M. For $p \in M$ we regard $\eta(P)$ as a point in the n-dimensional hyperbolic space $H^n(-1)$ canonically embedded in $\mathbb{E}_1^{n+1}$. We show

THEOREM I. *For $H \neq 0$ there exists a number $\tau = \tau(n, H) > 0$ with the following property: Let $M^n \subset \mathbb{E}_1^{n+1}$ be a spacelike hypersurface with constant mean curvature H. If $\eta(M)$ is contained in a geodesic ball of radius $\tau_1 < \tau$ in $H^n(-1)$ then M is not complete.*

To this end we demonstrate that there is an upper bound on the radius of a geodesic ball $B_\rho(x_0) \subset M$ if $\eta(M)$ is contained in a geodesic ball of radius τ in $H^n(-1)$ for τ sufficiently small.

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We will need the following.

LEMMA 1. *Let M be a smooth Riemannian manifold with Laplace operator Δ. Let S be a smooth function on M satisfying*

$$1 \leq \Delta S$$

and

$$\sigma \leq S \leq \tau.$$

Let $\Omega \subset M$ be relatively compact smoothly bounded domain. Then the first eigenvalue $\lambda_1 = \lambda_1(\Omega)$ of the problem

$$\Delta u + \lambda u = 0, \quad \textit{on } \Omega$$
$$u = 0 \quad \textit{on } \partial\Omega$$

satisfies

$$(\tau - \sigma)^{-1} \leq \lambda_1. \tag{1}$$

Proof. Let f be a smooth function on Ω and consider the problem

$$\Delta v = f, \quad \text{in } \Omega$$
$$v = 0, \quad \text{on } \partial\Omega.$$

It is well known that the solution is given by

$$-v(x) = \int_\Omega f(y) G(x, y) * 1(y) \tag{2}$$

where $G(x, y)$ is the (positive) Green's function and $*1$ is the volume element of M. In particular when $f \equiv 1$,

$$-\psi(x) = \int_\Omega G(x, y) * 1(y)$$

solves $\Delta\psi = 1$, $\psi|_{\partial\Omega} = 0$. Let S be as above and note that

$$\Delta(S - \psi) \geq 0, \quad \text{in } \Omega$$

and

$$S - \psi = S \leq \tau, \quad \text{on } \partial\Omega$$

Therefore by the maximum principle

$$S - \psi \leq \tau \quad \text{in } \Omega$$

and so

$$-\psi \leq \tau - \sigma \quad \text{in } \Omega.$$

Let u be an eigenfunction belonging to λ_1. It is well known that u is positive in Ω. We have by (2)

$$\begin{aligned} u(x) &= \lambda_1 \int_\Omega u(y) G(x, y) * 1(y) \\ &\leq \lambda (\max_\Omega u) \int_\Omega G(x, y) * 1(y) \\ &\leq \lambda_1 (\max_\Omega u)(\tau - \sigma). \end{aligned}$$

Choosing x_0 such that $u(x_0)$ is a maximum, one obtains (1).

Let M be a spacelike hypersurface of $\mathbb{E}_1^{n+1}$. We will always consider M with the induced Riemannian metric. For $x \in M$ let $B_\rho(x)$ denote the geodesic ball of radius ρ centered at x. Similarly for $\eta \in H^n(-1) \subset \mathbb{E}_1^{n+1}$, $\tilde{B}_\tau(\eta)$ will denote the geodesic ball of radius τ centered at η.

THEOREM II. *Let $M^n \subset \mathbb{E}_1^{n+1}$ be a spacelike hypersurface of constant mean curvature $H \neq 0$. Let $x \in M$ such that $B_\rho(x) \subset M$. Assume that $\eta(M) \subset \tilde{B}_\tau(\bar{\eta})$ for some $\bar{\eta} \in H^n(-1)$. Then $\lambda_1 = \lambda_1(B_\rho(x))$ satisfies*

$$nH^2(\cosh \tau - 1)^{-1} \leq \lambda_1.$$

In order to prove Theorem II, the following lemma is needed.

LEMMA 2. *Let M and η be as above. Then the differential $d\eta$ of η satisfies*

$$\|d\eta\|^2 \geq nH^2.$$

Proof. Using a standard argument, it is easily seen, using the smoothness of M, that $d\eta$ is self-adjoint. Choosing an orthonormal basis at an arbitrary point in M which diagonalizes, we can assume a matrix representation $d\eta =$ diagonal

$(k_1, \ldots, k_n)$ where k_j are the principal curvatures. We have:

$$\|d\eta\|^2 = \sum_j k_j^2 \equiv f(k_j)$$

$$nH = \sum_j k_j \equiv g(k_j)$$

Using, for example, Lagrange multipliers it is easily checked that the minimum of f on the level set $g = nH$ occurs when $k_j = H$ for all j.

Proof of Theorem II

By a result of Ishihara [I] the Gauss map η is a harmonic map of M into $H^n(-1)$. If $\eta = (\eta_1, \ldots, \eta_{n+1})$ then each component satisfies the Euler-Lagrange equation,

$$\Delta\eta_j = \|d\eta\|^2\eta_j. \tag{3}$$

We may assume, by first applying a Lorentz transformation to M if necessary, that $\bar{\eta} = (0, 0, \ldots, 0, 1)$. The assumption $\eta(M) \subset \tilde{B}_\tau(\bar{\eta})$ implies

$$1 \leq \eta_{n+1} \leq \cosh\tau.$$

By (3) we have

$$\Delta\eta_{n+1} = \|d\eta\|^2\,\eta_{n+1} > nH^2\eta_{n+1} > nH^2.$$

Define $s \equiv \eta_{n+1}(nH^2)^{-1}$. Then

$$1 < \Delta s$$

and

$$(nH^2)^{-1} \leq s \leq (nH^2)^{-1}\cosh\tau.$$

So by Lemma 1,

$$(nH^2)(\cosh\tau - 1)^{-1} \leq \lambda_1.$$

Proof of Theorem I

It is well known [CY] that the Ricci curvature of M is bounded below by $(-n^2H^2/4)$. Assume that $B_\rho(x) \subset M$ and $\eta(M) \subset \tilde{B}_\tau(\bar{\eta})$. Then applying the lower bound for λ_1 derived above along with an upper bound for λ_1 due to M. Gage, [G] we obtain the following inequalities:

When $n = 2$,

$$(2H^2)(\cosh \tau - 1)^{-1} \leq \lambda_1 \leq \frac{H^2}{4} + \frac{\pi^2}{\rho^2} - \frac{H^2}{4 \sinh^2 (H\rho)}$$

for $n = 3$

$$(3H^2)(\cosh \tau - 1)^{-1} \leq \lambda_1 \leq \tfrac{9}{8}H^2 + \frac{\pi^2}{\rho^2}$$

and for $n \geq 4$

$$(nH^2)(\cosh \tau - 1)^{-1} \leq \lambda_1 \leq \frac{(n-1)n^2H^2}{16} + \inf_{0<t<1} \left\{ \frac{\pi^2}{(1-t^2)\rho^2} + \frac{(n-1)(n-3)n^2H^2}{16 \sinh^2 (t\rho n(H/2))} \right\}$$

and the theorem follows.

REFERENCES

[C] E. CALABI, *Examples of Bernstein problems for some nonlinear equations*, Proc. Sym. Global Analysis, Univ. of Calif. Berkeley, 1968.

[Ch] S. S. CHERN, *On the curvatures on a piece of hypersurface in Euclidean space*, Abh. Math. Sem. Hamburg, *29*, 1964.

[CY] S. Y. CHENG and S. T. YAU, *Maximal hypersurfaces in the Lorentz-Minkowski spaces*, Annals of Math., *104*, 1976.

[G] M. GAGE, *Upper bounds for the first eigenvalue of the Laplace-Beltrami operator*, Ind. Univ. Math. J., *29*, No. 6, 1980.

[HOS] D. HOFFMAN, R. OSSERMAN and R. SCHOEN, *On the Gauss map of complete surfaces of constant mean curvature in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$, Comment. Math. Helv., *57*, 1982.

[I] T. ISHIHARA, *The harmonic Gauss map in a generalized sense*, J. London Math. Soc., *2*, No. 26, 1982.

[P] B. PALMER, *Constant mean curvature surfaces in Minkowski space,* preprint.
[T] A. E. TREIBERGS, *Entire spacelike hypersurfaces of constant mean curvature in Minkowski space,* Invent. Math., *66*, 1982.

The author would like to call the readers' attention to a recent paper of H. I. Choi and A. Treibergs, with the same title as the present one, in which the Gauss map of an entire spacelike surface is studied using its ideal boundary.

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