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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **66 (1991)**

PDF erstellt am: **23.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-50395

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Comment. Math. Helvetici 66 (1991) 150–154 0010–2571/91/010150–05\$1.50 + 0.20/0
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The semiregular polytopes

G. Blind and R. Blind

Abstract. A convex d-polytype in E^d is called semiregular, if its facets are regular and its vertices equivalent. A list of semiregular polytopes for $d \geq 4$ is known since 1900. Recently it has been proved by П. В. Макаров [сб. Вопр. дискр. геом., Мат. исслед. ИМ АН Молд. ССР вып. 103, С. 139–150, Кишинев 1988], that this list is complete for $d = 4$. We present here a simple proof for that this list is complete in any dimension.

1. Introduction

Let P be a convex d-polytope in E^d. P is called **regular-faced**, if all its facets are regular. P is called **semiregular,** if its facets are regular and if its vertices are equivalent (i.e. the group of symmetries of P acts transitively on the vertices of P). Clearly, every semiregular polytope is regular-faced.

For $d = 3$ the semiregular polytopes coincide with the Archimedean solids. For $d \geq 4$ Gosset established a list of semiregular polytopes in [6]. Various constructions of semiregular polytopes have been described since, but all the time it stayed open whether Gosset's enumeration is complete (see [7, p. 413]). Only in 1988 there is published a proof [8], that Gosset indeed found all the semiregular d-polytopes for $d = 4$. This proof uses the complete enumeration of regular-faced 3-polytopes in [9]; the vertex-figure of a semiregular polytope is then carefully examined.

In [1], [2] and [3] we completely enumerated the regular-faced d-polytopes for $d \geq 4$; a wide class of them, however, is only given by a method how to construct them. We shall show, how to deduce from this list of all regular-faced polytopes

THEOREM 1. *Gosset's list of semiregular d-polytopes is complete for d* ≥ 4.

2. Gosset' list of semiregular d-polytopes for d $\geq$ 4

Besides the regular polytopes Gosset found 7 semiregular polytopes, namely 3 for $d = 4$ and always one for $d = 5, \ldots, 8$. They are described in [6] by the sets of their facets together with some incidence properties of the following type

d	name in [6]	sets of facets	a $(d-3)$-face is contained in
4	tetroctahedric	5 octahedra 5 tetrahedra	2 octahedra 1 tetrahedron
	tetricosahedric	24 icosahedra 120 tetrahedra	$\left.\begin{array}{l}\text{2 icosahedra}\\ \text{1 tetrahedron}\end{array}\right\}$ or $\left\{\begin{array}{l}\text{1 icosahedron}\\ \text{3 tetrahedra}\end{array}\right.$
	octicosahedric	120 icosahedra 600 octahedra	1 icosahedron 2 octahedra
$5, \ldots, 8$		$(d-1)$-crosspolytopes $(d-1)$-simplices	2 $(d-1)$-crosspolytopes 1 $(d-1)$-simplex

3. Regular-faced d-polytopes for d ≥ 4

In [1], [2] and [3] we showed, that for $d \geq 4$ the following list of regular-faced d-polytopes is complete:

for $d \geq 5$: the regular polytopes,
the polytopes of Gosset's list,
the pyramid with basis a regular $(d-1)$-crosspolytope, and
the bipyramid with basis a regular $(d-1)$-simplex.

for $d = 4$: the regular polytopes,
the polytopes of Gosset's list,
the pyramid with basis an octahedron,
the bipyramid with basis a tetrahedron,
the pyramid and the bipyramid with basis an icosahedron,
the union of a tetroctahedron (see Gosset's list) and of a pyramid, whose basis is an octahedric facet of the tetroctahedron, and
the set $\mathscr{A}$ of polytopes arising from the regular 600-cell by cutting off vertices in the following way:

Let Z be the 600-cell and let $\{E_i\}_{i=1}^k$ be a set of vertices of Z, such that no two vertices of $\{E_i\}_{i=1}^k$ are adjacent, i.e. joined by an edge of Z. Since Z is regular, for every E_i the vertices adjacent to E_i are contained in a hyperplane, which determines a closed halfspace $H(E_i)$ not containing E_i. Then $Z \cap \cap_{i=1}^k H(E_i)$ is a convex polytope, which is regular-faced, because Z is simplicial and no two vertices of $\{E_i\}_{i=1}^k$ are adjacent. We say that $Z \cap \cap_{i=1}^k H(E_i)$ arises from Z by **maximally cutting off the vertices of** $\{E_i\}_{i=1}^k$. Thus the set $\mathscr{A}$ is the set of those regular-faced polytopes which arise from the 600-cell by maximally cutting off a set of vertices, where every two vertices are non-adjacent.

Let us remark that analogously we may maximally cut off a suitable set of vertices from the icosahedron.

4. Proof of the Theorem

Let P be a semiregular polytope, which is not regular and not contained in Gosset's list. Since P is regular-faced and since the vertices of P are equivalent, we immediately deduce from the list of all regular-faced polytopes that P is 4-dimensional and that $P \in \mathscr{A}$.

Now let P be any semiregular polytope with $P \in \mathscr{A}$. Let P arise from Z by maximally cutting off $k > 0$ vertices of Z. Then, since Z has 120 vertices, the number e of vertices of P is given by

$$(*) \quad e + k = 120.$$

The set of facets of P is a set of tetrahedra and exactly k icosahedra. Let a vertex E of P be incident with $m \geq 0$ icosahedric facets. Then the vertex-figure of P at E is a 3-polytope arising from an icosahedron by maximally cutting off m vertices where every two are non-adjacent. But such 3-polytopes exist only if $m \leq 3$. We remark that for $m = 3$ there exists only one such 3-polytope: its 3 pentagonal faces do not have a common vertex, but every two of them have a common edge; so in case $m = 3$ the 3 icosahedral facets through E do not have a common edge, but every two of them have a common 2-face.

The vertices of P are equivalent, hence every vertex of P is contained in m icosahedric facets; thus $m \geq 1$. Moreover, every icosahedric facet contains 12 vertices of P, so $e = 12k/m$. From this and (*) follows

$$\frac{12+m}{m}k = 120, \qquad 1 \leq m \leq 3.$$

The only integer solution of this equality is $m = 3$, $k = 24$; so we have $m = 3$.

Now let E be a given vertex of P and let I, I_1, I_2 be the $m = 3$ icosahedric facets through E. Then by the previous remark $I \cap I_1 \cap I_2$ is not an edge of I, but $I \cap I_1$ and $I \cap I_2$ are 2-faces of I. Thus, if $\mathscr{F}$ is the set of those 2-faces of I, which are also contained in another icosahedric facet, then no two elements of $\mathscr{F}$ have a common edge, but every vertex of I is contained in exactly two elements of $\mathscr{F}$. From this it is easily seen that $\mathscr{F}$ is uniquely determined for given $I \cap I_1$ and $I \cap I_2$, and so are the facets intersecting I in a 2-face. Proceeding in this way we see that P is uniquely determined for given I, I_1 and I_2.

Three icosahedric facets through a vertex correspond to vertices of Z, which are uniquely determined up to isometries of Z by the previous remark. So there exists at most one semiregular polytope in $\mathscr{A}$.

The tetricosahedric polytope of Gosset's list is known to be in $\mathscr{A}$ (see e.g. [5, p. 152f.]). Hence it is the only semiregular polytope in $\mathscr{A}$, which concludes the proof.

5. Concluding remarks

1. We tried to make the proof as self-contained as possible. The result would also follow immediately from $k = 24$ using Theorem 2 in [4].

2. It is well known (see e.g. [7, p. 413]) that if the definition of semi-regularity is slightly changed by substituting 'all vertex figures are congruent' for the transitivity of the symmetry group, then the Archimedean solids are no more the only 3-polytopes allowed, but there exists exactly one additional polytope. Further additional 3-polytopes exist, if instead of the congruence of the vertex figures we assume only, that every vertex is contained in the same number of facets (see [9]*). This is in contrast to the situation in higher dimensions:

In the proof of our Theorem we did not really use the transitivity of the symmetry group of a semiregular polytope, but only the fact that every vertex is contained in the same number of facets. Thus we have

THEOREM 2. *A regular-faced d-polytope, where every vertex is contained in the same number of facets, is semiregular for* $d \geq 4$.

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* e.g. $M_6 + M_9$ is such a polytope; note that thus it may happen, that some vertices of such a polytope are contained in a quadrangle and others are not.

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Received January 29, 1990