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Note on phantom phenomena and groups of self-homotopy equivalences

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§1. Introduction

The group $\operatorname{Aut}(X)$ of (pointed) homotopy classes of self-homotopy equivalences of a (pointed) space X has been extensively studied by homotopy-theorists. For a summary of results about $\operatorname{Aut}(X)$, see the survey article by Arkowitz [A].

Typically, computations of and general results about $\operatorname{Aut}(X)$ have been given for spaces X which are either finite-dimensional cell complexes or finite Postnikov spaces. In this note, we focus on the hybrid space $X = K(\mathbf{Z}, 2) \times S^3$ and obtain a complete computation of $\operatorname{Aut}(K(\mathbf{Z}, 2) \times S^3)$.[1]

It is convenient to study $\operatorname{Aut}(X)$ by placing it in a short exact sequence

$$WI(X) \rightarrowtail \operatorname{Aut}(X) \twoheadrightarrow \operatorname{Aut}(X)/WI(X), \tag{1.1}$$

where $WI(X)$ is the normal subgroup of $\operatorname{Aut}(X)$ consisting of the weak identities of X (see $[R_1]$). A complete analysis of (1.1) in the case $X = K(\mathbf{Z}, 2) \times S^3$ is possible thanks to results of Zabrodsky [Z], the author $[R_1]$ and Hopkins [H]. The analysis is interesting both because of the structure of the "phantom-like" subgroup $WI(K(\mathbf{Z}, 2) \times S^3)$ and the interaction of this subgroup with the rather pedestrian quotient group

$$\operatorname{Aut}(K(\mathbf{Z}, 2) \times S^3)/WI(K(\mathbf{Z}, 2) \times S^3).$$

It turns out, in fact, that

$$WI(K(\mathbf{Z}, 2) \times S^3) \cong \mathbf{R},$$

$$\operatorname{Aut}(K(\mathbf{Z}, 2) \times S^3)/WI(K(\mathbf{Z}, 2) \times S^3) \cong \mathbf{Z}/2 \times \mathbf{Z}/2$$

[1] Partial information on $\operatorname{Aut}(K(\mathbf{Z}, 2) \times S^3)$ was obtained in $[R_1]$.

and that Aut $(K(\mathbf{Z}, 2) \times S^3)$ is the semi-direct product of $\mathbf{R}$ and $\mathbf{Z}/2 \times \mathbf{Z}/2$ with respect to an action – made explicit in §2 – of $\mathbf{Z}/2 \times \mathbf{Z}/2$ on $\mathbf{R}$.

The rest of the note is organized as follows. §2 contains the details of the computation of Aut $(K(\mathbf{Z}, 2) \times S^3)$. Results of Hopkins on phantom maps ([H]), relevant to the computation in §2, suggest a certain direction for studying the group-theoretic structure of Aut (X). §3 consists then of a more or less random walk in this direction, leading to speculations about residual properties of Aut (X) and $\mathrm{Aut}_1(X)$, the subgroup of Aut (X) consisting of those self-equivalences[2] inducing the identity on all homotopy groups, at least when X is grouplike.

A preliminary version of the main result of this note may be found in [R_3].

I would like to thank Martin Bendersky for some helpful discussions of the material contained herein and Gilbert Baumslag for some useful group-theoretic information. I am also grateful to the Department of Mathematics of the University of Rochester for providing such a stimulating atmosphere while this note was being completed during the Fall 1990 semester.

§2. Computation of Aut $(K(\mathbf{Z}, 2) \times S^3)$

Given (pointed) spaces X, Y, we follow [R_1] in writing $Ph(X, Y)$ for the (pointed) homotopy classes of phantom maps from X to Y and $Ph(X)$ for $Ph(X, X)$. As in [R_1], we require all spaces to be nilpotent of finite type and with finite fundamental group. If $X = U \times V$ and Y is grouplike, there is a short exact sequence of groups

$$[U \wedge V, Y] \rightarrowtail [U \times V, Y] \twoheadrightarrow [U \vee V, Y]. \tag{2.1}$$

Hopkins observes ([H; Cor. 1.4]) that an element in $[U \times V, Y]$ lies in the normal subgroup $Ph(U \times V, Y)$ if and only if its "components" in $[U \vee V, Y]$, $[U \wedge V, Y]$ lie in $Ph(U \vee V, Y)$, $Ph(U \wedge V, Y)$ respectively.

Abbreviating

$$K = K(\mathbf{Z}, 2), \qquad S = S^3,$$

we begin our study of Aut $(K \times S)$ by examining the short exact sequence of groups

$$Ph(K \times S) \rightarrowtail [K \times S, K \times S] \twoheadrightarrow [K \times S, K \times S]/Ph(K \times S) \tag{2.2}$$

[2]As is customary, we often blur the distinction between a map and its homotopy class.

deriving from the standard grouplike structure on $K \times S$, which we write additively. Now,

$$\begin{aligned}[K \vee S, K \times S] &\cong [K, K \times S] \times [S, K \times S] \\ &\cong [K, K] \times [K, S] \times [S, S] \\ &\cong \mathbf{Z} \times [K, S] \times \mathbf{Z}; \end{aligned} \tag{2.3}$$

while

$$\begin{aligned}[K \wedge S, K \times S] &\cong [K \wedge S, K] \times [K \wedge S, S] \\ &\cong [K \wedge S, S]. \end{aligned} \tag{2.4}$$

According to [Z; Th.D],

$$\begin{aligned}[K, S] &\cong Ph(K, S) \cong \operatorname{Ext}(\mathbf{Q}, \mathbf{Z}) \cong \mathbf{R}, \\ [K \wedge S, S] &\cong Ph(K \wedge S, S) = 0. \end{aligned} \tag{2.5}$$

Combining (2.1), (2.3), (2.4) and (2.5), we infer that (2.2) reduces to the split short exact sequence

$$\mathbf{R} \rightarrowtail [K \times S, K \times S] \twoheadrightarrow \mathbf{Z} \times \mathbf{Z} \tag{2.6}$$

with trivial action of $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{R}$. For future use, we call attention to the isomorphism $Ph(K, S) \cong Ph(K \times S)(\cong \mathbf{R})$ which associates to φ in $Ph(K, S)$ the element Φ in $Ph(K \times S)$ defined by

$$\Phi : (k, s) \rightarrow (0, \varphi(k)). \tag{2.7}$$

In terms of the splitting (2.6), $\operatorname{Aut}(K \times S)$ is characterized as the subset of $[K \times S, K \times S]$ having each of its $\mathbf{Z}$-components equal to ± 1. Let C be the subgroup of $\operatorname{Aut}(K \times S)$ generated by the automorphisms ξ, η defined by

$$\xi : (k, s) \rightarrow (k, -s), \qquad \eta : (k, s) \rightarrow (-k, s).$$

In terms of the splitting (2.6), the $\mathbf{Z}$-components of ξ and η are $(1, -1)$ and $(-1, 1)$ respectively. Plainly,

$$C \cong \mathbf{Z}/2 \times \mathbf{Z}/2.$$

Our main result is the following:

THEOREM 2.1. Aut $(K \times S)$ *is the semidirect product of the normal subgroup* $WI(K \times S) \cong \mathbf{R}$ *and the subgroup* C, *with the (conjugation) action of* C *on* $WI(K \times S)$ *given by*

$$\xi * w = w^{-1} = \eta * w, \qquad w \in WI(K \times S).$$

Proof. It was proved in [R_1; Th.3.1] that for any grouplike space X, the map $Ph(X) \to WI(X)$ defined by

$$\Phi \to \Phi + 1_X$$

is an isomorphism of groups.[3] Thus, $WI(K \times S) \cong Ph(K \times S) \cong \mathbf{R}$ and any $w \in WI(K \times S)$ has the form

$$w : (k, s) \to (k, \varphi(k) + s) \tag{2.8}$$

for an element φ in $Ph(K, S)$ uniquely determined by w (see (2.7)). In terms of the splitting (2.6), $WI(K \times S)$ is therefore characterized as the subset of $[K \times S, K \times S]$ having each of its $\mathbf{Z}$-components equal to $+1$. In other words, $WI(K \times S)$ coincides with $\mathrm{Aut}_1(K \times S)$ – see the penultimate paragraph of §1 for the definition of the latter.

For α in Aut $(K \times S)$, there exists γ in C such that $\alpha \circ \gamma$ lies in $\mathrm{Aut}_1(K \times S) = WI(K \times S)$; moreover, γ is uniquely determined by α. We may therefore write

$$\alpha = w \circ \gamma, \qquad w \in WI(K \times S), \qquad \gamma \in C$$

with both w and γ uniquely determined by α, and so Aut $(K \times S)$ is, indeed, the semidirect product of $WI(K \times S)$ and C. It remains to establish that

$$\xi \circ w \circ \xi = w^{-1} = \eta \circ w \circ \eta, w \in WI(K \times S).$$

[3] The question of whether the map $Ph(X) \to WI(X)$ defined by $\Phi \to 1_X + \Phi$ is an isomorphism of groups when X is a cogroup was raised in [R_2]. This question has now been settled in the affirmative by Touhey [T].

First we compute

$$\begin{aligned}\xi \circ w \circ \xi : (k, s) &\to (k, -s)\\ &\to (k, \varphi(k) - s) \text{ (by (2.8))}\\ &\to (k, -(\varphi(k) - s)).\end{aligned}$$

But the map

$$(k, s) \to (k, -(\varphi(k) - s))$$

is homotopic to the map

$$(k, s) \to (k, s - \varphi(k))$$

which, in turn, is homotopic to the map

$$w^{-1} : (k, s) \to (k, -\varphi(k) + s)$$

as $[K \times S, K \times S]$ is abelian. [It would be sufficient to know that $Ph(K \times S)$ is a central subgroup of $[K \times S, K \times S]$.]

Next we compute

$$\begin{aligned}\eta \circ w \circ \eta : (k, s) &\to (-k, s)\\ &\to (-k, \varphi(-k) + s)\\ &\to (-(-k), \varphi(-k) + s),\end{aligned}$$

which is homotopic to the map

$$(k, s) \to (k, \varphi(-k) + s).$$

Allowing for a moment that φ is an H-map, we infer that the latter map is homotopic to the map

$$w^{-1} : (k, s) \to (k, -\varphi(k) + s).$$

The following lemma then completes the proof of Theorem 2.1.

LEMMA 2.1. *All elements of $Ph(K, S)$ are represented by H-maps, no matter which H-space structure is used on S.*

Proof. Given φ in $Ph(K, S)$, the unique multiplication m_K on K and any multiplication m_S on S, we must prove that $\varphi \circ m_K$ and $m_S \circ (\varphi \times \varphi)$ – both of which are clearly phantoms – are equal as elements of $[K \times K, S]$. To this end, it suffices to show that the components of $\varphi \circ m_K$ in $[K \vee K, S]$ and $[K \wedge K, S]$ coincide with the corresponding components of $m_S \circ (\varphi \times \varphi)$, bearing in mind that the components of both $\varphi \circ m_K$ and $m_S \circ (\varphi \times \varphi)$ are phantoms (see (2.1) et seq). That the components of $\varphi \circ m_K$ and $m_S \circ (\varphi \times \varphi)$ in $[K \vee K, S]$ coincide is evident since m_K and m_S are multiplications. On the other hand,

$$Ph(K \wedge K, S) = 0$$

by [Z; Th.D] since $H_2(K \wedge K; \mathbb{Q}) = 0$. Hence the components of $\varphi \circ m_K$ and $m_S \circ (\varphi \times \varphi)$ in $[K \wedge K, S]$ are both 0, and so coincide.

For reference in §3, we record a corollary to Theorem 2.1.

COROLLARY 2.1. Aut $(K \times S)$, *while solvable, is not residually nilpotent.*

Proof. To see that Aut $(K \times S)$ is not residually nilpotent, it suffices to show that the intersection of the terms $\Gamma_N(\mathrm{Aut}\,(K \times S))$, $N \geq 1$, in the lower central series of Aut $(K \times S)$ is non-trivial. From Theorem 2.1,

$$\Gamma_N(\mathrm{Aut}\,(K \times S)) = WI(K \times S), N \geq 1$$

and Corollary 2.1 is established.

§3. Aut (X) and residual properties

The point of departure for the discussion in this section is the short exact sequence of groups

$$Ph(X, Y) \rightarrowtail [X, Y] \twoheadrightarrow \varprojlim\, [X_\alpha, Y], \tag{3.1}$$

where X_α runs over the finite (connected) subcomplexes of the cell complex X and Y is grouplike.[4]

Hopkins remarks ([H; proof of Prop. 1.1]) that since each $[X_\alpha, Y]$ is nilpotent (according to a classical theorem of G. W. Whitehead), it is clear that $\varprojlim\, [X_\alpha, Y]$

[4]If Y is not grouplike, (3.1) is still a short exact sequence of sets.

is residually nilpotent. He then argues that, provided Y has the homotopy type of a finite cell complex, $Ph(X, Y)$ intersects the N^{th} term of the lower central series of $[X, Y]$ trivially for sufficiently large N (and therefore that $[X, Y]$ is residually nilpotent) but his argument seems to have a flaw. Nevertheless, later in [H], Hopkins conjectures that, again provided Y has the homotopy type of a finite cell complex, $[X, Y]$ is nilpotent – not merely residually nilpotent – and establishes this conjecture in many cases, for instance when the integral homology of Y is torsion-free ([H; Cor. 2.2]).[5]

More generally, if each $[X_\alpha, Y]$ possesses some group-theoretic property $\mathscr{P}$, then $\varprojlim [X_\alpha, Y]$ is residually $\mathscr{P}$ and $Ph(X, Y)$ is the sole possible obstruction to $[X, Y]$ being residually $\mathscr{P}$. As an example, referring to §2, $Ph(K, S)$ is the (genuine!) obstruction to the residual finiteness of $[K, S]$. This example illustrates

THEOREM 3.1. *$[X, Y]$ is residually finite $\Leftrightarrow Ph(X, Y) = 0$.*

Proof. We know, by the already-cited theorem of G. W. Whitehead, that $[X_\alpha, Y]$ is (finitely generated) nilpotent, hence residually finite (see, e.g., [B; Cor. 1.21]). It is then clear that $\varprojlim [X_\alpha, Y]$ is residually finite.

To complete the proof, it suffices to observe that a non-0 element in the (divisible) group $Ph(X, Y)$ cannot be detected in a finite homomorphic image of $[X, Y]$.

In a similar vein, the following question may be posed.

QUESTION 3.1. Is $[X, Y]$ residually finitely generated (or residually finitely presented) $\Leftrightarrow Ph(X, Y) = 0$.

The implications $\Leftarrow$ are certainly valid. However, finitely presented groups and, a fortiori, finitely generated groups, have complicated subgroup structures – for instance, any countable, abelian, divisible group embeds in some finitely presented group – so it is conceivable that the implications $\Rightarrow$ fail.

We now seek analogues of the foregoing for the group $\operatorname{Aut}(X)$ and begin by noting a variant of (3.1). For convenience, we impose the technical assumption that X has a cell complex structure for which each n-skeleton X_n is a finite cell complex.

PROPOSITION 3.1. *There is a short exact sequence of groups*

$$Ph(X, Y) \rightarrowtail [X, Y] \overset{P_*}{\twoheadrightarrow} \varprojlim [P_n X, P_n Y], \tag{3.1'}$$

[5] Added later: The conjecture is false in general. See V. K. Rao, $SO(n)$ is not homotopy nilpotent for $n \neq 2^m - 3,\ 2^m - 2$ (preprint).

where P_nX, P_nY are the n^{th} Postnikov approximations of X, Y and P_* is the epimorphism induced by associating to a map $f: X \to Y$ the compatible family of induced maps $P_nf: P_nX \to P_nY$.

Proof. We show that $Ph(X, Y) = \ker P_*$. First, let $f: X \to Y$ be such that $P_nf: P_nX \to P_nY$ is trivial for all n. From the commutative diagram

$$\begin{array}{ccccc} X_n & \hookrightarrow & X & \xrightarrow{f} & Y \\ & & \downarrow & & \downarrow \\ & & P_mX & \xrightarrow{P_mf} & P_mY, \end{array}$$

and the fact that $[X_n, Y] \xrightarrow{\cong} [X_n, P_mY]$ for $m \geq n$, we infer that $f|X_n$ is trivial for all n, hence that f is phantom.

Conversely, let $f: X \to Y$ be phantom and consider (3.2) for $m = n - 1$. If C is the cofiber of $X_n \hookrightarrow P_{n-1}X$, we have an exact sequence of groups

$$[C, P_{n-1}Y] \to [P_{n-1}X, P_{n-1}Y] \to [X_n, P_{n-1}Y].$$

Since the image of $P_{n-1}f$ in $[X_n, P_{n-1}Y]$ is trivial and since $[C, P_{n-1}Y]$ is trivial, we infer that $P_{n-1}f$ is trivial for all n, hence that P_*f is trivial.

Setting $X = Y$ in Proposition 3.1, we have

COROLLARY 3.1. *There is a short exact sequence of groups*

$$Ph(X) \rightarrowtail [X, X] \twoheadrightarrow \varprojlim [P_nX, P_nX].$$

An analogue of Corollary 3.1 for the group Aut (X) is readily available, namely

PROPOSITION 3.2. *There is a short exact sequence of groups*

$$WI(X) \rightarrowtail \operatorname{Aut}(X) \twoheadrightarrow \varprojlim \operatorname{Aut}(P_nX).$$

The proof of Proposition 3.2 is similar to that of Proposition 3.1 and is omitted. Note the contrast of the short exact sequence in Proposition 3.2 with that in (1.1).

To what extent does $WI(X)$ obstruct Aut (X) from satisfying a group-theoretic property $\mathscr{P}$ residually when each Aut (P_nX) satisfies $\mathscr{P}$?

The situation for residual finiteness is as follows. Here we need not bother to determine whether Aut (P_nX) is residually finite (it is) as we may simply appeal to [R_1; Th.3.2] to conclude the following exact analogue of Theorem 3.1.

THEOREM 3.2. $\mathrm{Aut}(X)$ *is residually finite* $\Leftrightarrow$ $WI(X)$ *is trivial.*

With regard to finite presentability, we recall that for any (not necessarily grouplike) X, a theorem of Wilkerson [W] and Sullivan [Su] implies that $\mathrm{Aut}(P_nX)$ is finitely presented. We ask

QUESTION 3.2. Is $\mathrm{Aut}(X)$ residually finitely presented $\Leftrightarrow$ $WI(X)$ is trivial?

The same caveat issued following Question 3.1 applies to Question 3.2.

Turning next to nilpotence, there does not seem to be an issue since, ordinarily, $\mathrm{Aut}(X)$ is far from being nilpotent even for X a finite cell complex or a finite Postnikov space. Corollary 2.1 shows that $\mathrm{Aut}(X)$ need not be residually nilpotent even if it accidentally occurs that $\varprojlim \mathrm{Aut}(P_nX)$ is nilpotent. Therefore, we shift attention to the subgroup, $\mathrm{Aut}_1(X)$, of $\mathrm{Aut}(X)$ consisting of those self-equivalences inducing the identity on all homotopy groups. For any (not necessarily grouplike) X, a theorem of Dror–Zabrodsky [DZ] asserts that $\mathrm{Aut}_1(P_nX)$ is nilpotent. We ask

QUESTION 3.3. Is $\mathrm{Aut}_1(X)$ residually nilpotent?

We point out a (very tenuous) link between the situations in [H] and Question 3.3. Let $[X, Y]_0$ denote the subgroup of $[X, Y]$ consisting of those f in $[X, Y]$ inducing the zero map on all homotopy groups. The short exact sequences in Proposition 3.1, Corollary 3.1 and Proposition 3.2 plainly all have versions in which $[\,,\,]$ is replaced by $[\,,\,]_0$ and $\mathrm{Aut}(\ )$ by $\mathrm{Aut}_1(\ )$. Now the map $[X, X]_0 \to \mathrm{Aut}_1(X)$ defined by

$$f \mapsto f + 1_X \tag{3.3}$$

is bijective; indeed there is a commutative diagram

$$\begin{array}{ccccc} Ph(X) & \rightarrowtail & [X, X]_0 & \twoheadrightarrow & \varprojlim [P_nX, P_nX]_0 \\ \downarrow & & \downarrow & & \downarrow \\ WI(X) & \rightarrowtail & \mathrm{Aut}_1(X) & \twoheadrightarrow & \varprojlim \mathrm{Aut}_1(P_nX) \end{array} \tag{3.4}$$

with the three vertical maps induced by (3.3), hence bijective. However, we emphasize that only the leftmost vertical map in (3.4) is asserted to be an isomorphism of groups.

A more conservative version of Question 3.3 would be

QUESTION 3.4. Is $\mathrm{Aut}_1(X)$ residually solvable?

Though Question 3.4 seems much easier to handle than Question 3.3 – the class of solvable groups being closed under group extensions (cf. Corollary 2.1) – it must be pointed out that there are examples of non-residually solvable groups G such that G admits a residually solvable quotient group with abelian kernel.

FINAL REMARK. I am informed by C. A. McGibbon and J. M. Møller that Proposition 3.2 and the fact that $WI(X)$ is abelian, divisible follow from results of A. K. Bousfield/D. M. Kan and C. U. Jensen, even without assuming X grouplike. Thus Theorem 3.2 and Questions 3.2, 3.3 and 3.4 may be formulated without this assumption on X. However (3.3), and hence (3.4), makes no sense without some sort of structure (grouplike, cogroup) on X.

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