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Simplicial approximation and low-rank trees

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Introduction

In [GS] a structure theorem is proved for certain group actions on Λ-trees, where Λ is a subgroup of $\mathbb{R}$. When Λ has $\mathbb{Q}$-rank 1, the theorem applies to an arbitrary action of a group Γ on a Λ-tree; it also applies when the $\mathbb{Q}$-rank is 2, provided that the action satisfies an "Ascending Chain Condition" [GS, 6.1] for segment stabilizers. The ascending chain condition holds if the action is free, and it holds in the especially important case where the action is small and the small subgroups of Γ are all finitely generated. (A group is *small* if it has no rank-2 free subgroup; an action on a tree is *small* if the stabilizers of non-degenerate segments are all small.)

It is shown in [GS] that by applying the structure theorem one can largely solve, in the case of $\mathbb{Q}$-rank ≤ 2, the problems about actions on Λ-trees ($\Lambda \leq \mathbb{R}$) raised in [Sh] and motivated by questions about degeneration of hyperbolic structures or by Lyndon's conjecture.

There are two questions that are not addressed in [GS] but are closely related to the questions answered there. An action without inversions of a finitely generated group Γ on a Λ-tree T defines a (translation) length function ℓ on Γ taking non-negative values in Λ. It is an open problem whether the given action can always be "simplicially approximated", in the sense that there is a sequence $(\ell_i)_{i \geq 0}$ of length functions defined by actions of Γ on $\mathbb{Z}$-trees, and a sequence $(n_i)_{i \geq 0}$ of positive integers, such that $\lim_{i \to \infty} \ell_i(\gamma)/n_i = \ell(\gamma)$ for every $\gamma \in \Gamma$. (In the language of [MS] or [CM] this says that the projectivized length function defined by the given action is in the closure of the set of projectivized length functions defined by simplicial actions.) A second question arises in the case that the given action is small: can one take the approximating length functions ℓ_i to be defined by small simplicial actions?

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The main result of this paper, Theorem 5.1, gives affirmative answers to these questions when Γ is finitely presented and Λ has $\mathbb{Q}$-rank at most 2, assuming, in the rank-2 case, that the action satisfies the ascending chain condition. In particular, Theorem 5.1 implies (Corollary 5.2) that the second question has an affirmative answer if Λ has $\mathbb{Q}$-rank at most 2 and the small subgroups of Λ are finitely generated. We also observe (Theorem 5.3) that the results remain true if Γ is assumed to be finitely generated, rather than finitely presented, but the given action is assumed to be free.

Of course the proofs of Theorems 5.1 and 5.3 rely heavily on the structure theorem of [GS]. On the other hand, Theorems 5.1 and 5.3 can be interpreted (via the Bass–Serre theory [Se]) as partial refinements of Theorems A, C and D of the introduction to [GS]. Likewise, Theorem B of the introduction to [GS] can easily be derived from Theorem 5.1 and the fact (see [FLP]) that the projective measured foliation space of a surface is compact. (Using a very different method, Skora has recently generalized Theorem B of [GS] to all subgroups of $\mathbb{R}$.)

The starting point for the proof of Theorems 5.1 and 5.3 is the observation that the structure theorem of [GS] allows one to approximate arbitrary actions satisfying the hypotheses of the theorem by actions which are "geometric" in the sense that they arise from actions on measured foliated singular surfaces. We prove a general result (Theorem 1.3) which applies in all $\mathbb{Q}$-ranks and allows one to approximate geometric actions by simplicial ones. This result is a refinement of Thurston's theorem that measured foliations of surfaces admit rational approximations. Theorem 1.3 seems to be of independent interest; it could play a role in generalizing Theorem 5.1 to the higher-rank case.

Section 1 contains some preliminary definitions and the statement of Theorem 1.3. The proof of the theorem is divided into a "soft" step, done in Section 3, and a "hard" step which occupies Section 4. The soft step is analogous to the observation that a measured foliation can be rationally approximated in terms of train-track coordinates; the hard step involves showing that convergence of generalized train track coordinates actually implies convergence of length functions. To prove that our approximation procedure preserves "smallness", we need a simple characterization of small geometric actions which is established in Section 2. In Section 5 we combine Theorem 1.3 with the results of [GS] to prove Theorems 5.1 and 5.3.

We assume that the reader is familiar with the paper [GS], to which the present paper is a sequel. We have given specific references to [GS] wherever appropriate, but have included relatively little explicit review. In the very few places where we depart from the terminology of the earlier paper we have been careful to explain the differences.

1. Simplicial approximation of geometric length functions: the statement

1.1. In the following discussion, Λ will denote a subgroup of $\mathbb{R}$, and Γ will denote an arbitrary group.

Suppose that we are given a 1-connected singular surface Σ (see [GS, 5.1]) and an action ρ of Γ on Σ by simplicial automorphisms. Let μ be a Λ-foliation [GS, 5.21] of Σ which is invariant under the action ρ. Then $(\Sigma, \mu, \Gamma, \rho)$ is by definition a 1-connected Λ-foliated singular surface with simplicial symmetry. Suppose in addition that $(\Sigma, \mu, \Gamma, \rho)$ is *uniform* in the sense that Σ is finite mod Γ [GS, 3.2 and 5.25]. Recall from [GS, 5.28] that $(\Sigma, \mu, \Gamma, \rho)$ has a well-defined leaf space $(T, \Gamma, \bar{\rho})$ which is a Λ-tree with symmetry; by definition this means that T is a Λ-tree and that $\bar{\rho}$ is an action of Γ on T.

We shall say that an action $\bar{\rho}$ of Γ on a Λ-tree is *geometric* if there exist a 1-connected singular surface Σ, an action ρ of Γ by simplicial automorphisms such that Σ is finite mod Γ, and an invariant Λ-foliation μ of Σ, such that $(T, \Gamma, \bar{\rho})$ is isomorphic to the leaf space of $(\Sigma, \mu, \Gamma, \rho)$.

As in [GS] we say that a group is *small* if it has no free subgroup of rank 2. Following the terminology of [CM], we shall say that an action of a group Γ on a Λ-tree is *small* if the stabilizer of every non-degenerate segment of T is a small subgroup of Γ.

1.2. A non-negative-valued function $\ell : \Gamma \to \Lambda$ will be called a *Λ-length function* if it is the length function [GS, 1.16] determined by some action without inversions of Γ on a Λ-tree. Note that a Λ-length function is in particular a non-negative real-valued function on Γ. It follows from [AB, 6.15–6.16] that a Λ-length function is in fact an $\mathbb{R}$-length function. On the other hand, it follows from [AB, Theorem 7.13(c)] that an $\mathbb{R}$-length function that takes values in Λ is a $\frac{1}{2}\Lambda$-length function. An $\mathbb{R}$-length function will often be called simply a *length function*.

A Λ-length function will be termed *geometric* (or *small*) if it is the length function determined by some geometric (or, respectively, small) action of Γ on a Λ-tree.

It follows from [CM, Theorem 3.7] that if ℓ is a small length function on a group Γ, then *any* action of Γ on a Λ-tree which determines the length function ℓ is a small action. In particular, a length function is small and geometric if and only if it is determined by some small geometric action on a Λ-tree.

1.3. The proof of the following theorem occupies the next three sections.

THEOREM. *Let ℓ be a geometric $\mathbb{R}$-length function on a group Γ. Then there exist a sequence (ℓ_i) of geometric $\mathbb{Z}$-length functions on Γ and a sequence (n_i) of* *positive integers, such that* $\lim_{i\to\infty}(\ell_i(g)/n_i) = \ell(g)$ *for every* $g \in \Gamma$. *Furthermore, if the geometric* $\mathbb{R}$*-length function* ℓ *is small then the geometric* $\mathbb{Z}$*-length functions* ℓ_i *may be taken to be small as well.*

In Section 3 we introduce the basic machinery used in the proof of Theorem 1.3, and reduce the proof to a technical result – Proposition 3.12 – which is proved in Section 4. The proof of the last sentence of Theorem 1.3 will involve a characterization of small geometric actions which is proved in Section 2.

2. A characterization of small geometric actions

In this section we establish a simple criterion for a geometric action of a group on an $\mathbb{R}$-tree, defined by a given action on an $\mathbb{R}$-foliated singular surface, to be small. The condition does not depend on the foliation, but only on the underlying action of the group on the singular surface.

2.1. PROPOSITION. *Let* $(\Sigma, \mu, \Gamma, \rho)$ *be a uniform* 1*-connected* $\mathbb{R}$*-foliated singular surface with simplicial symmetry. Let* $(T, \Gamma, \bar{\rho})$ *denote the leaf space of* $(\Sigma, \mu, \Gamma, \rho)$. *Then the geometric action* $\bar{\rho}$ *of* Γ *on the* $\mathbb{R}$*-tree* T *is small if and only if for every* 1*-simplex* τ *of* Σ, *the stabilizer* Γ_τ *is a small subgroup of* Γ.

2.2. In order to establish Proposition 2.1 in complete generality, without a countability assumption, we need the following lemma. Recall from [GS, 5.19] that if $(T, \Gamma, \bar{\rho})$ denotes the leaf space of a uniform 1-connected $\mathbb{R}$-foliated singular surface with simplicial symmetry $(\Sigma, \mu, \Gamma, \rho)$, then there is a natural Γ-equivariant map χ_μ of Σ onto T.

LEMMA. *Let* $(\Sigma, \mu, \Gamma, \rho)$ *be a uniform* 1*-connected* $\mathbb{R}$*-foliated singular surface with simplicial symmetry. Let* $(T, \Gamma, \bar{\rho})$ *denote the leaf space of* $(\Sigma, \mu, \Gamma, \rho)$. *Then every non-degenerate segement in* T *contains a point* z *such that the leaf* $L = \chi_\mu^{-1}(z)$ *contains no vertex of* Σ.

Proof. Set $\chi = \chi_\mu$. Let $[x, y]$ be any non-degenerate segment in T. Let a and b be points of Σ such that $\chi(a) = x$ and $\chi(b) = y$. It follows from [GS, 5.25, 5.17 and 5.20] that there is a topological arc A joining a to b in Σ such that $\chi \mid A$ is a monotone map of A onto $[x, y]$. Since A is a non-degenerate arc it must have a sub-arc A_0 which contains no vertices. Thus A_0 is contained in some global branch [GS, 5.3] of Σ, say $\mathscr{B}$. Since $\mathscr{B}$ is either a surface with points at infinity or a closed 1-simplex [GS, 5.3], it contains only countably many vertices. Hence there is a point

$z \in A_0$ such that $\chi^{-1}(z)$ contains no vertex of $\mathscr{B}$. But the definition of χ implies that $\chi^{-1}(z)$ is a leaf of μ and is therefore a connected set. Since the topological frontier of $\mathscr{B}$ consists of vertices it follows that $\chi^{-1}(z) \subset \mathscr{B}$. Hence $\chi^{-1}(z)$ contains no vertex of Σ. □

Proof of Proposition 2.1. Set $\chi = \chi_\mu : \Sigma \to T$. It follows from [GS, 5.19] that if τ is any 1-simplex of Σ then $\chi(\tau)$ is a non-degenerate segment of T. By equivariance we have $\Gamma_\tau \subset \Gamma_{\chi(\tau)}$. Thus if $\bar{\rho}$ is a small action, $\Gamma_{\chi(\tau)}$ is a small subgroup of Γ, and hence so is Γ_τ.

Now suppose that the stabilizer of every 1-simplex of Σ is a small subgroup of Γ. We must show that $\bar{\rho}$ is a small action; thus if $[x, y]$ is any non-degenerate segment in T, we must show that $\Gamma_{[x,y]}$ is small.

By Lemma 2.2, there is a point $z \in [x, y]$ such that the leaf $L = \chi^{-1}(z)$ contains no vertex of Σ. Thus for every 2-simplex σ of Σ, the set $L \cap \sigma$ either is empty or is a line segment in σ whose end points are interior to the edges of σ.

Recall from [GS, 5.1] that in the singular surface Σ, the valence of each 1-simplex τ, i.e. the number of 2-simplices incident to τ, is either 0 or 2. Hence L either is a point in the interior of a valence-0 1-simplex of Σ, or is homeomorphic to $\mathbb{R}$; and in the latter case, L has an induced triangulation in which the 1-simplices are the non-empty intersections of L with 2-simplices of Σ.

In the stabilizer $\Gamma_{[x,y]}$ there is a subgroup G of index at most 2 which fixes $[x, y]$ pointwise. In particular, G fixes z and hence leaves L invariant. In the case where L is homeomorphic to $\mathbb{R}$, G acts simplicially on L with respect to its induced triangulation. Since the full group of simplicial homeomorphisms of a triangulation of $\mathbb{R}$ is isomorphic to the infinite dihedral group D_∞, it follows that in this case G has a normal subgroup N which acts trivially on L, and such that G/N is isomorphic to a subgroup of D_∞. Now if σ is any 2-simplex intersecting L, the group N fixes (pointwise) the set $\sigma \cap L$; since the latter set is a line segment containing no vertex of σ it follows that N fixes σ. In particular, N is contained in the stabilizer of a 1-simplex of Σ and is therefore small. Since G/N is a subgroup of D_∞ and G has index at most 2 in $\Gamma_{[x,y]}$, it follows that $\Gamma_{[x,y]}$ is small in this case.

In the case that L is an interior point of a 1-simplex τ of Σ, the group G stabilizes τ and is therefore small. Again, since G has index at most 2 in $\Gamma_{[x,y]}$, it follows that $\Gamma_{[x,y]}$ is small. □

3. The space of length systems

In this section, Σ will denote a 1-connected singular surface. As in [GS, Section 7], we shall study measured foliations on Σ in terms of length systems.

3.1. Let μ be a measured foliation of Σ. According to the definitions given in [GS, Section 5], the underlying foliation of μ induces a standard k-prong decomposition of β for some $k \geq 2$. We shall call k the *order* of β with respect to μ. We shall say that μ is *strictly non-degenerate* if every non-compact local branch of Σ has infinite order with respect to μ. It is clear that if μ is invariant under a group Γ of simplicial automorphisms of Σ, and Σ is finite mod Γ, then μ is strictly non-degenerate.

Suppose that the measured foliation μ is tame in the sense of [GS, Section 5]: that is, no simplex is contained in a leaf, and the induced decomposition of every 2-simplex δ of Σ is a parallel family of open line segments in δ. (In particular, all singularities of a tame foliation must occur at vertices.) Then we can define a positive-valued function λ on the set of 1-simplices of Σ by defining $\lambda(\tau)$ to be the measure $\mu(\tau)$. It follows from the definitions that λ is a length system in the sense of [GS, 7.1–7.2]. (This means that the edges of each 2-simplex δ can be labeled τ, τ' and τ'' in such a way that $\lambda(\tau) = \lambda(\tau') + \lambda(\tau'')$. The edge τ is called the *long edge* of δ.) The order of any local branch β of Σ with respect to μ is equal to its order with respect to the length system ℓ in the sense of [GS, 7.2], i.e. the number of 2-simplices $\delta \subset \beta$ such that v is not incident to the distinguished edge of δ. In particular, if μ is strictly non-degenerate then λ is non-degenerate in the sense of [GS, 7.2] – that is, every 2-dimensional branch of Σ has order ≥ 2 with respect to λ, and every non-compact 2-dimensional branch has infinite order.

Moreover, it follows from [GS, 7.3] that this construction defines a bijective correspondence between tame measured foliations of Σ and non-degenerate length systems. If λ is a non-degenerate length system we shall denote the corresponding measured foliation by μ_λ.

3.2. It will be useful to extract combinatorial information from a length system. We define a *ferroviary structure* on Σ to be a function that assigns to each 2-simplex σ of Σ a 1-simplex $\tau < \sigma$, called the *distinguished edge* of τ. Each length system on Σ has an *underlying ferroviary structure* in which the distinguished edge of each 2-simplex is its long edge. (The underlying ferroviary structure of the length system corresponding to a given tame measured lamination is similar to a train track carrying a given measured lamination in Thurston's theory.) If a group Γ acts on Σ by automorphisms, and if ℓ is a Γ-invariant length system, then the underlying ferroviary structure of ℓ is also Γ-invariant; that is, if σ is a 2-simplex of Σ with distinguished edge τ, then $\gamma \cdot \tau$ is the distinguished edge of $\gamma \cdot \sigma$ for any $\gamma \in \Gamma$.

Non-degeneracy of a length system can be detected from its underlying ferroviary structure. If θ is a ferroviary structure on Σ, and β is a 2-dimensional local branch [GS, 5.2] of Σ at a vertex v of Σ, we define the *order* of β with respect to θ to be the number of 2-simplices $\delta \subset \beta$ such that v is not incident to the distinguished edge of δ. We shall say that θ is *non-degenerate* if for every vertex v of Σ, every 2-dimensional local branch at v has order ≥ 2, and every non-compact 2-dimensional local branch has infinite order. Thus when θ is the underlying ferroviary structure of a length system ℓ, the order of β with respect to θ is equal to its order with respect to ℓ. In particular, a length system is non-degenerate if and only if its underlying ferroviary structure is non-degenerate.

3.3. We shall see that the set of all length systems on Σ which have a given underlying ferroviary structure can be regarded in a useful way as a subset of a vector space. (This is similar to Thurston's parametrization of measured laminations by solutions of track equations.)

Suppose that we are given a singular surface Σ with a ferroviary structure θ. We let $\mathscr{U}$ denote the set of all 1-simplices of Σ, and we consider the vector space $\mathbb{R}^{\mathscr{U}}$ of all real-valued functions on $\mathscr{U}$.

Let σ be any 2-simplex of Σ. Let τ denote the distinguished edge of σ, and let τ' and τ'' denote its other two edges. Then

$$V_\sigma = V_\sigma(\theta) = \{\lambda \in \mathbb{R}^{\mathscr{U}} : \lambda(\tau) = \lambda(\tau') + \lambda(\tau'')\}$$

is a linear subspace of $\mathbb{R}^{\mathscr{U}}$.

We set $V(\theta) = V = \bigcap_\sigma V_\sigma$, where σ ranges over all 2-simplices of Σ. Then V is a linear subspace of $\mathbb{R}^{\mathscr{U}}$ which we call the *length space* associated to the given ferroviary structure on Σ. We let $C = C(\theta)$ denote the positive cone in V, i.e. the set of all $\lambda \in V$ which are (strictly) positive-valued functions on $\mathscr{U}$. According to the definitions, C is precisely the set of all length systems on Σ whose underlying ferroviary structure is θ.

3.4. Now suppose that a group Γ acts on Σ by automorphisms in such a way that Σ is finite mod Γ and θ is Γ-invariant. The action of Γ on Σ induces a linear action on $\mathbb{R}^{\mathscr{U}}$ given by $(g \cdot \lambda)(\tau) = \lambda(g^{-1} \cdot \tau)$; the subspace V is Γ-invariant. The elements of V (or $\mathbb{R}^{\mathscr{U}}$) which are fixed by the action of Γ form a linear subspace V_Γ (respectively, $\mathbb{R}^{\mathscr{U}}_\Gamma$). The set $C_\Gamma = C_\Gamma(\theta) = C \cap V_\Gamma$ consists of all Γ-invariant length systems on Σ with underlying ferroviary structure θ.

3.5. For the rest of this section we assume that θ is non-degenerate. Let λ be a point in C_Γ. Since θ is non-degenerate, λ is a non-degenerate, Γ-invariant length function; hence μ_λ is a tame, Γ-invariant measured foliation on Σ. Thus we have a 1-connected $\mathbb{R}$-foliated singular surface with simplicial symmetry $(\Sigma, \mu_\lambda, \Gamma, \rho)$, where ρ denotes the given action of Γ on Σ. Since Σ is finite modulo the simplicial action of Γ, it follows from 1.1 that $(\Sigma, \mu_\lambda, \Gamma, \rho)$ has a well-defined leaf space

$(T_\lambda, \Gamma, \rho_\lambda)$, where T_λ is an $\mathbb{R}$-tree and ρ_λ is a geometric action of Γ on T_λ. The geometric length function defined by this section will be denoted ℓ_λ.

3.6. If $\lambda \in C_\Gamma$ has its coordinates in a given subgroup Λ of $\mathbb{R}$, then $(\Sigma, \mu_\lambda, \Gamma, \rho)$ is obviously a Λ-foliated singular surface with simplicial symmetry [GS, 5.21]. This implies that ℓ_λ is a Λ-length function. (Indeed, according to [GS, 5.28] we may regard the leaf space $(T_\lambda, \Gamma, \rho_\lambda)$ as the completion of a canonically defined Λ-pre-tree with symmetry $(T_\lambda^0, \Gamma, \rho_\lambda^0)$, the pre-leaf space of $(\Sigma, \mu_\lambda, \Gamma, \rho)$. The Λ-completion of $(T_\lambda^0, \Gamma, \rho_\lambda^0)$ is a Λ-tree with symmetry whose $\mathbb{R}$-completion is again $(T_\lambda, \Gamma, \rho_\lambda)$; by [GS, 1.18], the length function defined by this Λ-tree with symmetry is ℓ_λ.)

3.7. For any $\lambda \in C_\Gamma$ and any positive real number α, we have $\mu_{\alpha\lambda} = \alpha\mu_\lambda$; hence $T_{\alpha\lambda}$ is the $\mathbb{R}$-tree which has the same underlying set as T_λ and in which the metric is obtained by multiplying the metric of T_λ by α. It follows that $\ell_{\alpha\lambda} = \alpha\ell_\lambda$.

3.8. Since Σ is finite mod Γ, the vector space $\mathbb{R}_\Gamma^{\mathscr{U}}$ is finite-dimensional; in fact, $\mathbb{R}_\Gamma^{\mathscr{U}}$ is canonically identified with $\mathbb{R}^{\mathscr{U}/\Gamma}$. An element λ of $\mathbb{R}_\Gamma^{\mathscr{U}}$ is identified with the point $(x_a)_{a \in \mathscr{U}/\Gamma}$ of $\mathbb{R}^{\mathscr{U}/\Gamma}$, where x_a is the value of λ on the orbit $a \in \mathscr{U}/\Gamma$.

Note that V_Γ is a linear subspace of $\mathbb{R}_\Gamma^{\mathscr{U}}$. As finite-dimensional real vector spaces, $\mathbb{R}_\Gamma^{\mathscr{U}}$ and V_Γ have natural topologies. The positive cone C_Γ is an open subset of V_Γ.

3.9. It follows from Proposition 2.1 that for a given $\lambda \in C_\Gamma$, the action ρ_λ is small if and only if for every 1-simplex τ of Σ, the stabilizer Γ_τ is a small subgroup of Γ. In particular, the condition that ρ_λ be small does not depend on the choice of the point $\lambda \in C_\Gamma$ but only on the action of Γ on Σ. We record this as a

PROPOSITION. *If ρ_λ is a small action for some $\lambda \in C_\Gamma$, then ρ_λ is a small action for every $\lambda \in C_\Gamma$.* □

(Of course Proposition 2.1 actually shows more than this: it shows that the condition that ρ_λ be small is not only independent of the point $\lambda \in C_\Gamma$ but is independent of the underlying ferroviary structure θ.)

3.10. An element λ of $\mathbb{R}_\Gamma^{\mathscr{U}}$ will be called *rational* if we have $\lambda(\tau) \in \mathbb{Q}$ for every $\tau \in \mathscr{U}$. Equivalently, $\lambda \in \mathbb{R}^{\mathscr{U}/\Gamma}$ is rational if and only if it has rational coordinates. A subspace of $\mathbb{R}_\Gamma^{\mathscr{U}}$ will be called *rational* if it is spanned by its rational elements, or equivalently if it can be defined by equations with rational coefficients in the coordinates of $\mathbb{R}^{\mathscr{U}/\Gamma}$. Note that any rational subspace has a dense subset consisting of rational elements.

3.11. The subspace V_Γ of $\mathbb{R}^{\mathscr{U}}_\Gamma$ is rational. Indeed, if $\sigma_1, \ldots, \sigma_k$ is a complete set of orbit representatives for the action of Γ on the set of 2-simplices of Σ, and if for each i we denote by o_i the orbit of the distinguished edge of σ_i, and by o_i' and o_i'' the orbits of the other two edges, then V_Γ is defined by the equations

$$x_{o_i} = x_{o_i'} + x_{o_i''} \qquad (1 \leq i \leq k)$$

in the coordinates of $\mathbb{R}^{\mathscr{U}/\Gamma}$.

3.12. For any point $\lambda \in C_\Gamma$ we shall let $W_\lambda = W_\lambda(\theta)$ denote the intersection of all rational subspaces of $\mathbb{R}^{\mathscr{U}}_\Gamma$ containing λ. Clearly W_λ is a rational subspace of $\mathbb{R}^{\mathscr{U}}_\Gamma$, and by 3.11 we have $W_\lambda \subset V_\Gamma$. Hence $W_\lambda \cap C \subset C_\Gamma$.

3.13. The following result will be proved in Section 4.

PROPOSITION. *Let λ_∞ be any element of C_Γ. Let $(\lambda_i)_{i \geq 0}$ be a sequence in $W_{\lambda_\infty} \cap C$ which converges to λ_∞ in the topology of $\mathbb{R}^{\mathscr{U}}_\Gamma$. Then for any $g \in \Gamma$ the sequence $(\ell_{\lambda_i}(g))$ converges to $\ell_{\lambda_\infty}(g)$ in $\mathbb{R}$.*

It is not obvious to us whether the conclusion remains true when (λ_i) is an arbitrary sequence in C_Γ which converges to λ_∞; that is, whether the assignment $\lambda \to \ell_\lambda(g)$ is continuous for every $g \in \Gamma$. The assumption that the λ_i lie in W_{λ_∞} is used in the last paragraph of the proof of Lemma 4.5 below, on which the proof of Proposition 3.13 depends.

Proof that Proposition 3.13 *implies Theorem* 1.3. Let ℓ be the length function defined by a geometric action $\bar{\rho}$ of a group Γ on a tree T. Then we may identify $(T, \Gamma, \bar{\rho})$ with the leaf space of $(\Sigma, \mu, \Gamma, \rho)$, where Σ is a 1-connected singular surface which is finite modulo a simplicial action ρ of Γ, and μ is a tame Γ-invariant measured foliation μ on Σ. As we observed in 3.1, μ must be strictly non-degenerate since Σ is finite modulo Γ.

We let $\mathscr{U}$ denote the set of 2-simplices of Σ.

According to 3.1, μ is defined by some non-degenerate length system λ on Σ. Since μ is Γ-invariant, so is λ. Let θ denote the underlying ferroviary structure of λ. Since λ is non-degenerate, θ is a non-degenerate ferroviary structure. We regard λ as an element of $C_\Gamma(\theta)$. By the above remarks, $W_\lambda(\theta)$ is a rational subspace of $\mathbb{R}^{\mathscr{U}}_\Gamma$ and thus has a dense subset consisting of rational points. Furthermore, λ belongs to the open subset $W_\lambda(\theta) \cap C(\theta)$ of $W_\lambda(\theta)$. Hence there is a sequence $(\lambda_i)_{i \geq 0}$ of rational points of $W_\lambda(\theta) \cap C(\theta)$ converging to λ. The λ_i, regarded as points of $\mathbb{R}^{\mathscr{U}/\Gamma}$, have rational coordinates. Hence we may write $\lambda_i = \lambda_i'/n_i$, where the n_i are positive integers and the λ_i' are points with integer coordinates in $\mathbb{R}^{\mathscr{U}/\Gamma}$. Clearly $\lambda_i' \in W_\lambda(\theta) \cap C(\theta)$.

Set $\ell_i = \ell_{\lambda_i'}$ for each i. By 3.6, ℓ_i is a geometric $\mathbb{Z}$-length function. By 3.7 we have $\ell_i(g)/n_i = \ell_{\lambda_i}(g)$. Proposition 3.13 implies that

$$\lim_{i\to\infty} \frac{\ell_i(g)}{n_i} = \lim_{i\to\infty} \ell_{\lambda_i}(g) = \ell(g)$$

for every $g \in \Gamma$.

If ℓ is a small geometric length function, then we may take the geometric action ρ_λ to be small. It then follows from Proposition 3.9 that the ρ_{λ_i} are small as well. Hence the geometric length functions ℓ_i are small. □

4. Comparing two notions of approximation

In this section we prove Proposition 3.13, which was shown in the last section to imply Theorem 1.3. Proposition 3.13 generalizes a result of Thurston's about rational approximation of measured foliations of surfaces; our proof is based largely on Thurston's techniques.

We are given a singular surface Σ and a fixed non-degenerate ferroviary structure θ. We use the notation of Section 3.

4.1. In the course of the proof we shall need to consider paths in Σ. We shall use slightly different conventions from those of [GS]. Here we define a *path* in Σ to be a continuous map of an arbitrary compact interval in $\mathbb{R}$ into Σ; in particular, the domain interval may consist of a single point, in which case the path is said to be *trivial*. A path is said to be *constant* if its image is a single point, and is called an *arc* if it is a 1-1 map; thus a trivial path is at once a constant path and an arc.

An (*affine*) *re-parameterization* of a path $\gamma \mid [a, b] \to \Sigma$ is a path of the form $\gamma \circ h$ where h is an orientation-preserving (affine) homeomorphism of some compact interval onto $[a, b]$. An *initial sub-path* of γ is the restriction of γ to an interval $[a, c]$ where $a \le c \le b$.

A path $\gamma : [a, b] \to \Sigma$ is called a *composition* of paths $\gamma_1, \dots, \gamma_n$ if there exist points $a = a_0, \dots, a_n = b$, such that $\gamma \mid [a_i, b_i]$ is an affine re-parametrization of γ_i for $i = 1, \dots, n$. In this case, we write (ambiguously) $\gamma = \gamma_1 * \cdots * \gamma_n$.

4.2. An affine path in a simplex σ of Σ will be called *proper* if it is non-constant and has its endpoints in the boundary of σ. By an *admissible path* we shall mean a piecewise-linear path in Σ of the form $\gamma = \varepsilon_1 * \cdots * \varepsilon_n$, where each ε_i is a proper affine path in a maximal simplex σ_i of Σ. (To say that σ_i is maximal means that it is not a face of a higher-dimensional simplex. Thus it is either a 2-simplex or a valence-0 1-simplex.) If we set $P_0 = \varepsilon_0(0)$ and $P_i = \varepsilon_i(1)$ for $i = 1, \ldots, n$, then P_i lies in an open simplex Δ_i of dimension ≤ 1 for $i = 0, \ldots, n$. Note that the sequences of simplices $(\sigma_1, \ldots, \sigma_n)$ and $(\Delta_0, \ldots, \Delta_n)$ are uniquely determined by γ. Two admissible paths which determine the same sequences $(\sigma_1, \ldots, \sigma_n)$ and $(\Delta_0, \ldots, \Delta_n)$ will be termed *combinatorially equivalent.*

Note that any admissible path γ is a piecewise-monotonic path in the sense of [GS, 5.9] with respect to any tame measured foliation μ of Σ. In particular γ has a well-defined μ-measure $\mu(\gamma)$.

4.3. LEMMA. *Let γ be an admissible path in Σ. Then the real-valued function $\lambda \to \mu_\lambda(\gamma)$ is continuous on C_Γ.*

Proof. It suffices to prove the lemma in the case where γ is a proper affine path in a maximal simplex σ of Σ. The assertion is trivial if σ is a 1-simplex of valence 0; thus we assume that σ is a 2-simplex. Let τ denote the distinguished edge of σ, and let τ' and τ'' denote its other edges. Let v, v' and v'' denote the vertices of σ opposite τ, τ' and τ'' respectively. For any $\lambda \in C$, let $f^\lambda : \sigma \to \mathbb{R}$ be an affine map (cf. [GS, 7.1]) which maps v, v' and v'' to $\lambda(\tau')$, $\lambda(\tau)$ and 0, respectively. If P and Q denote the initial and terminal points of γ, then there are linear functions $\phi, \psi : V_\Gamma \to \mathbb{R}$ such that $f^\lambda(P) = \phi(\lambda)$ and $f^\lambda(Q) = \psi(\lambda)$, for all $\lambda \in C_\Gamma$. Hence for any $\lambda \in C_\Gamma$ we have $\mu_\lambda(\gamma) = |\phi(\lambda) - \psi(\lambda)|$. This establishes the lemma. □

4.4. If λ is a point of C_Γ and if x and y are two points of a closed 1-simplex τ of Σ, we define the *λ-distance* between x and y, denoted $d_\lambda(x, y)$, to be their distance in the affine metric of length $\lambda(\tau)$ on τ. Equivalently, $d_\lambda(x, y)$ is the μ_λ-measure of the sub-segment of τ with endpoints x and y.

4.5. As in Section 3, we identify $\mathbb{R}^{\mathscr{U}}_\Gamma$ with $\mathbb{R}^{\mathscr{U}/\Gamma}$. A linear real-valued function on $\mathbb{R}^{\mathscr{U}}_\Gamma$ will be said to be *defined over* $\mathbb{Z}$ if it is a $\mathbb{Z}$-linear combination of the coordinate functions on $\mathbb{R}^{\mathscr{U}/\Gamma}$.

LEMMA. *Let λ_∞ be any element of C_Γ. Let γ be an admissible path in a leaf of μ_{λ_∞} such that $\gamma(0)$ is a vertex of Σ. Then there is a neighborhood U of λ_∞ in $W_{\lambda_\infty} \cap C$, such that for every $\lambda \in U$, there is an admissible path γ_λ which lies in a leaf of μ_λ and is combinatorially equivalent to γ; and γ_λ is unique modulo re-parametrization. Furthermore, if $\gamma(1)$ lies in the interior of a 1-simplex τ of Σ, and if v is an endpoint of τ, then there is a linear function $\omega : \mathbb{R}^{\mathscr{U}/\Gamma} \to \mathbb{R}$ defined over $\mathbb{Z}$, such that for every $\lambda \in U$ the λ-distance from $\gamma_\lambda(1)$ to v (along τ) is $\omega(\lambda)$.*

Proof. If the leaf carrying γ is a point, the assertion is trivial; thus we assume that this is not the case. We may therefore write $\gamma = \varepsilon_1 * \cdots * \varepsilon_n$, where each ε_i is a proper affine path in a 2-simplex σ_i of Σ. We argue by induction on n; if $n = 0$ then γ is a trivial path and the assertion is trivial. Now suppose that $n > 0$ and set $\gamma' = \varepsilon_1 * \cdots * \varepsilon_{n-1}$. We assume that there is an open neighborhood U' of λ_∞ in $W_{\lambda_\infty} \cap C$, such that for every $\lambda \in U'$, there is a unique admissible path γ'_λ which lies in a leaf of μ_λ and is combinatorially equivalent to γ'. Furthermore, if $\gamma'(1)$ lies in the interior of a 1-simplex of Σ, we let τ' denote this 1-simplex; and in this case we assume that for each endpoint v' of τ', there is a linear function $\omega' : \mathbb{R}^{\mathscr{U}/\Gamma} \to \mathbb{R}$ defined over $\mathbb{Z}$, such that $d_\lambda(\gamma'_\lambda(1), v') = \omega'(\lambda)$ for every $\lambda \in U'$.

We may write $\gamma = \gamma' * \varepsilon$, where $\varepsilon = \varepsilon_n$. We set $P = \gamma(1) = \varepsilon(1)$ and $P' = \gamma'(1) = \varepsilon(0)$. We also set $\sigma = \sigma_n$. Thus P and P' are boundary points of σ, and ε is the affine path in σ joining P' to P.

Let us first consider the case where P' is a vertex of Σ. Let τ denote the edge of σ opposite to P'. Since ε is contained in a leaf of μ_{λ_∞}, the point P must lie in the interior of τ, and τ must be the distinguished edge of σ in the ferroviary structure θ.

Let $\lambda \in U'$ be given. Since P' is a vertex and γ'_λ is combinatorially equivalent to γ', we have $\gamma'_\lambda(1) = P'$. There is a unique affine path ε_λ which is contained in a leaf of μ_λ and joins P' to a point of τ. Hence $\gamma_\lambda = \gamma'_\lambda * \varepsilon_\lambda$ is (modulo re-parametrization) the unique admissible path which lies in a leaf of μ_λ and is combinatorially equivalent to γ. If τ_1 denotes the edge of σ joining P' to a given vertex V of τ, then the λ-distance from v to $\gamma_\lambda(1) = \varepsilon_\lambda(1)$ is $\lambda(\tau_1)$. Hence in this case we can complete the induction by defining $U = U'$ and $\omega(\lambda) = \lambda(\tau_1)$.

Next consider the case that P' is an interior point of an edge τ' of σ, and that P is an interior point of another edge of σ, say τ. Since ε is carried by a leaf of μ_λ, either τ or τ' must be the distinguished edge of σ in the ferroviary structure θ. We denote by v' the common endpoint of τ and τ'. The induction hypothesis gives a linear function $\omega' : \mathbb{R}^{\mathscr{U}/\Gamma} \to \mathbb{R}$ defined over $\mathbb{Z}$, such that $d_\lambda(\gamma'_\lambda(1), v') = \omega'(\lambda)$ for every $\lambda \in U'$.

For any $\lambda \in U'$, consider the point $P'_\lambda = \gamma'_\lambda(1)$. There is a unique affine path ε_λ which is contained in a leaf of μ_λ and joins P'_λ to another boundary point P_λ of σ. If τ is the distinguished edge of σ then P_λ is an interior point of τ for any $\lambda \in U'$; in this subcase we set $U' = U$. If τ' is the distinguished edge of τ then P_λ will be an interior point of τ if and only if $\lambda(\tau) > \omega'(\lambda)$. The points $\lambda \in U'$ for which this inequality holds form an open set U in $W_\lambda \cap \mathrm{C}$, and we have $\lambda_\infty \in U$ since P is an interior point of τ. It is clear in both sub-cases that for any $\lambda \in U$, the path $\gamma_\lambda = \gamma'_\lambda * \varepsilon_\lambda$ is (modulo re-parametrization) the unique admissible path which lies in a leaf of μ_λ and is combinatorially equivalent to γ.

To complete the induction in this case we must show that for any vertex v of τ, there is a linear function $\omega : \mathbb{R}^{\mathscr{U}/\Gamma} \to \mathbb{R}$, defined over $\mathbb{Z}$, such that the λ-distance from $P_\lambda = \gamma_\lambda(1)$ to v is equal to $\omega(\lambda)$ for every $\lambda \in U'$. We clearly have $d_\lambda(P_\lambda, v') = d_\lambda(P'_\lambda, v') = \omega'(\lambda)$; and if v'' is the other endpoint of τ, we have $d_\lambda(P_\lambda, v'') = \lambda(\tau) - d_\lambda(P'_\lambda, v') = \lambda(\tau) - \omega'(\lambda)$. Thus we need only set $\omega = \omega'$ if $v = v'$, and define ω by $\omega(\lambda) = \lambda(\tau) - \omega'$ if $v = v''$. (The latter function is defined over $\mathbb{Z}$ since $\lambda(\tau)$ is one of the coordinate functions on $\mathbb{R}^{\mathscr{U}/\Gamma}$.)

Finally we consider the case that P' is an interior point of an edge τ' of σ, and that P is the vertex of σ opposite τ'. We fix a vertex v' of τ', and denote by τ_1 the edge joining P to v'. Again the induction hypothesis gives a linear function $\omega' : \mathbb{R}^{\mathscr{U}/\Gamma} \to \mathbb{R}$, defined over $\mathbb{Z}$, such that $d_\lambda(\gamma'_\lambda(1), v') = \omega'(\lambda)$ for every $\lambda \in U'$. Since the path ε is carried by a leaf of μ_{λ_∞}, the 1-simplex τ' must be the distinguished edge of σ, and we have $\lambda_\infty(\tau_1) = d(P', v') = \omega'(\lambda_\infty)$.

Thus λ_∞ is in the kernel of the linear map $\phi : \mathbb{R}^{\mathscr{U}/\Gamma} \to \mathbb{R}$ given by $\phi(\lambda) = \omega'(\lambda) - \lambda(\tau_1)$. Since ω' is defined over $\mathbb{Z}$, so is ϕ; hence $N = V_\Gamma \cap \ker \phi$ is a rational subspace of V_Γ. It therefore follows from the definition of W_{λ_∞} that $W_{\lambda_\infty} \subset N$, i.e. that ϕ vanishes identically on W_{λ_∞}. This means that for any $\lambda \in U'$ we have $\lambda(\tau_1) = \omega'(\lambda) = d(P'_\lambda, v')$. This implies that there is a unique affine path ε_λ in σ which joins P'_λ to P and is contained in a leaf of μ_λ. Hence $\gamma_\lambda = \gamma'_\lambda * \varepsilon_\lambda$ is the unique admissible path which is combinatorially equivalent to γ and is contained in a leaf of μ_λ. Thus we can complete the induction in this case by setting $U = U'$. □

4.6. ADDENDUM TO LEMMA 4.5. *If the admissible path γ is an arc, then the neighborhood U of λ_∞ can be chosen so that for every $\lambda \in U$ the path γ_λ is an arc.*

Proof. If U is the neighborhood of λ_∞ given by 4.5, we must show that λ_∞ has a neighborhood $U' \subset U$ such that γ_λ is an arc for every $\lambda \in U'$.

It is clear that an admissible path which is not an arc must have two distinct admissible initial sub-paths with the same terminal point. Hence for any $\lambda \in U$ the path γ_λ is an arc unless it has two admissible initial sub-paths γ'_λ and γ''_λ, combinatorially equivalent to distinct sub-paths γ' and γ'' of γ, with $\gamma'_\lambda(1) = \gamma''_\lambda(1)$. Since γ has only finitely many admissible sub-paths, it is enough to show that for any given sub-paths γ' and γ'' of γ there is a neighborhood $V = V_{\gamma'\gamma''} \subset U$ of λ_∞ such that $\gamma'_\lambda(1) \neq \gamma''_\lambda(1)$.

The hypothesis that γ is an arc guarantees that $\gamma'(1) \neq \gamma''(1)$. Hence if either $\gamma'(1)$ or $\gamma''(1)$ is a vertex of Σ, the combinatorial equivalence of γ'_λ, γ''_λ with γ', γ'' guarantees that $\gamma'_\lambda(1) \neq \gamma''_\lambda(1)$. The same argument applies if $\gamma'(1)$ and $\gamma''(1)$ lie in distinct open 1-simplices of Σ. Hence we may assume that $\gamma'(1)$ and $\gamma''(1)$ are interior to the same 1-simplex τ. We fix a vertex v of τ.

It follows from Lemma 4.5 that there exist a neighborhood $V_0 \subset U$ of λ_∞ and linear functions $\omega', \omega'' : \mathbb{R}_\Gamma^{\mathscr{U}} \to \mathbb{R}$ defined over $\mathbb{Z}$, such that $d_\lambda(\gamma'_\lambda(1), v) = \omega'(\lambda)$ and $d_\lambda(\gamma''_\lambda(1), v) = \omega''(\lambda)$ for every $\lambda \in V_0$. Since $\gamma'(1) \neq \gamma''(1)$ we have $\omega'(\lambda_\infty) \neq \omega''(\lambda_\infty)$. Hence for all λ in some neighborhood $V \subset V_0$ of λ_∞ we have $\omega'(\lambda) \neq \omega''(\lambda)$ and hence $\gamma'_\lambda(1) \neq \gamma''_\lambda(1)$. □

4.7. LEMMA. *Let λ_∞ be an element of C_Γ. Let $(\lambda_i)_{i \geq 0}$ be a sequence in $W_{\lambda_\infty} \cap C$ which converges to λ_∞ in the topology of $\mathbb{R}_\Gamma^{\mathscr{U}}$. Let γ_∞ be an admissible path in Σ which has its endpoints at vertices of Σ and is quasi-transverse* [GS, 5.15] *to μ_{λ_∞}. Then there exist an integer $i_0 \geq 0$ and a sequence $(\gamma_i)_{i \geq i_0}$ of admissible paths which are combinatorially equivalent to γ_∞, such that γ_i is quasi-transverse to μ_{λ_i} for each $i \geq i_0$, and*

$$\lim_{i \to \infty} \mu_{\lambda_i}(\gamma_i) = \mu_{\lambda_\infty}(\gamma_\infty).$$

Proof. According to the definition [GS, 5.15] of a quasi-transverse path, we may write γ in the form $\gamma^{(1)} * \cdots * \gamma^{(n)}$, where each $\gamma^{(j)}$ is either a strictly monotonic path or an arc in a leaf, and for each $j < n$ there exist a final sub-path $\alpha^{(j)}$ of $\gamma^{(j)}$ and an initial sub-path $\beta^{(j)}$ of $\gamma^{(j+1)}$ such that one of the following alternatives holds:

(i) $|\alpha^{(j)}|$ and $|\beta^{(j)}|$ lie in distinct local branches at $\gamma^{(j)}(1)$; or
(ii) there is a standard neighborhood Z of $\gamma^{(j)}(1)$ in a 2-dimensional local branch of Σ such that
  (a) $|\alpha^{(j)}|$ and $|\beta^{(j)}|$ are interior to different sectors of Z, or
  (b) one of the $|\alpha^{(j)}|$ and $|\beta^{(j)}|$ lies in a prong of Z and the other is interior to a sector not containing that prong.

(Note in particular that when $\gamma^{(j)}(1)$ is not a vertex, Z has exactly two sectors and (a) always holds.)

We shall denote by J_1 (respectively J_2) the set of all indices j such that $\gamma^{(j)}$ is an arc in a leaf (respectively, a strictly monotonic path). Since γ is admissible, we may take all the $\gamma^{(j)}$ to be admissible, and for $j = 1, \ldots, n$ we may take $\alpha^{(j)}$ and $\beta^{(j)}$ to be proper affine paths in 2-simplices $\omega^{(j)}$ and $\psi^{(j)}$ of Σ. Furthermore, we may take $\alpha^{(j)} = \beta^{(j-1)} = \gamma^{(j)}$ for each $j \in J_2$.

It follows from Lemma 4.5 and its Addendum 4.6 that for all sufficiently large i, and for each $j \in J_1$, there is an admissible arc $\gamma_i^{(j)}$ which is combinatorially equivalent to $\gamma^{(j)}$ and is contained in a leaf of μ_{λ_i}.

For each $j \in J_2$ we set $\gamma_i^{(j)} = \gamma^{(j)}$. Note that for all sufficiently large i, the path $\gamma_i^{(j)} = \gamma^{(j)}$ is strictly monotonic with respect to μ_{λ_i}. (Indeed, by Lemma 4.3 we have $\lim_{i \to \infty} \mu_{\lambda_i}(\gamma^{(j)}) = \mu_\lambda(\gamma^{(j)}) \neq 0$, and hence $\gamma^{(j)}$ is not contained in a leaf of μ_{λ_i} when i is large; since $\gamma^{(j)}$ is a proper affine path in a 2-simplex, this means that it is strictly monotonic.)

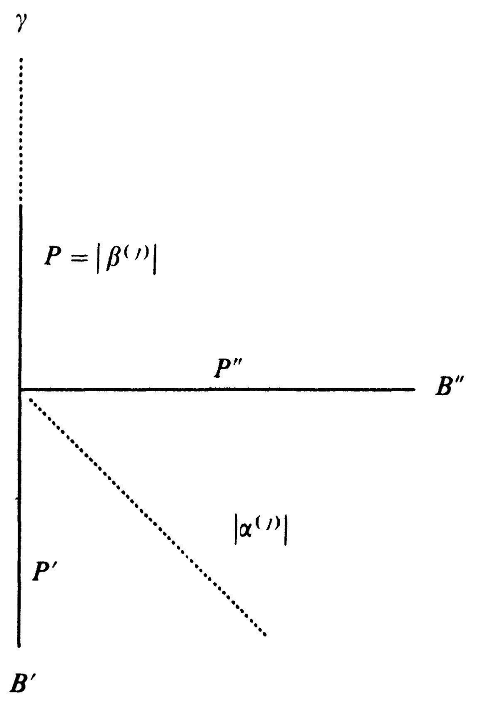
Lemma 4.3 also implies that $\lim_{i \to \infty} \mu_{\lambda_i}(\gamma_i^{(j)}) = \mu_{\lambda_\infty}(\gamma_\infty^{(j)})$ for $j = 1, \ldots, n$.

We set $\gamma_i = \gamma_i^{(1)} * \cdots * \gamma_i^{(n)}$ for large i. Then $\lim_{i\to\infty} \mu_{\lambda_i}(\gamma_i) = \mu_{\lambda_\infty}(\gamma_\infty)$. Since the $\gamma_i^{(j)}$ are admissible and combinatorially equivalent to the $\gamma^{(j)}$, it is clear that the γ_i are admissible and combinatorially equivalent to γ.

It remains to show γ_i is quasi-transverse when i is large. For this purpose we consider an arbitrary index $j < n$. For all large i we let $\alpha_i^{(j)}$ (respectively $\beta_i^{(j)}$) denote the unique final (resp. initial) sub-path of $\gamma^{(j)}$ (resp. $\gamma^{(j+1)}$) which is a proper affine path in $\omega^{(j)}$ (resp. $\psi^{(j)}$). In order to show that γ_i is quasi-transverse (for large i), it suffices to show that either (i) or (ii) holds with $\alpha_i^{(j)}$ and $\beta_i^{(j)}$ in place of $\alpha^{(j)}$ and $\beta^{(j)}$. Since $\gamma_i^{(j)}$ and $\gamma_i^{(j+1)}$ are combinatorially equivalent to $\gamma^{(j)}$ and $\gamma^{(j+1)}$, it is clear that if $\alpha^{(j)}$ and $\beta^{(j)}$ satisfy (i), then $\alpha_i^{(j)}$ and $\beta_i^{(j)}$ also satisfiy (i) for all i for which they are defined.

Now suppose that $\alpha^{(j)}$ and $\beta^{(j)}$ satisfy (ii), and let Z be a standard neighborhood of $v = \gamma^{(j)}(1)$ in a 2-dimensional local branch of Σ such that (a) or (b) holds. Since γ is admissible, we may take Z itself to be a local branch of Σ at the vertex v; the link C of v in Z is a PL 1-manifold.

Suppose that (b) holds; then by symmetry we may assume that $P = |\beta^{(j)}|$ is a prong of Z, and that $|\alpha^{(j)}|$ is interior to a sector not containing P. Thus we have $j \in J_2$ and $j + 1 \in J_1$. There exist distinct prongs P' and P'', meeting C in points B' and B'', such that $\alpha^{(j)}(0)$ and $B = \beta^{(j)}(1)$ lie in distinct components of $C - \{B', B''\}$. (See figure.)



When i is large, $P_i = |\beta_i^{(j)}|$ is the unique prong P_i of the foliation μ_{λ_i} at v which is contained in the same 2-simplex as P, and $B_i = \beta_i^{(j)}(1)$ is the point of intersection of P_i with C. An application of Lemma 4.5 to $\beta^{(j)}$ (or an easy direct argument) shows that B_i converges to B (in the topology of the 1-manifold C) as $i \to \infty$.

Similarly, for large i there is a unique prong P'_i (resp. P''_i) of the foliation μ_{λ_i} at v which is contained in the same 2-simplex as P' (resp. P''); this prong meets C in a single point B'_i (resp. B''_i), and B'_i (resp. B''_i) converges to B' (resp. B'') as $i \to \infty$. Hence for large enough i, the points $\alpha_i^{(j)}(0)$ and $B_i = \beta_i^{(j)}(1)$ lie in distinct components of $C - \{B'_i, B''_i\}$. This means that (b) still holds if we replace $\alpha^{(j)}$ and $\beta^{(j)}$ by $\alpha_i^{(j)}$ and $\beta_i^{(j)}$.

A very similar argument shows that if $\alpha^{(j)}$ and $\beta^{(j)}$ satisfy (a) then $\alpha_i^{(j)}$ and $\beta_i^{(j)}$ also satisfy (a) for large i. □

Proof of Proposition 3.13. For $0 \le i \le \infty$, set $\mu_i = \mu_{\lambda_i}$, let T_i denote the leaf space of μ_i, d_i the distance function in T_i, and χ_i the natural map from Σ to T_i. According to [GS, Theorem 5.20], the image under χ_i of the 0-skeleton of Σ is an $\mathbb{R}$-pre-tree $T_i^0 \subset T_i$. Let $g \in \Gamma$ be given. According to [GS, 1.16], we have $\ell_\infty(g) = \min_{x \in T^0_\infty} d_\infty(x, g \cdot x)$. Let us fix a point $x^0_\infty \in T^0_\infty$ such that $d_\infty(x^0_\infty, g \cdot x^0_\infty) = \ell_\infty(g)$. It follows from [MS, Theorem II.2.3] that the segments $[g^{-1} \cdot x^0_\infty, x^0_\infty]$, $[x^0_\infty, g \cdot x^0_\infty] \subset T_\infty$ intersect only at the point x^0_∞. Hence $x^0_\infty \in [g^{-1} \cdot x^0_\infty, g \cdot x^0_\infty]$, and $d_\infty(g^{-1} \cdot x^0_\infty, g \cdot x^0_\infty) = 2l_\infty(g)$.

Since $x^0_\infty \in T^0_\infty$, there is a vertex v of Σ such that $\chi_\infty(v) = x^0_\infty$. For $0 \le i < \infty$ we set $x_i^0 = \chi_i(v) \in T_i^0$.

We claim that there is a path γ_∞ from $g^{-1} \cdot v$ to $g \cdot v$, quasi-transverse to μ_∞, such that $|\gamma_\infty|$ contains some vertex v' in the leaf $\chi_\infty^{-1}(x^0_\infty)$ of μ_∞. Indeed, according to [GS, Proposition 5.25], there exist a path η_- from $g^{-1} \cdot v$ to v, and a path η_+ from v to $g \cdot v$, which are quasi-transverse to μ_∞. By [GS, 5.17], η_- and η_+ are parametrized arcs in Σ. It follows from [GS, 5.20] that χ_∞ maps $|\eta_-|$ and $|\eta_+|$ monotonically onto $[g^{-1} \cdot x^0_\infty, x^0_\infty]$ and $[x^0_\infty, g \cdot x^0_\infty]$ respectively. Hence $|\eta_-| \cap |\eta_+| \subset \chi_\infty^{-1}(x^0_\infty)$. Since η_- and η_+ are quasi-transverse to μ_∞, it follows from [GS, 5.16] that $|\eta_-| \cap |\eta_+|$ is a (possibly degenerate) arc in $\chi_\infty^{-1}(x^0_\infty)$. If $|\eta_-| \cap |\eta_+|$ is degenerate, we may set $\gamma_\infty = \eta_- * \eta_+$ and $v' = v$; it follows from the definitions that γ_∞ is quasi-transverse. If $|\eta_-| \cap |\eta_+|$ is non-degenerate, let v' denote the endpoint $\neq v$ of $|\eta_-| \cap |\eta_+|$. It follows from quasi-transversality that either Σ has at least two local branches at v', or some local branch v' has order at least 3 with respect to μ; since μ is tame it follows that v is a vertex of Σ. Furthermore, if ζ_- denotes the sub-arc of η_- with endpoints $g^{-1} \cdot v$ and v', and ζ_+ denotes the sub-arc of η_+ with endpoints v' and $g \cdot v$, then it again follows from the definitions that $\gamma_\infty = \zeta_- * \zeta_+$ is quasi-transverse to μ_∞. This proves the claim.

Since v and v' lie in the same leaf of μ_∞, it follows from Lemma 4.5 that they lie in the same leaf of μ_i for all sufficiently large i. Thus $\chi_i(v') = \chi_i(v) = x_i^0$ when i is large.

According to Lemma 4.7, there exist an integer $i_0 \geq 0$, and a sequence of admissible paths $(\gamma_i)_{i \geq i_0}$ in Σ, such that γ_i is quasi-transverse to μ_i and combinatorially equivalent to γ_∞, and

$$\lim_{i \to \infty} \mu_i(\gamma_i) = \mu_\infty(\gamma_\infty).$$

Since γ_i is combinatorially equivalent to γ_∞, it joins $g^{-1} \cdot v$ to $g \cdot v$ and passes through v'. Since γ_i is quasi-transverse to μ_i, it follows from [GS, 5.20] that γ_i is a parametrized arc and that χ_i maps $|\gamma_i|$ homeomorphically onto $[g^{-1} \cdot x_i^0, g \cdot x_i^0]$. Hence $x_i^0 = \chi_i(v') \in [g^{-1} \cdot x_i^0, g \cdot x_i^0]$ for large i.

This means that $[g^{-1} \cdot x_i^0, x_i^0] \cap [x_i^0, g \cdot x_i^0] = \{x_i^0\}$. By [MS, proof of Lemma II.2.4] this implies that either x_i^0 is fixed by g or $[g^{-1} \cdot x_i^0, x_i^0] \cup [x_i^0, g \cdot x_i^0]$ is contained in the axis [MS, II.2] of g; hence by [MS, Theorem II.2.3] we have $[g^{-1} \cdot x_i^0, g \cdot x_i^0] = 2\ell_i(g)$. Using [GS, 5.19], we see that

$$\begin{aligned} 2 \lim_{i \to \infty} \ell_i(g) &= \lim_{i \to \infty} [g^{-1} \cdot x_i^0, g \cdot x_i^0] = \lim_{i \to \infty} \mu_i(\gamma_i) \\ &= \mu_\infty(\gamma_\infty) = [g^{-1} \cdot x_\infty^0, g \cdot x_\infty^0] = 2\ell_\infty(g). \end{aligned}$$ □

5. Simplicial approximation of length functions in low $\mathbb{Q}$-ranks

In this section we prove our main results, which allow one to obtain simplicial approximations of certain length functions defined over subgroups of $\mathbb{R}$ of $\mathbb{Q}$-rank at most 2.

5.1. THEOREM. *Let ℓ be a length function on a finitely presented group Γ. Suppose that ℓ takes values in a subgroup Λ of $\mathbb{R}$ whose $\mathbb{Q}$-rank is at most* 2. *If Λ has $\mathbb{Q}$-rank* 2, *assume that ℓ is defined by an action of Γ on an $\mathbb{R}$-tree which satisfies the Ascending Chain Condition* [GS, 6.1]. *Then there exist a sequence (ℓ_k) of geometric $\mathbb{Z}$-length functions on Γ and a sequence (n_k) of positive integers such that* $\lim_{k \to \infty} (\ell_k(g)/n_k) = \ell(g)$ *for every $g \in \Gamma$. Furthermore, if ℓ is a small length function then the $\mathbb{Z}$-length functions ℓ_k may be taken to be small as well.*

Proof. Let us fix an action of Γ on an $\mathbb{R}$-tree which defines the length function ℓ. According to [AB, Theorem 7.13(a)] or [CM, Proposition 3.1], we may assume the action to be minimal in the sense that no proper sub-tree is Γ-invariant. Since ℓ takes values in Λ, it follows from [AB, Theorem 7.13(c)] that the tree on which Γ acts is the real completion $\mathbb{R}T$ of a Λ-tree T, and that the action of Γ on $\mathbb{R}T$ is obtained by completing an action ρ of Γ on T. After possibly enlarging Λ we may assume Λ to be a $\mathbb{Q}$-vector space; then Γ acts on T without inversions. By [AB, 6.15–6.16], ℓ may be interpreted as the length function defined by the action ρ.

If Λ has $\mathbb{Q}$-rank 2 then by hypothesis we may take the action of Γ on $\mathbb{R}T$ to satisfy the ascending chain condition. It is then clear that the action ρ of Γ on T also satisfies the ascending chain condition.

The Λ-tree with symmetry (T, Γ, ρ) is now seen to satisfy the hypotheses of [GS, Proposition 2.6] (in the case where Λ has $\mathbb{Q}$-rank 1) or [GS, Proposition 6.4] (in the case where Λ has $\mathbb{Q}$-rank 2). Hence there exists a strongly convergent direct system (T_i, Γ, ρ_i) of Λ-trees with symmetry such that all the actions ρ_i are geometric, and (T, Γ, ρ) is the limit of the system. Let ℓ^i denote the length function on Γ defined by ρ_i. By [GS, 1.28], the sequence (ℓ^i) of functions on Γ converges strongly to ℓ; that is, for every $g \in \Gamma$ we have $\ell^i(g) = \ell(g)$ for all sufficiently large i.

Now apply Theorem 1.3 to each ℓ_i. This gives, for each i, a sequence (ℓ_j^i) of $\mathbb{Z}$-length functions on Γ and a sequence (n_j^i) of positive integers, such that

$$\lim_{j\to\infty} \frac{\ell_j^i(g)}{n_j^i} = \ell^i(g)$$

for every $g \in \Gamma$.

Let us now index the set of all elements of Γ by positive integers: $\Gamma = \{g_r : r \geq 1\}$. For each positive integer k we choose an index i_k so that for all $i \geq i_k$ and for $r = 1, \ldots, k$ we have $\ell^i(g_r) = \ell(g_r)$. Likewise, for each k we choose j_k so that for all $j \geq j_k$ and for $r = 1, \ldots, k$ we have

$$\left| \frac{\ell_j^i(g_r)}{n_j^i} - \ell(g_r) \right| < 1/k.$$

We take the sequences $(i_k)_{k\geq 0}$ and $(j_k)_{k\geq 0}$ to be monotone increasing.

It follows at once that the sequence (ℓ_k) of $\mathbb{Z}$-length functions defined by $\ell_k = \ell_{j_k}^{i_k}$ satisfies

$$\lim_{k\to\infty} \frac{\ell_k(g)}{n_k} = \ell(g)$$

for every $g \in \Gamma$, where $n_k = n_{j_k}^{i_k}$.

If ℓ is a small length function then it follows from [CM, Theorem 3.7] that the action of Γ on T is small. The action ρ is then automatically small, since the stabilizer of a segment $[x, y] \subset T$ is also the stabilizer of the segment $[x, y]_{\mathbb{R}T}$ in the given tree. This implies that the actions ρ_i are small; for by the definition of the limit of a strongly convergent direct system, we have for each i a Γ-equivariant morphism of Λ-trees $\phi_i : T_i \to T$. The definition of a morphism of trees implies that the image of any segment in T_i contains some segment in T; hence the stabilizer in Γ of any segment in T_i is contained in the stabilizer of a segment in T and is therefore small.

Thus the ℓ^i are small length functions in this case. Hence by Theorem 1.3 we may take the length functions ℓ^i_j to be small. In particular the ℓ_k are then small. $\square$

5.2. If all small subgroups of Γ are all finitely generated, then any small action of Γ on an $\mathbb{R}$-tree satisfies the ascending chain condition. Hence Theorem 5.1 has the

COROLLARY. *Let Γ be a finitely presented group whose small subgroups are all finitely generated. Let ℓ be a small length function on Γ which takes values in a subgroup Λ of $\mathbb{R}$ whose $\mathbb{Q}$-rank is at most 2. Then there exist a sequence (ℓ_k) of small geometric $\mathbb{Z}$-length functions of Γ and a sequence (n_k) of positive integers, such that* $\lim_{k\to\infty} (\ell_k(g)/n_k) = \ell(g)$ *for every $g \in \Gamma$.* $\square$

5.3. Our final result is a variant of Theorem 5.1: the group is assumed only to be finitely generated (rather than finitely presented), but the action must be assumed to be free.

THEOREM. *Let ℓ be a length function defined by a free action of a finitely generated group Γ on an $\mathbb{R}$-tree. Suppose that ℓ takes values in a subgroup Λ of $\mathbb{R}$ whose $\mathbb{Q}$-rank is at most 2. Then there exist a sequence (ℓ_k) of small geometric $\mathbb{Z}$-length functions on Γ and a sequence (n_k) of positive integers, such that* $\lim_{k\to\infty} (\ell_k(g)/n_k) = \ell(g)$ *for every $g \in \Gamma$.*

Proof. This is proved in the same way as Theorem 5.1, except that in the rank-1 case we use [GS, 2.2] in place of [GS, 2.6], and in the rank-2 case we use [GS, 6.3] in place of [GS, 6.4]. $\square$

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