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On a simplicial complex associated with tilting modules

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Introduction

Let Λ be a finite-dimensional associative algebra over an algebraically closed field, and denote by mod Λ the category of finite-dimensional Λ-modules. We fix the number of pairwise non-isomorphic simple Λ-modules to be $n + 1$.

Denote by $\mathscr{E}$ a set of fixed representatives for the isomorphism classes of indecomposable Λ-modules T satisfying the following conditions:

(i) The projective dimension of T is at most 1.

(ii) T does not extend itself, *i.e.* $\mathrm{Ext}^1_\Lambda(T, T) = 0$.

Following Ringel, we define a simplicial complex $\mathscr{C}_\Lambda$ on the set $\mathscr{E}$ of vertices: $(T_0, \ldots, T_r)$ is an r-simplex if $\mathrm{Ext}^1_\Lambda(T_0 \oplus \cdots \oplus T_r, T_0 \oplus \cdots \oplus T_r) = 0$. Ringel told us that $\mathscr{C}_\Lambda$ is a triangulated ball for certain hereditary algebras. Our goal is to prove the following result:

THEOREM. *If $\mathscr{E}$ is finite, the geometric realization of $\mathscr{C}_\Lambda$ is an n-dimensional ball.*

We wish to thank C. Ringel for drawing our attention to $\mathscr{C}_\Lambda$ and N. A'Campo for discussing with us the topological aspects of the question.

1. The Bongartz completion

1.1. Recall from [3], [5] that a Λ-module T is a *tilting module* if it satisfies:

(i) $\mathrm{projdim}_\Lambda T \leq 1$,

(ii) $\mathrm{Ext}^1_\Lambda(T, T) = 0$,

(iii) There is an exact sequence

$$0 \to \Lambda \to T' \to T'' \to 0,$$

with modules T', T'' that belong to the full subcategory add T of mod Λ whose objects are direct summands of T^N for some N.

The simplest example of a tilting module is Λ itself, and for some algebras, e.g. the selfinjective ones, there are no others (aside from those obtained by changing the multiplicities of the indecomposable direct summands). Bongartz proved in [2] that a module T satisfying (i) and (ii) is a tilting module if and only if the number of its pairwise non-isomorphic indecomposable direct summands equals the number $n+1$ of isomorphism classes of simple modules. He also showed that any module T satisfying (i) and (ii) is a direct summand of a tilting module. We recall his construction: write $T = \bigoplus_{i=0}^{r} T_i^{\lambda_i}$ as a direct sum of pairwise non-isomorphic indecomposables $T_0, \ldots, T_r$ with multiplicities $\lambda_0, \ldots, \lambda_r$. Choose an exact sequence

$$0 \to \Lambda \to X \to \bigoplus_{i=0}^{r} T_i^{\mu_i} \to 0$$

with the property that, for any $k = 0, \ldots, r$, the induced map

$$\operatorname{Hom}_\Lambda\left(T_k, \bigoplus_{i=0}^{r} T_i^{\mu_i}\right) \to \operatorname{Ext}_\Lambda^1(T_k, \Lambda), \tag{$*$}$$

is surjective. Then $T \oplus X$ is the desired tilting module.

Of course the condition $(*)$ does not determine X uniquely. But it is easy to see that possible choices for X only differ by direct summands in add T, up to isomorphism. Hence T determines a multiplicity-free tilting module $\tilde{T} = \bigoplus_{i=0}^{n} T_i$, which is unique up to isomorphism. We call $T_B = T_{r+1} \oplus \cdots \oplus T_n$ the *Bongartz completion* of T.

1.2. Let $T_0, \ldots, T_n$ be pairwise non-isomorphic indecomposables, and suppose that $\bigoplus_{i=0}^{n} T_i$ is a tilting module.

PROPOSITION. *The following statements are equivalent*:

(a) $\bigoplus_{i=r+1}^{n} T_i$ *is the Bongartz completion of* $\bigoplus_{i=0}^{r} T_i$.

(b) *For* $j = r+1, \ldots, n$, *there is no surjection from any module in* add $(T_0 \oplus \cdots \oplus T_{j-1} \oplus T_{j+1} \oplus \cdots \oplus T_n)$ *to* T_j.

Proof. Let $\bigoplus_{i=r+1}^{n} T_i$ be the Bongartz completion of $\bigoplus_{i=0}^{r} T_i$, and suppose there is a surjection $f: \bigoplus_{i \neq j} T_i^{\nu_i} \to T_j$ for some $j > r$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \Lambda & \longrightarrow & T_j^{\rho_j} \oplus \bigoplus_{i \neq j} T_i^{\rho_i} & \longrightarrow & \bigoplus_{i=0}^{r} T_i^{\mu_i} & \to 0 \\
& \| & & \uparrow {\scriptstyle \begin{bmatrix} f \cdot & 0 \\ 0 & \cdot f \end{bmatrix}} & & \| \quad\quad \uparrow & \\
0 \longrightarrow & \Lambda & \xrightarrow[g]{} & \left(\bigoplus_{i \neq j} T_i^{\nu_i} \right)^{\rho_j} \oplus \bigoplus_{i \neq j} T_i^{\rho_i} & \longrightarrow & X & \longrightarrow 0.
\end{array}$$

The first row is an exact sequence used to construct the Bongartz completion, and the existence of g follows from the projectivity of Λ. The square on the right yields another exact sequence:

$$0 \to \bigoplus_{i \neq j} T_i^{\nu_i \rho_j + \rho_i} \to \bigoplus_{i=0}^{n} T_i^{\rho_i} \oplus X \to \bigoplus_{i=0}^{r} T_i^{\mu_i} \to 0,$$

which must split. But then T_j is isomorphic to some T_i for $i \neq j$, and this is impossible.

As to the converse, we choose an exact sequence

$$0 \to \Lambda \to \bigoplus_{i=0}^{n} T_i^{\alpha_i} \xrightarrow{h} \bigoplus_{i=0}^{n} T_i^{\beta_i} \to 0.$$

For any $j > r$ with $\beta_j > 0$, the composition of h with the canonical projection from $\bigoplus_{i=0}^{n} T_i^{\beta_i}$ to $T_j^{\beta_j}$ must be retraction by (b). So we can choose another such sequence with $\beta_j = 0$ for $j > r$. As our sequence then satisfies (*), $\bigoplus_{i=r+1}^{n} T_i$ must be the Bongartz completion of $\bigoplus_{i=0}^{r} T_i$.

Remark. The same arguments show that $T = \bigoplus_{i=0}^{n} T_i$ is a projective tilting module if and only if there is no surjection from any modules in $\mathrm{add}\,(T_0 \oplus \cdots \oplus T_{j-1} \oplus T_{j+1} \oplus \cdots \oplus T_n)$ to T_j, for $j = 0, \ldots, n$.

1.3. Let $T_0, \ldots, T_{n-1}$ be pairwise non-isomorphic indecomposables of projective dimension 1 at most, and assume that $\mathrm{Ext}^1_\Lambda(T, T) = 0$ for $T = \bigoplus_{i=0}^{n-1} T_i$. Denote by T_n the Bongartz completion of T.

The following result has been obtained independently by Happel in [4]. In case Λ is hereditary, it was proved in [7] and later in [6].

PROPOSITION. *There is at most one indecomposable T_n' not isomorphic to T_n such that $T \oplus T_n'$ is a tilting module. If such a T_n' exists, there is an exact sequence*

$$0 \to T_n \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T_n' \to 0.$$

We first have to recall the definitions of a source map and a sink map used in [7]. Closely related concepts have been introduced in [1]. Let $X_1, \ldots, X_r$ be pairwise non-isomorphic indecomposables and let Y be a module not having any direct summands in add X, where $X = \bigoplus_{i=1}^r X_i$.

A map $f: Y \to \bigoplus_{i=1}^r X_i^{\lambda_i}$ is a *source map from* Y to add X if

(i) for any X' in add X, any map from Y to X' factors through f, and

(ii) f is minimal with respect to property (i); *i.e.* if $\alpha \circ f$ still has property (i) for an endomorphism α of $\bigoplus_{i=1}^r X_i^{\lambda_i}$, then α is an automorphism.

Source maps exist and are unique up to isomorphism. If a map $g: Y \to \bigoplus_{i=1}^r X_i^{\mu_i}$ has property (i), it is isomorphic to $\begin{bmatrix} f \\ 0 \end{bmatrix}: Y \to \bigoplus_{i=1}^r X_i^{\lambda_i} \oplus X'$ for any source map f, where X' lies in add X.

Sink maps from add X *to* Y are defined by dualizing the definition of source maps.

Proof of the proposition. Let T_n' be an indecomposable not isomorphic to T_n such that $T \oplus T_n'$ is a tilting module. By the preceding proposition, there is a surjection from some module in add T to T_n'. In particular, any sink map

$$g: \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T_n',$$

from add T to T_n' is surjective. Consider the exact sequence

$$0 \to Z \xrightarrow{f} \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \xrightarrow{g} T_n' \to 0,$$

where $Z = \ker g$.

Since g is a sink map, f lies in the radical of mod Λ; *i.e.*, its restriction to any indecomposable direct summand of Z is never a section. Moreover, any map from Z to T_j factors through f, since we have $\mathrm{Ext}^1(T_n', T_j) = 0$, for $j = 0, \ldots, n-1$. Therefore Z has no direct summand that belongs to add T. As g lies in the radical of mod Λ, f is a source map from Z to add T.

Obviously the projective dimension of Z is 1 at most, and by construction we have $\mathrm{Ext}_\Lambda^1(T_j, Z) = 0$, for $j = 0, \ldots, n-1$. Considering maps from our sequence to Z and T_j, respectively, and using that $\mathrm{projdim}_\Lambda T_n' \le 1$, we find that $\mathrm{Ext}_\Lambda^1(Z, Z) = 0$ and $\mathrm{Ext}_\Lambda^1(T_j, Z) = 0$, for $j = 0, \ldots, n-1$. As Z does not belong to add T, $T \oplus Z$ is a tilting module.

If there were a surjection from some T' in add T to Z, it would induce a surjection from $\mathrm{Ext}_\Lambda^1(T_n', T')$ to $\mathrm{Ext}^1(T_n', Z)$, since $\mathrm{projdim}_\Lambda T_n' \le 1$. But this is impossible, as the first group is zero and our sequence does not split. By the preceding proposition, we know that Z is isomorphic to T_n^λ for some $\lambda \ge 1$, and we may suppose $Z = T_n^\lambda$.

We now want to show that $\lambda = 1$. Let $h : T_n \to T'$ be a source map from T_n to add T. The map

$$\begin{bmatrix} h & & 0 \\ & \ddots & \\ 0 & & h \end{bmatrix} : T_n^\lambda \to T'^\lambda,$$

still has the first property of a source map, and it is therefore isomorphic to

$$\begin{bmatrix} f \\ 0 \end{bmatrix} : T_n^\lambda \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \oplus T'',$$

for some T'' in add T. Comparing cokernels, we find that $(\operatorname{coker} h)^\lambda$ is isomorphic to $T'' \oplus T'_n$, which implies $\lambda = 1$, by Krull–Schmidt.

Finally, since $f : T_n \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i}$ is a source map, its cokernel T'_n is determined uniquely, up to isomorphism, by T_n. Our proposition is proved.

Remark. There exist modules T as in the proposition whose only completion is the Bongartz completion T_n. Indeed, if $\bigoplus_{i=0}^{n} P_i$ is a projective tilting module, at least one of the modules $\bigoplus_{i \neq j} P_i$ has this property, since chains of injections in the radical of mod Λ between projectives have bounded length.

2. Proof of the theorem

2.1. We associate a quiver K with the complex $\mathscr{C}_\Lambda$ defined in the introduction in the following way: the vertices of K are the n-simplices of $\mathscr{C}_\Lambda$. For each $(n-1)$-simplex $(T_0, \dots, T_{n-1})$ which is face of two n-simplices, K contains an arrow $\sigma = (T_0, \dots, T_n) \to \sigma' = (T_0, \dots, T_{n-1}, T'_n)$, where T_n is the Bongartz completion of $\bigoplus_{i=0}^{n-1} T_i$. For any simplex τ of $\mathscr{C}_\Lambda$, we let K_τ denote the full subquiver of K whose vertices are then n-simplices of $\mathscr{C}_\Lambda$ containing τ.

LEMMA. *Let τ be a simplex of $\mathscr{C}_\Lambda$. If there is a path $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_s$ in K with σ_1, σ_s in K_τ, then the whole path lies in K_τ.*

Proof. Recall that, for a tilting module T, the category $\mathscr{T}(T)$ of torsion modules with respect to T is the full subcategory of mod Λ whose objects are quotients of T^N for some N. Set $\mathscr{T}(\sigma) = \mathscr{T}(\bigoplus_{i=0}^{n} T_i)$ for $\sigma = (T_0, \dots, T_n)$.

If K contains an arrow $\sigma = (T_0, \dots, T_n) \to \sigma' = (T_0, \dots, T_{n-1}, T'_n)$, there is an exact sequence

$$0 \to T_n \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T'_n \to 0,$$

by 1.3, and therefore any module in $\mathcal{T}(\sigma')$ belongs to $\mathcal{T}(\sigma)$. However by 1.2, T_n does not lie in $\mathcal{T}(\sigma')$. Moreover, for any path $\sigma \to \sigma' \to \cdots \to \sigma''$ in K, $\mathcal{T}(\sigma'')$ lies in $\mathcal{T}(\sigma')$ and thus does not contain T_n.

The lemma follows by applying these considerations to $\sigma = \sigma_k \to \sigma' = \sigma_{k+1} \to \cdots \to \sigma'' = \sigma_s$ in case $\sigma_1 \to \cdots \to \sigma_s$ does not lie in K_τ, where k is the maximal index for which $\sigma_1 \to \cdots \to \sigma_k$ is in K_τ. Then τ contains T_n, by the choice of k, but σ_s cannot.

2.2. Applying the lemma to an n-simplex we find:

PROPOSITION. *K does not contain oriented cycles.*

This allows us to define an *order relation* for the n-simplices of $\mathscr{C}_\Lambda : \sigma \le \sigma'$ if there is an oriented path $\sigma = \sigma_1 \to \sigma_2 \to \cdots \to \sigma_s = \sigma'$ in K.

Remarks. (a) The *Hasse diagram* of this order relation is the quiver whose vertices are the n-simplices of $\mathscr{C}_\Lambda$ and which contains an arrow $\sigma \to \sigma'$ if $\sigma \le \sigma'$, $\sigma \ne \sigma'$ and $\sigma \le \sigma'' \le \sigma'$ implies either $\sigma'' = \sigma$ or $\sigma'' = \sigma'$. Applying the lemma to an $(n-1)$-simplex which is face of two n-simplices, it is easy to see that the Hasse diagram coincides with K.

(b) Our order relation is in general distinct from the one defined by: $\sigma \le \sigma'$ if $\mathcal{T}(\sigma) \supseteq \mathcal{T}(\sigma')$. The projective and the injective tilting module of a hereditary algebra of infinite representation type furnish an example. We don't know, however, whether the Hasse diagrams coincide.

2.3. Suppose now that $\mathscr{E}$ is finite. Number the n-simplices $\sigma_1, \sigma_2, \ldots, \sigma_M$ of $\mathscr{C}_\Lambda$ in such a way that $\sigma_i \le \sigma_j$ implies $i \le j$. For $N \le M$, let $\mathscr{B}_N$ be the union of $\sigma_1, \sigma_2, \ldots, \sigma_N$.

The following proposition implies our theorem.

PROPOSITION. *The geometric realization of $\mathscr{B}_N$ is an n-ball, for all N.*

Proof. The result is true for $n = 0$, as a local algebra admits no modules of projective dimension 1.

For $n > 0$, we proceed by induction on N, the case $N = 1$ being obvious. Suppose that the geometric realization of $\mathscr{B}_{N-1}$ is an n-ball for some $N \ge 2$. Our goal is to show that the intersection $\sigma_N \cap \mathscr{B}_{N-1}$, which lies in the boundary of $\mathscr{B}_{N-1}$, is a union of $(n-1)$-faces of σ_N. Then the geometric realization of $\mathscr{B}_N$ is either an n-sphere or an n-ball, according as $\sigma_N \cap \mathscr{B}_{N-1}$ is the whole boundary of σ_N or not. The case of a sphere can be ruled out, as we know that $\mathscr{B}_N$ has a non-empty boundary by the remark in 1.3.

The intersection $\sigma_N \cap \mathscr{B}_{N-1}$ contains at least one $(n-1)$-face of σ_N, and hence $\mathscr{B}_N$ is connected. Indeed, σ_N is distinct from the unique minimal n-simplex of $\mathscr{C}_\Lambda$, whose vertices are the indecomposable projectives (remark 1.2). Any predecessor of σ_N in K, and in particular the tail of any arrow in K whose head in σ_N, belongs to $\mathscr{B}_{N-1}$.

Now let $\tau = (T_0, \ldots, T_r)$ be a simplex in $\sigma_N \cap \mathscr{B}_{N-1}$, and let $\bigoplus_{i=r+1}^n T_i$ be the Bongartz completion of $\bigoplus_{i=0}^r T_i$. By proposition 1.2, the n-simplex $\sigma = (T_0, \ldots, T_n)$ is the unique minimal vertex of K_τ. Note that σ_N is a vertex of K_τ. As any path in K from σ to σ_N lies in K_τ by lemma 2.1, and since any predecessor of σ_N belongs to $\mathscr{B}_{N-1}$, there is an $(n-1)$-simplex in $\sigma_N \cap \mathscr{B}_{N-1}$ containing τ.

Remark. If $\mathscr{C}_\Lambda$ is infinite, the same argument shows that the geometric realization of a union $\sigma_1 \cup \cdots \cup \sigma_M$ is an n-ball, provided that the full subquiver of K whose vertices are $\sigma_1, \ldots, \sigma_M$ is closed under predecessors in K.

3. Examples

3.1. Let Q be the quiver $\cdot \rightrightarrows \cdot$ and Λ its quiver algebra. Denote by P_m and I_m the preprojective and preinjective indecomposables, respectively, given by

$$P_m = k^m \underset{\begin{bmatrix}0 & & 0\\ 1 & & \\ & \ddots & \\ & & 1\end{bmatrix}}{\overset{\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\\ 0 & \cdots & 0\end{bmatrix}}{\rightrightarrows}} k^{m+1} \qquad I_m = k^{m+1} \underset{\begin{bmatrix}0 & 1 & & 0\\ \vdots & & \ddots & \\ 0 & 0 & & 1\end{bmatrix}}{\overset{\begin{bmatrix}1 & 0 & & 0\\ & \ddots & & \vdots\\ 0 & & 1 & 0\end{bmatrix}}{\rightrightarrows}} k_m$$

for $m \geq 0$. These are the only indecomposables that do not extend themselves. As $\mathscr{E}$ is infinite, our theorem does not apply. In fact, the complex $\mathscr{C}_\Lambda$ has two connected components:

$$P_0 - P_1 - P_2 - \cdots$$

$$\cdots I_2 - I_1 - I_0.$$

The arrows of K are:

$$(P_m, P_{m+1}) \to (P_{m+1}, P_{m+2})$$

and

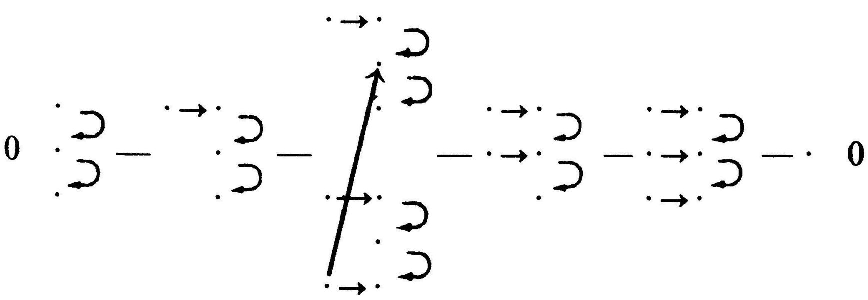
$$(I_{m+2}, I_{m+1}) \to (I_{m+1}, I_m),$$

for $m \geq 0$. They all correspond to almost split sequences.

3.2. Let Λ be the quiver algebra of $Q = 1 \to 2 \to 3 \leftarrow 4$, and denote by $\overline{ij}$ a representative of the indecomposable whose support are the vertices $i, i+1, \ldots, j$, for $1 \leq i \leq j \leq 4$. We only draw K as it contains all information necessary to build $\mathscr{C}_\Lambda$.

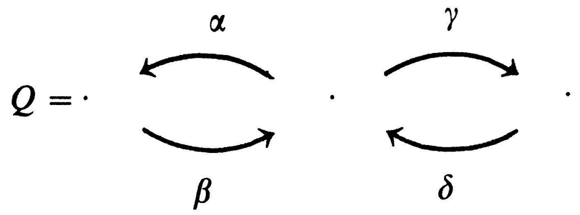
$$\begin{tikzcd}
 & & (\overline{12}, \overline{13}, \overline{14}, \overline{22}) \arrow[dr] \arrow[ddrr, bend left=60] & & \\
(\overline{13}, \overline{22}, \overline{23}, \overline{24}) \arrow[r] & (\overline{13}, \overline{14}, \overline{22}, \overline{24}) \arrow[r] \arrow[ur] & (\overline{14}, \overline{22}, \overline{24}, \overline{44}) \arrow[r] & (\overline{12}, \overline{14}, \overline{22}, \overline{44}) \arrow[dr] & \\
(\overline{13}, \overline{23}, \overline{24}, \overline{34}) \arrow[r] \arrow[u] & (\overline{13}, \overline{14}, \overline{24}, \overline{34}) \arrow[r] \arrow[u] \arrow[dr] & (\overline{14}, \overline{24}, \overline{34}, \overline{44}) \arrow[r] \arrow[u] & (\overline{11}, \overline{14}, \overline{34}, \overline{44}) \arrow[r] & (\overline{11}, \overline{12}, \overline{14}, \overline{44}) \\
(\overline{13}, \overline{23}, \overline{33}, \overline{34}) \arrow[r] \arrow[u] & (\overline{11}, \overline{13}, \overline{33}, \overline{34}) \arrow[r] & (\overline{11}, \overline{13}, \overline{14}, \overline{34}) \arrow[r] \arrow[ur] & (\overline{11}, \overline{12}, \overline{13}, \overline{14}) \arrow[ur] &
\end{tikzcd}$$

3.3. Consider the quiver $Q = \cdot \xrightarrow{\alpha} \cdot \circlearrowleft \beta$, let I be the two-sided ideal in the quiver-algebra kQ generated by β^3, and set $\Lambda = kQ/I$. Then C_Λ is an interval:

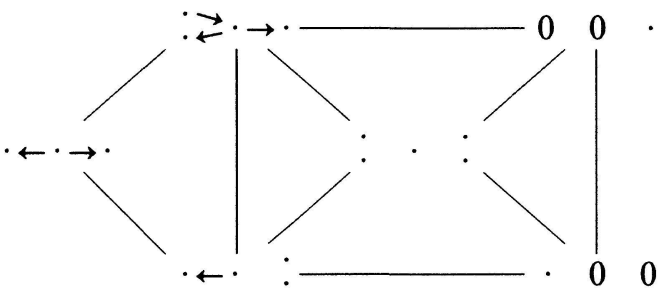


To picture representations, we represent each basis vector by a dot. The linear map $V(\gamma) : V(i) \to V(j)$ corresponding to an arrow $\gamma : i \to j$ sends a dot in $V(i)$ to the sum of the heads of all arrows of type γ starting at the dot, and to zero if there is no such arrow.

3.4. Finally, we give an example of an algebra Λ of infinite representation type and for which the complex $\mathscr{C}_\Lambda$ is finite. Let Q be the quiver



and I the two-sided ideal in kQ generated by $\alpha\beta$ and $\gamma\delta$. The complex $\mathscr{C}_\Lambda$ for the algebra $\Lambda = kQ/I$ is the following:



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