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On Cheeger's inequality

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In [Ch], Cheeger proved the following general lower bound for the first eigenvalue λ_1 of a closed Riemannian manifold:

THEOREM ([Ch]):

$$\lambda_1 \geq \frac{1}{4} h^2,$$

where

$$h = \inf_N \frac{\text{area}\,(N)}{\min\,(\text{vol}\,(A), \text{vol}\,(B))}$$

where N runs over (possibly disconnected) hypersurfaces of M which divide M into two pieces A and B, and where area *denotes $(n-1)$-dimensional volume, and* vol *denotes n-dimensional volume, where $n = \dim(M)$.*

$h(M)$ is called the Cheeger constant of M.

Cheeger's inequality is quite straightforward to prove, and is essentially the co-area formula of geometric measure theory. It is therefore surprising that the inequality plays such a crucial role in the study of the geometry of the Laplace operator, see [Bu3]. Indeed, one has the following general upper bound for λ_1 in terms of h, due to Peter Buser [Bu]:

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THEOREM ([Bu]):

$$\lambda_1 \le c_1 h + c_2 h^2,$$

where c_1, c_2 depend only on a lower bound on the Ricci curvature of M.

Thus, from a qualitative point of view, λ_1 and h are essentially the same thing, in the sense that one tends to zero if and only if the other does (in the presence of bounded curvature).

We observe that Cheeger's inequality is true, and is proved in exactly the same way, when M is a complete, non-compact manifold, or a manifold with boundary and either Dirichlet or Neumann boundary conditions, provided one interprets λ_1 and h correctly.

It has therefore been an interesting question to understand, in a general way, how sharp Cheeger's inequality really is. A major problem in coming to terms with this question has been that, for the most part, Cheeger's inequality is the only generally useful method known for estimating λ_1 from below.

In this paper, we will explore this question in three ways. First of all, by a celebrated theorem of Selberg [Se], there are general lower bounds

$$\lambda_1(S_p) \ge \frac{3}{16}$$

for certain arithmetic Riemann surfaces S_p, which we will discuss below. Selberg raised the question of whether

$$\lambda_1(S_p) \ge \frac{1}{4}$$

for these surfaces, and it was suggested in [Bi] that perhaps one could demonstrate this by showing that $h(S_p) \ge 1$ for these surfaces.

We will show that this is not the case, and indeed $h(S_p)$ is so small for these surfaces that one cannot even obtain Selberg's $\frac{3}{16}$ bound via Cheeger's constant:

THEOREM 1.1. *For $p \equiv 1 \pmod 4$,*

$$h(S_p) \le \frac{3\log(3)}{2\pi}\left(\frac{p-1}{p+1}\right).$$

Note that $3\log(3)/2\pi$ has a value of approximately .52455. The value of $(1/4)(.52455)^2$ is approximately .068788, a little bit bigger than 1/16.

Secondly, we will show:

THEOREM 2.1. *There exist two isospectral Riemann surfaces S_1 and S_2 whose Cheeger constants satisfy*

$$h(S_1) \neq h(S_2).$$

This too answers a question raised in [Bi].

Both of these results lie in the category of surfaces with boundary geometry – and indeed the examples have constant curvature – 1. For our third result, we will leave this category to study the spectral geometry of manifolds of 2 and 3 dimensions with no curvature assumptions. We will show:

THEOREM 3.1. *For $n = 2$ or 3, there is a constant $K(n)$ such that, if M is a compact n-manifold satisfying*

$$\lambda_1 > K(n) \frac{\|\mathrm{Ricc}\|_2}{\sqrt{\mathrm{Vol}\,(M)}},$$

then the Cheeger constant of M is bounded above and below in terms of the spectrum of M.

We give some numerical estimates for $K(n)$ below. In a separate paper [BPP], we show by example that the number $K(n)$ cannot be made arbitrarily small.

According to Cheeger's inequality, λ_1 is bounded below by h, so the content of Theorem 3.1 is to give an upper bound for λ_1 in terms of h analogous to Buser's inequality, where the constants involved depend only on spectral data, rather than pointwise curvature bounds. Indeed, Theorem 3.1 may be thought of as a version of Buser's Inequality, with L^p curvature bounds for $p > n/2$, $n = \dim(M)$, replacing pointwise curvature bounds. The dimension restriction enters from the fact that L^2 bounds are available from the spectrum, so one requires that $2 > n/2$.

The first two results answer questions which were raised by Frederic Bien in [Bi]. We would like to thank him for his prodding, which encouraged us to write the present paper.

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1. Selberg's theorem

Let $\Gamma = PSL(2, \mathbb{Z})$, and let

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ n) \right\}$$

be the congruence subgroup of Γ of level n. It is easily seen that $\Gamma/\Gamma_n \cong PSL(2, \mathbb{Z}/n)$. Then Γ (and hence Γ_n) acts on the hyperbolic plane $\mathbb{H}$, with quotient a finite area Riemann surface with singularities, whose fundamental domain is the well-known figure shown in Figure 1.

For all n, $\mathbb{H}/\Gamma_n$ is a finite orbifold covering of this surface, and for $n \neq 2$ or 3, $\mathbb{H}/\Gamma_n$ has no singularities.

It was shown by Selberg [Se] that $\lambda_1(\mathbb{H}/\Gamma_n) \geq \frac{3}{16}$ for all n, and he further conjectured that $\lambda_1(\mathbb{H}/\Gamma_n) \geq \frac{1}{4}$.

Selberg's Theorem can be "compactified" in a number of ways, to provide families of compact Riemann surfaces with large λ_1. For our purposes, one of the most interesting of these compactifications is a recent result of Burger, Buser, and Dodziuk [BBD], which proceeds in the following way:

Let us take a Riemann surface S with an even number of cusps, and pair off the cusps in some arbitrary way. Then, for each ε, we may perturb the metric on S slightly, to obtain a new Riemann surface S_ε, which is compact and bounded by geodesic circles of length ε. We may then glue corresponding cusps together to obtain a closed surface S_ε.

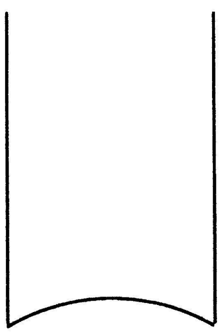

Figure 1. The fundamental domain.

It is more-or-less evident that, as ε tends to zero $h(\tilde{S}_\varepsilon)$ tends to $h(S)$. To see this, observe that as $\varepsilon \to 0$, the necks in $\tilde{S}_\varepsilon$ become arbitrarily long, so that the optimal way of dividing $\tilde{S}_\varepsilon$ into two pieces is to divide S into two pieces, and then snip off the appropriate thin necks. Any other method would have to involve a curve which passed through the whole length of the neck, and hence contribute too much to the numerator in the ratio defining h.

It is less obvious that $\lambda_1(\tilde{S}_\varepsilon)$ tends to $\lambda_1(S)$ as ε tends to 0. This is shown in [BBD].

If we now set $S_p = H/\Gamma_p$, we will now estimate $h(S_p)$ from above:

THEOREM 1.1. *Let $p \equiv 1 \pmod 4$.*
Then

$$h(S_p) \leq \left(\frac{3\log(3)}{2\pi} \cdot \frac{(p-1)}{(p+1)}\right).$$

Proof. We will first pick two generators

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for $PSL(2, \mathbb{Z})$. Note that these two generators are the "geometric" generators for the fundamental domain F shown in Figure 1 – that is, they correspond to elements of $\pi_1(S)$ which identify the edges of F.

We may now describe S_n in the following graph-theoretic way: Consider the graph G_n whose vertices are given by elements of $PSL(2, \mathbb{Z}/n)$, and whose edges are given by left-multiplication by U and V. This is a trivalent graph, where every vertex has two edges corresponding to U and one corresponding to V.

To obtain S_n, we will take one copy of F for each vertex of G_n, and glue boundary components of F according to the edges of G_n.

We will now try to decompose S_n in the following way: we will write

$$S_n = A_n \cup B_n,$$

where A_n and B_n are unions of copies of F. This will be accomplished by cutting S_n along boundary components of F. Since we want the cuttings to be of finite length, we will only cut along edges corresponding to V.

To record this information in a useful way, we observe that if W is a matrix in $SL(2, \mathbb{Z})$, then multiplication by U does not change the bottom row of W, while V flips top and bottom rows with a sign change. Thus we are led to the graph G_n', described as follows: the vertices of G_n' are equivalence classes of row vectors in $\mathbb{Z}/n \times \mathbb{Z}/n$, with $(a, b) \sim (-a, -b)$, and the greatest common divisor of a and b relatively prime to n. Furthermore, (a, b) and (c, d) are joined by an edge if

$$\det\begin{pmatrix} a & b \\ b & d \end{pmatrix} \equiv \pm 1 \pmod{n}.$$

We show G_n' for $n = 5$ in Figure 2. Note that each vertex of G_n' has exactly n edges leading from it.

In order to visualize G_n', we note that G_5' is the 1-skeleton of the icosahedron. In general, G_n' is dual to the 1-skeleton of a polygonal division of a surface into regular n-gons, so that G_3' is the 1-skeleton of a tetrahedron, G_4' is the 1-skeleton of an octahedron, and so on.

We will now estimate $h(G_p')$ for p a prime number.

LEMMA 1. *For* $p \equiv 1 \pmod 4$,

$$\frac{p^2 - 2p + 5}{4(p-1)} \leq h(G_p') \leq \frac{(p-1)p}{2(p+1)}.$$

Proof. We begin with the following algebraic:

LEMMA 2. *Given* (a, b) *and* (a', b') *with*

$$\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \neq 0,$$

there exist two distinct paths of length 2 *joining* (a, b) *and* (a', b') *in* G_p'.

Proof. A path of length 2 joining (a, b) and (a', b') is given by a vector (c, d) satisfying:

$$\text{(a)} \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm 1 \pmod{p}$$

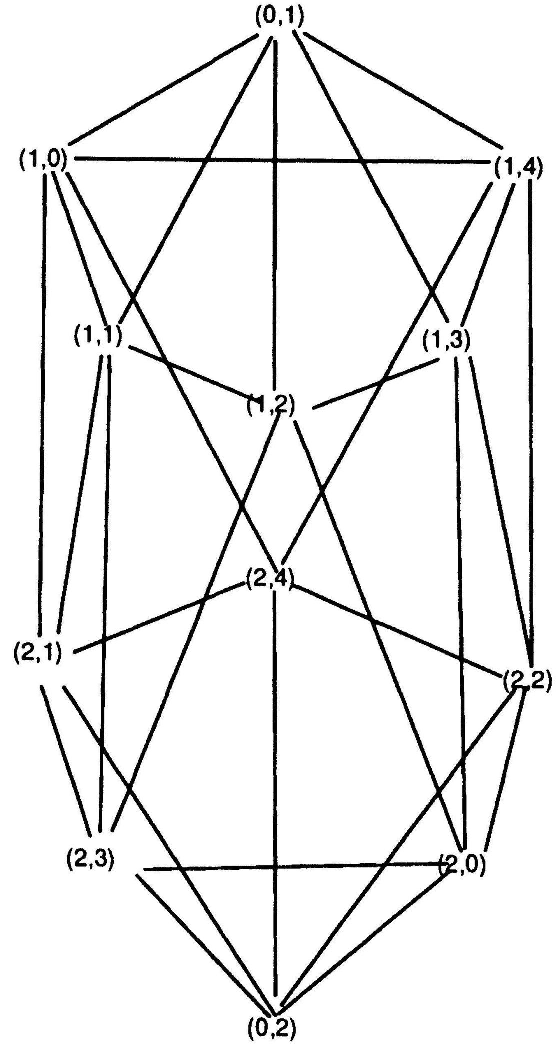

Figure 2. The graph G'_5.

and

(b) $\det\begin{pmatrix} c & d \\ a' & b' \end{pmatrix} \equiv \pm 1 \pmod p$.

Two such paths given by (c, d) and (c', d') will be distinct unless

$(c, d) = \pm(c', d')$.

Since

$$\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = \alpha \neq 0,$$

any vector (c, d) may be written as

$$(c, d) = k_1(a, b) + k_2(a', b'),$$

so that

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = k_2 \det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$$
$$= k_2\alpha,$$

while

$$\det\begin{pmatrix} c & d \\ a' & b' \end{pmatrix} = k_1 \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = k_1\alpha,$$

so that choosing

$$k_1 = \pm\frac{1}{\alpha} \qquad k_2 = \pm\frac{1}{\alpha}$$

gives four possible choices for (c, d), which represent two distinct paths in G'_p.

This completes the proof of Lemma 2.

Now let us decompose G_p into two sets A and B by removing a collection of edges E, and suppose that $\#(A) \leq \#(B)$. We wish to estimate $\#(E)/\#(A)$ from below.

For each element $(a, b) \in A$, and for each element $(a', b') \in B$ not a multiple of (a, b), the Lemma establishes that these are two paths of length 2 joining (a, b) to (a', b'). In each of these two paths, one of the two edges must lie in E. Furthermore, each edge lies in at most $2(p-1)$ different sets of paths of length 2. It follows that

$$\#(E) \geq \frac{2\#(A)\left(\#(B) - \left(\frac{p-1}{2}\right) + 1\right)}{2(p-1)}$$

so that

$$\frac{\#(E)}{\#(A)} \geq \frac{2\left(\#(B) - \left(\frac{p-1}{2}\right) + 1\right)}{2(p-1)}$$
$$\geq \frac{(p^2 - 2p + 5)}{4(p-1)},$$

since $\#(B) \geq \#(G_p)/2 = (p^2-1)/4$.

This establishes the lower bound of the lemma.

To establish the upper bound, we will assume $p \equiv 1 \pmod 4$, and divide G_p into two sets A and B as follows: Let

$$A = \{(0, a) : a \text{ is a square} \pmod p\} \cup \{(b, c) : b \neq 0 \text{ is a square} \pmod p\}$$

and

$$B = \{(0, a) : a \text{ is not a square} \pmod p\}$$
$$\cup \{(b, c) : b \neq 0 \text{ is not a square} \pmod p\}.$$

Note that $\#(A) = \#(B) = (p^2 - 1)/4$.

Let E be the number of edges joining an element of A with an element of B. Then:

CLAIM:

$$\#(E) = \frac{(p-1)}{4} \cdot p\left(\frac{p-1}{2}\right).$$

Proof. No element of A of the form $(0, a)$ is joined with an element of B of the form (b, c), since

$$\det\begin{pmatrix} 0 & a \\ b & c \end{pmatrix} = -ab$$

is not a square $\pmod p$. Similarly, $(0, a)$ is not joined to an element of the form $(0, a')$, since

$$\det\begin{pmatrix} 0 & a \\ 0 & a' \end{pmatrix} = 0.$$

On the other hand, every element of A of the form (b, c), $b \neq 0$, is joined to exactly $\left(\frac{p-1}{2}\right)$ elements of B, since if $\det\begin{pmatrix} b' & c' \\ b & c \end{pmatrix} = 1$, then the vertices joining (b, c) are the vectors of the form $(b', c') + k \cdot (b, c) = (b' + k \cdot b, c' + k \cdot c)$, and, since $b \neq 0$, each equivalence class $\pmod p$ occurs as the first coordinate of such a vector exactly once.

It follows that

$$\frac{\#(E)}{\#(A)} = \frac{\dfrac{(p-1)\cdot p}{4} \cdot \dfrac{(p-1)}{2}}{\dfrac{(p^2-1)}{4}}$$

$$= \frac{(p-1)p}{2(p+1)},$$

and so $h(G_p) \leq (p-1)p/2(p+1)$, as desired.

To prove the theorem, we may now divide S_p into two pieces in the following way: let $\mathscr{A}$ be the union of the fundamental domains corresponding to matrices in $PSL(2, \mathbb{Z}/p)$ whose bottom row lies in A, and $\mathscr{B} = S_p - \mathscr{A}$. Then $\mathscr{A}$ and $\mathscr{B}$ are separated by a geodesic curve (possibly with several components) consisting of one arc for each element of E. This arc is isometric to the bottom arc in Figure 1, and the length of this arc is easily calculated by elementary hyperbolic trigonometry to be $\log(3)$.

On the other hand,

$$\text{area}((\mathscr{A})) = \text{area}(F) \cdot \#(A) \cdot p,$$

since each vertex of G_p corresponds to p copies of F in S_p, and

$$\text{area}(F) = \frac{\pi}{3},$$

so that

$$h(S_p) \leq \frac{\log(3)}{\pi/3} \cdot \frac{h(G_p)}{p} \leq \frac{3\log(3)}{\pi} \frac{(p-1)}{2(p+1)},$$

as desired.

2. Isospectral surfaces

In this section, we will prove:

THEOREM 2.1. *There exists a pair of isospectral Riemann surfaces S_1 and S_2 with $h(S_1) \neq h(S_2)$.*

We begin the proof with the analogous statement for graphs. Consider the graphs G_1 and G_2 shown in Figures 3 and 4.

These graphs are the Cayley graphs for coset spaces G/H_1 and G/H_2 respectively, where the $G = PSL(3, \mathbb{Z}/2)$,

$$H_1 = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix},$$

with generators

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

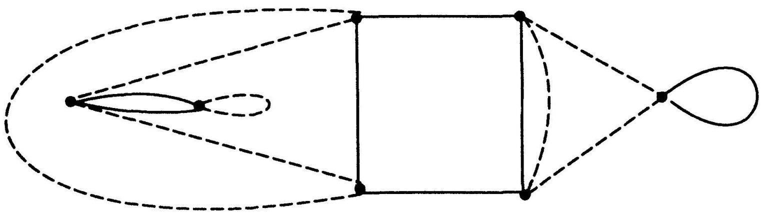

Figure 3. The graph G_1.

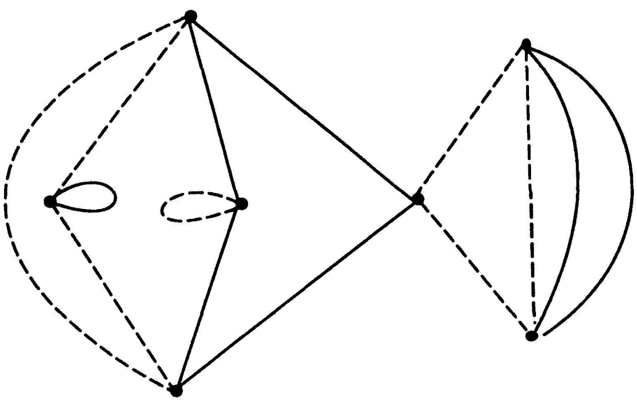

Figure 4. The graph G_2.

representing the solid lines, and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

representing the dotted lines, see [Bu2] for details.

This triple of groups was used in [BT] to provide examples of isospectral surfaces of genus 3 and 4, and by Buser in [Bu2] to provide examples of flat surfaces which are isospectral and topologically planar. The drawings in Figures 3 and 4 came from [Bu2].

The fact that these graphs are isospectral comes from Sunada's Theorem [Su], or can be verified directly.

We now observe the following distinction between the two graphs: graph G_2 can be disconnected into two pieces, one of which contains 2 vertices and the other of which contains four vertices, by removing one vertex, while the graph G_1 cannot be so disconnected.

Now consider a Riemann surface S_0 as shown in Figure 5, which is built out of two Y-pieces as shown in Figure 6.

Here the bottom boundary component has length ε, assumed small, while the top two components are of some sizeable length (say, for instance, at least 10ε). It is easy to arrange this so that every geodesic of S_0 other than the one of length ε has length at least, say, 3ε.

We now form two surfaces S_1 and S_2, which are coverings of the surface S_0, and are obtained in the follow way from the graphs G_1 and G_2: we open up S_0 along the two curves A and B to obtain a surface S which is conformally S^2 with four disks removed. At each vertex in the graph G_i $(i = 1, 2)$, we place a copy of S, and then join boundary components corresponding to A whenever the corresponding vertices are joined by a solid edge, and similarly for B.

According to Sunada's theorem [Su], the surfaces S_1 and S_2 are now isospectral.

We claim that $h(S_1) \neq h(S_s)$. To see this, we first observe that

$$h(S_2) \leq \frac{\varepsilon}{10\pi},$$

since S_2 may be disconnected into two pieces, the smallest of which contains five Y-pieces, by cutting one curve of length ε, and each Y-piece has area 2π.

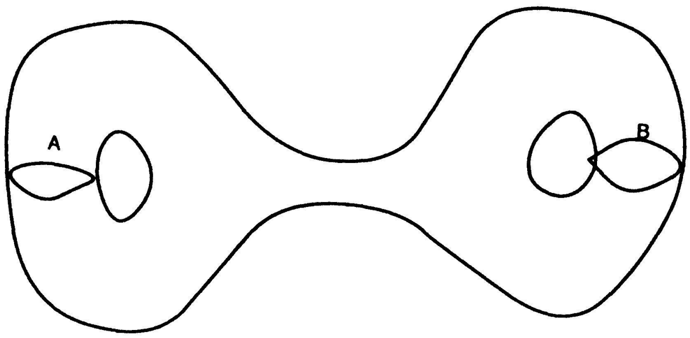

Figure 5. The surface S_0.

On the other hand, we have that

$$h(S_1) \geq \frac{\varepsilon}{8\pi},$$

which can be seen as follows: the most efficient way of dividing S_1 into two pieces by a geodesic curve of length ε has the smaller piece consisting of 3 Y-pieces. One can get a somewhat better Cheeger constant by cutting along a curve of constant mean curvature homotopic to this geodesic, rather than the geodesic itself, but this curve cannot cut off an area larger than four Y-pieces. Thus the best Cheeger constant that can be achieved by cutting along only one curve is $\varepsilon/8\pi$. But if one cuts along two curves, the length must be at least 2ε, while the smallest piece can be at most 14π. Thus, $h(S_1)$ is at most $\varepsilon/8\pi$.

This completes the proof of Theorem 2.1.

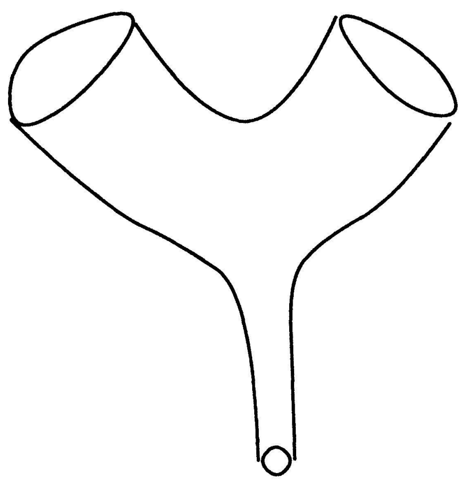

Figure 6. One Y-piece.

3. A bound for the Cheeger constant

In this section, we will prove:

THEOREM 3.1. *For $n = 2$ or 3, there is a constant $K(n, 2)$ such that, if M is a compact n-manifold satisfying*

$$\lambda_1(M) > K(n, 2)\frac{\|\mathrm{Ricc}\|_2}{\sqrt{\mathrm{Vol}\,(M)}},$$

then $h(M)$ is bounded above and below by the spectrum of M.

Our proof gives a value of $K(2, 2)$ of approximately 58.16359, and a value of $K(3, 2)$ of approximately 236.65428. In [BPP], we show by example that $K(n, p)$ cannot be arbitrarily small.

Note that, from Cheeger's inequality, $h(M)$ is bounded above by $2\sqrt{\lambda_1}$. Thus, the non-trivial part of Theorem 3.1 is to bound $h(M)$ from below in terms of spectral data. In fact, we will prove:

THEOREM 3.2. *Given n and $p > n/2$, there is a constant $K(n, p)$ such that if M is an n-manifold satisfying*

$$\lambda_1(M) > K(n, p)\frac{\|\mathrm{Ricc}\|_p}{\mathrm{Vol}\,(M)^{1/p}},$$

then $h(M)$ is bounded from below in terms of $\mathrm{Vol}\,(M)$, $\lambda_1(M)$, *and* $\|\mathrm{Ricc}\|_p$.

We remark that Theorem 3.1 follows from Theorem 3.2 by noting that $\mathrm{Vol}\,(M)$ is the a_0 term in the heat expansion of M, and hence a spectral invariant, while for manifolds of dimension <6, $\|\mathrm{Ricc}\|_2$ is bounded by the a_2 term in the heat expansion.

We begin our discussion by first considering the function

$$g(x) = \frac{e^x - 1}{x^2}$$

which occurs in the volume and eigenvalue estimates below. It is easily seen that as $x \to 0^+$ and as $x \to +\infty$, we have that $g(x) \to \infty$. Since $g'(x)$ has a unique zero in $(0, \infty)$, it follows that there is a positive number x_0 at which $g(x)$ attains its minimum. This value is given approximately by

$$x_0 = 1.594625$$

and

$$g(x_0) = 1.55441386.$$

We do not know a closed-form expression for either x_0 or $g(x_0)$.

The idea of the proof of Theorem 3.2 can now be described as follows: Suppose that D is a judiciously chosen domain in M, and denote by D_ε the tubular neighborhood

$$D_\varepsilon = \{x \in M : \operatorname{dist}(x, D) < \varepsilon\}$$

about D, with boundary ∂D_ε.

Suppose that the volume of $D_\varepsilon - D$ is not too big, and $\operatorname{vol}(D)$ and ε are not too small. Then we may construct test functions $f_{1,\varepsilon}$ and $f_{2,\varepsilon}$ by

$$\begin{aligned} f_{1,\varepsilon} &= 1 && \text{on } D \\ &= 1 - \frac{2}{\varepsilon} \operatorname{dist}(x, D) && \text{for } \operatorname{dist}(x, D) \le \frac{\varepsilon}{2} \\ &= 0 && \text{for } \operatorname{dist}(x, D) > \frac{\varepsilon}{2} \end{aligned}$$

and

$$\begin{aligned} f_{2,\varepsilon} &= 1 && \text{on } M - D_\varepsilon \\ &= \frac{2}{\varepsilon} \operatorname{dist}(x, D) - 1 && \text{for } \frac{\varepsilon}{2} \le \operatorname{dist}(x, D) \le \varepsilon \\ &= 0 && \text{for } \operatorname{dist}(x, D) < \frac{\varepsilon}{2} \end{aligned}$$

Then $f_{1,\varepsilon}$ and $f_{2,\varepsilon}$ are functions with disjoint support whose Rayleigh quotients are bounded by

$$\frac{\int_M \|\operatorname{grad}(f_{1,\varepsilon})\|^2}{\int_M f_{1,\varepsilon}^2} \le \frac{4}{\varepsilon^2} \left(\frac{\operatorname{Vol}(D_\varepsilon - D)}{\operatorname{Vol}(D)} \right)$$

and

$$\frac{\int_M \|\operatorname{grad}(f_{2,\varepsilon})\|^2}{\int_M f_{2,\varepsilon}^2} \le \frac{4}{\varepsilon^2}\left(\frac{\operatorname{Vol}(D_\varepsilon - D)}{\operatorname{Vol}(M - D_\varepsilon)}\right)$$

respectively.

If $\operatorname{vol}(D_\varepsilon) \le \operatorname{vol}(M) - \operatorname{vol}(D)$, then we have that

$$\lambda_1(M) \le \frac{4}{\varepsilon^2}\left(\frac{\operatorname{vol}(D_\varepsilon - D)}{\operatorname{vol}(D)}\right),$$

by the minimax characterization of λ_1.

The strategy is now to choose D and ε so that if $h(M)$ is too small, then the right-hand side of the equation will be smaller than the left-hand side. This will then give an implicit bound for $h(M)$ from below.

This is essentially the strategy of the argument of Buser in [Bu].

In order to implement this strategy, we will need an effective way of estimating the volume of D_ε from above. In the situation of [Bu], where one assumes pointwise curvature bounds, this is handled by the Heintze–Karcher Theorem [HK]. In our case, we will need the following estimate, due to Gallot [Gal], which is an L^p version of the Heintze–Karcher Theorem:

THEOREM [Gal]. *Let Ω be a domain in M with boundary $\partial\Omega = H$ a hypersurface. Denote by Ω_R the domain consisting of all points at distance at most R from Ω. Then*

$$\begin{aligned}
&\operatorname{Vol}(\Omega_{R+\varepsilon}) - \operatorname{Vol}(\Omega_R)\\
&\quad \le (e^{B(p)\alpha\varepsilon} - 1)\Bigg[\operatorname{Vol}(\Omega_R) - \operatorname{Vol}(\Omega) + (B(p)\alpha)^{-1}\operatorname{Vol}(\partial\Omega)\\
&\qquad + \frac{(n-1)^{2p-1}}{(B(p)\alpha)^{2p}}\int_{\partial\Omega}[\eta_+(x)]^{2p-1}\,d\,\text{area} + \int_{\Omega_{R+\varepsilon}-\Omega}\left(\frac{r_-}{\alpha^2} - 1\right)_+^p d\,\text{vol}\Bigg], \qquad (1)
\end{aligned}$$

where p is any number $> n/2$, $B(p)$ is an explicit constant given by

$$B(p) = \left(\frac{2p-1}{p}\right)^{1/2}(n-1)^{1-1/(2p)}\left(\frac{p-1}{p-n/2}\right)^{1/2-1/(2p)},$$

η_+ denotes the positive part of the mean curvature of H, α is any constant, r_- is the negative part of the Ricci curvature, and

$$\left|\frac{r_-}{\alpha^2} - 1\right|_+ = \sup\left(\frac{r_-}{\alpha^2} - 1, 0\right),$$

See [Gal] for a discussion of notation.

Note that $|r_-/\alpha^2 - 1|_+ \leq |\mathrm{Ricc}|/\alpha^2$.

We will apply (1) in the following way: let H be a hypersurface which realizes the Cheeger constant (see [Bu] for a discussion of the existence of such a minimizer), and let Ω be the component of $M - H$ which has the smallest volume, making an arbitrary choice if both components have the same volume. Then H is a hypersurface with

$$\mathrm{area}\,(H) = h \cdot \mathrm{Vol}\,(\Omega)$$

and

$$|\eta(H)| \leq h,$$

with equality if H does not divide M into two pieces of equal size.

From here on, we will always let H and Ω denote these choices.

In order to illustrate our line of argument, and also because we will need part (b) below later, we will prove:

LEMMA 3.1. *Let κ and c be positive numbers, and let M be a manifold satisfying one of the two following conditions:*

Either

(a) *The Ricci curvature is bounded below by κ*

or

(b) *The volume of Ω is bounded below by $c \cdot \mathrm{Vol}\,(M)$.*

Then, for $p > n/2$, h is bounded below in terms of the spectrum of M and $\|\mathrm{Ricc}\|_p$. In case (a), *h is bounded below by λ_1, p, and κ, while in case* (b), *h is bounded below in terms of λ_1, $\mathrm{Vol}\,(M)$, $\|\mathrm{Ricc}\|_p$, and c.*

Note that case (a) is a weak version of Buser's inequality.

Proof. We apply inequality (1) with $R = 0$. We then have

$$\operatorname{Vol}(\Omega_\varepsilon) - \operatorname{Vol}(\Omega) \le (e^{B(p)\alpha\varepsilon} - 1)\Bigg[(B(p)\alpha)^{-1}h \cdot \operatorname{Vol}(\Omega) + \frac{(n-1)^{2p-1}}{(B(p)\alpha)^{2p}} h^{2p} \cdot \operatorname{Vol}(\Omega) + \int_M \left|\frac{r_-}{\alpha^2} - 1\right|_+^p d\,\mathrm{vol}\Bigg] \tag{2}$$

and

$$\frac{4}{\varepsilon^2}\frac{\operatorname{Vol}(\Omega_\varepsilon) - \operatorname{Vol}(\Omega)}{\operatorname{Vol}(\Omega)} \le \frac{4(e^{B(p)\alpha\varepsilon} - 1)}{\varepsilon^2}\Bigg[\left(\frac{B(p)}{\alpha}\right)^{-1} h + (B(p)\alpha)^{-2p}(n-1)^{2p-1}h^{2p} + \frac{\int_{\Omega_{R_\varepsilon} - \Omega}\left|\frac{r_-}{\alpha^2} - 1\right|_+^p}{\operatorname{Vol}(\Omega)}\Bigg]. \tag{3}$$

Let us choose $\varepsilon = x_0/B(p)\alpha$, so that $B(p)\alpha\varepsilon = x_0$. We may then eliminate ε from the above, so that the right-hand side of inequality (3) becomes

$$4B^2(p)g(x_0)\left[\frac{\alpha}{B(p)}h + \frac{(n-1)^{2p-1}}{B(p)^{2p}\alpha^{2p-2}}h^{2p} + \frac{\int_{\Omega_{R_\varepsilon} - \Omega}\left|\frac{r_-}{\alpha^2} - 1\right|_+^p}{\operatorname{Vol}(\Omega)}\right]. \tag{4}$$

Let us first consider case (a). In this case, we may choose α so large that the third term in (4) is 0.

In inequality (2), we may then find a constant h_0 such that if $h < h_0$, then

$$\operatorname{Vol}(\Omega_\varepsilon) - \operatorname{Vol}(\Omega) < \frac{1}{2}\operatorname{Vol}(\Omega).$$

Similarly, in inequality (3), we may find h_1 such that if $h < h_1$, then

$$\frac{4}{\varepsilon^2}\frac{\operatorname{Vol}(\Omega_\varepsilon) - \operatorname{Vol}(\Omega)}{\operatorname{Vol}(\Omega)} < \frac{\lambda_1}{2}.$$

On the other hand, by the minimax characterization of λ_1, we have

$$\begin{aligned}\lambda_1 &\le \max\left(\frac{4}{\varepsilon^2}\frac{\mathrm{Vol}(\Omega_\varepsilon)-\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Omega)}, \frac{4}{\varepsilon^2}\frac{\mathrm{Vol}(\Omega_\varepsilon)-\mathrm{Vol}(\Omega)}{\mathrm{Vol}(M)-\mathrm{Vol}(\Omega_\varepsilon)}\right)\\ &\le \max\left(\frac{4}{\varepsilon^2}\frac{\mathrm{Vol}(\Omega_\varepsilon)-\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Omega)}, \frac{4}{\varepsilon^2}\frac{\mathrm{Vol}(\Omega_\varepsilon)-\mathrm{Vol}(\Omega)}{\mathrm{Vol}(M)-(3/2)\,\mathrm{Vol}(\Omega)}\right)\\ &\le \max\left(\frac{4}{\varepsilon^2}\frac{\mathrm{Vol}(\Omega_\varepsilon)-\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Omega)}, \frac{4}{\varepsilon^2}\frac{\mathrm{Vol}(\Omega_\varepsilon)-\mathrm{Vol}(\Omega)}{(1/2)\,\mathrm{Vol}(\Omega)}\right)\\ &\le 2\left(\frac{4}{\varepsilon^2}\frac{\mathrm{Vol}(\Omega_\varepsilon)-\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Omega)}\right),\end{aligned}$$

using that $\mathrm{Vol}(M) \ge 2\,\mathrm{Vol}(\Omega)$.

Therefore, if $h < \min(h_0, h_1)$, we have a contradiction. This establishes (a).

To establish (b), we argue similarly, except that we can no longer make the third term in (4) disappear by choosing α large. We can, however, replace

$$\int_{\Omega_\varepsilon - \Omega}\left|\frac{r_-}{\alpha^2} - 1\right|_+^p$$

by

$$\frac{\int_M |\mathrm{Ricc}|^p}{\alpha^{2p}}.$$

We now have the two inequalities

$$\begin{aligned}\mathrm{Vol}(\Omega_\varepsilon) - \mathrm{Vol}(\Omega) \le (e^{x_0} - 1)\Big[&(B(p)\alpha)^{-1}h \cdot \mathrm{Vol}(\Omega)\\ &+ \frac{(n-1)^{2p-1}}{(B(p)\alpha)^{2p}}h^{2p}\cdot \mathrm{Vol}(\Omega) + \alpha^{-2p}\|\mathrm{Ricc}\|_p^p\Big]\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\frac{4}{\varepsilon^2}\frac{\mathrm{Vol}(\Omega_\varepsilon)-\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Omega)}\\ &\quad\le 4(g(x_0)B(p)^2)\Big[(\alpha/B(p))h + (B(p))^{-2p}(\alpha)^{2-2p}(n-1)^{2p-1}h^{2p}\\ &\qquad + \alpha^{2-2p}\frac{\|\mathrm{Ricc}\|_p^p}{c\cdot \mathrm{Vol}(M)}\Big].\end{aligned} \tag{6}$$

We may now choose α sufficiently large so that the third right-hand term in (5) is less than (1/3) Vol (Ω), while the third right-hand term in (6) is less than $\lambda_1/3$. Then, as before, we may find h_0 and h_1 such that if $h < h_0$ and $h < h_1$, right hand sides of (5) and (6) are less than (1/2) Vol (Ω) and $(1/2)\lambda_1$ respectively. The proof of (b) now concludes in the same way as the proof of (a).

The difficulty in proving Theorem 3.2 is now clearly that we have no a priori control over Vol (Ω), and hence the denominators in the third terms may go to zero. We will remedy this by choosing R in the inequality (1) so that Vol (Ω_R) is large. To do this, we will not need to choose a value for R, but only for δ, where Vol $(\Omega_R) = (1+\delta^2)$ Vol (Ω).

Applying (1) to these choices, we have

$$\begin{aligned}\frac{4}{\varepsilon^2}\left[\frac{\operatorname{Vol}(\Omega_{R+\varepsilon}-\Omega_R)}{\operatorname{Vol}(\Omega_R)}\right] &\le \frac{4(e^{B(p)\alpha\varepsilon}-1)}{\varepsilon^2}\left[\frac{\delta^2}{1+\delta^2}+(B(p)\alpha)^{-1}\frac{h}{1+\delta^2}\right.\\ &\quad\left.+\frac{(n-1)^{2p-1}}{(B(p)\alpha)^{2p}}\frac{h^{2p}}{(1+\delta^2)}+\frac{1}{\alpha^{2p}}\frac{\|\mathrm{Ricc}\|_p^p}{(1+\delta^2)\operatorname{Vol}(\Omega)}\right]\\ &= 4(B^2(p))g(x_0)\left[\alpha^2\frac{\delta^2}{1+\delta^2}+\frac{\alpha}{B(p)}\frac{h}{1+\delta^2}\right.\\ &\quad\left.+\frac{(n-1)^{2p-1}}{B(p)^{2p}\alpha^{2p-2}}\frac{h^{2p}}{1+\delta^2}+\frac{1}{\alpha^{2p-2}}\frac{\|\mathrm{Ricc}\|_p^p}{(1+\delta^2)\operatorname{Vol}(\Omega)}\right].\end{aligned} \tag{7}$$

It now remains to choose α and δ in a reasonable way. We will do this in such a way as to minimize the sum of the two terms not involving h. To do this, we will need the following elementary

LEMMA 3.2. *For A and B positive, the minimum of*

$$\alpha^2 A+\frac{1}{\alpha^{2p-2}}B$$

is

$$B^{1/p}A^{1-1/p}((p-1)^{1/p})\left(\frac{p}{p-1}\right),$$

and occurs when

$$\alpha^2=\left[(p-1)\frac{B}{A}\right]^{1/p}.$$

Applying this to (7), we see that the sum of the first and last terms is minimized by

$$4(B^2(p))g(x_0)(p-1)^{1/p}\left(\frac{p}{p-1}\right)\frac{\|\mathrm{Ricc}\|_p}{\delta^{2/p}\,\mathrm{Vol}\,(\Omega)^{1/p}}\frac{\delta^2}{(1+\delta^2)},$$

for

$$\alpha^2=(p-1)^{1/p}\frac{\|\mathrm{Ricc}\|_p}{\delta^{2/p}\,\mathrm{Vol}\,(\Omega)^{1/p}}.$$

Setting

$$Q(p,n)=4(B^2(p))g(x_0)(p-1)^{1/p}\left(\frac{p}{p-1}\right),$$

we may rewrite the minimum as

$$Q(p,n)\frac{\|\mathrm{Ricc}\|_p}{\delta^{2/p}\,\mathrm{Vol}\,(\Omega)^{1/p}}\frac{\delta^2}{(1+\delta^2)}. \tag{8}$$

We want to make (8) less than λ_1, which will be achieved when

$$\delta^{2/p}\,\mathrm{Vol}\,(\Omega)^{1/p}\geq\frac{Q(p,n)\|\mathrm{Ricc}\|_p}{\lambda_1},$$

using the fact that

$$\frac{\delta^2}{(1+\delta^2)}<1,$$

so that

$$\alpha^2=(p-1)^{1/p}\frac{\lambda_1}{Q(p,n)}.$$

Notice that α does not depend on $\mathrm{Vol}\,(\Omega)$, while $\delta\to\infty$ as $\mathrm{Vol}\,(\Omega)\to 0$. Notice also that the sum of the two remaining terms is

$$4B^2(p)g(x_0)\left[\frac{\alpha}{B(p)}\frac{h\cdot\mathrm{Vol}\,(\Omega)}{(1+\delta^2)\,\mathrm{Vol}\,(\Omega)}+\frac{(n-1)^{2p-1}}{(B(p)^{2p}\alpha^{2p-2})}\frac{h^{2p}\cdot\mathrm{Vol}\,(\Omega)}{(1+\delta^2)\,\mathrm{Vol}\,(\Omega)}\right],$$

so that the coefficients of $h \cdot \mathrm{Vol}(\Omega)$ and $h^{2p} \cdot \mathrm{Vol}(\Omega)$ depend on $\delta^2 \mathrm{Vol}(\Omega)$, and not on δ^2 alone.

In order to make use of (8), we must have that

$$\mathrm{Vol}(\Omega_{R+\varepsilon}) \le \mathrm{Vol}(M) - \mathrm{Vol}(\Omega_R),$$

or, in other words,

$$\mathrm{Vol}(\Omega_{R+\varepsilon}) - \mathrm{Vol}(\Omega_R) \le \mathrm{Vol}(M) - 2\,\mathrm{Vol}(\Omega_R). \tag{9}$$

But

$$\begin{aligned}\mathrm{Vol}(\Omega_{R+\varepsilon}) - \mathrm{Vol}(\Omega_R) &\le (e^{B(p)\alpha\varepsilon} - 1)\Bigg[\mathrm{Vol}(\Omega_R) - \mathrm{Vol}(\Omega) + \frac{1}{B(p)\alpha} h\,\mathrm{Vol}(\Omega) \\ &\qquad + \frac{(n-1)^{2p-1}}{[B(p)\alpha]^{2p}} h^{2p}\,\mathrm{Vol}(\Omega) + \frac{\|\mathrm{Ricc}\|_p^p}{\alpha^{2p}}\Bigg] \\ &= (e^{x_0} - 1)\left[\delta^2\,\mathrm{Vol}(\Omega) + \cdots + \frac{\|\mathrm{Ricc}\|_p^p}{\alpha^{2p}}\right],\end{aligned}$$

where "$\cdots$" denotes terms which are small when h and $\mathrm{Vol}(\Omega)$ are small.

Substituting

$$\alpha^2 = (p-1)^{1/p} \frac{\|\mathrm{Ricc}\|_p}{\delta^{2/p}\,\mathrm{Vol}(\Omega)^{1/p}},$$

we find that (9) holds when

$$(e^{x_0} - 1)\left[\delta^2\,\mathrm{Vol}(\Omega) + \cdots + \frac{\delta^2\,\mathrm{Vol}(\Omega)}{p-1}\right] \le \mathrm{Vol}(M) - 2(1+\delta^2)\,\mathrm{Vol}(\Omega),$$

or

$$\delta^2\,\mathrm{Vol}(\Omega)\left[(e^{x_0} - 1)\left(1 + \frac{1}{p-1}\right) + 2 + \cdots\right] \le \mathrm{Vol}(M) - 2\,\mathrm{Vol}(\Omega).$$

Now suppose that

$$\lambda_1 > \frac{Q(p,n)\|\mathrm{Ricc}\|_p}{\mathrm{Vol}(M)^{1/p}}\left[(e^{x_0} - 1)\left(\frac{p}{p-1}\right) + 2\right]^{1/p}.$$

We may then find a value for $\delta^2 \operatorname{Vol}(\Omega)$ such that (7) is less than λ_1 and (9) holds, unless either $\operatorname{Vol}(\Omega)$ is bounded from below or the " $\cdots$ " terms are bounded from below. In the first case, Lemma 3.1 gives us a lower bound for h. In the second case, we then have lower bounds for two expressions of the form

$$(\text{const})h \cdot \operatorname{Vol}(\Omega) + (\text{const}')h^{2p} \cdot \operatorname{Vol}(\Omega).$$

Using the upper bound for h by Cheeger's inequality, we then have a lower bound for $\operatorname{Vol}(\Omega)$. We then also have a lower bound for h. Note that since we also have a value for $\delta^2 \operatorname{Vol}(\Omega)$, we now have a bound for δ as well.

This concludes the proof of Theorem 3.2, and hence also Theorem 3.1.

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