Comment. Math. Helvetici 70 (1995) 248–266 0010-2571/95/020248-19$1.50 + 0.20/0

One sided branching in Anosov foliations

SÉRGIO R. FENLEY[1]

Abstract. We study topologically transitive Anosov flows in 3-manifolds. We show that if one of the stable or unstable foliations in the universal cover does not have Hausdorff leaf space (branching occurs) then both have branching in the positive and negative directions.

1. Introduction

Generally speaking there are two well established techniques for studying Anosov flows. First there is the widely developed regularity theory of stable and unstable foliations [Gh2, BFL] yielding powerful rigidity results. The second technique consists of specifying a property of the fundamental group of the manifold (for instance being solvable, in codimension one Anosov flows [P12, P13]) which determines the flow up to topological equivalence. In dimension 3, the above flows are always topologically equivalent to suspensions or geodesic flows, hence the underlying manifolds cannot be hyperbolic.

On the other hand Goodman [Go] and Christy [Ch] constructed many examples of Anosov flows in hyperbolic 3-manifolds. The examples were obtained by doing Dehn surgery on closed orbits of suspension Anosov flows. This idea was then greatly extended by Fried [Fr], who proved that any topologically transitive Anosov flow in dimension 3 has a Birkoff section and consequently is obtained by doing Dehn surgery on finitely many closed orbits of a suspension of a pseudo-Anosov homeomorphism a closed surface [Ca-Bl].

In order to study Anosov flows in hyperbolic 3-manifolds, a technique that has proven fruitful is the analysis of the joint topological structure of the stable and unstable foliations in the universal cover [Fe1, Fe2]. This technique was introduced in a remarkable paper of Verjovsky [Ve] which studies codimension one Anosov flows. This technique has also been used by Barbot to study transversely projective Anosov flows [Ba1, Ba2], Anosov flows in graph manifolds [Ba3, Ba4] and incompressible tori in 3-manifolds supporting Anosov flows [Ba5].

[1] Research partially supported by NSF grants DMS-9201744 and DMS-9306059.

The main issue in this theory is whether branching occurs in the stable and unstable foliations in the universal cover. In this article we deal with a basic question concerning the structure of such branching. Let $\mathscr{F}^s$ and $\mathscr{F}^u$ be the stable and unstable 2-dimensional foliations (here called Anosov foliations) associated to the Anosov flow Φ in M^3. Let $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^u$ be the respective lifts to the universal cover. The leaf spaces of $\tilde{\mathscr{F}}^s$ and $\tilde{\mathscr{F}}^u$, are 1-dimensional manifolds, a priori not Hausdorff. If they are Hausdorff then they will be homeomorphic to the set of real numbers **R** and the foliation ($\tilde{\mathscr{F}}^s$ or $\tilde{\mathscr{F}}^u$) is said to be an **R**-foliation, otherwise we say the foliation has branching. If $\tilde{\mathscr{F}}^s$ (or $\tilde{\mathscr{F}}^u$) is an **R**-foliation we say that $\mathscr{F}^s$ (or $\mathscr{F}^u$) is **R**-covered. Suspensions and geodesic flows are always **R**-covered.

One fundamental fact in this topological theory is the following: if one of $\mathscr{F}^s$ or $\mathscr{F}^u$ is **R**-covered then both are **R**-covered (hence the flow is **R**-covered) [Fe1, Ba2]. In addition the joint topological structure of $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^u$ in $\tilde{M}$ can be, up to isotopy, of only two simple types [Fe1, Ba2]. The types correspond to the types of suspensions and geodesic flows.

It is easy to prove that if the flow is **R**-covered then it is topologically transitive [So, Ba2]. The main problem in the subject was to decide whether topologically transitive implies **R**-covered. This implication was claimed by Verjovsky [Ve] in the seventies, but later 2 gaps were found in his arguments. Since then the **R**-covered property has been proved by Ghys [Gh1] when the manifold is a Seifert fibered space and by Barbot [Ba1] when the fundamental group is solvable. In both cases this is an essential step in showing conjugation to a canonical model. Barbot [Ba3, Ba4] showed the **R**-covered property for many flows in graph manifolds and this property was also proved in [Fe1] for flows obtained by Dehn surgery on closed orbits of suspensions and geodesic flows.

It was widely expected that all topologically transitive Anosov flows in dimension 3 are **R**-covered. However in a striking development, Bonatti and Langevin [Bo-La] have recently constructed a counterexample to this. Then in the past year Brunella [Br] constructed a large class of counterexamples.

Given this result, it now becomes essential to understand the structure of topologically transitive, non **R**-covered Anosov flows. In this article we start this program by analysing the simplest question in this direction. As $\tilde{M}$ is simply connected, the foliations $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^u$ are transversely orientable. We say that $\tilde{\mathscr{F}}^s$ or (or $\tilde{\mathscr{F}}^u$) has *one sided branching* if its leaf space is not **R**, but the branching occurs only in the positive (or negative) transversal direction. The main result of this paper states that there is a symmetry in branched Anosov flows, that is, one sided branching cannot occur:

MAIN THEOREM. *Let Φ be a topologically transitive Anosov flow in M^3. If one of $\tilde{\mathscr{F}}^s$ or $\tilde{\mathscr{F}}^u$ has branching then both $\tilde{\mathscr{F}}^s$ and $\tilde{\mathscr{F}}^u$ have branching in the positive and negative directions.*

The first part of the proof involves showing that if one of the foliations has one sided branching, then so does the other. This uses only the analysis of the orbit space. Using the full 3-dimensional picture we then derive a contradiction to one sided branching.

This article concerns the study of the *global* structure of branching in $\tilde{\mathscr{F}}^s$ (or $\tilde{\mathscr{F}}^u$). In a forthcoming paper [Fe3] we study the *local* structure of branching. We say that $F \in \tilde{\mathscr{F}}^s$ (or $\tilde{\mathscr{F}}^u$) is a branching leaf, if it is a non Hausdorff point in its leaf space. In [Fe3] we study the structure of the set $\mathscr{B}_F$ of non separated leaves from a branching leaf F. We show in [Fe3] that $\mathscr{B}_F$ has a simple, well understood structure.

When studying branching one essential object is the following: given $G \in \tilde{\mathscr{F}}^u$, consider the set $\mathscr{I}^u(G)$ of stable leaves intersecting G, see detailed definition in section 3. Branching in $\tilde{\mathscr{F}}^s$ is equivalent to a particular relation between sets $\mathscr{I}^u$ for some leaves in $\tilde{\mathscr{F}}^u$, see lemma 4.4. Another important idea is that of "perfect fits" of leaves in the universal cover, see section 4. These two tools are used both here and in [Fe3]. Aside from this, the techniques and results are different in the 2 articles.

This article is organized as follows. The next section contains needed background material. Section 3 analyses product regions, a very useful condition implying the **R**-covered property. In the following section we prove that one sided branching in one of the foliations implies one sided branching in the other one also. In section 5 we show this produces a contradiction.

2. Preliminaries

Let $\Phi_t : M \to M$ be a nonsingular flow in a closed Riemannian manifold M. The flow Φ is *Anosov* if there is a continuous decomposition of the tangent bundle TM as a Whitney sum $TM = E^0 \oplus E^s \oplus E^u$ of $D\Phi_t$ invariant subbundles and there are constants $\mu_0 \geq 1$, $\mu_1 > 0$ so that:

(i) E^0 is one dimensional and tangent to the flow,
(ii) $\|D\Phi_t(v)\| \leq \mu_0 e^{-\mu_1 t}\|v\|$ for any $v \in E^s$, $t \geq 0$,
(iii) $\|D\Phi_{-t}(v)\| \leq \mu_0 e^{-\mu_1 t}\|v\|$ for any $v \in E^u$, $t \geq 0$,

In this article we restrict to M of dimension 3. Then E^s, E^u are one dimensional and integrate to one dimensional foliations $\mathscr{F}^{ss}$, $\mathscr{F}^{uu}$ called the strong stable and strong unstable foliations of the flow. Furthermore, the bundles $E^0 \oplus E^s$ and $E^0 \oplus E^u$ are also integrable [An] producing 2-dimensional foliations $\mathscr{F}^s$, $\mathscr{F}^u$ which are the stable and unstable foliations of the flow. The flow is said to be *orientable* when both $\mathscr{F}^s$, $\mathscr{F}^u$ are transversely orientable.

The leaves of $\mathscr{F}^s$, $\mathscr{F}^u$ are either topological planes, annuli or Möebius bands. The last two correspond exactly to leaves containing closed orbits of Φ. The flow is *topologically transitive* if the nonwandering set is the whole manifold [Sm]. Equivalently (i) the periodic orbits form a dense subset of M, or (ii) there is a dense orbit, or (iii) every leaf of $\mathscr{F}^s$, or $\mathscr{F}^u$ is dense [An, Pl1, Sm]. No closed transversal to either of the foliations is null homotopic [No].

Let $\pi : \tilde{M} \to M$ be the universal covering space of M. This notation will be fixed throughout the article. The Anosov foliations $\mathscr{F}^s$, $\mathscr{F}^u$ lift to foliations $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^u$ in $\tilde{M}$. The leaves of $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^u$ are topological planes, so $\tilde{M}$ is homeomorphic to $\mathbf{R}^3$ [Pa]. The induced flow in $\tilde{M}$ is denoted by $\tilde{\Phi}$.

Let $\mathcal{O}$ be the orbit space of $\tilde{\Phi}$ obtained by collapsing flow lines to points and let $\Theta : \tilde{M} \to \mathcal{O}$ be the projection map. A key property which will be repeatedly used is that $\mathcal{O}$ is Hausdorff and homeomorphic to $\mathbf{R}^2$ [Fe1]. This is a significant simplification since now much of the analysis can be done in dimension 2 instead of dimension 3. We stress that $\mathcal{O}$ is only a topological object. There is no natural metric in $\mathcal{O}$ since the flow direction contracts and expands distances in $\tilde{M}$. The foliations $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^u$ induce two transverse 1-dimensional foliations in $\mathcal{O}$, which will also be denoted by $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^u$. By an abuse of notation we will many times identify sets in $\tilde{M}$ or orbits of $\tilde{\Phi}$ to their respective images in $\mathcal{O}$.

Lew $W^s(x)$ be the leaf of $\mathscr{F}^s$ containing x and similarly define $W^u(x)$, $W^{ss}(x)$, $W^{uu}(x)$, $\tilde{W}^s(x)$, $\tilde{W}^u(x)$, $\tilde{W}^{ss}(x)$ and $\tilde{W}^{uu}(x)$. General references for Anosov flows are [An], [Bo], and [Sm].

3. Product regions

The following definitions will be useful. If L is a leaf of $\tilde{\mathscr{F}}^s$ or $\tilde{\mathscr{F}}^u$, then a *half leaf* of L is a connected component A of $L - \gamma$, where γ is any full orbit in L. The closed half leaf is $\bar{A} = A \cup \gamma$ and its boundary is $\partial A = \gamma$. If L is a leaf of $\tilde{\mathscr{F}}^s$ or $\tilde{\mathscr{F}}^u$ then a *flow band* B defined by orbits $\alpha \neq \beta$ in L is the connected component of $L - \{\alpha, \beta\}$ which is not a half leaf of L. The closed flow band associated to it is $\bar{B} = B \cup \{\alpha, \beta\}$ and its boundary is $\partial B = \{\alpha, \beta\}$.

We first describe the two topological types possible for the joint structure of $\tilde{\mathscr{F}}^s$ and $\tilde{\mathscr{F}}^u$ in the case of $\mathbf{R}$-covered Anosov flows. First identify the orbit space $\mathcal{O}$ (homeomorphic to $\mathbf{R}^2$) to

$$H = \{(x, y) \in \mathbf{R}^2 \mid -1 < x < 1\}.$$

In the *product* structure $\tilde{\mathscr{F}}^s$ is identified to the foliation by horizontal segments in H and $\tilde{\mathscr{F}}^u$ is the foliation by vertical lines in H. In the *skewed* structure $\tilde{\mathscr{F}}^s$ is the foliation by horizontal segments in H and $\tilde{\mathscr{F}}^u$ is a foliation by (bounded) parallel segments making an angle $\neq \pi/2$ with the horizontal. In this case, any two distinct leaves or half leaves of $\tilde{\mathscr{F}}^u$ do not intersect the same set of leaves of $\tilde{\mathscr{F}}^s$ and vice versa. Suspensions have product type and geodesic flows have skewed type. The basic result about the $\mathbf{R}$-covered case is the following :

THEOREM 3.1 [Fe1, Ba2]. *Let Φ be an Anosov flow in M^3. If one of $\mathscr{F}^u$ or $\mathscr{F}^s$ is $\mathbf{R}$-covered then Φ is $\mathbf{R}$-covered and its structure is up to isotopy (in $\mathcal{O}$) either product or skewed.*

Furthermore we will use the following result, announced by Solodov [So] and proved by Barbot [Ba2].

THEOREM 3.2 [Ba2, So]. *Let Φ be an Anosov flow in M^3. Suppose that any leaf of $\tilde{\mathscr{F}}^s$ intersects every leaf of $\tilde{\mathscr{F}}^u$. Then Φ is topologically equivalent to a suspension Anosov flow.*

Let now Φ be an Anosov flow in M^3, not a priori $\mathbf{R}$-covered. Choose transversal orientations to $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^u$, assumed to agree with the lifts of the transversal orientations to $\mathscr{F}^s$, $\mathscr{F}^u$ if any of these is transversely oriented.

For $p \in \tilde{M}$, let $\tilde{W}^s_+(p)$ be the half leaf of $\tilde{W}^s(p)$ defined by the orbit $\tilde{\Phi}_{\mathbf{R}}(p)$ and the positive transversal orientation to $\tilde{\mathscr{F}}^u$ at p. Similarly define $\tilde{W}^s_-(p)$, $\tilde{W}^u_+(p)$ and $\tilde{W}^u_-(p)$.

A fundamental fact for us is the following: since $\tilde{M}$ is simply connected then any leaf L in $\tilde{\mathscr{F}}^s$ or $\tilde{\mathscr{F}}^u$ separates $\tilde{M}$. The *front* of L is the component of $\tilde{M} - L$ defined by the positive transversal orientation to L. Similarly define the *back* of L. For $p \in \tilde{M}$ let $\tilde{W}^{ss}_+(p)$ be the component of $\tilde{W}^{ss}(p) - \{p\}$ defined by the positive transversal orientation to $\tilde{\mathscr{F}}^u$ at p. Similarly define $\tilde{W}^{ss}_-(p)$, $\tilde{W}^{uu}_+(p)$ and $\tilde{W}^{uu}_-(p)$.

The key object for the study of one sided branching is the following:

DEFINITION 3.3. Given $p \in \tilde{M}$ (or $p \in \mathcal{O}$), let

$$\mathscr{I}^u_+(p) = \{F \in \tilde{\mathscr{F}}^s \mid F \cap \tilde{W}^u_+(p) \neq \varnothing\}.$$

Similarly define $\mathscr{I}^u_-(p)$, $\mathscr{I}^s_+(p)$ and $\mathscr{I}^s_-(p)$. Let also

$$\mathscr{L}^u_+(p) = \bigcup_{F \in \mathscr{I}^u_+(p)} F.$$

Notice that $\mathscr{L}^u_+(p)$ is an open subset of $\tilde{M}$ and that $\tilde{W}^s(p) \subset \partial\mathscr{L}^u_+(p)$. Similarly define $\mathscr{L}^u_-(p)$, $\mathscr{L}^s_+(p)$ and $\mathscr{L}^s_-(p)$.

DEFINITION 3.4. A positive unstable product region of $\tilde{\Phi}$ is defined by a strong stable segment $\beta \subset F \in \tilde{\mathscr{F}}^s$ so that

$$\forall p, q \in \beta, \qquad \mathscr{I}^u_+(p) = \mathscr{I}^u_+(q).$$

Similarly define negative unstable product regions and stable product regions.

PROPOSITION 3.5. *Let Φ be a topologically transitive Anosov flow in M^3. If there is a product region of $\tilde{\Phi}$ in $\tilde{M}$, then Φ is* **R**-*covered and of product type. Therefore Φ is topologically equivalent to a suspension Anosov flow.*

Proof of 3.5. Since the hypothesis is a statement about the structure in the universal cover we may lift to a regular finite cover where the lifted flow is orientable. We may also assume that there is a stable product region defined by a strong unstable segment $\beta \subset G \in \tilde{\mathscr{F}}^u$. Changing transversal orientation to $\tilde{\mathscr{F}}^u$ if necessary suppose this is a positive stable product region.

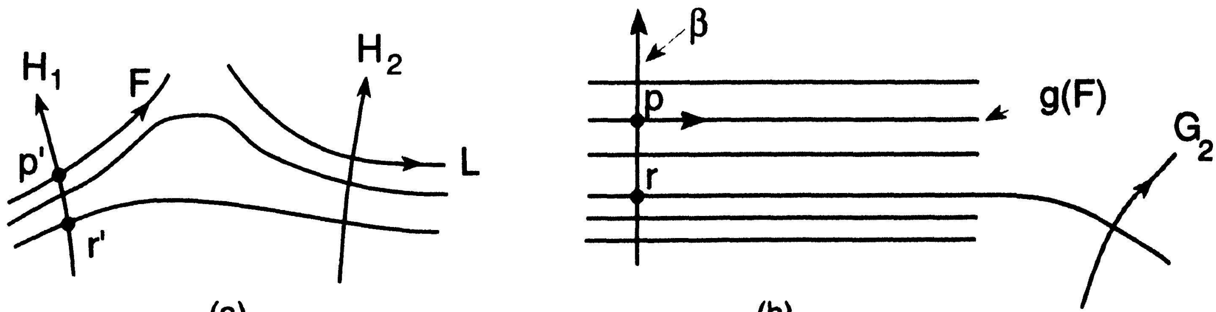

Figure 1. Contradiction to a product region.

If Φ is not **R**-covered then by theorem 3.1, $\mathscr{F}^s$ is not **R**-covered and there are $F, L \in \tilde{\mathscr{F}}^s$ which form a *branching pair*, that is, they are not separated in the leaf space of $\tilde{\mathscr{F}}^s$. Suppose they are not separated in their negative sides (so the branching is in the positive direction), the other case being similar. First notice that if F, L are not separated then they do not intersect a common unstable leaf. In fact there is no transversal (to $\tilde{\mathscr{F}}^s$) from F to L. Let

$$H_1, H_2 \in \tilde{\mathscr{F}}^u, \qquad \text{with } H_1 \cap F \neq \emptyset \quad H_2 \cap L \neq \emptyset.$$

Switching the roles of F and L if necessary we may assume that H_2 is in front of

H_1. As $\pi(F)$ is dense in M, let g be a covering translation of $\tilde{M}$ so that $g(F) \cap \beta \neq \emptyset$ and let $p \in g(F) \cap \beta, p' = g^{-1}(p)$. Assume H_1 is $\tilde{W}^u(p')$. Let $q' \in H_2 \cap L$ and $G_2 = g(H_2)$. Since F, L are not separated on the negative side, choose $r' \in \tilde{W}^{uu}_-(p')$ near enough p' so that $\tilde{W}^s(r') \cap H_2 \neq \emptyset$ and $r = g(r') \in \beta$. After applying g:

$$G_2 \cap \tilde{W}^s_+(r) \neq \emptyset, \qquad \text{so } G_2 \in \mathscr{I}^s_+(r).$$

But $H_2 \cap L \neq \emptyset$, implies that $H_2 \cap F = \emptyset$, hence $G_2 \cap \tilde{W}^s(p) = \emptyset$, so $G_2 \notin \mathscr{I}^s_+(p)$ and $\mathscr{I}^s_+(p) \neq \mathscr{I}^s_+(r)$. As a result β does not define a product region, contradiction. This shows that Φ is $\mathbf{R}$-covered. Furthermore there are distinct half stable leaves intersecting the same set of unstable leaves. Therefore the flow cannot have skewed type, hence by theorem 3.1, it has product type.

The last assertion of the theorem follows from theorem 3.2. □

4. One sided branching

If $F \in \tilde{\mathscr{F}}^s$ and $G \in \tilde{\mathscr{F}}^u$ then F and G intersect in at most one orbit, since two intersections would force a tangency of $\tilde{\mathscr{F}}^s$ and $\tilde{\mathscr{F}}^u$. This is easiest seen in $\mathcal{O}$, as $\tilde{\mathscr{F}}^s$ and $\tilde{\mathscr{F}}^u$ are then 1-dimensional foliations of the plane.

We say that leaves $F, L \in \tilde{\mathscr{F}}^s$ and $G, H \in \tilde{\mathscr{F}}^u$ form a *rectangle* if F intersects both G and H and so does L. We also say that S intersects F *between* G and H is $S \cap F$ is contained in the flow band in F defined by $F \cap G$ and $F \cap H$.

LEMMA 4.1. *Let $F, L \in \tilde{\mathscr{F}}^s$ and $G, H \in \tilde{\mathscr{F}}^u$ forming a rectangle. If $S \in \tilde{\mathscr{F}}^u$ intersects F between G and H then it also intersects L between G and H.*

Proof of 4.1. It is easier to understand the proof in $\mathcal{O} \cong \mathbf{R}^2$. Let S' be the half leaf of S defined by $S \cap F$ and contained in the same side of F that L is. There are 4 cases to consider:

(i) S' stays in the region bounded by F, L, G and H. This is a compact region in $\mathcal{O}$. By the Poincaré–Bendixson theorem, S′ has to limit on either a closed curve or a point, both impossible since $\tilde{\mathscr{F}}^u$ (as seen in $\mathcal{O}$, which is homeomorphic to $\mathbf{R}^2$) is a foliation of the plane.
(ii) S' cannot intersect G or H as they are all unstable leaves.
(iii) If $S' \cap F \neq \emptyset$, then S intersects F in at least 2 orbits, contradiction.
(iv) Therefore S' intersects L and since both G and H separate $\tilde{M}$, then $S' \cap L$ is between G and H. □

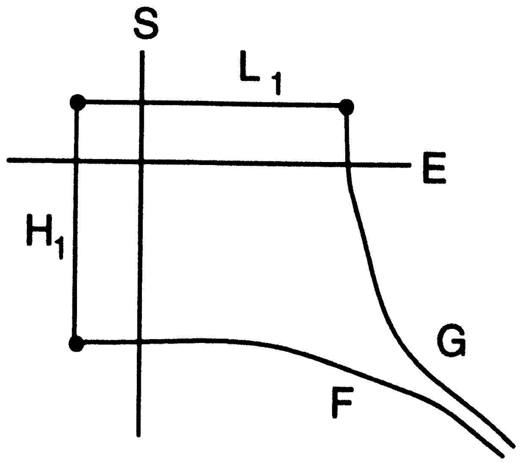

Figure 2. Perfect fits in the universal cover.

The following definition will be essential for all the analysis that follows.

DEFINITION 4.2. Two leaves F, G; $F \in \tilde{\mathscr{F}}^s$ and $G \in \tilde{\mathscr{F}}^u$, form a perfect fit if $F \cap G = \emptyset$ and there are half leaves F_1 of F and G_1 of G and also flow bands $L_1 \subset L \in \tilde{\mathscr{F}}^s$ and $H_1 \subset H \in \tilde{\mathscr{F}}^u$, (see fig. 2) so that:

$$\bar{L}_1 \cap \bar{G}_1 = \partial L_1 \cap \partial G_1, \qquad \bar{L}_1 \cap \bar{H}_1 = \partial L_1 \cap \partial H_1, \qquad \bar{H}_1 \cap \bar{F}_1 = \partial H_1 \cap \partial F_1,$$

$$\forall S \in \tilde{\mathscr{F}}^u \quad S \cap L_1 \neq \emptyset \Leftrightarrow S \cap F_1 \neq \emptyset$$

and

$$\forall E \in \tilde{\mathscr{F}}^s \quad E \cap G_1 \neq \emptyset \Leftrightarrow E \cap H_1 \neq \emptyset.$$

Notice that the flow bands L_1, H_1 (or leaves L, H) are not uniquely determined given the perfect fit (F, G). We will also say that F and G are *asymptotic* in the sense that if we consider stable leaves near F and on the side containing G they will intersect G and vice versa.

LEMMA 4.3 (uniqueness of perfect fits). *Let $F \in \tilde{\mathscr{F}}^s$. Then there is at most one unstable leaf G making a perfect fit with a given half leaf of F and on a given side of F.*

Proof of 4.3. Let $F \in \tilde{\mathscr{F}}^s$ and suppose that $G_1, G_2 \in \tilde{\mathscr{F}}^u$ form a perfect fit with the same half leaf of F and both are in the same side of F. Let L_i, H_i, $i = 1, 2$ be flow bands defining the perfect fit (F, G_i). Let $p_i \in F \cap \partial H_i$. By Reeb stability [Re] there are open, strong unstable segments τ_i, with $p_i \in \tau_i$ and so that the set of stable leaves intersecting τ_1 and τ_2 is the same. Let $E \in \tilde{\mathscr{F}}^s$ with $E \cap \tau_i \neq \emptyset$ and E on the side of F containing G_i. If E is near enough F it follows (again by Reeb stability) that $E \cap H_i \neq \emptyset$ and hence $E \cap G_i \neq \emptyset$. Notice that E is between F and L_i.

Furthermore since G_1 and G_2 form perfect fits with the same half leaf of F then $E \cap G_1$ and $E \cap G_2$ are on the same side of H_1 (and also on the same side of H_2).

Suppose first that G_1 intersects E between H_2 and G_2. Then the leaves E, L_2, G_2, H_2 form a rectangle in $\mathcal{O}$, so by the previous lemma $G_1 \cap E \neq \emptyset \Rightarrow G_1 \cap L_2 \neq \emptyset$. But this implies that $G_1 \cap F \neq \emptyset$ contradicting the fact that (F, G_1) is a perfect fit. By the same reasoning, G_2 intersecting E between H_1 and G_1 is also ruled out. Therefore $G_1 = G_2$ and the lemma is proved. $\square$

If (L, G) forms a perfect fit and g is any orientation preserving covering translation with $g(L) = L$, then $g(G) = G$. This follows from uniqueness of perfect fits and the fact that, as g acts by homeomorphisms in $\mathcal{O}$, it takes perfect fits to perfect fits.

If p, q are in the same strong stable or strong unstable leaf then $[p, q]$ denotes the closed segment in that leaf from p to q.

We say that $\mathcal{J}^s_+(p)$ and $\mathcal{J}^s_+(q)$ are comparable and will denote this by $\mathcal{J}^s_+(p) \sim \mathcal{J}^s_+(q)$, if one of them is cntained in the other. Then we write $\mathcal{J}^s_+(p) < \mathcal{J}^s_+(q)$ if the former is strictly contained in the latter. Similarly define $\leq$, $>$ and $\geq$. The symbol $\not\sim$ means not comparable.

LEMMA 4.4. *Let Φ be an Anosov flow in M^3. Then $\tilde{\mathcal{F}}^s$ has branching in the positive direction if and only if there is $F \in \tilde{\mathcal{F}}^s$ and $p, q \in F$ with $\mathcal{J}^u_+(p) \not\sim \mathcal{J}^u_+(q)$.*

Proof of 4.4. Suppose first that $\tilde{\mathcal{F}}^s$ has branching in the positive direction. Let $E, L \in \tilde{\mathcal{F}}^s$ which are not separated in their negative sides. Then there are $F_a \in \tilde{\mathcal{F}}^s$ with $F_a \to E \cup L$ (in the leaf space of $\tilde{\mathcal{F}}^s$) when $a \to 0$. Fix $p' \in E$, $q' \in L$. For a small enough F_a intersects both $\tilde{W}^u_-(p')$ and $\tilde{W}^u_-(q')$. Let p_a and q_a respectively be points in this intersection. Then $E \in \mathcal{J}^u_+(p_a) - \mathcal{J}^u_+(q_a)$ and $L \in \mathcal{J}^u_+(q_a) - \mathcal{J}^u_+(p_a)$, hence these sets are not comparable and $\tilde{W}^s(p_a) = \tilde{W}^s(q_a) = F_a$ as desired.

For the converse, let $p, q \in F \in \tilde{\mathcal{F}}^s$ satisfying the hypothesis. Parametrize $\tilde{W}^{uu}_+(p)$ by arclength as p_t, $t \in (0, +\infty)$. Similarly parametrize $\tilde{W}^{uu}_+(q)$ as q_s, $s \in (0, \infty)$. As $\tilde{W}^s(p) = \tilde{W}^s(q)$, then for t small $\tilde{W}^s(p_t) \in \mathcal{J}^u_+(q)$. Let

$$t' = \inf\{t \mid \tilde{W}^s(p_t) \notin \mathcal{J}^u_+(q)\}.$$

Similarly define s'. For $t \in (0, t')$, let $\varphi(t) \in (0, +\infty)$ with $\tilde{W}^s(p_t) = \tilde{W}^s(q_{\varphi(t)})$. As leaves of $\tilde{\mathcal{F}}^s$ separate $\tilde{M}$ it is easy to check that $\varphi : (0, t') \to (0, s')$ is an orientation preserving homeomorphism. When $t \to t'$:

$$\tilde{W}^s(p_t) = \tilde{W}^s(q_{\varphi(t)}) \to \tilde{W}^s(p_{t'}) \cup \tilde{W}^s(q_{s'}) \quad \text{and} \quad \tilde{W}^s(p_{t'}) \neq \tilde{W}^s(q_{s'}),$$

so there is branching in the positive direction as desired. $\square$

Therefore if there is no positive branching in $\tilde{\mathscr{F}}^s$ then a priori various sets $\mathscr{J}^u_+$ will be comparable. This fact will be repeatedly used in the proof of the next therorem.

THEOREM 4.5. *Let Φ be a topologically transitive Anosov flow in M^3. If one of $\tilde{\mathscr{F}}^s$ or $\tilde{\mathscr{F}}^u$ has one sided branching, then both have one sided branching.*

Proof of 4.5. The proof will be done using various intermediate lemmas. Up to finite cover, assume that Φ is orientable. Reversing the flow direction and changing the transversal orientation to $\tilde{\mathscr{F}}^s$ is necessary, assume that $\tilde{\mathscr{F}}^s$ has one sided branching and only in the negative direction. By theorem 3.1 since $\mathscr{F}^s$ is not **R**-covered then $\mathscr{F}^u$ is also not **R**-covered.

Fix $F \in \tilde{\mathscr{F}}^s$ and let $p, q \in F, q \in \tilde{W}^{ss}_+(p)$. By the previous lemma, $\mathscr{J}^u_+(p) \sim \mathscr{J}^u_+(q)$.

LEMMA 4.6. $\mathscr{J}^u_+(p) \neq \mathscr{J}^u_+(q)$.

Proof of 4.6. Suppose that they are equal. If for all $z \in [p, q]$

$$\mathscr{J}^u_+(z) = \mathscr{J}^u_+(p),$$

then $[p, q]$ is the base segment of a positive unstable product region and by proposition 3.5, it follows that Φ is **R**-covered, contrary to hypothesis.

Let $E \in \mathscr{J}^u_+(p)$. Then $E \in \mathscr{J}^u_+(q)$ and therefore F, E, $\tilde{W}^u(p)$ and $\tilde{W}^u(q)$ define a rectangle. By lemma 4.1 it follows that for any $z \in [p, q]$, $\tilde{W}^u_+(z) \cap E \neq \emptyset$, hence $E \in \mathscr{J}^u_+(z)$ and consequently $\mathscr{J}^u_+(z) \geq \mathscr{J}^u_+(p)$.

Choose now $z \in [p, q]$ with $\mathscr{J}^u_+(z) > \mathscr{J}^u_+(p)$. Since $\mathscr{L}^u_+(p)$ is connected, there is a unique leaf $E \in \mathscr{J}^u_+(z)$ which is in the boundary of $\mathscr{L}^u_+(p)$. We remark that $E \neq F$. Furthermore

$$E \cap (\tilde{W}^u(p) \cup \tilde{W}^u(q)) = \emptyset.$$

Since Φ is topologically transitive $\pi(\tilde{W}^u_+(p))$ is dense in M [Fe1]. Let g be a covering translation with $g(\tilde{W}^u_+(p)) \cap F \neq \emptyset$ and so that $g(\tilde{W}^u(p))$ is in the front of $\tilde{W}^u(q)$. Then $\mathscr{J}^u_+(g(z)) > \mathscr{J}^u_+(g(p))$. Since $F \cap \tilde{W}^u_+(g(p)) \neq \emptyset$, let $w \in F \cap \tilde{W}^u_+(g(z))$. Notice that $g(E) \in \mathscr{J}^u_+(w)$. But $g(E) \notin \mathscr{J}^u_+(g(p))$, hence $g(E) \notin \mathscr{J}^u_+(z)$ (since $\tilde{W}^u(g(p))$ separates $\tilde{M}$). Similarly $E \notin \mathscr{J}^u_+(g(z))$ so it is not in $\mathscr{J}^u_+(w)$. Since $z, w \in F$, this contradicts proposition 4.4. $\square$

If necessary switch the transverse orientation to $\tilde{\mathscr{F}}^u$ and exchange the roles of p and q to ensure that $\mathscr{J}^u_+(p) < \mathscr{J}^u_+(q)$. We will then prove this implies that $\tilde{\mathscr{F}}^u$

has only negative branching. Let F_p be the leaf in the boundary of $\mathscr{L}^u_+(p)$ so that $F_p \cap \tilde{W}^u_+(q) \neq \emptyset$.

LEMMA 4.7. *$\tilde{W}^u(p)$ and F_p form a perfect fit.*

Proof of 4.7. Let $z \in [p,q]$ be the point closest to q in $[p,q]$ with $\tilde{W}^u(z) \cap F_p = \emptyset$. Any leaf in the back of F_p, near enough F_p belongs to $\mathscr{I}^u_+(p)$ as $F_p \subset \partial\mathscr{L}^u_+(p)$. Hence $\tilde{W}^u(z)$ and F_p form a perfect fit. If $p \neq z$ then since $[p,z]$ does not define a product region, there is $w \in [p,z]$ with $\mathscr{I}^u_+(w) > \mathscr{I}^u_+(p)$. Let $L \in \mathscr{I}^u_+(w) - \mathscr{I}^u_+(p)$. By lemma 4.4, $w, q \in F$, together with leaves L and F_p produce positive branching in $\tilde{\mathscr{F}}^s$, contradiction. □

By uniqueness of perfect fits F_p depends only on p and not on q. It now follows that there is a strict ordering in the sets $\mathscr{I}^u_+(z), z \in \tilde{W}^{ss}(p)$:

LEMMA 4.8. *For any $z, w \in \tilde{W}^{ss}(p)$ with $w \in \tilde{W}^{ss}_+(z)$ then $\mathscr{I}^u_+(z) < \mathscr{I}^u_+(w)$.*

Proof of 4.8. As before $\mathscr{I}^u_+(w) \sim \mathscr{I}^u_+(z)$. Equality is disallowed by lemma 4.6. If $\mathscr{I}^u_+(w) < \mathscr{I}^u_+(z)$ choose a covering translation g so that $g(\tilde{W}^u_+(w)) \cap F \neq \emptyset$ and $g(\tilde{W}^u(w))$ is in the back of $\tilde{W}^u(p)$. As before this is disallowed by proposition 4.4. □

As in the proof of lemma 4.7 it is easy to see that if $w \in F$ then there is $F_w \in \tilde{\mathscr{F}}^s$ so that F_w is in the front of $\tilde{W}^u(w)$ and so that $(F_w, \tilde{W}^u_+(w))$ forms a perfect fit. For fixed $q \in F$, the leaves F_w, $w \in \tilde{W}^{ss}_-(q)$ will intersect $\tilde{W}^{uu}_+(q)$ in a nested fashion. We stress that a priori this set of intersecting points might be a proper subset of $\tilde{W}^{uu}_+(q)$.

LEMMA 4.9. *Let $F_1, F_2 \in \tilde{\mathscr{F}}^s$, intersecting a common unstable leaf $G \in \tilde{\mathscr{F}}^u$. Suppose that F_1 is in the front of F_2. Let $u_i \in F_i \cap G$ with $u_1 \in \tilde{W}^{uu}_+(u_2)$. Then $\mathscr{I}^s_+(u_1) > \mathscr{I}^s_+(u_2)$ and there is $H \in \tilde{\mathscr{F}}^u$ making a perfect fit with F_2 and so that $H \cap \tilde{W}^u_+(u_1) \neq \emptyset$.*

Proof of 4.9. There are 4 cases:

Case 1. $\mathscr{I}^s_+(u_1) \not\sim \mathscr{I}^s_+(u_2)$.

Let $H_1, H_2 \in \tilde{\mathscr{F}}^u$ with $H_1 \in \mathscr{I}^u_+(u_1) - \mathscr{I}^s_+(u_2)$ and $H_2 \in \mathscr{I}^s_+(u_2) - \mathscr{I}^s_+(u_1)$. Let g be a covering translation so that

$$g(F) \cap G \neq \emptyset \quad \text{and} \quad g(F) \cap H_2 \neq \emptyset.$$

Let $q \in g(F) \cap G$ and $r \in \tilde{W}^{ss}_+(q) \cap H_2$. Then $F_1 \in \mathscr{I}^u_+(q) - \mathscr{I}^u_+(r)$. Translating back by g we get $g^{-1}(F_1) \in \mathscr{I}^u_+(g^{-1}(q)) - \mathscr{I}^u_+(g^{-1}(r))$. This contradicts the previous lemma, as $g^{-1}(r) \in \tilde{W}^{ss}_+(g^{-1}(q)) \subset F$.

Case 2. $\mathscr{I}^s_+(u_1) < \mathscr{I}^s_+(u_2)$.
There is $H_2 \in \mathscr{I}^s_+(u_2) - \mathscr{I}^s_+(u_1)$. Ruled out as in case 1.

Case 3. $\mathscr{I}^s_+(u_1) = \mathscr{I}^s_+(u_2)$.

Since Φ is not $\mathbf{R}$-covered it follows $[u_1, u_2]$ is not the basis segment of a stable product region. As in proposition 4.4, there is $u_3 \in [u_1, u_2]$ with $\mathscr{I}^s_+(u_3) > \mathscr{I}^s_+(u_1)$. Then apply the proof of case 2 to get a contradiction.

Hence it follows that:

Case 4. $\mathscr{I}^s_+(u_1) > \mathscr{I}^s_+(u_2)$.
Let now $H \in \tilde{\mathscr{F}}^u$ in the boundary of $\mathscr{L}^s_+(u_2)$ so that $H \cap \tilde{W}^u_+(u_1) \neq \emptyset$. The same proof as in lemma 4.7 shows that H and F_2 form a perfect fit as desired. □

End of the proof of theorem 4.5. The proof of case 1 in the previous lemma, together with proposition 4.4, properly applied to $\tilde{\mathscr{F}}^u$, rules out the existence of positive branching in $\tilde{\mathscr{F}}^u$.

In fact the analysis in the theorem shows the following:

COROLLARY 4.10. *Let Φ be a topologically transitive, one sided branching Anosov flow so that $\tilde{\mathscr{F}}^s$ and $\tilde{\mathscr{F}}^u$ have branching only in the negative direction. Then*:

For each $G_1, G_2 \in \tilde{\mathscr{F}}^u$ ($L_1, L_2 \in \tilde{\mathscr{F}}^s$) so that they intersect a common stable leaf (respectively unstable leaf) and so that G_2 is in front of G_1 (L_2 is in front of L_1), then there is a unique $E \in \tilde{\mathscr{F}}^s$ ($S \in \tilde{\mathscr{F}}^u$) making a perfect fit with G_1 (L_1) and so that $E \cap G_2 \neq \emptyset$ ($S \cap L_2 \neq \emptyset$).

Proof of 4.10. Consider the pair G_1, G_2. There is a covering translation g with $g(F)$ intersecting both G_1 and G_2. The result then follows from lemma 4.8. The other statement is exactly as in lemma 4.9. □

5. Two sided branching

In the last section our only tool was the structure of the foliations in the 2-dimensional orbit space $\mathcal{O}$. We will now use the full 3-dimensional picture of $\tilde{M}$ in order to rule out one sided branching.

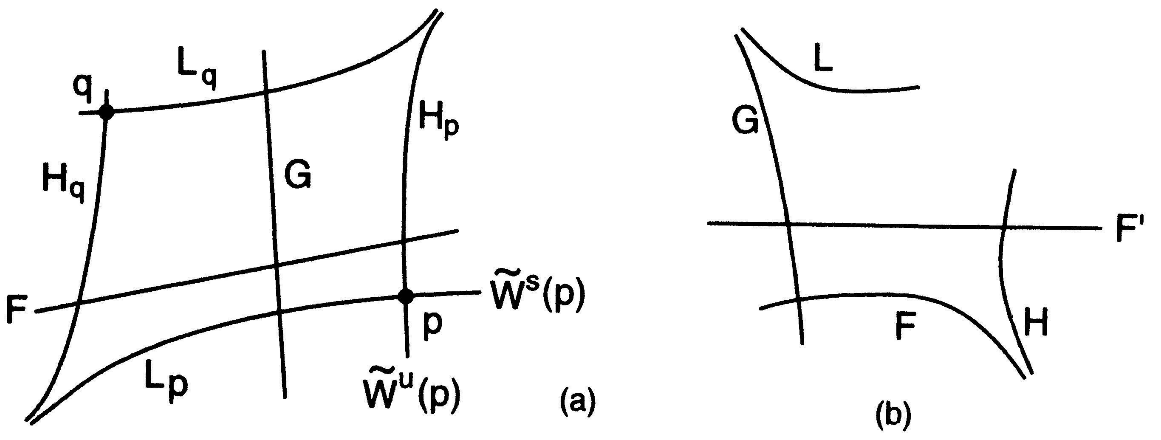

Figure 3. a. A lozenge. b. Closing up of a lozenge.

DEFINITION 5.1. Lozenges – Let $p, q \in \tilde{M}, p \notin \tilde{W}^u(q), p \notin \tilde{W}^s(q)$. Let $H_p(L_p)$ be the half leaf of $\tilde{W}^u(p)$ $(\tilde{W}^s(p))$ defined by $\Phi_{\mathbf{R}}(p)$ and contained in the same side of $\tilde{W}^s(p)$ $(\tilde{W}^u(p))$ as q. Similarly define H_q, L_q. Then p, q form a lozenge, fig. 3a, if H_p, L_q and H_q, L_p respectively form perfect fits.

We say that p, q (or $\tilde{\Phi}_{\mathbf{R}}(p)$, $\tilde{\Phi}_{\mathbf{R}}(q)$) are the corners of the lozenge. If the lozenge with corner p is contained in the back of $\tilde{W}^s(p)$ then p is a corner of type $(+, *)$, otherwise it is of type $(-, *)$. Similarly using $\tilde{W}^s(p)$, define types $(*, +)$, $(*, -)$. The sides of the lozenge are H_p, L_p, H_q and L_q.

PROPOSITION 5.2. *Let Φ be a topologically transitive Anosov flow in M^3. Assume that both $\tilde{\mathscr{F}}^s, \tilde{\mathscr{F}}^u$ only have negative branching. Then any $p \in \tilde{M}$ is the $(-, -)$ corner of a lozenge.*

Proof of 5.2. Let $F = \tilde{W}^s(p)$, $G = \tilde{W}^u(p)$. By corollary 4.10 there are $L \in \tilde{\mathscr{F}}^s$ and $H \in \tilde{\mathscr{F}}^u$ making a perfect fit with G and F respectively and so that L and H are in the positive sides of G and F respectively. All that is left to prove is that $L \cap H \neq \emptyset$. Since H forms a perfect fit with F, then G and H intersect a leaf $F' \in \tilde{\mathscr{F}}^s$ near F and in the positive side of F fig. 3, b. By corollary 4.10, there is $L' \in \tilde{\mathscr{F}}^s$ making a perfect fit with G so that $L' \cap H \neq \emptyset$. Uniqueness of perfect fits implies that $L = L'$ as desired. $\square$

DEFINITION 5.3. A (3, 1) ideal quadrilateral $\mathscr{D}$ in $\tilde{M}$ (or in $\mathcal{O}$) is a region determined by leaves $F, E \in \tilde{\mathscr{F}}^s$, $G, H \in \tilde{\mathscr{F}}^u$, for which there are half leaves E_0 or E and H_0 of H so that

$$(F, G),\ (F, H_0) \text{ and } (E_0, G) \text{ form perfect fits and } \partial E_0 = \partial H_0.$$

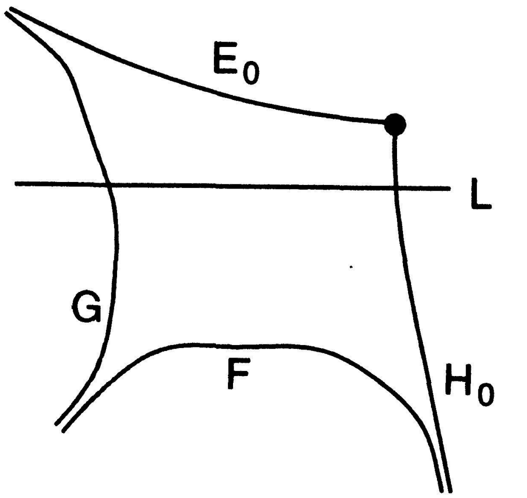

Figure 4. A (3, 1) ideal quadrilateral in the universal cover.

It follows that for any $L \in \tilde{\mathscr{F}}^s, L \cap G \neq \emptyset \Leftrightarrow L \cap H_0 \neq \emptyset$ and similarly for unstable leaves, see fig. 4. There are 3 ideal vertices and one actual vertex (the orbit $\partial E_0 = \partial H_0$) in the quadrilateral.

PROPOSITION 5.4. *If Φ is a topologically transitive Anosov flow with one sided branching then there are* (3, 1) *ideal quadrilaterals in* $\tilde{M}$.

Proof of 5.4. Assume that $\tilde{\mathscr{F}}^s, \tilde{\mathscr{F}}^u$ branch only in the negative direction. Let $C, F \in \tilde{\mathscr{F}}^s$ which are not separated and let $S_1, S_2 \in \tilde{\mathscr{F}}^u$ intersecting C, F respectively. Assume that S_2 is in front of S_1. Consider $C' \in \tilde{\mathscr{F}}^s$ with $C' \cap S_i \neq \emptyset, i = 1, 2$. Let $p_i \in C' \cup S_i$. There is a unique $p \in \tilde{W}^{ss}_{-}(p_2)$, $G = \tilde{W}^u(p)$, so that $G \cap F = \emptyset$, but for any $p' \in [p, p_2]$ with $p' \neq p$, then $\tilde{W}^u(p') \cap F \neq \emptyset$, see fig. 5. By lemma 4.7, G and F form a perfect fit. Let $q \in S_2 \cup F$. By proposition 5.2, q is the $(-, -)$ corner of a lozenge. Let H be the leaf in the unstable boundary of the lozenge which makes a perfect fit with F.

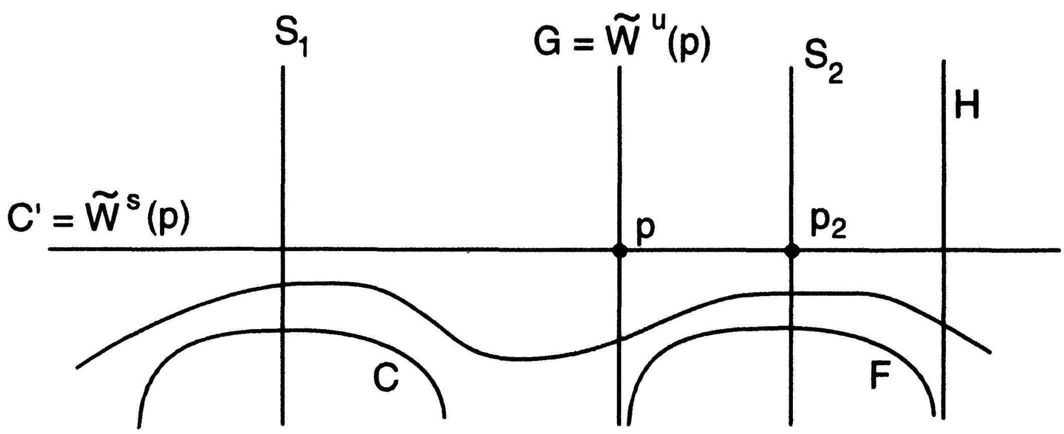

Figure 5. Producing a (3, 1) ideal quadrilateral.

Notice that C' intersects both G and H. By corollary 4.10 there is a leaf $E \in \tilde{\mathscr{F}}^s$ which intersects H and makes a perfect fit with G. Let $\gamma = H \cap E$ and let $H_0 = \tilde{W}^u_-(\gamma)$ and $E_0 = \tilde{W}^s_-(\gamma)$. Then F, G, E_0, H_0 are the sides of a $(3, 1)$ ideal quadrilateral.

We say that a leaf F of $\tilde{\mathscr{F}}^s$ or $\tilde{\mathscr{F}}^u$ (or an orbit γ of $\tilde{\Phi}$) is "periodic" if there is a non trivial covering translation g with $g(F) = F$ $(g(\gamma) = \gamma)$. Recall that F is periodic if and only if $\pi(F)$ contains a periodic orbit of Φ.

The proof of the main theorem will be completed by:

PROPOSITION 5.5. *Let Φ be a topologically transitive, one sided branched Anosov flow in M^3. Then $\tilde{M}$ has no $(3, 1)$ ideal quadrilateral.*

Proof of 5.5. Assume that $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ only branch in the negative direction. Notice that Φ is orientable, since orientation reversing covering translations would induce branching in both directions. Using corollary 4.10 it is easy to show that the only type of $(3, 1)$ ideal quadrilateral that can possibly occur is the one produced by the previous proposition, that is, one with a $(+, +)$ actual vertex. We prove that these cannot exist either.

Suppose then that $\mathscr{D}$ is one such quadrilateral, with sides $F \in \tilde{\mathscr{F}}^s, G \in \tilde{\mathscr{F}}^u$, E_0, H_0, with $E_0 \subset E \in \tilde{\mathscr{F}}^s$ and $H_0 \subset H \in \tilde{\mathscr{F}}^u$. If F is periodic then there is a non trivial covering translation g with $g(F) = F$. As Φ is orientable and (F, G) forms a perfect fit, it follows that G is periodic and $g(G) = G$. In the same way $g(E) = E$ and $g(H) = H$. Let $\delta = E \cap H$. By the above $g(\delta) = \delta$. Let $\delta_1 \subset F$ with $g(\delta_1) = \delta_1$. Notice that

$$\emptyset \neq \tilde{W}^u_+(\delta_1) \cap E_0 = \delta' \neq \delta.$$

But $g(\tilde{W}^u_+(\sigma_1)) = \tilde{W}^u_+(\delta^1)$ implies that $g(\delta') = \delta'$. Then δ and δ' would be periodic orbits in E, contradiction. Hence none of F, G, E, H is periodic.

Let now $q \in F$, $x = \pi(q)$, $\gamma = \tilde{\Phi}_{\mathbf{R}}(q)$ and $\alpha = \pi(\gamma)$ an orbit of Φ. Then α is not a closed orbit nor is it asymptotic to a closed orbit in the forward direction. Consider a forward limit point z of α and let $z_i \in \alpha$ with $z_i \to z$, $z_i = \Phi_{t_i}(x)$, $t_i \to +\infty$.

Choose p a lift of z to $\tilde{M}$ and let $p_i \to p$ be coherent lifts of z_i to $\tilde{M}$. Then there are covering translations g_i, with $p_i \in g_i(F), p_i = g_i(\tilde{\Phi}_{t_i}(q))$. Let $F_i = g_i(F)$ and similarly define G_i, H_i, E_i, which form the boundary of the $(3, 1)$ ideal quadrilateral $\mathscr{D}_i = g_i(\mathscr{D})$. Since $\tilde{W}^u_+(q) \cap E \neq \emptyset$, let $u = \tilde{W}^{uu}_+(q) \cap E$ and let $\beta_* = [q, u]$. Then $\beta_i = g_i(\tilde{\Phi}_{t_i}(\beta_*))$ is a strong unstable segment from p_i to E_i, so that

the interior of β_i (as a segment) does not intersect E_i. $\quad (*)$

Furthermore β_i has length l_i and $l_i \to \infty$ as $i \to \infty$. As $\tilde{\mathscr{F}}^{uu}$ is a continuous foliation, then $\beta_i \to \beta = \tilde{W}^{uu}_+(p)$, having infinite length.

After truncating finitely many terms suppose that all z_i are in a box $\mathscr{C}$ foliated by sheets of $\mathscr{F}^s$ and segments of $\mathscr{F}^{uu}$. Furthermore assume that

$$d(p_i, p_k) \ll 1, \quad \forall i, k, \quad l_i \gg 1, \quad \forall i. \tag{**}$$

If z_i and z_k are in the same local sheet of $\mathscr{F}^s$ in $\mathscr{C}$ and $i \neq k$, then the curve α_0, obtained by going from z_i to z_k along α and then in the local sheet of $W^s(z_i)$ in $\mathscr{C}$ from z_k to z_i; is homotopic to a curve transversal to $\mathscr{F}^s$ hence not null homotopic in its leaf. Therefore $W^s(z_i)$ is not simply connected, contains a periodic orbit and F is periodic, a contradiction.

Then up to subsequence there are 2 cases:

(1) For all $i < k$, F_i is in the back of F_k.

Since $\mathscr{D}_i$ and $\mathscr{D}_k$ are (3, 1) ideal quadrilaterals, this implies that E_i is in front of E_k, G_i is in front of G_k and H_i is in the back of H_k, see fig. 6. Furthermore G_i intersects both E_k and F_k and so does H_i.

By (**), it follows that p is inside the (3, 1) ideal quadrilateral $\mathscr{D}_i$, for all i. Hence $\emptyset \neq \beta \cap E_i = r_i$ and let $a_i > 0$ be the length of the segment of β from p to r_i. Since E_i is in front of E_k if $i < k$, then a_i is a decreasing sequence, so there is $a > 0$, with $a_i < a$ for all i. Let $\beta_i(0, s)$ be the open segment of $\tilde{W}^{uu}_+(p_i)$ with one endpoint p_i and length s. For i big enough $\beta_i(0, 2a)$ is very near $\beta(0, 2a)$. Hence the interior of β_i will intersect E_i, contradiction to (*). In this case we do not use the one sided branching property.

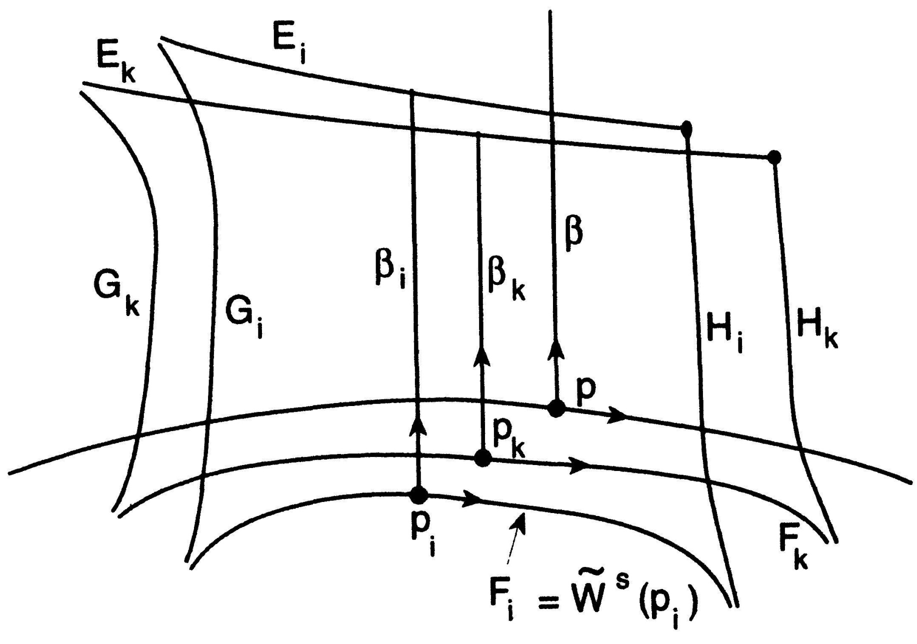

Figure 6. Eliminating upper convergence of (3, 1) ideal quadrilaterals.

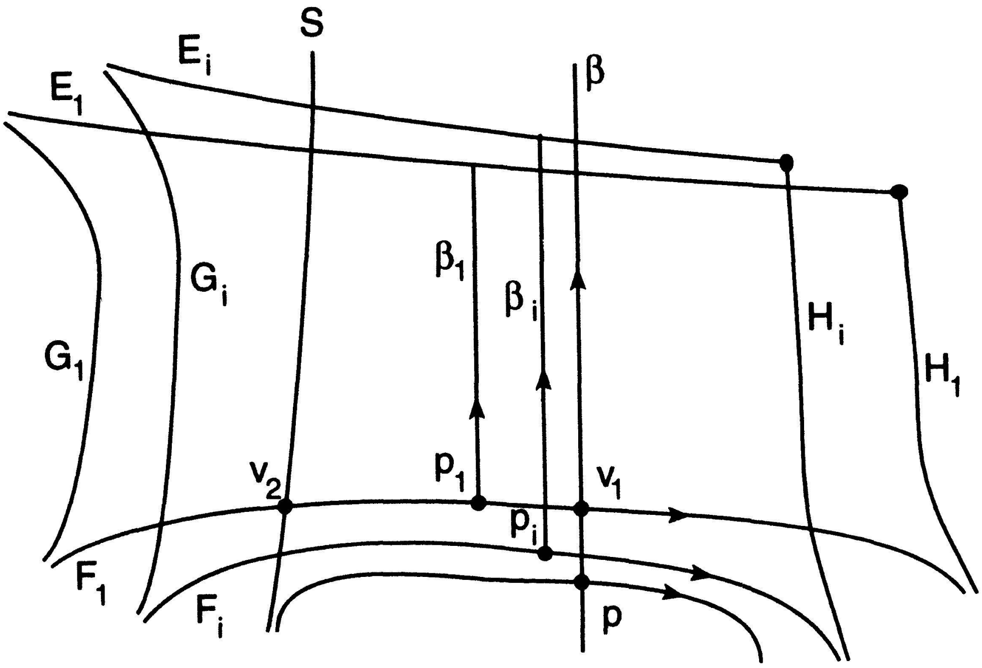

Figure 7. Eliminating lower convergence of (3, 1) ideal quadrilaterals in the one sided branching case.

(2) For all $i < k$, F_i is in the front of F_k.

In this case E_i is in the back of E_k, G_i in the back of G_k and H_i is in the front of H_k. The argument above does not apply because the E_i are not trapped above, so there is no contradiciton to length of β_i going to infinity.

Let $v_1 = \beta \cap F_1$. By (**), $v_1 \in \mathscr{D}_i$ for all $i > 1$. Hence any such G_i intersects F_1. As G_k is in front of G_i for $k > i$ and all G_i are in the back of $\tilde{W}^u(p)$, then $G_i \to S$, with $S \cap \tilde{W}^s(p) = \varnothing$, so $S \neq \tilde{W}^u(p)$. Let $v_2 = \tilde{W}^{ss}(v_1) \cap S$.

CLAIM. $\mathscr{I}^s_+(v_1) = \mathscr{I}^s_+(v_2)$.

(a) If $L \in \mathscr{I}^s_+(v_2)$, then as $G_i \to S$, it follows that $L \cap G_i \neq \varnothing$, for i big enough. As $\mathscr{D}_i$ is a (3, 1) ideal quadrilateral it follows that $L \cap H_i \neq \varnothing$, hence $L \cap \tilde{W}^u_+(p) \neq \varnothing$ so $L \in \mathscr{I}^s_+(v_i)$.

(b) Conversely if $L \in \mathscr{I}^s_+(v_1)$ then $L \cap \beta \neq \varnothing$. Choose i big enough so that $\beta_i \cap L \neq \varnothing$, hence $L \cap G_i \neq \varnothing$ and as S separates $\tilde{M}$, it follows that $L \cap S \neq \varnothing$, hence $L \in \mathscr{I}^s_+(v_2)$, proving the claim.

Since Φ has one sided branching and $v_2 \in \tilde{W}^s(v_1)$, lemma 4.6 shows that $\mathscr{I}^s + (v_2) \neq \mathscr{I}^s_+(v_1)$, contradicting the claim. This finishes the proof. $\square$

Acknowledgements

We thank Joe Christy, Lee Mosher and Bill Thurston for stimulating conversations about branching in Anosov foliations. We also thank Etienne Ghys for several comments and detailed suggestions which greatly helped improve the presentation of this article.

REFERENCES

[An] D. V. ANOSOV, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. *90* (1969).

[Ba1] T. BARBOT, *Géométrie transverse des flots d'Anosov*, Thesis, École Norm. Sup. Lyon, 1992.

[Ba2] T. BARBOT, *Caractérization des flots d'Anosov en dimension* 3 *par leurs feuilletages faibles*, to appear in Erg. Th. Dyn. Sys.

[Ba3] T. BARBOT, *Flots d'Anosov sur les variétées graphées au sens de Waldhausen I: Morceaux fibrés de bfeuilletages d'Anosov*, to appear.

[Ba4] T. BARBOT, *Flots d'Anosov sur les variétées graphées au sens de Waldhausen II: Caractérization des examples de Handel-Thurston*, to appear.

[Ba5] T. BARBOT, *Mise en position optimale d'un tore par rapport à un flot d'Anosov*, to appear.

[BFL] Y. BENOIST, P. FOULON and F. LABOURIE, *Flots d'Anosov à distributions stable et instable différentiables*, Jour. Amer. Math. Soc. *5* (1992) 33–74.

[Bo–La] C. BONATTI and R. LANGEVIN, *Un example de flot d'Anosov transitif transverse a un tore et non conjugue a une suspension*, preprint, 1992.

[Bo] R. BOWEN, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture notes in Mathematics *470* Springer Verlag, 1975.

[Br] M. BRUNELLA, *On the discrete Godbillon-Vey invariant and Dehn surgery on geodesic flows*, preprint 1994.

[Ca–Bl] A. CASSON and S. BLEILER, *Automorphisms of surfaces after Nielsen and Thurston*, London Math. Soc. Student Texts *9*, Cambridge Univ. Press, 1988.

[Ch] J. CHRISTY, *Intransitive Anosov flows on* 3*-manifolds*, to appear in C.B.M.S. lecture series.

[Fe1] S. FENLEY, *Anosov flows in* 3*-manifolds*, Ann. of Math *139* (1994) 79–115.

[Fe2] S. FENLEY, *Quasigeodesic Anosov flows and homotopic properties of closed orbits*, to appear in Jour. Diff. Geom.

[Fe3] S. FENLEY, *The structure of branching in Anosov flows of* 3*-manifolds*, preprint.

[Fr] D. FRIED, *Transitive Anosov flows and pseudo-Anosov maps*, Topology *22* (1983) 299–303.

[Gh1] E. GHYS, *Flots d'Anosov sur les* 3*-variétés fibrés en cercles*, Ergod. Th. Dyn. Sys. *4* (1984) 67–80.

[Gh2] E. GHYS, *Flots d'Anosov dont les feuilletages stables sont différentiables*, Ann. Sci. Ecole Norm. Sup. (4) *20* (1987) 251–270.

[Go] S. GOODMAN, *Dehn surgery and Anosov flows*, in Proceedings of the geometric dynamics conference, Lecture notes in mathematics *1007*, Springer, 1983.

[No] S. P. NOVIKOV, *Topology of foliations*, Trans. Moscow Math. Soc. *14* (1963) 268–305.

[Pa] F. PALMEIRA, *Open manifolds foliated by planes*, Ann. of Math., *107* (1978) 109–131.

[Pl1] J. PLANTE. *Anosov flows*, Amer. Jour. Math. *94* (1972) 729–754.

[Pl2] J. PLANTE, *Anosov flows, transversely affine foliations and a conjecture of Verjovsky*, J. London Math. Soc. (2) *23* (1981) 359–362.

[Pl3] J. PLANTE, *Solvable groups acting on the line*, Trans. Amer. Math. Soc. *278* (1983) 401–414.

[Re] G. REEB, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Sci. Indust., *1182*, Hermann, Paris, 1982.

[Sm] S. SMALE, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. *73* (1967) 747–817.
[So] V. V. SOLODOV, *On the universal cover of Anosov flows*, Research announcement.
[Ve] A. VERJOVSKY, *Codimension one Anosov flows*, Bol. Soc. Mat. Mex. *19* (1977) 49–77.

Mathematical Sciences Research
Institute and University
of California, Berkeley,
CA 94720
USA

Received January 3, 1994; August 12, 1994