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Autor(en): **Knop, Friedrich**

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On the set of orbits for a Borel subgroup

FRIEDRICH KNOP*

1. Introduction

Let $X = G/H$ be a homogeneous variety for a connected complex reductive group G and let B be a Borel subgroup of G. In many situations, it is necessary to study the B-orbits in X. An equivalent setting of this problem is to analyze H-orbits in the flag variety G/B.

The probably best known example is the Bruhat decomposition of G/B where one takes $H = B$. Another well-studied situation is the case where H is a symmetric subgroup, i.e., the fixed point group of an involution of G. Then H-orbits in G/B play a very important role in representation theory. They are the main ingredients for the classification of irreducible Harish-Chandra modules (see e.g. the surveys [Sch], [Wo]).

In this paper, we introduce two structures on the set of all B-orbits. The first one is not really new, namely an action of a monoid W^* on the set $\mathfrak{B}(X)$ of all B-stable closed subvarieties of X. As a set, W^* is the Weyl group W of G but with a different multiplication. That has already been done by Richardson and Springer [RS1] in the case of symmetric varieties and the construction generalizes easily. As an application we obtain a short proof of a theorem of Brion [Br1] and Vinberg [Vin]: If B has a open orbit in X then B has only finitely many orbits. Varieties with this property are called *spherical*. All examples mentioned above are of this type.

The second structure which we are introducing is an action of the Weyl group W on a certain subset of $\mathfrak{B}(X)$. Let me remark that in the most important case, X spherical, $\mathfrak{B}(X)$ is just the set of B-orbit closures and the W-action will be defined on all of it.

We give two methods to construct this action. In the first, we define directly the action of the simple reflections s_α of W. This is done by reduction to the case $\operatorname{rk} G = 1$ and then by a case-by-case consideration. The advantage of this method is that it is very concrete and works in general. The problem is to show that the

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s_α-actions actually define a W-action. For that the braid relations have to be verified which I don't know how to do directly.

The second method doesn't have this problem but it is more complicated, less explicit, and works only in the spherical case. It is based on a construction of Lusztig and Vogan [LV]. Let $\mathscr{H}_q$ be the Hecke algebra attached to the Weyl group W. Then Lusztig and Vogan define an $\mathscr{H}_q$-module $\mathscr{C}_q$ which is closely related to $\mathfrak{B}(X)$. In this paper we look only at the case $q = 1$. Then $\mathscr{H}_1$ is just the group algebra of W, hence $\mathscr{C}_1$ is a W-module. We show that, after some modifications, $\mathscr{C}_1$ becomes a permutation representation with $\mathfrak{B}(X)$ as basis. Hence, this defines a W-action on $\mathfrak{B}(X)$.

It should be noted that one could generalize Lusztig-Vogan's construction of the $\mathscr{H}_q$-module to all spherical varieties. Then specializing $q = 0$ or $q = \infty$ gives the W^*-action (see [RS2] 7.4 in the symmetric case). Hence, $\mathscr{H}_q$ unifies both the W- and the W^*-action. However, in this paper I don't pursue this line any further.

Actually, there is a third method to construct the W-action, but so far it works only on an even smaller subset of $\mathfrak{B}(X)$. It consists in relating B-orbits via conormal bundles to the cotangent bundle of X. The advantage of this construction is that one obtains more information. Observe, that $\mathfrak{B}(X)$ contains a distinguished element, namely X itself. We are able to determine its isotropy group $W_{(X)}$:

$$W_{(X)} = W_X \ltimes W_{P(X)}.$$

Here W_X is the Weyl group of X. It was defined by Brion [Br2] for spherical varieties and generalized in [Kn1], [Kn2], [Kn4]. The group $W_{P(X)}$ is the Weyl group of a certain parabolic subgroup attached to X. If X is symmetric then W_X is just the little Weyl group and $P(X)$ the complexification of a minimal parabolic subgroup. As opposed to the symmetric case, the definition of W_X is in general very complicated. Hence, it is one of the main virtues of the W-action on $\mathfrak{B}(X)$ that one obtains a relatively easy construction of W_X.

Finally, let me mention that many statements hold over an arbitrary algebraically closed ground field of characteristic $p \geq 0$. The monoid action goes through and also the W-action, at least if X is spherical and $p \neq 2$. There are counterexamples for $p = 2$.

Notation. All varieties are defined over an algebraically closed field k. Let p be its characteristic exponent, i.e., $p = 1$ if char $k = 0$ and $p = \text{char}\, k$ otherwise. The algebra of regular functions on a variety X is denoted by $k[X]$.

Throughout this paper, G will denote a connected reductive group. Let $B \subseteq G$ be a Borel subgroup with unipotent radical U and maximal torus T. Let Δ be the set of roots and let Δ^+ be the subset of positive roots corresponding to B. Let $\Sigma \subseteq \Delta^+$

be the set of simple roots. For $\alpha \in \Sigma$ let s_α be the simple reflection in the Weyl group $W = N_G(T)/T$ of G and P_α the corresponding minimal parabolic subgroup of G containing B. Conjugation is denoted by ${}^gH = gHg^{-1}$.

For any B-module V let $V^{(B)}$ be the set of non-zero semiinvariant vectors vectors. For $v \in V^{(B)}$ let $\chi_v \in \chi(B) = \chi(T)$ be the character with which B acts on the line kv. For any abelian group A and positive integer n let $A_{(n)} := A \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}]$.

2. The action of the Richardson-Springer monoid

For a B-variety Z we define the following objects:

$c(Z) := \operatorname{trdeg} k(Z)^B/k = \min_{x \in Z} \operatorname{codim}_Z Bx$ (the *complexity* of Z)

$\chi(Z) := \{\chi_f \in \chi(B) \mid f \in k(Z)^{(B)}\}$ (the *character group* of Z)

$\operatorname{rk} Z := \operatorname{rk} \chi(Z)$ (the *rank* of Z)

$u(Z) := \max_{x \in Z} \dim Ux.$

These are all invariants of Z under B-birational morphisms.

2.1. LEMMA. *The relation* $\dim Z = c(Z) + \operatorname{rk} Z + u(Z)$ *holds.*

Proof. By replacing Z by an open subset we may assume that the orbit spaces Z/U and $Z/B = (Z/U)/T$ exist. Then we have $\dim Z/B = c(Z)$ and $\dim Z = \dim Z/U + u(Z)$. Moreover, the image of T in $\operatorname{Aut} Z/U$ is of dimension $\operatorname{rk} Z$. This implies $\dim Z/U = \dim Z/B + \operatorname{rk} Z$.

2.2. THEOREM. *Let X be a G-variety and $Z \subseteq X$ a G-stable subvariety. Then* $c(Z) \le c(X)$, $\operatorname{rk} Z \le \operatorname{rk} X$, $u(Z) \le u(X)$, *and if* $c(Z) = c(X)$ *and* $\operatorname{rk} Z = \operatorname{rk} X$ *then* $Z = X$.

Proof. Let $\tilde{X} \to X$ be the normalization and $\tilde{Z} \subseteq \tilde{X}$ a component of the preimage of Z which maps onto Z. Then $c(Z) = c(\tilde{Z})$, $\chi(Z) \subseteq \chi(\tilde{Z})$, $u(Z) = u(\tilde{Z})$, $c(X) = c(\tilde{X})$, $\chi(X) = \chi(\tilde{X})$, and $u(X) = u(\tilde{X})$. Hence we may assume X to be normal. By Sumihiro's Theorem (see [KKLV]) we may then assume that X is G-isomorphic to a subvariety of a projective space. For every $f \in k(Z)^{(B)}$ there is $q = p^l$ and $\tilde{f} \in \mathcal{O}_{X,Z}^{(B)}$ such that $\tilde{f}|_Z = f^q$ ([Kn2] 2.3). Because of $\chi_{\tilde{f}} = q\chi_f$ we get $\chi(Z) \subseteq \chi(X)_{(p)}$. We also obtain that $k(Z)^B$ is a purely inseparable extension of the residue field of $\mathcal{O}_{X,Z}^B \subseteq k(X)^B$. Hence $c(Z) \le c(X)$. The inequality $u(Z) \le u(X)$ follows from the lower semicontinuity of the function $x \mapsto \dim Ux$.

Finally, there is a G-variety $\tilde{X}$ and a proper birational morphism $\tilde{X} \to X$ such that $u(\tilde{Z}) = u(\tilde{X})$ for every $\tilde{Z} \subseteq \tilde{X}$ which is G-stable ([Kn2] 2.13). Choose a $\tilde{Z}$ which is mapped onto Z. Then $u(\tilde{Z}) = u(\tilde{X})$ implies $\dim \tilde{Z} = \dim \tilde{X}$, hence $Z = X$. □

In [RS1], Richardson and Springer defined a new product $w * w'$ on the Weyl group W which turns it into a monoid. It can be described as follows: For $w \in W$ consider the Schubert cell $X_w = \overline{BwB} \subseteq G$. Every closed $B \times B$-stable subvariety of G is of this type. Therefore, one can define $w * w'$ by

$$X_{w * w'} = X_w X_{w'}.$$

We denote the set W equipped with this product by W^*. It is easy to see that W^* is generated by $\{s_\alpha \mid \alpha \in \Sigma\}$ with the relations $s_\alpha * s_\alpha = s_\alpha$ for all $\alpha \in \Sigma$ and the braid relations for all $\alpha, \beta \in \Sigma$.

For a G-variety X let $\mathfrak{B}(X)$ be the set of all non-empty, closed, irreducible, B-stable subsets of X. Let $w \in W$ and $Z \in \mathfrak{B}(X)$. Then $X_w Z$ is the image of $X_w \times^B Z$ under the proper morphism $G \times^B Z \to X$. Hence $X_w Z \in \mathfrak{B}(X)$ and

$$w * Z := X_w Z$$

defines a W^*-action on $\mathfrak{B}(X)$. For a parabolic subgroup P containing B let w_P be the longest element of its Weyl group. Then $P = X_{w_P}$ and we get $w_P * Z = PZ$ for every $Z \in \mathfrak{B}(X)$. Note in particular, $s_\alpha * Z = P_\alpha Z$.

Next we study the behavior of $c(Z)$, $\chi(Z)$, and $u(Z)$ under the W^*-action. In the next theorem, $\ell(w)$ is the length of $w \in W$.

2.3. THEOREM. *Let X be a G-variety, let $w \in W$ and $Z \in \mathfrak{B}(X)$. Then* $c(Z) \le c(w * Z)$, $\operatorname{rk} Z \le \operatorname{rk} w * Z$, $u(Z) \le u(w * Z)$, *and* $\dim Z \le \dim w * Z \le \dim Z + \ell(w)$.

Proof. By induction on $l(w)$ it suffices to consider the case of a simple reflection $w = s_\alpha$. Consider the surjective morphisms

$$\begin{array}{l} P_\alpha \times^B Z \xrightarrow{\Psi} P_\alpha Z = s_\alpha * Z \\ \quad\downarrow \\ P_\alpha / B \cong \mathbf{P}^1 \end{array}$$

If $Z = P_\alpha Z$ then there is nothing to prove. Otherwise, $\dim P_\alpha Z = \dim Z + 1$. In particular, Ψ has finite degree. Let $B_\alpha := B \cap s_\alpha B s_\alpha^{-1}$. Then $P_\alpha \times^B Z$ contains $B \times^{B_\alpha} s_\alpha Z$ as an open subset. Hence,

$$c(P_\alpha Z) = \operatorname{trdeg}_k k(P_\alpha Z)^B = \operatorname{trdeg}_k k(P_\alpha \overset{B}{\times} Z)^B$$
$$= \operatorname{trdeg}_k k(s_\alpha Z)^{B_\alpha} = \operatorname{trdeg}_k k(Z)^{B_\alpha} \geq \operatorname{trdeg}_k k(Z)^B = c(Z).$$

Every $f \in k(Z)^{(B)}$ defines a rational function $\bar{f}$ on $B \times^{B_\alpha} s_\alpha Z$ by $\bar{f}(tu, s_\alpha z) = f(s_\alpha^{-1} t s_\alpha z)$. Then $\bar{f} \in k(P_\alpha \times^B Z)^{(B)}$ with $\chi_{\bar{f}} = s_\alpha \chi_f$. Hence, $s_\alpha \chi(Z) \subseteq \chi(P_\alpha \times^B Z)$. Because Ψ is of finite degree, $\chi(P_\alpha Z) \hookrightarrow \chi(P_\alpha \times^B Z)$ is of finite index. This shows $\operatorname{rk} Z \leq \operatorname{rk} P_\alpha Z$. Finally, by semicontinuity of orbit dimensions, we get

$$u(Z) = u(1 \times Z) \leq u(P_\alpha \overset{B}{\times} Z) = u(P_\alpha Z). \qquad \square$$

The combination of Theorem 2.3 for $w = w_G$ with Theorem 2.2 yields:

2.4. COROLLARY. *Let* $Z \in \mathfrak{B}(X)$. *Then* $c(Z) \leq c(X)$, $\operatorname{rk} Z \leq \operatorname{rk} X$, *and* $u(Z) \leq u(X)$.

2.5. COROLLARY. *Let H be either B or U. Then for any H-stable subvariety Z of a G-variety X the following inequality holds*

$$\operatorname{trdeg}_k k(Z)^H \leq \operatorname{trdeg}_k k(X)^H.$$

Proof. We may assume that Z is closed. Then for $H = B$ the assertion follows directly from Corollary 2.4. Consider $H = U$. Because B normalizes U, the general U-orbits in Z and BZ have the same dimension. This implies $\operatorname{trdeg}_k k(Z)^U \leq \operatorname{trdeg}_k k(BZ)^U = c(BZ) + \operatorname{rk} BZ$. Now the assertion follows from Corollary 2.4. $\square$

Recall, that X is called *spherical* if B has a dense open orbit in X. This is equivalent to $k(X)^B = k$, i.e., $c(X) = 0$.

2.6. COROLLARY. *Every spherical variety contains only finitely many B-orbits.*

Proof. Let $Z \in \mathfrak{B}(X)$ be minimal with infinitely many B-orbits. By Corollary 2.4 $c(Z) = 0$ which implies that Z contains a dense orbit Bz. But then one of the components of $Z \backslash Bz$ contains infinitely many orbits contradicting the minimality of Z. $\square$

Remark. The last statements are not new. Vinberg [Vin] proved Corollary 2.5 in characteristic zero using an entirely different method. Independently, Brion [Br1] used the same method to prove Corollary 2.6 (also char $k = 0$). Later on, Grosshans [Gro] generalized the method used by Brion and Vinberg to arbitrary characteristic, such that Corollary 2.5 could be deduced in full generality. Then, Matsuki [Ma] has given another proof for homogeneous spherical varieties which uses ideas which are similar to those in the proof given above.

The character group is actually a property of a general orbit:

2.7. PROPOSITION. *For any B-variety Z there is a non-empty, open subset Z_0 such that $\chi(Z)=\chi(Bx)$ for all $x \in Z_0$.*

Proof. There is a non-empty open subset $Z_1 \subseteq Z$ such that the orbit space Z_1/U exists. Furthermore, Z_1 can be chosen B-stable ([DR] 1.6). By replacing Z with Z_1/U and B by $T=B/U$ we may assume that B is a torus. Then Z contains a non-empty, B-stable, open, affine subset Z_2 in which all orbits are closed. Then the B-action on Z_2 is the same as a $\chi(B)$-grading of $k[Z_2]$ and $\chi(Z_2)$ is the group generated by those characters which actually occur. Because $k[Bx]$ is a quotient of $k[Z_2]$ for every $x \in Z_2$ this shows $\chi(Bx) \subseteq \chi(Z_2)=\chi(Z)$. Conversely, choose $f_1,\ldots,f_s \in k[Z_2]^{(B)}$ such that $\chi(Z)$ is generated by their characters. Then $\chi(Bx) \supseteq \chi(Z)$ for every $x \in Z_2$ such that $f_1(x)\ldots f_s(x) \neq 0$. $\square$

3. Isotropy groups

Fix a minimal parabolic subgroup $P=P_\alpha$ with Levi part L and unipotent radical P_u. We want to study the relation between the B-orbits in a P-orbit Px. Let $\mathbf{P}:=P/B \cong \mathbf{P}^1$ and consider the canonical morphism

$$\pi : P \overset{B}{\times} X \cong \mathbf{P} \times X \longrightarrow X.$$

Then the P_x-orbits in $\mathbf{P}$ correspond to P-orbits in $\mathbf{P} \times Px$, which correspond to the B-orbits in Px, the correspondence being given by

$$P_x\bar{g} \leftrightarrow P(\bar{g},x)=P(\bar{e},g^{-1}x) \leftrightarrow Bg^{-1}x. \qquad (*)$$

here $\bar{g}=gB \in \mathbf{P}$. The isotropy groups are in the following relation:

$$(P_z)_{\bar{g}}=P_{\bar{g}} \cap P_x={}^g(P_{\bar{e}} \cap P_{g^{-1}x})={}^gB_{g^{-1}x},$$

Set $H=P_x$ and let $\bar{H}$ be its image in $\operatorname{Aut}\mathbf{P} \cong PGL_2$. Then we get a cartesian diagram $(x'=g^{-1}x)$:

$$\begin{array}{ccc} {}^{q}B_{x'} & \twoheadrightarrow & \bar{H}_{\bar{g}} \\ \cup & & \cup \\ P_x & \twoheadrightarrow & \bar{H} \end{array}$$

3.1. LEMMA. *Let* $x \in X$.

(a) *The equivalences* $c(Px) = c(Bx) \Leftrightarrow c(Px) = 0 \Leftrightarrow \dim \bar{H} \geq 1$ *hold.*

(b) *If* $c(Px) = c(Bx) + 1$ *then* $s_\alpha \chi(Px)_{(p)} = \chi(Px)_{(p)}$ *and* $\chi(Bx)_{(p)} \subseteq \chi(Px)_{(p)}$.

Proof. (a) follows from the correspondence $(*)$.

(b) By (a), $\dim \bar{H} = 0$ hence all P_x-orbits in $\mathbf{P}$ are finite. Therefore, for general $\bar{g} \in \mathbf{P}$ the isotropy subgroup $\bar{H}_{\bar{g}}$ is trivial. This implies $(P_x)_{\bar{g}}$ is normal in P and contained in $Z(L)Pu$. Hence, $B_y = (P_x)_{\bar{g}}$ with $y = g^{-1}x$ and $\alpha \in \chi(By)_{(p)} = \chi(Px)_{(p)}$ which implies the s_α-stability. Furthermore, $B_y \subseteq B_x$ which implies the inclusion. □

We consider now the case $\dim \bar{H} \geq 1$. For $G_0 := PGL_2(k)$ define

$$s_0 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad T_0 := \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \qquad U_0 := \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix},$$
$$B_0 := T_0 \cdot U_0, \qquad N_0 := N_{G_0}(T_0) = T_0 \cup s_0 T_0$$

It is easy to see that all positive dimensional subgroups of G_0 are conjugated to either G_0, $S \cdot U_0$ (where $S \subseteq T_0$), T_0, or N_0. Hence, we can choose an isomorphism $\varphi : \mathbf{P} \xrightarrow{\sim} \mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$ inducing $\Phi : P \to \operatorname{Aut} \mathbf{P} \xrightarrow{\sim} G_0$ such that $\Phi(P_x)$ is one of the subgroups above.

Choose $n \in N_P(T) - T$ and let $\bar{q} \in \mathbf{P}$ be one of its fixed points. Then we can find $a \in P$ with $\varphi(a\bar{e}) = \infty$, $\varphi(a\bar{n}) = 0$, and $\varphi(a\bar{q}) = 0$, and $\varphi(a\bar{q}) = 1$. Let $x_\infty := a^{-1}x$ and $x_0 := (an)^{-1}x = n^{-1}x_\infty$ and $x_1 := (aq)^{-1}x = q^{-1}x_\infty$. Furthermore, let $H := P_x$ and H_∞, H_0, H_1 the isotropy group of H at $\bar{a}$, $a\bar{n}$ and $a\bar{q}$ respectively. Observe the equalities

$$H_\infty = {}^{a}B_{x_\infty}, \qquad H_0 = {}^{an}B_{x_0}, \qquad H_1 = {}^{aq}B_{x_1}.$$

Now we go through the different cases in more detail. We are interested in the character group up to p-torsion $\chi(Bx)_{(p)}$ and in the group of components $\kappa(H_i) := H_i/H_i^0$.

3.2. LEMMA. *With* $H = P_x$ *one of the following cases holds*:

$\Phi(H) = G_0$. *Then* $Bx = Px$ *and* $\chi(Bx) \subseteq \chi(T)^{\langle s_\alpha \rangle}$.

$\Phi(H) = S \cdot U_0$. *Then* Px *contains two* B-*orbits, namely* Bx_∞ (*closed*) *and* Bx_0 (*open*).

There is a short exact sequence

$$1 \longrightarrow K \longrightarrow \kappa(H_0) \longrightarrow \kappa(H_\infty) = \kappa(H) \longrightarrow 1.$$

where K is a finite elementary abelian p-group. For the character groups holds

$$s_\alpha \chi(Bx_0)_{(p)} = \chi(Bx_\infty)_{(p)}.$$

If S is finite then $\chi(Bx_0) \otimes \mathbb{Q}$ is s_α-stable and in case S is trivial then even $\chi(Bx_0)_{(p)}$ is s_α-stable. If S is not trivial then K is contained in the commutator subgroup of $\kappa(H_0)$. If $S = T_0$ then K is trivial.

$\Phi(H) = T_0$. Then Px contains three B-orbits, namely Bx_1 (open), Bx_0, and Bx_∞ (both closed). There is an exact sequence

$$1 \longrightarrow K \longrightarrow \kappa(H_1) \longrightarrow \kappa(H_\infty) = \kappa(H_0) \longrightarrow 1,$$

where K is cyclic of order prime to p. For the character groups holds

$$s_\alpha \chi(Bx_0)_{(p)} = \chi(Bx_\infty)_{(p)} \subseteq \chi(Bx_1)_{(p)} = s_\alpha \chi(Bx_1)_{(p)}.$$

Furthermore, $\chi(Bx_1)_{(p)} / \chi(Bx_\infty)_{(p)} \cong \mathbb{Z}_{(p)}$.

$\Phi(H) = N_0$. Then Px contains two B-orbits, namely Bx_1 (open), and $Bx_\infty = Bx_0$ (closed). There are short exact sequences

$$1 \longrightarrow \kappa(H_\infty) \longrightarrow \kappa(H) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

$$1 \longrightarrow K \longrightarrow \kappa(H_1) \longrightarrow \kappa(H) \longrightarrow 1$$

where K is cyclic of order prime to p. Let $I := {}^{a^{-1}}\ker(H \to N_0)$. Then

$$\chi(Bx_\infty)_{(p)} \subseteq \chi(B/I)_{(p)} \supseteq \chi(Bx_1)_{(p)}$$

with quotient $\mathbb{Z}_{(p)}$ and $(\mathbb{Z}/2\mathbb{Z})_{(p)}$ respectively. All of these groups are s_α-stable.

Proof. $\Phi(H) = G_0$: Then $Bx = Px$ implies $\chi(Bx) \subseteq \chi(P) \subseteq \chi(T)^{\langle s_\alpha \rangle}$.

$\Phi(H) = S \cdot U_0$: We get the cartesian diagram

$$\begin{array}{ccccc} {}^{an}B_{x_0} & = & H_0 & \twoheadrightarrow & S \\ \cup & & \cup & & \cup \\ {}^{a}B_{x_\infty} = & H_\infty & = H & \twoheadrightarrow & S \cdot U_0 \end{array}$$

This implies in particular, ${}^{n}B_{x_0} \hookrightarrow B_{x_\infty}$. Because both groups differ only by a unipotent group we get the claimed equalities of character groups.

Let $\bar{U}$ be a maximal unipotent subgroup of H^0. It acts transitively on $H_\infty/H_0 \cong U_0$, hence $H_\infty = H_0\bar{U}$. This implies the claim on the component groups with $K = \kappa(\bar{U} \cap H_0)$. Then K it is elementary abelian because it is a subgroup of $\bar{U}/(\bar{U} \cap H_0)^0$ which is connected, unipotent, and one-dimensional hence isomorphic to $\mathbf{G}_\alpha$.

If S is finite then $B^0_{x_0} \subseteq Z(L)P_u$. Hence, $\alpha \in \chi(Bx_0) \otimes \mathbb{Q}$ which implies that $\chi(Bx_0) \otimes \mathbb{Q}$ is s_α-stable. If S is trivial then $B_{x_0} \subseteq Z(L)P_u$ and by the same reasoning we get that $\chi(Bx_0)_{(p)}$ is s_α-stable. If S is not trivial, then it acts on U_0, hence on its covering $U/(U \cap H_0)^0 \cong \mathbf{G}_a$ by multiplication with a non-trivial character. This implies that K consists of commutators. In particular, if $S = T_0$ then K must be trivial.

$\Phi(H) = T_0$: As above we get

$$H_\infty = H_0 = H; \qquad H_1 = \ker(H \twoheadrightarrow T_0) = H \cap \ker \Phi \subseteq Z(L)P_u.$$

Thus we get ${}^{n}B_{x_0} = B_{x_\infty} \supseteq {}^{q}B_{x_1}$ and $\alpha \in \chi(Bx_1)$. This implies the assertion on the character groups. There is a one-dimensional tòrus S contained in H_∞ with $H_\infty = H_1 S$. This implies the exact sequence for κ where $K = \kappa(H_1 \cap S) \subset S$ is cyclic of order prime to p.

$\Phi(H) = N_0$: There are two cartesian diagrams

$$\begin{array}{ccc} H_\infty & \twoheadrightarrow & T_0 \\ \hookuparrow & & \hookuparrow \\ H & \twoheadrightarrow & N_0 \end{array} \qquad \begin{array}{ccc} H_1 & \twoheadrightarrow & \langle s_0 \rangle \\ \hookuparrow & & \hookuparrow \\ H & \twoheadrightarrow & N_0 \end{array}$$

which imply the exact sequences for κ where as above K is isomorphic to a subgroup of $\mathbf{G}_m$. Furthermore, we have $B_\infty/I \cong T_0$, ${}^{q}B_1/I \cong \langle s_0 \rangle$. This implies the inclusions among the character groups. We show that all of them are s_α-stable. This is clear for $\chi(B/I)$ because $I \subseteq Z(L)P_u$. There is $\tilde{n} \in {}^{a^{-1}}H$ such that $\Phi(a\tilde{n}a^{-1}) = s_0$. Then $\tilde{n} \in nZ(L)P_u$ and $\tilde{n}$ normalizes B_{x_∞}. This implies that $\chi(Bx_\infty)$ is s_α-stable. It also implies that $B_{x_\infty}U$ contains $\alpha^\vee(\mathbf{G}_m)$, where $\alpha^\vee : \mathbf{G}_m \to T$ such that $s_\alpha \alpha^\vee = -\alpha^\vee$ and $\langle \alpha, \alpha^\vee \rangle = 2$. Then $\Phi(a\alpha^\vee(-1)a^{-1}) = 1$, hence $\alpha^\vee(-1) \in IU$. It follows that s_α acts trivially on $\chi(B/I)_{(p)} \otimes \mathbb{Z}/2\mathbb{Z}$. Hence every subgroup of index two, in particular $\chi(Bx_1)_{(p)}$, is s_α-stable. $\square$

3.3. COROLLARY. *Let X be a G-variety, $Z \in \mathfrak{B}(X)$ and $w \in W$ with $c(w * Z) = c(Z)$. Let $z \in Z$ and $y \in w * Z$ be points in general position. Then B_z/B^0_z is isomorphic to a subquotient of B_y/B^0_y.*

Proof. It suffices to consider $w = s_\alpha$. Then the assertion follows by inspection from Lemma 3.2. □

3.4. COROLLARY. *Let X be homogeneous and spherical with open B-orbit Bx_0. Assume U_{x_0} is connected. The U_x is connected for every $x \in X$.*

Proof. U_x is connected if and only if B_x/B_x^0 has no p-torsion. Now apply Corollary 3.3 with $w = w_G$.

4. The action of the Weyl group

In this section, I construct an action of the Weyl group with its usual multiplication on a subset of $\mathfrak{B}(X)$, namely on

$$\mathfrak{B}_0(X) := \{Z \in \mathfrak{B}(X) \mid c(Z) = c(X)\}.$$

Observe, that $\mathfrak{B}_0(X)$ is stable for the W^*-action (Theorem 2.3). In particular, $PZ \in \mathfrak{B}_0(X)$ whenever P is a parabolic subgroup of G and $Z \in \mathfrak{B}_0(X)$.

Let for the moment H be any connected algebraic group and X any H-variety. Recall, that a *sheet for H* is an irreducible component of one of the locally closed subsets $\{x \in X \mid \dim Hx = d\}$, $d = 0, 1, \ldots, \dim X$. Obviously, there are only finitely many sheets.

If X is spherical then $\mathfrak{B}_0(X) = \mathfrak{B}(X)$ and consists precisely of the B-orbit closures (Corollary 2.6). In general, we have

4.1. PROPOSITION. *Every $Z \in \mathfrak{B}_0(X)$ is the closure of some sheet for B. In particular, $\mathfrak{B}_0(X)$ is finite.*

Proof. Because Z is the disjoint union of the locally closed subsets $Z \cap S$, S a sheet, there must be one such that $S \cap Z$ is open in Z. This implies $Z \subseteq \bar{S}$. Because all B-orbits in S have the same dimension, we get $\dim S - \dim Z = c(S) - c(Z) \leq c(X) - c(X) = 0$. Hence $Z = \bar{S}$. □

For a parabolic subgroup $P \subseteq G$ and a P-stable closed subvariety $Y \subseteq X$ with $c(Y) = c(X)$ let

$$\mathfrak{B}_0(Y, P) := \{Z \in \mathfrak{B}_0(X) \mid PZ = Y\}.$$

Then $\mathfrak{B}_0(X)$ is the disjoint union of its subsets of the form $\mathfrak{B}_0(Y, P)$.

For every simple root α we construct an action of s_α on $\mathfrak{B}_0(X)$. We are going to define this action on each set $\mathfrak{B}_0(Y, P_\alpha)$ separately. Choose an identification

$P_\alpha/B = \mathbf{P}^1$ inducing $\Phi : P \twoheadrightarrow PGL_2$. Then there is an open P-stable subset $Y^0 \subseteq Y$ such that for all $y \in Y_0$ the group $\Phi(P_y)$ is either finite or conjugated to G_0, $S \cdot U_0$, T_0 or N_0 (notation as in section 3). If $\Phi(P_y)$ is finite then every proper B-stable subvariety Z of Y with $PZ = Y$ has smaller complexity (see Lemma 3.1a). Hence, $\mathfrak{B}(Y, P_\alpha) = \{Y\}$. Otherwise, each Py breaks up into at most three B-orbits giving rise to at most three elements in $\mathfrak{B}(Y, P_\alpha)$.

In case $\Phi(P_y) \sim T_0$ and $c(Y) \geq 1$ it may happen that the union of the closed B-orbits in $Py, y \in Y^0$ has either two components Z_0, Z_∞ or form a single irreducible component $Z_{0\infty}$. For example, consider $G = G_0 \cong PGL_2$ and $X_1 = G_0/T_0 \times \mathbf{A}^1$. Then $\mathfrak{B}_0(X_1, G_0)$ has three elements. Now consider the involution σ of X_1 defined by $\sigma(gT_0, x) = (gs_0T_0, 1-x)$ and let $X = X_1/\sigma$. Then the generic isotropy group is still T_0, but Z_0 and Z_∞ map to one component $Z_{0\infty}$.

Now, the s_α-action on $\mathfrak{B}_0(Y, P_\alpha)$ is defined by the following table.

$\Phi(P_y)$	$\mathfrak{B}_0(Y, P_\alpha)$	s_α-action		
finite	$\{Y\}$	$s_\alpha \cdot Y = Y$		
G_0	$\{Y\}$	$s_\alpha \cdot Y = Y$		
$S \cdot U_0$	$\{Y, Z\}$	$s_\alpha \cdot Y = Z$	$s_\alpha \cdot Z = Y$	
T_0	$\{Y, Z_0, Z_\infty\}$	$s_\alpha \cdot Y = Y$	$s_\alpha \cdot Z_0 = Z_\infty$	$s_\alpha \cdot Z_\infty = Z_0$
	or $\{Y, Z_{0\infty}\}$	$s_\alpha \cdot Y = Y$	$s_\alpha \cdot Z_{0\infty} = Z_{0\infty}$	
N_0	$\{Y, Z\}$	$s_\alpha \cdot Y = Y$	$s_\alpha \cdot Z = Z$	

One of the main results of this paper is

4.2. THEOREM. *Let X be a G-variety. Assume one of the following conditions holds*:

(a) char $k = 0$.

(b) *X is spherical and* char $k \neq 2$.

(c) *X is spherical and U_x is connected for every $x \in X$.*

(d) *X is spherical and G_x is contained in a Borel subgroup for every $x \in X$.*

Then the s_α-actions on $\mathfrak{B}_0(X)$ define an action of W.

The cases (b), (c), and (d) will be proved in section 5. The proof for part (a) is given in section 7. Observe, that it suffices to check (c) for the general points of each G-orbit (Corollary 3.4).

When char $k \neq 2$, then I conjecture that one gets a W-action for all X, spherical or not. Actually, it would be possible to prove this by using "brute force", i.e., rather nasty case-by-case considerations in rank two (see the remark after Lemma 7.3). Note, that without further conditions for char $k = 2$ the assertion of the Theorem is definitely wrong (see the example after Theorem 5.10).

Let $\tilde{W}$ be the free group generated by $\{s_\alpha \mid \alpha \in \Sigma\}$ with the relations $s_\alpha^2 = 1$. Then clearly, we have a $\tilde{W}$-action on $\mathfrak{B}_0(X)$ denoted by $w \cdot Z$. There is a surjective homomorphism $\tilde{W} \twoheadrightarrow W$ and the theorem claims that the $\tilde{W}$-action factors through W. One of the main features of this $\tilde{W}$-action is that it preserves character groups:

4.3. THEOREM. *Let* $Z \in \mathfrak{B}_0(X)$ *and* $w \in \tilde{W}$. *Then* $\chi(w \cdot Z)_{(p)} = w\chi(Z)_{(p)}$. *In particular,* $\operatorname{rk} w \cdot Z = \operatorname{rk} Z$.

Proof. It suffices to prove this for $w = s_\alpha$. Let $Y = P_\alpha Z$. In case $\Phi(P_y)$ is finite, the assertion follows from Proposition 2.7 and Lemma 3.1b. Otherwise, it follows from Proposition 2.7 and the explicit calculations in Lemma 3.2. □

Remark. The inversion of p is really necessary in every positive characteristic. Take for example $G = GL_2$ and $X = \mathbf{A}^2 \times {}^p\mathbf{P}^1$. Here ${}^p\mathbf{P}^1$ is the projective line with the Frobenius twisted G-action. Then $s_\alpha \cdot X = X$, but $\chi(X) = \mathbb{Z}p\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ is not s_α-stable.

5. The spherical case

In the whole section, X is a spherical G-variety. In this case $\mathfrak{B}_0(X) = \mathfrak{B}(X)$ and we identify the set of B-orbits with $\mathfrak{B}(X)$. We want to modify the construction of a representation of the Hecke algebra due to Lusztig-Vogan [LV]. Under condition (b), (c), or (d) of Theorem 4.2 this will define an action of W on $\mathfrak{B}(X)$.

Fix a prime number $l \neq p$ and an algebraic closure $\mathbb{F}$ of the prime field $\mathbb{F}_l$. let H be an algebraic group and X and H-variety. Then let $\mathfrak{S}(X, H)$ be the category of constructible H-equivariant sheaves of $\mathbb{F}$-vector spaces on X. Denote its associated Grothendieck group by $S(X, H)$. For a sheaf $\mathscr{F} \in \mathfrak{S}(X, H)$ let $[\mathscr{F}]$ be its class in $S(X, H)$. For an H-stable subvariety Z let $\mathbb{F}_Z$ the sheaf which is the constant sheaf with fiber $\mathbb{F}$ on Z and zero outside. If $\varphi : X \to Y$ is an H-equivariant morphism then there is a homomorphism

$$\varphi_! : S(X, H) \to S(Y, H) : [\mathscr{F}] \mapsto \sum_i (-1)^i [\mathbf{R}^i \varphi_! \mathscr{F}].$$

If H is a subgroup of an algebraic group G then we have an induction functor

$$\operatorname{ind}_H^G : \mathfrak{S}(X, H) \to \mathfrak{S}(G \overset{H}{\times} X, G) : \mathscr{F} \mapsto G \overset{H}{\times} \mathscr{F} := (p_* q^* \mathscr{F})^H$$

where $p : G \times X \to G \times^H X$ is the quotient and $q : G \times X \to X$ the projection. Then ind_H^G is an equivalence of categories where restriction to the fiber over $1H \in G/H$ is inverse to it.

For three H-varieties X_i and sheaves $\mathscr{F}_1 \in \mathfrak{S}(X_1 \times X_2, H)$, $\mathscr{F}_2 \in \mathfrak{S}(X_2 \times X_3, H)$ define

$$\mathscr{F}_1 \circ \mathscr{F}_2 := p_{13!}[p_{12}^*\mathscr{F}_1 \otimes p_{23}^*\mathscr{F}_2)] \in S(X_1 \times X_3, H).$$

where $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the projections. The following theorem is well known.

5.1. THEOREM. *Let X_i, $i = 1, 2, 3, 4$, be four H-varieties.*

(i) *The product $\circ$ induces a bilinear homomorphism*

$$S(X_1 \times X_2, H) \times S(X_2 \times X_3, H) \to S(X_1 \times X_3, H).$$

(ii) *Assume $X_1 = X_2$ and let $\Delta \subseteq X_1 \times X_2$ be the diagonal. Then $[\mathbb{F}_\Delta] \circ F = F$ for all $F \in S(X_1 \times X_3, H)$.*

(iii) *Let $F_i \in S(X_i \times X_{i+1})$ for $i = 1, 2, 3$. Then $(F_1 \circ F_2) \circ F_3 = F_1 \circ (F_2 \circ F_3)$.*

Proof. (i) and (ii) are easy. For (iii) see the argument in [Fu] 16.1.1. □

5.2. COROLLARY. *With this operation, $R = S(X_1 \times X_1, H)$ becomes a ring and $M = S(X_1 \times X_2, H)$ a left R-module.*

Now return to our situation that G is a connected reductive group and X a spherical G-variety. We consider the case $X_1 = G/B$ and $X_2 = X$. Then $X_1 \times X_i = G/B \times X_i = G \times^B X_i$ implies $S(X_1 \times X_1, G) \xrightarrow{\sim} S(G/B, B)$ and $S(X_1 \times X_2, G) \xrightarrow{\sim} S(X, B)$. Therefore, $R = S(G/B, B)$ is a ring and $M = S(X, B)$ is an R-module. The task is to make R and M as explicit as possible.

We start by describing a basis. Let $Bz \subseteq X$ be a B-orbit and $\rho : B_z/B_z^0 \to GL(\mathbb{F}^\rho)$ a representation on a finite dimensional $\mathbb{F}$-vector space $\mathbb{F}^\rho$. We can form the sheaf $\mathbb{F}_{Bz}^\rho$ which is $\operatorname{ind}_{B_z}^B \mathbb{F}^\rho$ on $Z = B/B_z$, extended to X by zero. We denote the class of $\mathbb{F}_{Bz}^\rho$ in $S(X, B)$ by $[z, \rho]$. Then $S(X, B)$ is freely generated by the set of all $[z, \rho]$ where Bz runs through all B-orbits and ρ through all irreducible representations.

In particular, if all isotropy groups B_z are connected then a basis of $S(X, B)$ consists of all $[z, 1] = [\mathbb{F}_{Bz}]$. This happens e.g. for $X = G/B$. Therefore, if we let $[w] := [\bar{w}, 1]$ for $w \in W$ then R is the free abelian group with basis $[w]$, $w \in W$. The next lemma gives an easy method for computing the product with $[s_\alpha]$ (cf. [LV] 3.3).

5.3. LEMMA. *Consider the morphism $\mu : P_\alpha \times^B X \twoheadrightarrow X : [p, x] \mapsto px$ and let $\mathscr{F} \in \mathfrak{S}(X, B)$. Then*

$$[s_\alpha] \circ [\mathscr{F}] = \mu_![\operatorname{ind}_B^{P_\alpha} \mathscr{F}] - [\mathscr{F}].$$

Proof. In $S(G/B, B)$, the equality $[s_\alpha] + [1] = [\mathbb{F}_{P_\alpha/B}]$ holds. The formula follows from

$$\mathbb{F}_{P_\alpha/B} \otimes \operatorname{ind}_B^G \mathscr{F} = \operatorname{ind}_B^{P_\alpha} \mathscr{F}. \qquad \square$$

The following theorem describes the action of $[s_\alpha]$ on $S(X, B)$ explicitly. It is the analogue of [LV] 3.5.

5.4. LEMMA. *Using the notation Lemma* 3.2, *let* $Px \subseteq X$, $z \in \{x_0, x_1, x_\infty\}$ *and* ρ *an irreducible representation of* B_z/B_z^0. *Then* $[s_\alpha]$ *acts on* $S(X, B)$ *according to the following table.*

$\Phi(P_z)$	z	$[s_\alpha] \circ [z, \rho]$	
G_0	x_∞	$[x_\infty, \rho]$	
$S \cdot U_0$	x_0	$-[x_0, \rho]$	$\rho\,\|_K$ *non-trivial*
		$[x_\infty, \rho]$	*otherwise*
	x_∞	$[x_0, \rho]$	
T_0	x_1	$-[x_1, \rho]$	
	x_0	$[x_\infty, \rho] + [x_1, \rho]$	
	x_∞	$[x_0, \rho] + [x_1, \rho]$	
N_0	x_1	$-[x_1, \rho]$	$\rho\,\|_K$ *non-trivial*
		$-[x_1, \epsilon\rho]$	*otherwise, where* $\epsilon : B_z \to N_o \twoheadrightarrow \{\pm 1\} \subseteq \mathbb{F}^\times$
	x_∞	$[x_\infty, \rho] + [x_1, \rho']$	*where* $\rho' := \operatorname{ind}_{B_z^0}^{P_z} \rho$

Proof. Let $\mathscr{F}$ be the sheaf on $\mathbf{P} := P_a/B$ which equals $H \times^{H_{\bar{e}}} \mathbb{F}^\rho$ on $H\bar{e}$ and zero outside. Let $\mathscr{H}^i := H^i(\mathbf{P}, \mathscr{F})$ considered as an $\kappa(H)$-module. Then Lemma 5.3 translates into

$$[s_\alpha] \circ [z, \rho] = \sum_{i=0}^{2} (-1)^i [P_\alpha \overset{H}{\times} \mathscr{H}^i] - [z, \rho].$$

Therefore, everything boils down to calculate the $\mathscr{H}^i$ which is either very easy or follows from the following lemma.

5.5. LEMMA. *Let* $\mathbf{G}$ be either $\mathbf{G}_a$ *or* $\mathbf{G}_m$ *and let* K *be a finite subgroup of* $\mathbf{G}$. *Then* $\mathbf{G}/K \cong \mathbf{G}$ *and there exists an open embedding* $j : \mathbf{G}/K \hookrightarrow \mathbf{P}^1$. *For a character* $\chi : K \to \mathbb{F}^\times$ *let* $\mathscr{F}$ *be the sheaf* $\mathbf{G} \times^K \mathbb{F}^\chi$ *on* $\mathbf{G}/K$. *Assume* $\chi \neq 1$. *Then* $H^i(\mathbf{P}^1, j_! \mathscr{F}) = 0$ *for all* $i \geq 0$.

Proof. Clearly, $H^0(\mathbf{P}^1, j_!\mathscr{F}) = 0$. Observe $H^2(\mathbf{P}^1, j_!\mathscr{F}) = H^2_c(\mathbf{G}/K, \mathscr{F})$ by definition of cohomology with compact support. Then $H^2_c(\mathbf{G}/K, \mathscr{F}) \cong \mathit{Hom}(\mathscr{F}, \mathbb{F}_{\mathbf{G}/K})^\vee$ by Poincaré duality ([Mi] V.2.1). Because $\chi \neq 1$, the latter group vanishes.

Now for H^1 to vanish it suffices that the Euler-Poincaré characteristic of $j_!\mathscr{F}$ vanishes. This follows easily from the expression of the Euler-Poincaré characteristic in local terms: $\chi = 2 - \Sigma_x c_x$, where c_x is the conductor of $j_!\mathscr{F}$ at $x \in \mathbf{P}^1$ (see [Mi] V.2.12). In case $\mathbf{G} = \mathbf{G}_a$ there is only one ramification point ∞ with ramification groups $G_0 = G_1 = K$ and $G_i = 1$ for $i \geq 2$. Hence $c_\infty = 2$. If $\mathbf{G} = \mathbf{G}_m$ there are two ramification points but the ramification is tame, hence the sum of the conductors is again two. □

The following Corollary is well known:

5.6. COROLLARY. *The map $w \mapsto [w]$ induces an algebra-isomorphism $\iota : \mathbb{Z}[W] \xrightarrow{\sim} S(G/B, B)$. In particular, $S(X, B)$ is a W-module.*

Proof. Take $X = G/B$ and $z = \bar{w} = wB$. Then all B-orbits in X have the same rank. Hence, in the table of Lemma 5.4 only the case $\Phi(P_z) = T_0 \cdot U_0$ occurs. It follows,

$$[s_\alpha] \circ [w] = [s_\alpha w].$$

This shows by induction on the length that ι is multiplicative. □

5.7. COROLLARY. *The W-action on $S(X, B)$ preserves the subgroup $S_r(X, B)$ generated by all classes $[z, \rho]$ with* $\operatorname{rk} Bz \geq r$.

Proof. This follows by inspection from Lemma 5.4. □

In particular, W will act on the associated graded module $\operatorname{gr} S(X, B)$. This is almost a permutation representation. To get rid of the signs, we define $\bar{S}(X, B) := \operatorname{gr} S(X, B) \otimes \mathbb{Z}/2\mathbb{Z}$. The class of $[z, \rho]$ in $\bar{S}(X, B)$ will be denoted by $[\![z, \rho]\!]$.

5.8. THEOREM. *There is an action of the Weyl group W on the set of all isomorphism classes $[\![z, \rho]\!]$ of simple B-equivariant constructible sheaves of $\mathbb{F}$-vector spaces, such that*

(a) *Its $\mathbb{F}_2$-permutation representation is $\bar{S}(X, B)$.*
(b) *If $w[\![z, \rho]\!] = [\![z', \rho']\!]$ then* $\dim \rho = \dim \rho'$.
(c) *If $w[\![z, \rho]\!] = [\![z', \rho']\!]$ then $w\chi(Bz) \otimes \mathbb{Q} = \chi(Bz') \otimes \mathbb{Q}$. In particular,* $\operatorname{rk} Bz = \operatorname{rk} Bz'/S'$.
(d) *Assume* $\operatorname{char} k = 0$ *or* $\dim \rho = 1$. If $w[\![z, \rho]\!] = [\![z, \rho]\!]$ *then* $w\chi(Bz)_{(p)} = \chi(Bz')_{(p)}$.

Proof. (a) holds by definition. For the rest we may assume that w is a simple reflection. For (b) observe that ρ is at most tensored by a character. For (c) and (d), we may use Theorem 4.3 and the fact that $s_\alpha \cdot Bz = Bz'$ except in one case: $\Phi(P_z) = S \cdot U_0$ and $\rho|_K$ is non-trivial. By Lemma 3.2, this cannot happen under the assumptions of (d). In any case, $\chi(Bz) \otimes \mathbb{Q}$ is s_α-stable. □

Now we are able to prove part of Theorem 4.2:

5.9. THEOREM. *Assume X satisfies one of the conditions* (b), (c) *or* (d) *of Theorem* 4.2. *Then the s_α-actions on $\mathfrak{B}(X)$ extend to a W-action.*

Proof. Let $\mathfrak{B}$ be the set of all $[\![z, 1]\!]$ which is in bijection to $\mathfrak{B}(X)$. First I claim that we can arrange $\mathfrak{B}$ to be W-stable. By Lemma 5.4, the only bad case is $\Phi(P_z) = N_0$ with the appearance of ϵ. If (b), char $k \neq 2$, then we may choose $\mathbb{F}$ to be of characteristic two which forces $\epsilon = 1$. If char $k = 2$ and if (c), U_z is connected, then B_z/B_z^0 has odd order. Hence, case $\Phi(P_z) = N_0$ doesn't occur. The same happens under (d), G_z is contained in a connected solvable subgroup, since then N_0 is not a subquotient of G_z. This shows the claim.

It remains to check that the action of W on $\mathfrak{B}$ coincides with the $\tilde{W}$-action on $\mathfrak{B}(X)$. According to Lemma 5.4, the only bad case is $\Phi(P_z) = S \cdot U_0$ with $\rho|_K$ non-trivial which doesn't occur since already ρ is trivial. □

In any case, if we start off with the trivial representation for ρ then at most characters of order two appear. These correspond to double covers of orbits. Hence we get

5.10. THEOREM. *Let X be a spherical variety. Then there is a canonical action of W on the set of equivariant double covers of B-orbits in X. If* char $k \neq 2$, *then this action is compatible with the W-action on the set of B-orbits.*

In char $k = 2$ in general some extra condition is needed as the example $X = PGL_3/SO_3$ shows. This variety is spherical with five B-orbits. Two of them have rank two. These are interchanged by one simple reflection and fixed by the other. Hence, this doesn't define a W-action. The B-isotropy group of the open orbit has order two, hence that orbit has a non-trivial double cover, while the isotropy group of the other orbit of rank two is connected ($\cong \mathbf{G}_a$) One checks that $W \cong S_3$ acts on these three objects, namely the two orbits and the double cover.

6. The orbits of maximal rank in characteristic zero

This section is independent of the preceding one. Here, the G-variety X may be arbitrary but we consider only a subset of $\mathfrak{B}_0(X)$ namely

$$\mathfrak{B}_{00}(X) := \{Z \in \mathfrak{B}(X) \mid c(Z) = c(X);\ \mathrm{rk}\, Z = \mathrm{rk}\, X\}.$$

It is $\tilde{W}$-stable by Theorem 4.3 and contains X as an element. If $Z \subseteq X$ is a closed B-stable subvariety then, regarded as an element of $\mathfrak{B}(X)$, we denote it sometimes by (Z). This may avoid confusion.

6.1. LEMMA. *The group $\tilde{W}$ acts transitively on $\mathfrak{B}_{00}(X)$, i.e., $\mathfrak{B}_{00}(X) = \tilde{W} \cdot (X)$.*

Proof. Assume $(Z) \in \mathfrak{B}_0(X)$ is of maximal rank and of maximal dimension in its $\tilde{W}$-orbit. Then Z is P_α-stable for all α by Lemma 3.2, hence G-stable. Then Theorem 2.2 implies $Z = X$. □

Hence, for the description of $\mathfrak{B}_{00}(X)$ we need the isotropy group $\tilde{W}_{(X)}$. For this assume from here to the end of the paper $\mathrm{char}\, k = 0$. Then I defined in [Kn4] a certain subgroup W_X of $\mathrm{Aut}\, \chi(X)$, the little Weyl group of X. According to [Kn3] 6.5 it has a canonical lift to W as follows: Let $\rho \in \chi(B)$ be the half-sum of the positive roots. Then for every $w \in W_X$ there is a unique $w' \in W$ inducing w on $\chi(X)$ with $w'\rho - \rho \in \chi(X)$. Using this lift we regard W_X as a subgroup of W.

Furthermore, we defined in [Kn4] §2 the parabolic subgroup

$$P(X) := \{g \in G \mid gBz = Bz \text{ for general } z \in X\}.$$

Let $W_{P(X)}$ be its Weyl group. It is generated by all s_α with $\Phi((P_\alpha)_z) = G_0$ in the notation of section 3. Then the other main result of this paper is

6.2. THEOREM. *Assume* $\mathrm{char}\, k = 0$. *Then the $\tilde{W}$-action on $\mathfrak{B}_{00}(X)$ factors through W. The isotropy group of (X) is $W_{(X)} = W_X \ltimes W_{P(X)}$.*

Let me start with some reductions. First, we may assume that X is the only G-stable subvariety in $\mathfrak{B}_{00}(X)$. Next, recall from [Kn4] that X is called *non-degenerate* if $P(X)$ is determined by $\chi(X)$ in the following sense: For every root α appearing in $\mathrm{Lie}\, R_u P(X)$ there is $\chi \in \chi(X)$ such that $\langle \chi, \alpha^\vee \rangle \neq 0$. This means that $P(X)$ is the largest parabolic subgroup P such that every character in $\chi(X)$ extends to a character of P. By [Kn4] §5, there exists a principal $\mathbf{G}_m$-bundle $\pi : L \to X$ with G-action such that L is non-degenerate as a $G \times \mathbf{G}_m$-variety. Then $\mathfrak{B}_{00}(X) \xrightarrow{\sim} \mathfrak{B}_{00}(L) : Z \mapsto \pi^{-1}(Z)$, $W_{P(L)} = W_{P(X)}$ and $W_L = W_X$ ([Kn4] 7.5). Hence, may replace X by L and may assume that X is non-degenerate.

Also we may remove all singularities from X. Then the cotangent bundle T_X^* is a vector bundle over X. It is equipped with a projection $\pi : T_X^* \to X$ and the moment map $\Phi : T_X^* \to \mathfrak{g}^*$ defined by $[\Phi(\alpha)](\xi) = \alpha(\xi_{\pi(\alpha)})$. This gives rise to the composed morphism

$$\Psi : T_X^* \to \mathfrak{g}^* \to \mathfrak{g}^*//G = \mathfrak{t}^*/W.$$

The last equality is the Chevalley isomorphism.

With $\mathfrak{u} := \operatorname{Lie} U$ we define

$$C := \Phi^{-1}(\mathfrak{u}^\perp) = \{\alpha \in T_X^* \mid z := \pi(\alpha), \alpha(\mathfrak{u}z) = 0\}.$$

Then C is the union of the conormal bundles of the U-orbits in X. Therefore, if Z is a U-sheet then $\pi^{-1}(Z) \cap C$ is a vector bundle over Z with total dimension $\dim X + m$ where $m = \dim Z - u(Z) = c(Z) + \operatorname{rk} Z$ is the number of parameters (Lemma 2.1). By Corollary 2.4, we have $m \leq c(X) + \operatorname{rk} X$ with equality if and only if the closure of Z is in $\mathfrak{B}_{00}(X)$. Define $N(X) := \dim X + c(X) + \operatorname{rk} X = 2\dim(X) - u(X)$. Thus we have proved

6.3. PROPOSITION. *The dimension of C is $N(X)$ and $Z \mapsto C_Z$ defines a bijection between $\mathfrak{B}_{00}(X)$ and the set of irreducible components of C of dimension $N(X)$.*

Consider the morphism

$$\tilde{T}_X := T_X^* \underset{\mathfrak{t}^*/W}{\times} \mathfrak{t}^* \to \mathfrak{g}^* \underset{\mathfrak{g}^*/W}{\times} \mathfrak{t}^*.$$

The projection $\mathfrak{g}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^* \to \mathfrak{g}^*$ admits a section over $\mathfrak{u}^\perp$, namely

$$\sigma : \mathfrak{u}^\perp \to \mathfrak{g}^* \underset{\mathfrak{t}^*/W}{\times} \mathfrak{t}^* : \lambda \mapsto (\lambda, \lambda|_{\mathfrak{t}^*}).$$

Then the preimage $\tilde{C}$ of $\sigma(\mathfrak{u}^\perp)$ in $\tilde{T}_X$ is isomorphic to C. Let $\tilde{C}_Z$ be its component corresponding to $Z \in \mathfrak{B}_{00}(X)$.

The point is now, that there is an action of W on $\tilde{T}_X$ induced by the action on the second factor $\mathfrak{t}^*$. In particular, $\tilde{W}$ is acting via W on the set $\mathcal{I}$ of irreducible components of $\tilde{T}_X$.

6.4. THEOREM. *Let X be non-degenerate.*

(a) *For $Z \in \mathfrak{B}_{00}X)$ let $\iota(Z) := G \cdot \tilde{C}_Z$. Then $\iota(Z)$ is an irreducible component of $\tilde{T}_X$. In particular, this defines a map $\iota : \mathfrak{B}_{00} \to \mathcal{I}$.*

(b) *The map ι is bijective and $\tilde{W}$-equivariant.*

Before we enter into the proof let me first show how to derive Theorem 6.2.

Proof of Theorem 6.2. Part (b) implies immediately that the $\tilde{W}$-action on $\mathfrak{B}_{00}(X)$ factors through W. The assertion on the isotropy group is almost the definition of W_X in [Kn4] §3: $\iota(X)$ is the component of $\tilde{T}_X$ containing $\tilde{C}_X$. Then observe that the objects denoted by C and $\hat{T}_X$ in [Kn4] are open subsets of C_X and $\iota(X)$. Denote the isotropy group of $\iota(X)$ in W by W_0. The image of $\iota(X)$ in $\mathfrak{t}^*$ is the k-subspace $\mathfrak{a}^*$ generated by $\chi(X) \subseteq \mathfrak{t}^*$ ([Kn4] 3.2). This implies

$$C_W(\mathfrak{a}^*) \subseteq W_0 \subseteq N_W(\mathfrak{a}^*).$$

Because X is non-degenerate, we have $C_W(\mathfrak{a}^*) = W_{P(X)}$. The images of W_0 and W_X in $GL(\mathfrak{a}^*)$ are the same by definition (see [Kn4] p. 315). Hence $W_0 = W_X \ltimes W_{P(X)}$. □

Theorem 6.4 will follow from the next three lemmas.

6.5. LEMMA. *All irreducible components of $\tilde{T}_X$ have the same dimension. They map onto T_X^* and W acts transitively on $\mathcal{I}$.*

Proof. Because $\mathfrak{t}^*/W$ is an affine space, $\tilde{T}_X$ is a complete intersection in $T_X^* \times \mathfrak{t}^*$. This implies that all irreducible components of $\tilde{T}_X$ have the same dimension. Since all fibers of $\tilde{T}_X \to T_X^*$ are W-orbits every component of $\tilde{T}_X$ is mapped finite to one onto T_X^*. This implies also that W acts transitively on $\mathcal{I}$.

6.6. LEMMA. *Let $Z \in \mathfrak{B}_{00}(X)$. Then $\iota(Z) = G \cdot \tilde{C}_Z$ is an irreducible component of $\tilde{T}_X$ and $\iota : \mathfrak{B}_{00}(X) \to \mathcal{I}$ is $\tilde{W}$-equivariant.*

Proof. First assume that the Lemma is true for groups of semisimple rank one. We show the general case. Because $\tilde{W}$ acts transitively on $\mathfrak{B}_{00}(X)$ it suffices to show that for every simple reflection $s_\alpha \in W$ and $Z \in \mathfrak{B}_{00}(X)$:

(i) $\iota(X) \in \mathcal{I}$.
(ii) $\iota(Z) \in \mathcal{I} \Rightarrow \iota(s_\alpha \cdot Z) \in \mathcal{I}$.
(iii) $\iota(s_\alpha \cdot Z) = s_\alpha \iota(Z)$.

Part (i) is precisely the content of [Kn4] 3.2. Let $P_u := R_u P_\alpha$, $L := P/P_u$, and $W_\alpha = W_{P_\alpha} = \langle s_\alpha \rangle$. There is a nonempty, smooth open subset $Y \subseteq P_\alpha Z$ such that a quotient morphism $Y \to V := Y/P_u$ exists and is smooth. Furthermore, Y can be chosen to be P_α-stable ([DR] 1.6). Then V is an L-variety and $\mathfrak{B}_{00}(V)$ is defined with respect to the L-action. Define $Z' := Z \cap Y/P_u \subseteq V$ which is in $\mathfrak{B}_{00}(V)$. By the definition of the $\tilde{W}$-action, we get $(s_\alpha \cdot Z) \cap Y/P_u = s_\alpha \cdot Z'$. Now let

$$S := \{\alpha \in T_X^* \mid x = \pi(\alpha) \in Y, \alpha(\mathfrak{p}_u x) = 0\}.$$

Then there is a morphism $S \to T_V^*$ which is easily seen to be the quotient morphism by P_u. Moreover, there is a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & T^*_V \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\ \mathfrak{g}^* \supseteq \mathfrak{p}_u^\perp & \xrightarrow{\mathrm{res}_\mathfrak{p}} & \mathfrak{l}^* \end{array}$$

This shows that the preimage of $C_{Z'} \subseteq T^*_V$ is an open subset of C_Z namely $C_Z \cap \pi^{-1}(Y)$.

Now we show (ii): Assume $\iota(Z) \in \mathscr{I}$. By Lemma 6.5, this is equivalent to $G \cdot C_Z = T^*_X$. Because $P_u^- P$ is dense in G and $PC_Z \subseteq \bar{S}$ we conclude that

$$P_u^- S \text{ is dense in } T^*_X. \qquad (*)$$

Since X is non-degenerate the general element of $\Phi(T^*_X) \subseteq \mathfrak{g}^*$ is semisimple ([Kn1] 5.4). By $(*)$ and the commutative diagram above, the same holds for $\Phi(T^*_V)$, i.e., V is non-degenerate. Hence, we may use that the Lemma is true for the group L. Therefore, $L \cdot C_{s_\alpha \cdot Z'} = T^*_V$ which implies that $P \cdot C_{s_\alpha \cdot Z}$ is dense in $\bar{S}$. With $(*)$ we get $G \cdot C_{s_\alpha \cdot Z} = T^*_X$ which means $\iota(s_\alpha \cdot Z) \in \mathscr{I}$.

Finally observe that there is the following diagram of W_α-equivariant morphisms:

$$\begin{array}{ccc} S \times_{\mathfrak{t}^*/W_\alpha} \mathfrak{t}^* & \longrightarrow & T^*_X \times_{\mathfrak{t}^*/W} \mathfrak{t}^* \\ \downarrow & & \\ T^*_V \times_{\mathfrak{t}^*/W_\alpha} \mathfrak{t}^* & & \end{array}$$

Therefore $\iota(s_\alpha \cdot Z') = s_\alpha \iota(Z')$ implies (iii). This finishes the reduction and it remains to prove the Lemma when the semisimple rank of G is one.

We may assume that the quotient X/G exists. Then it is easy to see that it suffices to consider only a general fiber, i.e., we may assume that $X = G/H$ is homogeneous. Let H_0 be the image of H in $G/Z(G) \cong PGL_2$. If H_0 is finite or conjugated to G_0, T_0, or N_0 (notation as in section 3) then $\mathfrak{B}_{00}(X) = \{X\}$ and $G \cdot C_X = T^*_X$ by [Kn4] 3.2. Furthermore, either $H_0 = G_0$ and $W_{P(X)} \neq 1$ or $W_X \neq 1$ (see [Kn1] 9.1, or direct computation). Therefore, $\tilde{T}_X$ is irreducible, i.e., $\mathscr{I} = \{\tilde{T}_X\}$. This shows the Lemma in these cases.

It remains the case $H_0 \sim S \cdot U_0$. Because X is non-degenerate, H does not contain the Borel subgroup of (G, G). This means that $\mathfrak{a}^*$ does not consist of s_α-fixed points. Choose $B \subseteq G$ such that its image in PGL_2 is opposite to B_0. Then we have $T^*_X = G \times^H \mathfrak{h}^\perp$ with $\mathfrak{h}^\perp = \mathfrak{a}^* \oplus \mathfrak{u}_0$. Hence $\tilde{T}_X$ has two components consisting of all $[(g, \xi + \mathfrak{u}_0], \xi)$, and$([g, \xi + \mathfrak{u}_0], s_\alpha(\xi))$ respectively. Furthermore, $\tilde{C}_X$ and $\tilde{C}_Z$ (with $Z = s_\alpha \cdot X$) contains $([b, \xi], \xi)$ and $([bs_\alpha, \xi + \mathfrak{u}_0], s_\alpha \xi)$ respectively. This shows that $G \cdot C_X$ and $G \cdot C_Z$ are dense in T^*_X and that $\tilde{C}_X$ and $\tilde{C}_Z$ lie in different components of $\tilde{T}_X$. This finishes the proof of the lemma. □

6.7. LEMMA. *Let* $Z \in \mathfrak{B}_{00}(X)$. *Then* $G \cdot \tilde{C}_Z = G \cdot \tilde{C}_X$ *implies* $Z = X$.

Proof. It suffices to show that there is a non-empty G-stable open subset T' of $\imath(X) = G \cdot \tilde{C}_X$ such that $T' \cap \tilde{C}$ is irreducible. By [Kn4] §3, there is an open G-stable subset $\hat{T}_X$ of $\imath(X)$ such that there is a factorization

$$\hat{T}_X \xrightarrow{\hat{\Phi}} G/L \times \mathfrak{a}^r \xrightarrow{\gamma} \mathfrak{g}^* \underset{\mathfrak{t}^*/W}{\times} \mathfrak{t}^*.$$

where L is the Levi part of $P(X)$ and $\mathfrak{a}^r$ is the open subset of elements of $\mathfrak{a}^*$ which have L as centralizer. The morphism γ is defined by $\gamma(gL, \lambda) = (g\lambda, \lambda)$. Now assume that $(g\lambda, \lambda)$ is in the image of $\mathfrak{u}^\perp$. Then there is $u \in U$ such that $ug\lambda \in \mathfrak{t}^*$. Furthermore, $ug\lambda = \lambda$. Hence $\lambda \in \mathfrak{a}^r$ implies $ug \in C_G(\mathfrak{a}^*) = L$. This shows that $\tilde{C} \cap \hat{T}_X$ is the preimage of the irreducible subset $U\bar{e} \times \mathfrak{a}^r \subseteq G/L \times \mathfrak{a}^r$. Furthermore, the morphism $\hat{\Phi}$ has irreducible generic fibers (see [Kn4] 3.4). This implies that there is $T' \subseteq \hat{T}_X$ such that $T' \cap \tilde{C}$ is irreducible. □

Proof of Theorem 6.4. Lemma 6.6 establishes a $\tilde{W}$-equivariant map $\mathfrak{B}_{00}(X) \to \mathscr{I}$ where Z is mapped to $G \cdot \tilde{C}_Z$. This map is surjective (Lemma 6.5) and injective (Lemma 6.7). □

The method of proof gives more namely a description of the W-action on $\mathfrak{B}_{00}(X)$ in terms of the G-action on X. For this we need:

6.8. LEMMA. *Let X be non-degenerate and $Z = w \cdot X \in \mathfrak{B}_{00}(X)$. Then $Bw\mathfrak{a}^r$ is dense in $\Phi(C_Z) \subseteq \mathfrak{g}^*$.*

Proof. If we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ we have $\Phi(C_Z) \subseteq \mathfrak{u}^\perp = \mathfrak{t}^* \oplus \mathfrak{u}$. Let $f \in k(Z)^{(B)}$. Then the 1-form df is a section of $C_Z \to Z$. We have $\langle \Phi(df), \xi \rangle = \chi_f(\xi)$ for all $\xi \in \mathfrak{t}$. This shows that $\Phi(C_Z) \subseteq \mathfrak{a}' \oplus \mathfrak{u}$ where $\mathfrak{a}' \subseteq \mathfrak{t}^*$ is the subspace spanned by $\chi(Z)$. Hence, $\mathfrak{a}' = w\mathfrak{a}^*$ by Theorem 4.3. Furthermore, the projection $\Phi(C_Z) \to w\mathfrak{a}^*$ is surjective.

Let $\xi \in w\mathfrak{a}^r \oplus \mathfrak{u}$ and $\xi_s + \xi_n$ be its Jordan decomposition. Then there is $b \in B$ such that $b\xi_s \in w\mathfrak{a}^r$. Let also $\xi \in \Phi(C_Z)$. Then $\xi \in \overline{G\mathfrak{a}^*}$ because X is non-degenerate ([Kn4] 3.3). In particular, $\dim C_G(\xi) \geq \dim L = \dim C_G(\xi_s)$. This can only happen if $\xi = \xi_s$ which shows $\Phi(C_Z) \cap \mathfrak{a}^r \oplus \mathfrak{u} = B\mathfrak{a}^r$. □

For $n \in N_G(T)$ let $\bar{n} \in W$ be its image in the Weyl group.

6.9. THEOREM. *For every $Z \in \mathfrak{B}_{00}(X)$ choose any non-empty open B-stable subset Z^ω. Then there exists $x \in X$ such that $nx \in (\bar{n} \cdot X)^\omega$ for all $n \in N_G(T)$.*

Proof. We may again assume that X is non-degenerate. Consider the diagonal embedding $\mathfrak{t}^* \hookrightarrow \mathfrak{g}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$ and its preimage $\Xi \subseteq \tilde{T}_X$. Then $N_G(T)$ acts on Ξ diagonally: $n(\alpha, \lambda) := (n\alpha, \bar{n}\lambda)$. Let $T' \subseteq G \cdot \tilde{C}_X$ as in the proof of Lemma 6.7. Then we get $n(\Xi \cap T') \subseteq \tilde{C} \cap \bar{n}T' \subseteq \tilde{C}_{\bar{n} \cdot X}$. Therefore, $(\alpha, \lambda) \in \Xi \cap T'$ implies $n\alpha \in C_{\bar{n} \cdot X}$, hence $nx \in \bar{n} \cdot X$ for $x = \pi(\alpha)$. By Lemma 6.8, $B\Xi \cap \tilde{C}_Z$ is dense in $\tilde{C}_Z$ for any $Z \in \mathfrak{B}_{00}(X)$. Hence we can choose $x \in X$ such that nx lies in the chosen open B-stable subset Z^ω of $Z = \bar{n} \cdot X$. □

In case of a spherical variety there is a canonical choice for $Z^\omega \subseteq Z$, namely the open B-orbit. Hence, we obtain:

6.10. COROLLARY. *Let X be a spherical variety. Then there exists a point x in the open B-orbit of X such that $\bar{n} \cdot X = \overline{Bnx}$ for all $n \in N_G(T)$.*

This means that the W-action on $\mathfrak{B}_{00}(X)$ is induced by some carefully chosen $N_G(T)$-orbit. When X is a symmetric variety, then one can choose x such that $W_{(X)}$ is precisely the image of $N_G(T) \cap G_x$ in W. This means, every $z \in \mathfrak{B}_{00}(X)$ contains precisely one component of the orbit $N_G(T)x$. This can not be achieved in general, for the simple reason that the isotropy group G_x may not contain W_X as a subquotient (e.g. if G_x is solvable and W_X is not.) If $c(X) > 0$ then it may even happen that all components of $N_G(T)x$ lie in different B-orbits. Take for example $X = G$ on which G acts by left translation. Then $Bn_1x = Bn_2x$ implies $\bar{n}_1 = \bar{n}_2$ for every $x \in X$ and $n_1, n_2 \in N_G(T)$. In this case $\mathfrak{B}_{00}(G) = \{G\}$.

7. Characteristic zero

In this chapter I want to finish off the proof that the $\tilde{W}$-action on $\mathfrak{B}_0(X)$ factors through W, provided $\operatorname{char} k = 0$. We start with some reductions.

For every set $I \subseteq \Sigma$ of simple roots let P_I be the corresponding parabolic subgroup and $W_I \subseteq W$, $\tilde{W}_I \subseteq \tilde{W}$ the subgroups generated by $\{s_\alpha \mid \alpha \in I\}$. Then W_I is the Weyl group of $L_I := P_I/R_uP_I$.

7.1. PROPOSITION. *For $P = P_I$ let $(Y) \in \mathfrak{B}_0(X)$ be such that Y is P-stable. Let Py be a general P-orbit of Y. Consider $X_I := Py/R_uP = P/P_yR_uP$ as an L_I-variety. Then there is a natural $\tilde{W}_I$-equivariant surjective map*

$$\iota_I : \mathfrak{B}_0(X_I) \longrightarrow \mathfrak{B}_0(Y, P) \subseteq \mathfrak{B}_0(X).$$

Proof. First observe, that there is a bijection between $\mathfrak{B}_0(X_I)$ and closed B-stable subvarieties Z of Py such that $c(Z) = c(Py)$. Now choose $Y^0 \subseteq Y$ non-empty, open such that the orbit space Y^0/P exists. Moreover, we can choose Y^0 that small such that for $y \in Y^0$ and for $Z \subseteq Py$ a closed B-stable subvariety there is a unique $\bar{Z} \in \mathfrak{B}_0(Y, P)$ such that Z is an irreducible component of $\bar{Z} \cap Py$. This defines the surjective map $\mathfrak{B}_0(X_I) \to \mathfrak{B}_0(Y, P)$. The $\tilde{W}_I$-equivariance follows directly from the definition. □

7.2. LEMMA. *It suffices to prove Theorem* 4.2(a) *for G semisimple of rank two and $X = G/H$ homogeneous where H is connected.*

Proof. As a normal subgroup, the kernel of $\tilde{W} \to W$ is generated by the braid relations

$$s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots$$

Because each of them involves only two reflections it suffices to check them for the L_I-action on X_I where $I = \{\alpha, \beta\}$ and where Y runs through all P_I-stable closed subvarieties of X with $c(Y) = c(X)$. Dividing out the center of L_I doesn't change $\mathfrak{B}_0$ and makes the group semisimple. Finally, the natural map $\mathfrak{B}_0(G/H^0) \to \mathfrak{B}_0(G/H)$ is surjective and equivariant. Hence it suffices to consider connected isotropy subgroups. □

From now on assume we are in the situation of the Lemma. For $r \in \mathbb{N}$ define

$$\mathfrak{B}_{0r} := \{Z \in \mathfrak{B}_0(X) \mid \operatorname{rk} Z = \operatorname{rk} X - r\}.$$

This set is $\tilde{W}$-stable and empty unless $r = 0$, 1, or 2. The case $r = 0$ is handled in Theorem 6.2 (that is where $\operatorname{char} k = 0$ comes in). For $r = 2$ remember the correspondence of B-orbits on G/H and H-orbits on G/B (section 3). If $\mathfrak{B}_{02}(X)$ is not empty then H must contain a maximal torus T of G. Then each $Z \in \mathfrak{B}_{02}(X)$ corresponds to a subset $\overline{HV} \subseteq G/B$ where V is an irreducible component of $(G/B)^T$. Hence V is one point which implies $c(X) = c(Z) = 0$, i.e., X is spherical. This case is handled in Theorem 5.9.

That leaves the case $r = 1$. Assume $\mathfrak{B}_{01}(X) \neq \emptyset$. Then H must contain a one-dimensional torus S such that some $Z \in \mathfrak{B}_{01}(X)$ corresponds to $\overline{HV}$ where V is a component of $(G/B)^S$. Again we may exclude the case that X is spherical. Then $c(Z) \geq 1$ implies that V has positive dimension. Therefore, if we choose H and S such that $S \subseteq T$ then S is the connected kernel of some root β. Let $L = C_G(S)$. Then $S = H \cap L$ because otherwise H has a dense orbit in HV. In particular, S is a maximal torus of H. This implies that every $Z \in \mathfrak{B}_{01}(X)$ corresponds to some $\overline{HV}$.

For $w \in W$ let $\bar{w} := wB \in G/B$ and $V_w := Lw^{-1} \subseteq G/B$. This induces a bijection between $W/\langle s_\beta \rangle$ and the set of irreducible components $(G/B)^S$. Let $Z_w \in \mathfrak{B}(X)$ be the subvariety associated to $\overline{HV_w}$. Now Theorem 4.2(a) follows from the next lemma.

7.3. LEMMA. *The assignment $w \mapsto Z_w$ induces a $\tilde{W}$-equivariant surjective mapping $W/\langle s_\beta \rangle \to \mathfrak{B}_{01}(X)$.*

Proof. As already mentioned, every $Z \in \mathfrak{B}_{01}(X)$ is of the form Z_w. Because we assumed that $\mathfrak{B}_{01}(X)$ is not empty at least one Z_w is in it. Therefore, it suffices to show for every simple root α and $w \in W$:

Assume $Z_w \in \mathfrak{B}_{01}(X)$. Then $Z_{s_\alpha w}$ is in $\mathfrak{B}_{01}(X)$ and equals $s_\alpha \cdot Z_w$.

For this consider the projection $\pi : G/B \to G/P_\alpha$. Then V_w is a fiber of π if and only if $L \subseteq w^{-1}P_\alpha w$. Otherwise, π is injective on V_w.

In the first case we have $\underline{w^{-1}s_\alpha w} \in L$, hence $\overline{(s_\alpha w)^{-1}} = w^{-1}s_\alpha ww^{-1} \in \overline{V_w}$. This implies $Z_{s_\alpha w} = Z_w$. Let $y := \pi(w^{-1})$. Then H_y induces a finite group action on the fiber V_w. Hence $s_\alpha \cdot Z_w = Z_w$ by definition.

Now assume $\pi : V_w \hookrightarrow G/P_\alpha$. Then as above $V_{s_\alpha w}$ has the same image in G/P_α. Let y be a general point in this image. Then $S \subseteq H_y$ acts non trivially in the fiber $\pi^{-1}(y)$ with two fixed points lying in V_w and $V_{s_\alpha w}$, respectively. This implies that HV_w contains a dense H-orbit if and only if $\pi(HV_w)$ does. Hence $Z_w \in \mathfrak{B}_0(X)$ implies $Z_{s_\alpha w} \in \mathfrak{B}_0(X)$. Now a simple case-by-case consideration shows, that either $Z_{s_\alpha w} = Z_w = s_\alpha \cdot Z_w$ or s_α interchanges Z_w and $Z_{s_\alpha w}$.

Remark. As already mentioned, the only part of the proof which uses char $k = 0$ was Theorem 6.2. Therefore, to prove that for a given ground field k with char $k \neq 2$, the $\tilde{W}$-action on $\mathfrak{B}_0(X)$ factors through W it suffices to prove this for the subset $\mathfrak{B}_{00}(X)$ where $X = G/H$ is homogeneous and not spherical, H connected and G is semisimple of rank two.

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Department of Mathematics,
Rutgers University,
New Brunswick,
NY 08903
USA
E-mail address: knop@math.rutgers.edu

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