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Classification of certain infinite simple C^*-algebras, II

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1 Introduction

The main purpose of this paper is to complete the construction of classifiable models of separable, purely infinite, simple C^*-algebras that was started in [15]. The class $\mathscr{C}$ of classifiable models has, almost by its definition (see [15, Definition 5.5], and also Definition 5.1 below), the property that if A and B are two unital C^*-algebras in $\mathscr{C}$ with $(K_0(A), [1], K_1(A))$ isomorphic to $(K_0(B), [1], K_1(B))$, as abelian groups with a distinguished element of K_0, then A and B are isomorphic. Non-unital C^*-algebras in $\mathscr{C}$ are always stable and are classified by the two K-groups alone (see [15, Theorem 5.7]).

The class $\mathscr{C}$ is closed under inductive limits, which is proved in [15, Theorem 5.9], and in a stronger form here in Theorem 5.4. The C^*-algebras in $\mathscr{C}$ are natural models for separable, purely infinite, simple C^*-algebras with the same K-theory in, at least, two different ways. First, $\mathscr{C}$ has nice properties as a class, such as the one mentioned above. Secondly, there is a necessary and sufficient condition of a rather intrinsic nature, that a C^*-algebra be isomorphic to its model in $\mathscr{C}$ (see Theorem 5.6) if this exists. In fact, as we shall show, every separable, purely infinite, simple C^*-algebra has a classifiable model. Recall that it was proved in [15, Theorem 8.2], that for every triple (G_0, g_0, G_1), where G_0 and G_1 are countable abelian groups and $g_0 \in G_0$, such that G_1 is torsion free, there is a unitial C^*-algebra A in $\mathscr{C}$ with $(K_0(A), [1], K_1(A))$ isomorphic to (G_0, g_0, G_1). To complete the construction of classifiable models we must show that the condition that G_1 be torsion free can be removed. This is done in Theorem 5.6.

The construction involved is an elaboration of simultaneous but independent work, [2] and [10], in which simple inductive limits of direct sums of matrix algebras over C^*-algebras of the form $\mathcal{O}_{2n} \otimes C(\mathbb{T})$ are proved to be classified by K-theory. The C^*-algebra $\mathcal{O}_n$ is the Cuntz algebra, and this C^*-algebra is so far known to be classifiable in the above sense when n is even. Since $K_1(\mathcal{O}_n \otimes C(\mathbb{T}))$ is isomorphic to $\mathbb{Z}/(n-1)\mathbb{Z}$, these results bring C^*-algebras with torsion into the picture, but it is not proved in [2] or in [10] that the simple inductive limits they consider actually belong to $\mathscr{C}$. The Cuntz algebras $\mathcal{O}_n$ have models Q_n in $\mathscr{C}$ by [15] because $K_1(\mathcal{O}_n) = 0$, and it is known that $\mathcal{O}_{2n}$ is isomorphic to Q_{2n}. We shall prove here (in Corollary 5.5) that every simple, inductive limit of direct sums of matrix algebras over $Q_n \otimes C(\mathbb{T})$ belongs to $\mathscr{C}$, and that these C^*-algebras therefore are classified by K-theory. This, together with techniques from [15], brings C^*-algebras with arbitrary K_1-groups into the classifiable class $\mathscr{C}$. In short, the class $\mathscr{C}$ exhausts the invariant.

2 Preliminaries

We shall in this chapter in part recall and in part develop the theory needed for the constructions which follow later on.

2.1 *The decoy Cuntz algebras*

The Cuntz algebra $\mathcal{O}_n, n \geq 2$, is defined in [5] as the universal C^*-algebra generated by isometries $s_1, \dots, s_n$ satisfying $1 = s_1 s_1^* + \cdots + s_n s_n^*$. Cuntz proved in [5] that $\mathcal{O}_n$ is simple and purely infinite, and in [6] he proved that $K_0(\mathcal{O}_n)$ is isomorphic to $\mathbb{Z}/(n-1)\mathbb{Z}$ and that $K_1(\mathcal{O}_n) = 0$. The class of the unit in $\mathcal{O}_n$ generates $K_0(\mathcal{O}_n)$.

It is proved in [15, Theorem 8.2] that what is called there the classifiable class $\mathscr{C}$ of purely infinite, simple C^*-algebras (see also Section 1 above) contains unital C^*-algebras $Q_n, n \geq 2$, with $K_0(Q_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}, K_1(Q_n) = 0$ and with $K_0(Q_n)$ generated by [1]. Moreover, by the properties of $\mathscr{C}$, there is for each $n \geq 2$ only one C^*-algebra – up to isomorphism – in $\mathscr{C}$ with this K-theory. It is proved in [14] that Q_n is isomorphic to $\mathcal{O}_n$ (equivalently, $\mathcal{O}_n$ belongs to $\mathscr{C}$) if n is even. It is not known (at present) if this also holds for odd n. The C^*-algebras Q_n might be called the decoy Cuntz algebras (with the idea being to replace the decoy with the real thing).

The decoy Cuntz algebra Q_n is, like $\mathcal{O}_n$, generated by a UHF-algebra B of type n^∞ together with an isometry s such that $b \mapsto sbs^*$ defines an endomorphism of B. This endomorphism is a perturbation of the Bernoulli shift by an approximately inner automorphism. It is here that the construction of Q_n differs from that of $\mathcal{O}_n$, which involves the Bernoulli shift itself. By [15, Proposition 3.7] (possibly after replacing s with us for some unitary $u \in B$) there exists a generating nest $\{B_k\}$ of subalgebras of B such that for each k, B_k is isomorphic to a full matrix algebra of order a power of n, and sB_ks^* and s^*B_ks are contained in B_{k+1}.

We may assume that B_1 is isomorphic to M_n and that ss^* is a one-dimensional projection in B_1. In that case there are partial isometries $b_1, \ldots, b_n \in B_1$ such that $\Sigma\, b_j ss^* b_j^* = 1$. This defines an embedding of $\mathcal{O}_n$ into Q_n by mapping s_j onto $b_j s$. We shall have occasion to refer to this embedding later.

2.2 *The Bott element associated with two almost commuting unitaries*

Two communting unitaries u and v in a unital C^*-algebra A determine in a canonical way a $*$-homomorphism $\varphi : C(\mathbb{T}^2) \to A$. Define $b(u, v) \in K_0(A)$ to be $K_0(\varphi)(b)$, where $b \in K_0(C(\mathbb{T}^2))$ is the Bott element.

Terry Loring observed in [11] that there are real-valued continuous functions $f, g, h \in C(\mathbb{T})$ such that if z_1, z_2 are the canonical unitary generators of $C(\mathbb{T}^2)$, then

$$e = \begin{pmatrix} f(z_2) & h(z_2)z_1 + g(z_2) \\ \bar{z}_1 h(z_2) + g(z_2) & 1 - f(z_2) \end{pmatrix}$$

is a projection in $M_2(C(\mathbb{T}^2))$, and

$$b = [e] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

It follows that the image of e in $M_2(A)$, with respect to two commuting unitaries u, v in A as above, i.e.,

$$e(u, v) = \begin{pmatrix} f(v) & h(v)u + g(v) \\ u^*h(v) + g(v) & 1 - f(v) \end{pmatrix} \in M_2(A), \tag{2.2.1}$$

is a projection, and

$$b(u, v) = [e(u, v)] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]. \tag{2.2.2}$$

Loring observed further that there is a universal constant $\delta_0 > 0$ such that if $u, v \in A$ are unitaries with $\|uv - vu\| \le \delta_0$, then the spectrum of the self-adjoint element $e(u, v)$ does not contain $\frac{1}{2}$. This allows us to associate a Bott element to a pair of almost commuting unitaries u and v as

$$b(u, v) = [\chi_{[1/2, \infty)}(e(u, v))] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]. \tag{2.2.3}$$

Note that if $\{u_t\}$, $t \in [0, 1]$, is a continuous path of unitaries such that $\|u_t v - v u_t\| \leq \delta_0$ for all t, then $b(u_0, v) = b(u_1, v)$. It follows easily from the definition of the Bott element that

$$b\left[\begin{bmatrix} u_1 & & & 0 \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{bmatrix}, \begin{bmatrix} v_1 & & & 0 \\ & v_2 & & \\ & & \ddots & \\ 0 & & & v_n \end{bmatrix}\right] = \sum_{j=1}^{n} b(u_j, v_j) \tag{2.2.4}$$

when $\|u_j v_j - v_j u_j\| \leq \delta_0$. From these two observations and standard homotopy tricks one easily deduces that

$$b(u, v) = -b(v, u), \tag{2.2.5}$$

and that

$$b(u_1 u_2 \cdots u_n, v) = b(u_1, v) + \cdots + b(u_n, v), \tag{2.2.6}$$

if $u_1, \ldots, u_n$ are unitaries such that $\|u_j v - v u_j\| \leq \delta_0 / n$.

The following existence theorem for $C(\mathbb{T}^2)$ will be used several times in this paper.

THEOREM 2.2.1. *Let D be a stable, purely infinite, simple C^*-algebra. It follows that for every pair of group homomorphisms $\alpha_0 : K_0(C(\mathbb{T}^2)) \to K_0(D)$ and $\alpha_1 : K_1(C(\mathbb{T}^2)) \to K_1(D)$ there is an injective *-homomorphism $\varphi : C(\mathbb{T}^2) \to D$ such that $K_0(\varphi) = \alpha_0$ and $K_1(\varphi) = \alpha_1$.*

The theorem can be rephrased as follows: If D is a unital, purely infinite, simple C^*-algebra, if $g_0 \in K_0(D)$ and if $g_1, h_1 \in K_1(D)$, then there are commuting unitaries u, v in D such that $[u] = g_1$, $[v] = h_1$ and $b(u, v) = g_0$, and such that $C^*(u, v) \cong C(\mathbb{T}^2)$.

Proof. We prove the rephrased version. Set

$$G = \mathbb{Z}[1/2] \oplus \mathbb{Z}, \qquad G^+ = \{(t, n) \in G \mid t > 0\} \cup \{(0, 0)\}, \qquad u = (1, 0).$$

Then (G, G^+, u) is a dimension group with a distinguished order unit, and there is a unital AF-algebra B which has that dimension group as its invariant. Notice that B is simple because G is a simple dimension group. It is proved in [9] that there is a unital embedding $\psi : C(\mathbb{T}^2) \to B$ with $K_0(\psi)(b) = (0, 1) \in G$.

Find mutually orthogonal, non-zero projections p_1, p_2, p_3 in D with $p_1 + p_2 + p_3 = 1$ and $[p_1] = 0$ in $K_0(D)$. By the existence theorem for AF-algebras (see [15, Lemma 7.2]) there is a unital *-homomorphism $\psi' : B \to p_1 D p_1$ with $K_0(\psi')((t, n)) = ng_0$. Combining this with the embedding of $C(\mathbb{T}^2)$ into B and the fact that $K_1(B) = 0$ we get commuting unitaries u_1, v_1 in $p_1 D p_1$ with $b(u_1, v_1) = g_0$, and $[u_1] = [v_1] = 0$ in $K_1(D)$. Find unitaries $u_2 \in p_2 D p_2$ and $v_3 \in p_3 D p_3$ such that $[u_2] = g_1$ and $[v_3] = h_1$, and set $u = u_1 + u_2 + p_3$, $v = v_1 + p_2 + v_3$. Then, clearly, $[u] = g_1$ and $[v] = h_1$ and by (2.2.4) we find that $b(u, v) = g_0$. □

The following theorem is a fundamental ingredient in our proof of the uniqueness theorem for $Q_n \otimes C(\mathbb{T})$.

THEOREM 2.2.2. ([3, Theorems 8.1 and 9.1]) *For every* $\varepsilon > 0$ *there is a* $\delta \in (0, \delta_0)$ *so that the following is true. Let D be a unital, purely infinite, simple* C^**-algebra and let* u, v *be unitaries in D. Suppose that* $[u] = 0$ *in* $K_1(D)$, *that* $\|uv - vu\| \le \delta$ *and that* $b(u, v) = 0$ *in* $K_0(D)$. *Then there is a continuous path* $\{u_t\}$, $t \in [0, 1]$, *of length less than* $5\pi + 1$, *consisting of unitaries in D such that* $u_0 = 1$, $u_1 = u$ *and* $\|u_t v - v u_t\| \le \varepsilon$ *for all* $t \in [0, 1]$.

2.3 *Approximate unitary equivalence of two unitaries*

The purpose of this section is to prove a sharper version of the following theorem.

THEOREM 2.3.1. ([7]) *Let D be a unital, purely infinite, simple* C^**-algebra, and let* u, v *be unitaries in D. If* $[u] = [v]$ *in* $K_1(D)$ *and* $\operatorname{sp}(u) = \operatorname{sp}(v) = \mathbb{T}$, *then u and v are approximately unitarily equivalent in D (i.e., there is a sequence* $\{w_n\}$ *of unitaries in D such that* $w_n u w_n^* \to v$).

LEMMA 2.3.2. *Let D be a unital, purely infinite, simple* C^**-algebra, and let* $u \in D$ *be a unitary with full spectrum. Then for every* δ *in* $(0, \delta_0)$, *where* δ_0 *is as in Section 2.2, every* $g_0 \in K_0(D)$, *and every* $g_1 \in K_1(D)$ *there exists a unitary* $v \in D$ *with* $\|uv - vu\| \le \delta$, $[v] = g_1$, *and* $b(u, v) = g_0$.

Proof. It follows from Theorem 2.2.1 that there are commuting unitaries $u_0, v_0 \in D$ with full spectrum such that $[u_0] = [u]$, $[v_0] = g_1$ and $b(u_0, v_0) = g_0$. We conclude from Theorem 2.3.1 that there is a unitary $w \in D$ such that $\|w u_0 w^* - u\| \le \delta/2$. With $v = w v_0 w^*$, we get $\|uv - vu\| \le \delta$, $[v] = g_1$, and

$$b(u, v) = b(w u_0 w^*, w v_0 w^*) = b(u_0, v_0) = g_0.$$ □

THEOREM 2.3.3. *Let D be a unital, purely infinite, simple C^*-algebra and let $u, v \in D$ be two unitaries with full spectrum such that $[u] = [v]$ in $K_1(D)$. Then for every $g \in K_1(D)$ there is a continuous path $\{w_t\}$, $t \in [1, \infty)$, of unitaries in D such that $[w_1] = g$ and $w_t u w_t^* \to v$ as $t \to \infty$.*

Proof. Let $\delta = \delta_n$ correspond to $\varepsilon = 1/n$ in Theorem 2.2.2. By Theorem 2.3.1 there exists a sequence $\{z_n\}$ of unitaries in D such that

$$\|z_n u z_n^* - v\| \leq \tfrac{1}{4} \min \{n^{-1}, \delta_n\}.$$

By Lemma 2.3.2, there are unitaries $x_n \in D$ with

$$\|x_n u x_n^* - u\| \leq \tfrac{1}{4} \min \{n^{-1}, \delta_n\},$$

$$[x_n] = g - [z_n],$$

$$b(x_{n+1}, u) = -b(x_n^* z_n^* z_{n+1}, u).$$

Set $z_n x_n = w_n$. Then $[w_n] = g$,

$$\|(w_n^* w_{n+1})u - u(w_n^* w_{n+1})\| \leq \min \{n^{-1}, \delta_n\},$$

and $b(w_n^* w_{n+1}, u) = 0$. Hence Theorem 2.2.2 yields a continuous path $\{w_t\}$, $t \in [1, \infty)$, of unitaries in D such that w_n is as above when $n \in \mathbb{N}$, and for $t \in [n, n+1]$,

$$\|w_t wuw_t^* - v\| \leq \|w_t u w_t^* - w_n u w_n^*\| + \|w_n u w_n^* - v\|$$
$$\leq n^{-1} + \frac{1}{2} n^{-1} = \frac{3}{2} n^{-1}.$$

This proves that $w_t u w_t^* \to v$ as $t \to \infty$. □

We conclude this section by quoting the following "tail lemma".

LEMMA 2.3.4. ([3, Theorem 11.3]) *Let D be a unital, purely infinite, simple C^*-algebra, and let $\{u_t\}$ and $\{v_t\}$, $t \in [0, 1]$, be two continuous paths of unitaries in D such that $[u_t] = [v_t]$ in $K_1(D)$ and all u_t and v_t have full spectrum. Let $\delta > 0$ be given. Then there is a continuous path $\{w_t\}$, $t \in [0, 1]$, of unitaries in D such that $\|w_t u_t w_t^* - v_t\| \leq \delta$ for all $t \in [0, 1]$. If $u_0 = v_0$, then $\{w_t\}$ can be chosen such that $w_0 = 1$.*

2.4 *The universal coefficient theorem*

Let A_n here denote either $\mathcal{O}_n$ or Q_n. Suppose that $\varphi, \psi : A_n \otimes C(\mathbb{T}) \to D$ are $*$-homomorphisms, where D is an arbitrary C^*-algebra. We shall determine when $KK(\varphi) = KK(\psi)$ in terms of specific data associated with the $*$-homomorphisms φ and ψ. By the universal coefficient theorem of Rosenberg and Schochet, [13], if

$$K_*(\varphi) = K_*(\psi) : K_*(A_n \otimes C(\mathbb{T})) \to K_*(D),$$

then $KK(\varphi) - KK(\psi) = (\varepsilon_0, \varepsilon_1)$, where

$$\varepsilon_j \in \operatorname{Ext}(K_j(A_n \otimes C(\mathbb{T})), K_{1-j}(D)).$$

The Ext-elements ε_0 and ε_1 can be calculated as follows. Consider the C^*-algebra E of pairs (f, a) with $f : [0, 1] \to D$ continuous, $a \in A_n \otimes C(\mathbb{T}), f(0) = \varphi(a)$ and $f(1) = \psi(a)$. By means of the map

$$\pi : E \to A_n \otimes C(\mathbb{T}), \qquad (f, a) \mapsto a,$$

there is then associated to the pair (φ, ψ) an extension of C^*-algebras:

$$0 \to SD \to E \xrightarrow{\pi} A_n \otimes C(\mathbb{T}) \to 0. \tag{2.4.1}$$

If $K_*(\varphi) = K_*(\psi)$, then the six-term exact sequence of K-groups of this extension has zero index maps and therefore yields two short exact sequences:

$$0 \to K_1(D) \xrightarrow{\alpha_0} K_0(E) \xrightarrow{K_0(\pi)} K_0(A_n \otimes C(\mathbb{T})) \to 0, \tag{2.4.2}$$

$$0 \to K_0(D) \xrightarrow{\alpha_1} K_1(E) \xrightarrow{K_1(\pi)} K_1(A_n \otimes C(\mathbb{T})) \to 0. \tag{2.4.3}$$

These are the elements ε_0 and ε_1.

We wish to identify ε_0 and ε_1 more explicitly. The following construction from [4] will be used in the proof of the following proposition and in Section 4. Let CD denote the C^*-algebra of all continuous bounded functions $f : [1, \infty) \to D$, and let C_0D denote the ideal in CD consisting of those $f \in D$ for which $f(t) \to 0$ as $t \to \infty$. There is a canonical map $j : D \to CD/C_0D$, mapping an element $x \in D$ into the constant function $f(t) = x$ modulo C_0D. The induced group homomorphism $K_*(j) : K_*(D) \to K_*(CD/C_0D)$ is injective. It has been proved by G. Nagy in [12] that $K_*(j)$ is an ismomorphism when D is stable. (Moreover, $K_*(j)$ is an isomorphism when D is a purely infinite simple C^*-algebra, but the two facts about $K_*(j)$ quoted above will suffice for the purposes of this paper.)

PROPOSITION 2.4.1. *Suppose that D is a unital, purely infinite, simple C^*-algebra and that $\varphi, \psi : A_n \otimes C(\mathbb{T}) \to D$ are injective unital *-homomorphisms. Let $B_1 \subseteq A_n$ be as described in Section 2.1, and let z denote the canonical unitary generator of $C(\mathbb{T})$. Assume that $\varphi|B_1 \otimes 1 = \psi|B_1 \otimes 1$ and that $\|\varphi(1 \otimes z) - \psi(1 \otimes z)\| \leq \delta_0/2$, where δ_0 is as stipulated in Section 2.2. Then $K_*(\varphi) = K_*(\psi)$, $u = \varphi(s \otimes 1)^*\psi(s \otimes 1)$ is unitary, and if $v = \varphi(1 \otimes z)$, then $\|uv - vu\| \leq \delta_0$. Let $\hat{v} \in E$ be a short path of unitaries in D connecting $\varphi(1 \otimes z)$ to $\psi(1 \otimes z)$. Then, with α_0 and α_1 as in* (2.4.2) *and* (2.4.3),

$$n\alpha_0([u]) = (n-1)[1_E], \qquad n\alpha_1(b(u, v)) = (n-1)[\hat{v}].$$

In particular, $KK(\varphi) = KK(\psi)$ if and only if $[u] \in (n-1)K_1(D)$ and $b(u, v) \in (n-1)K_0(D)$.

Proof. That $K_*(\varphi) = K_*(\psi)$ follows from the fact that the class of the unit in $A_n \otimes C(\mathbb{T})$ generates its K_0-group and the class of $1 \otimes z$ generates its K_1-group, together with the assumptions that φ and ψ are unital and that $\|\varphi(1 \otimes z) - \psi(1 \otimes z)\| \leq \delta_0/2 < 2$, whence $[\varphi(1 \otimes z)] = [\psi(1 \otimes z)]$. Because $ss^* \in B_1$, we get $\varphi(ss^* \otimes 1) = \psi(ss^* \otimes 1)$, and it follows that u is unitary. It is easily checked that $\|uv - vu\| \leq \delta_0$.

Assume to begin with that $\psi(1 \otimes z) = \varphi(1 \otimes z)\ (=v)$, and let us relax at the same time the assumptions on D, and assume only that D is unital. Set $D \cap \{v\}' = D_0$, and let $\varphi_0, \psi_0 : A_n \to D_0$ denote the *-homomorphisms $a \mapsto \varphi(a \otimes 1)$, $a \mapsto \psi(a \otimes 1)$. Then $K_0(\varphi_0) = K_0(\psi_0)$ because φ_0 and ψ_0 are both unital. Consider the extension of C^*-algebras analogous to (2.4.1) for the pair (φ_0, ψ_0),

$$0 \to SD_0 \to E_0 \xrightarrow{\pi_0} A_n \to 0,$$

and its induced exact sequence of K-groups,

$$0 \to K_1(D_0) \xrightarrow{\alpha_0'} K_0(E_0) \xrightarrow{K_0(\pi_0)} K_0(A_n) \to 0.$$

Since $\varphi_0|B_1 = \psi_0|B_1$ there is an embedding $\iota : B_1 \to E_0$ such that $\pi_0 \circ \iota = \mathrm{id}_{B_1}$. (Map $b \in B_1$ into the pair (f, b) where f denotes the constant function with value $\varphi_0(b)$.) We can now apply [15, Lemma 6.9], and conclude that

$$n\alpha_0'([u]) = nK_0(\iota)([1] - [ss^*]) = K_0(\iota)((n-1)[1]) = (n-1)[1_{E_0}].$$

Considering the extension (2.4.1) arising from (φ, ψ), we have that $v \in E$ (when v is identified with the pair (f, v) with f the constant map) and $E_0 \subseteq E$. Define $*$-homomorphisms $D_0 \otimes C(\mathbb{T}) \to D$ and $E_0 \otimes C(\mathbb{T}) \to E$ by $a \otimes 1 \mapsto a$ and $1 \otimes z \mapsto v$. These lead to the commutative diagram.

$$\begin{tikzcd}
0 \arrow[r, dash] & SD_0 \otimes C(\mathbb{T}) \arrow[r] \arrow[d] & E_0 \otimes C(\mathbb{T}) \arrow[r, "\pi_0 \otimes \mathrm{id}"] \arrow[d] & A_n \otimes C(\mathbb{T}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & SD \arrow[r] & E \arrow[r, "\pi"] & A_n \otimes C(\mathbb{T}) \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact. By functoriality, from the first square we obtain the following commutative diagram of groups:

$$\begin{tikzcd}
K_1(D_0) \otimes K_j(C(\mathbb{T})) \arrow[r, "\alpha_0' \otimes \mathrm{id}"] \arrow[d] & K_0(E_0) \otimes K_j(C(\mathbb{T})) \arrow[d] \\
K_{1-j}(D_0 \otimes C(\mathbb{T})) \arrow[r] \arrow[d] & K_j(E_0 \otimes C(\mathbb{T})) \arrow[d] \\
K_{1-j}(D) \arrow[r, "\alpha_j"] & K_j(E)
\end{tikzcd}$$

Consider the composition of the two maps in the left column. For $j = 0$, the element $[u] \otimes [1] \in K_1(D_0) \otimes K_0(C)\mathbb{T}))$ is first mapped onto $[u \otimes 1] \in K_1(D_0 \otimes C(\mathbb{T}))$, and then onto $[u] \in K_1(D)$. For $j = 1$, $[u] \otimes [z] \in K_1(D_0) \otimes K_1(C(\mathbb{T}))$ is first mapped onto $b(u \otimes 1, 1 \otimes z) \in K_0(D_0 \otimes C(\mathbb{T}))$, as we shall show below, and then to $b(u, v) \in K_0(D)$. Consider next the composition of $\alpha_0' \otimes \mathrm{id}$ with the two maps in the right column, and recall that $\alpha_0'(n[u]) = (n-1)[1_{E_0}]$. For $j = 0$, $(n[u]) \otimes [1]$ is mapped to $((n-1)[1_{E_0}]) \otimes [1]$ and then on to $(n-1)[1_E] \in K_0(E)$. For $j = 1$, $(n[u]) \otimes z$ is first mapped onto $((n-1)[1_{E_0}]) \otimes [z]$, then to $(n-1)[1 \otimes z]$, and finally to $(n-1)[v] \in K_1(E)$. Because the diagram is commutative, we conclude that $n\alpha_0([u]) = (n-1)[1_E]$ and $n\alpha_1(b(u, v)) = (n-1)[v]$ when $\varphi(1 \otimes z) = \psi(1 \otimes z)$.

To prove the assertion made above concerning the image of $[u] \otimes [z]$, let us examine the map $K_1(D_0) \otimes K_1(C(\mathbb{T})) \to K_0(D_0 \otimes C(\mathbb{T}))$. Upon identifying SA with $C_0(\mathbb{R}) \otimes A$, this map is the composition of the following three natural maps:

$$\begin{aligned}
K_1(D_0) \otimes K_1(C(\mathbb{T})) &\to K_0(C_0(\mathbb{R}) \otimes D_0) \otimes K_1(C(\mathbb{T})) \\
&\to K_1(C_0(\mathbb{R}) \otimes D_0 \otimes C(\mathbb{T})) \to K_0(D_0 \otimes C(\mathbb{T})).
\end{aligned}$$

Let s be a unitary in $C_0(\mathbb{R})$, with a unit adjoined, whose class in $K_1(C_0(\mathbb{R}))$ is a generator. Denote by $\langle .,. \rangle$ the natural pairings

$$K_j(A) \otimes K_{1-j}(B) \to K_1(A \otimes B).$$

Then, by the definitions of the isomorphisms $K_1(D_0) \to K_0(C_0(\mathbb{R}) \otimes D_0)$ and $K_1(C_0(\mathbb{R}) \otimes D_0 \otimes C(\mathbb{T})) \to K_0(D_0 \otimes C(\mathbb{T}))$, and by associativity of the tensor product, we get

$$\begin{aligned}[u] \otimes [z] &\mapsto b(s \otimes 1, 1 \otimes u) \otimes [z] \\ &\mapsto \langle b(s \otimes 1 \otimes 1, 1 \otimes u \otimes 1), [1 \otimes 1 \otimes z] \rangle \\ &= \langle [s \otimes 1 \otimes 1], b(1 \otimes u \otimes 1, 1 \otimes 1 \otimes z) \rangle \\ &\mapsto b(u \otimes 1, 1 \otimes z).\end{aligned}$$

Consider now the case that $\|\varphi(1 \otimes z) - \psi(1 \otimes z)\| \le \delta_0/2$ and D is simple and purely infinite. Theorem 2.3.3 gives us a continuous path $\{w_t\}$, $t \in [1, \infty)$, of unitaries in $D \cap \varphi(B_1')$ such that $w_1 = 1$ and $w_t \psi(1 \otimes z) w_t^* \to \varphi(1 \otimes z)$. Set $\bar{\varphi} = j \circ \varphi : A_n \otimes C(\mathbb{T}) \to CD/C_0D$, where CD/C_0D and $j : D \to CD/C_0D$ are as described above, and let $\bar{\psi} : A_n \otimes C(\mathbb{T}) \to CD/C_0D$ be given by

$$\bar{\psi}(x) = \{(\mathrm{Ad} w_t \circ \psi)(x)\}_{t \in [1,\infty)} + C_0 D.$$

Then $\bar{\varphi}(1 \otimes z) = \bar{\psi}(1 \otimes z)\,(= \bar{v})$, and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SD & \longrightarrow & E & \longrightarrow & A_n \otimes C(\mathbb{T}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & S(CD/C_0D) & \longrightarrow & CE/C_0E & \longrightarrow & A_n \otimes C(\mathbb{T}) & \longrightarrow & 0 \end{array}$$

where the second row is the extension associated with the pair $(\bar{\varphi}, \bar{\psi})$. From the first square we obtain the following cummutative diagram of K-groups with injective vertical maps:

$$\begin{array}{ccc} K_{1-j}(D) & \xrightarrow{\alpha_j} & K_j(E) \\ \downarrow & & \downarrow \\ K_{1-j}(CD/C_0D) & \xrightarrow{\bar{\alpha}_j} & K_j(CE/C_0E) \end{array}$$

Set $\bar{\varphi}(s \otimes 1)^* \bar{\psi}(s \otimes 1) = \bar{u}$. From the previous paragraphs we deduce that $n\bar{\alpha}_0([\bar{u}]) = (n-1)[1]$ in $K_0(CE/C_0E)$, and $n\bar{\alpha}_1(b(\bar{u}, \bar{v})) = (n-1)[\bar{v}]$ in $K_1(CE/C_0E)$. To complete the desired calculations of α_0 and α_1, just check that $[u] \in K_1(D)$ is mapped onto $[\bar{u}] \in K_1(CD/C_0D)$, that $b(u, v) \in K_0(D)$ is mapped onto $b(\bar{u}, \bar{v}) \in K_0(CD/C_0D)$, and that $[\hat{v}] \in K_1(E)$ is mapped onto $[\bar{v}] \in K_1(CE/C_0E)$.

The universal coefficient theorem yields, as mentioned above, that $KK(\varphi) = KK(\psi)$ if and only if the extensions (2.4.2) and (2.4.3) split.

Since $K_j(A_n \otimes C(\mathbb{T})) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ for $j = 0, 1$, and these groups are generated by $[1]$, respectively $[1 \otimes z]$, the extensions split if and only if these two generators lift to elements in $K_j(E)$ of order $n-1$. Since $K_0(\pi)([1]) = [1]$ and $K_1(\pi)([\hat{v}]) = [1 \otimes z]$, this is the case if and only if there are elements $g_j \in K_j(D)$ with

$$(n-1)([1]) - \alpha_0(g_1)) = 0, \qquad (n-1)([\hat{v}] - \alpha_1(g_0)) = 0.$$

But

$$(n-1)([1] - \alpha_0(g_1)) = \alpha_0(n[u] - (n-1)g_1),$$

$$(n-1)([\hat{v}] - \alpha_1(g_0)) = \alpha_1(nb(u, v) - (n-1)g_0).$$

Because α_0 and α_1 are injective, it follows that g_0, g_1 can be found if and only if $[u] \in (n-1)K_1(D)$ and $b(u, v) \in (n-1)K_0(D)$. □

3 A uniqueness theorem for $Q_n \otimes C(\mathbb{T})$

It will be proved that every pair of injective, unital *-homomorphisms from $Q_n \otimes C(\mathbb{T})$ into a unital, purely infinite, simple C^*-algebra that define the same element in KK-theory are approximately unitarily equivalent. The proof of this follows the lines of [15], [2] and [10], but significant sharpenings of results from these papers are necessary and will be carried out in the lemmas below. We shall try to make as much use of the already established results as possible without disrupting the present exposition unduly.

Let $\{B_k\}$ and $b_1, \ldots, b_n \in B_1$ be as in Section 2.1. Suppose that D is a unital, purely infinite, simple C^*-algebra and that $\varphi : Q_n \to D$ is a *-homomorphism. Set, as in [15],

$$\sigma(x) = \sum_{j=1}^{n} \varphi(b_j s) x \varphi(b_j s)^*, \qquad x \in D. \tag{3.1}$$

Observe that if $x \in D$, $\delta > 0$, and $v \in U(D)$ are such that $\|x\| \leq 1$, $\|v\varphi(s) - \varphi(s)v\| \leq \delta$, $\|vx - xv\| \leq \delta$, and $\|v\varphi(b) - \varphi(b)v\| \leq \delta \|b\|$ for all $b \in B_1$, then $\|v - v_0\| \leq \delta$ for some $v_0 \in D \cap \varphi(B_1)'$ with $\|v_0\| \leq 1$, and hence

$$\|\sigma(x)v - v\sigma(x)\| \leq 2\delta + \|\sigma(x)v_0 - v_0\sigma(x)\| \leq 11\delta.$$

We therefore get by induction that

$$\|\sigma^k(x)v - v\sigma^k(x)\| \leq 11^k\delta \|x\| \tag{3.2}$$

for all $x \in D$.

LEMMA 3.1. (cf. [15, Lemma 6.5]) *For every $\varepsilon > 0$ and $k \in \mathbb{N}$ there are $m \geq k$ and $\delta \in (0, \delta_0/2)$, where δ_0 is as defined in Section 2.2, such that the following holds.*

Let D be a unital, purely infinite, simple C^*-algebra and let $\varphi, \psi : Q_n \to D$ be unital $*$-homomorphisms such that $\varphi|B_m = \psi|B_m$ and such that $(u =)\ \varphi(s)^*\psi(s)$ belongs to $U_0(D \cap \varphi(B_{m-1})')$. Let $v \in D$ be a unitary in $D \cap \varphi(B_m)'$ such that $\|v\varphi(s) - \varphi(s)v\| \leq \delta$ and $\|v\psi(s) - \psi(s)v\| \leq \delta$, and such that $b(u, v) = 0$ in $K_0(D \cap \varphi(B_{m-1})')$. Then there exists a unitary w in $D \cap \varphi(B_k)'$ such that $\|w\varphi(s)w^* - \psi(s)\| < \varepsilon$ and $\|wv - vw\| < \varepsilon$.

Proof. Notice that $\|uv - vu\| \leq 2\delta \leq \delta_0$ so that $b(u, v)$ is defined. The proof is a modification of the proofs of [15, Lemmas 6.4 and 6.5], and we shall just describe what changes must be made to those proofs. Let N denote the least integer so that $(5\pi + 1)/N \leq \varepsilon/2$ and let r ($\geq k$) denote the integer appearing in the first paragraph of the proof of [15, Lemma 6.5]. Set $r + 3N - 1 = m$. (The size of $\delta > 0$ will be determined in the course of the proof.)

By Theorem 2.2.2, if δ is chosen small enough, then there is a continuous path $\{u_t\}$, $t \in [0, 1]$, of unitaries in $D \cap \varphi(B_{m-1})'$ connecting 1 to u such that

$$\|u_t v - v u_t\| \leq \left(\sum_{j=1}^{N} 11^j\right)^{-1} \delta_0$$

for all $t \in [0, 1]$. Set

$$x = \sigma(u)\sigma^2(u) \cdots \sigma^N(u),$$

$$x_t = \sigma(u_t)\sigma^2(u_t) \cdots \sigma^N(u_t),$$

and observe that $\{x_t\}$ is a continuous path of unitaries in $D \cap \varphi(B_{m-N-1})'$

connecting 1 to x (use [15, Lemma 6.3]). It follows from (3.2) that

$$\|x_t v - vx_t\| \le \sum_{j=1}^{N} 11^j \|u_t v - vu_t\| \le \delta_0,$$

$$\|xv - vx\| \le \sum_{j=1}^{N} 11^j \|uv - vu\|.$$

Hence $b(x, v) = b(1, v) = 0$ in $K_0(D \cap \varphi(B_{m-N-1})')$. Moreover, to a given $\delta_1 > 0$ if δ is small enough, then another application of Theorem 2.2.2 yields a new continuous path $\{y_t\}$ of unitaries in $D \cap \varphi(B_{m-N-1})'$ of length at most $5\pi + 1$ such that $y_0 = 1, y_1 = x$, and $\|y_t v - vy_t\| \le \delta_1$ for all $t \in [0, 1]$. Choose a division $0 = t_0 < t_1 < \cdots < t_N = 1$ of the interval $[0, 1]$ such that, with $z_j = y_{t_j}$, $j = 0, \ldots, N-1$,

$$\|z_{j+1} - z_j\| \le (5\pi + 1)/N \le \varepsilon/2.$$

Then $z_0 = 1, z_N = x$, and $\|z_j v - vz_j\| \le \delta_1$ for all j.

Set $z_1 = w_1$ and $z_j^* z_{j+1} = w_{j+1}$. Then $x = w_1 w_2 \cdots w_N$, $\|w_j - 1\| \le \varepsilon/2$, and $\|w_j v - vw_j\| \le 2\delta_1$. Continue the proofs of [15, Lemmas 6.4 and 6.5] using these unitaries $w_1, \ldots, w_N$ (and with w equal to x above) to obtain unitaries $v_0, v_1, \ldots, v_N$ in $D \cap \varphi(B_r)'$ such that $\|v_j \lambda(v_{j-1})^* - \lambda(u)\| \le \varepsilon/2$ (where $\lambda(x) = \varphi(s)x\varphi(s)^*$) and $\|v_j v - vv_j\| \le \delta_2$ for some $\delta_2 > 0$ which depends on δ_1 and tends to zero as δ_1 tends to zero. The proof of [15, Lemma 6.5] will then produce a unitary w in $D \cap \varphi(B_k)'$ (called v in [15]) such that $\|w\varphi(s)w^* - \psi(s)\| < \varepsilon$ and $\|wv - vw\| < \varepsilon$ if δ and hence δ_1 are sufficiently small. □

LEMMA 3.2. *Let D be a unital C^*-algebra and let $\varphi : Q_n \otimes C(\mathbb{T}) \to D$ be a unital $*$-homomorphism. Set $\varphi(1 \otimes z) = v$ and $\varphi(s \otimes 1) = t$. Let $m \in \mathbb{N}$ and let w be a unitary in $D \cap \varphi(B_m \otimes 1)'$ with $\|wv - vw\| \le \delta_0$. Then t^*wt is a unitary in $D \cap \varphi(B_{m-1} \otimes 1)'$, $\|t^*wtv - vt^*wt\| \le \delta_0$, and $b(w, v) = nb(t^*wt, v)$ in $K_0(D \cap \varphi(B_{m-1} \otimes 1)')$.*

Proof. The map $\mu : x \to t^*xt$ is a unital $*$-homomorphism from $D \cap \varphi(B_m \otimes 1)'$ into $D \cap \varphi(B_{m-1} \otimes 1)'$, because $tt^* \in \varphi(B_1 \otimes 1) \subseteq \varphi(B_m \otimes 1)$ and $t\varphi(B_{m-1} \otimes 1)t^* \subseteq \varphi(B_m \otimes 1)$; cf. Section 2.1 (see also [15, Lemma 6.1]). We see in particular that $t^*wt = \mu(w)$ is a unitary in $D \cap \varphi(B_{m-1} \otimes 1)'$. Since v and t commute,

$$\|t^*wtv - vt^*wt\| = \|t^*(wv - vw)t\| = \|wv - vw\| \le \delta_0.$$

If p is a projection in $\varphi(B_{m-1}\otimes 1)$, then tpt^* is a projection in $\varphi(B_m\otimes 1)$ and $[p]=n[tpt^*]$ in $K_0(\varphi(B_m\otimes 1))$. Hence, for every projection e in $D\cap\varphi(B_m\otimes 1)'$,

$$[ep]=n[etpt^*]=n[t^*etp]$$

in $K_0(D)$, which implies that $[e]=n[t^*et]$ in $K_0(D\cap\varphi(B_{m-1})')$. This again implies that

$$nK_0(\mu)=K_0(\iota):K_0(D\cap\varphi(B_m\otimes 1)')\to K_0(D\cap\varphi(B_{m-1}\otimes 1)').$$

Denote by $b(\cdot,\cdot)$ and $b'(\cdot,\cdot)$ the Bott elements in $K_0(D\cap\varphi(B_{m-1})')$ and $K_0(D\cap\varphi(B_m)')$. Then

$$\begin{aligned} b(w,v) &= K_0(\iota)(b'(w,v))=nK_0(\mu)(b'(w,v))\\ &= nb(t^*wt,t^*vt)=nb(t^*wt,v). \end{aligned}$$

□

LEMMA 3.3. *Suppose that D is a unital, purely infinite, simple C^*-algebra and that $\varphi,\psi:Q_n\otimes C(\mathbb{T})\to D$ are injective, unital *-homomorphisms with $KK(\varphi)=KK(\psi)$. Then for every $m\in\mathbb{N}$ and every $\delta\geq 0$ there is a unitary $w\in D$ such that if*

$$u=(\operatorname{Ad}w\circ\varphi)(s\otimes 1)^*\psi(s\otimes 1),\qquad v=(\operatorname{Ad}w\circ\varphi)(1\otimes z),$$

then $\|uv-vu\|\leq\delta_0$ and

(i) $(\operatorname{Ad}w\circ\varphi)|B_m\otimes 1=\psi|B_m\otimes 1$,
(ii) $u\in U_0(D\cap\varphi(B_{m-1}\otimes 1)')$,
(iii) $\|v-\psi(1\otimes z)\|\leq\delta$,
(iv) $b(u,v)=0$ *in* $K_0(D\cap\varphi(B_{m-1}\otimes 1)')$.

Proof. It follows from the uniqueness theorem for Q_n, [15, Theorem 6.12], that there is a unitary $w_1\in D$ such that (i) and (ii) hold with w_1 in the place of w and with $m+1$ instead of m. Since $KK(\operatorname{Ad}w_1\circ\varphi)=KK(\varphi)$, we may, upon replacing φ with $\operatorname{Ad}w_1\circ\varphi$, suppose that $\varphi|B_{m+1}\otimes 1=\psi|B_{m+1}\otimes 1$ and that $\varphi(s\otimes 1)^*\psi(s\otimes 1)\in U_0(D\cap\varphi(B_m\otimes 1)')$.

We have in particular that $K_1(\varphi)=K_1(\psi)$, and so, for every projection $p\in B_{m+1}$, the following holds in $K_1(D)$:

$$\begin{aligned} [\varphi(1\otimes z)\varphi(p\otimes 1)+(1-\varphi(p\otimes 1))] &= [\varphi(p\otimes z+1-p\otimes 1)]\\ &= [\psi(p\otimes z+1-p\otimes 1)]\\ &= [\psi(1\otimes z)\varphi(p\otimes 1)+(1-\varphi(p\otimes 1))]. \end{aligned}$$

This shows that $[\varphi(1\otimes z)]=[\psi(1\otimes z)]$ in $K_1(D\cap\varphi(B_{m+1}\otimes 1)')$, and Theorem 2.3.3 therefore yields a unitary w_2 in $U_0(D\cap\varphi(B_{m+1}\otimes 1)')$ with

$$\|(\operatorname{Ad} w_2\circ\varphi)(1\otimes z)-\psi(1\otimes z)\|\le\min\{\delta/2,\delta_0/4\}\qquad(=\delta_1).$$

Observe that (i) and (ii) hold with w_2 and $m+1$ in place of w and m. If we replace φ with $\operatorname{Ad} w_2\circ\varphi$ we obtain that $\varphi|B_{m+1}\otimes 1=\psi|B_{m+1}\otimes 1$, $\varphi(s\otimes 1)^*\psi(s\otimes 1)\in U_0(D\cap\varphi(B_m\otimes 1)')$, and $\|\varphi(1\otimes z)-\psi(1\otimes z)\|\le\delta_1$.

Proposition 2.4.1 yields that the Bott element $b(\varphi(s\otimes 1)^*\psi(s\otimes 1),\varphi(1\otimes z))$ belongs to $(n-1)K_0(D)$. It also belongs to $K_0(D\cap\varphi(B_m\otimes 1)')$, and because n^k and $n-1$ are relatively prime for every $k\in\mathbb{N}$, we conclude that

$$(n-1)K_0(D)\cap K_0(D\cap\varphi(B_m\otimes 1)')=(n-1)K_0(D\cap\varphi(B_m\otimes 1)').$$

The Bott element therefore belongs to $(n-1)K_0(D\cap\varphi(B_m\otimes 1)')$, and so is equal to $(n-1)g_0$ for some $g_0\in K_0(D\cap\varphi(B_m\otimes 1)')$.

Use Lemma 2.3.2 to find a unitary x in $U_0(D\cap\varphi(B_m\otimes 1)')$ with $\|x\varphi(1\otimes z)-\varphi(1\otimes z)x\|\le\delta_1/n$ and $b(x,\varphi(1\otimes z))=-g_0$ in $K_0(D\cap\varphi(B_m\otimes 1)')$, and set $w=x^n$. Then (i) and (ii) are satisfied, and (iii), holds because

$$\|v-\psi(1\otimes z)\|\le\|w\varphi(1\otimes z)-\varphi(1\otimes z)w\|+\|\varphi(1\otimes z)-\psi(1\otimes z)\|\le 2\delta_1\le\delta.$$

Finally, from Lemma 3.2 and (2.2.6) we get

$$\begin{aligned}
b(u,v)&=b(u,\varphi(1\otimes z))\\
&=b(w,\varphi(1\otimes z))+b(\varphi(s\otimes 1)^*w^*\varphi(s\otimes 1),\varphi(1\otimes z))\\
&\quad+b(\varphi(s\otimes 1)^*\psi(s\otimes 1),\varphi(1\otimes z))\\
&=nb(x,\varphi(1\otimes z))-nb(\varphi(s\otimes 1)^*x\varphi(s\otimes 1),\varphi(1\otimes z))\\
&\quad+b(\varphi(s\otimes 1)^*\psi(s\otimes 1),\varphi(1\otimes z))\\
&=-ng_0+g_0+(n-1)g_0=0
\end{aligned}$$

in $K_0(D\cap\varphi(B_{m-1})')$. □

THEOREM 3.4. *Suppose that D is a unital, purely infinite, simple C^*-algebra and that $\varphi,\psi:Q_n\otimes C(\mathbb{T})\to D$ are injective, unital $*$-homomorphisms. Then φ and ψ are approximately unitarily equivalent if and only if $KK(\varphi)=KK(\psi)$.*

Proof. The "only if" part follows from [15, Proposition 5.4], because $KK(Q_n \otimes C(\mathbb{T}), D) = KL(Q_n \otimes C(\mathbb{T}), D)$ (this holds as $K_*(Q_n \otimes C(\mathbb{T}))$ is finitely generated; cf. above). To prove the "if" part it sufficies to show that for every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there is a unitary w in D such that

(i) $(\mathrm{Ad}\, w \circ \varphi)|B_k \otimes 1 = \psi|B_k \otimes 1$,

(ii) $\|(\mathrm{Ad}\, w \circ \varphi)(s \otimes 1) - \psi(s \otimes 1)\| \leq \varepsilon$,

(iii) $\|(\mathrm{Ad}\, w \circ \varphi)(1 \otimes z) - \psi(1 \otimes z)\| \leq 2\varepsilon$.

Let $m \geq k$ and $\delta > 0$ be as in Lemma 3.1, corresponding to the given k and ε. By Lemma 3.3 we may assume that φ and ψ agree on $B_m \otimes 1$, and that

$$\|\varphi(1 \otimes z) - \psi(1 \otimes z)\| \leq \min\{\varepsilon, \delta/2\},$$

$$(u =)\ \varphi(s \otimes 1)^* \psi(s \otimes 1) \in U_0(D \cap \varphi(B_{m-1})'),$$

$$b(u, \varphi(1 \otimes z)) = 0 \qquad \text{in } K_0(D \cap \varphi(B_{m-1})').$$

Set $\varphi(1 \otimes z) = v$ and observe that $v\varphi(s \otimes 1) = \varphi(s \otimes 1)v$ and that

$$\|v\psi(s \otimes 1) - \psi(s \otimes 1)v\| \leq 2\|\varphi(1 \otimes z) - \psi(1 \otimes z)\| \leq \delta.$$

Lemma 3.1 yields a unitary w in $D \cap \varphi(B_k)'$ satisfying (ii) and such that $\|w\varphi(1 \otimes z) - \varphi(1 \otimes z)w\| \leq \varepsilon$. It follows that (i) holds, and that (iii) holds is seen from

$$\begin{aligned}\|(\mathrm{Ad}\, w \circ \varphi)(1 \otimes z) - \psi(1 \otimes z)\| &\leq \|w\varphi(1 \otimes z) - \varphi(1 \otimes z)w\| \\ &\quad + \|\varphi(1 \otimes z) - \psi(1 \otimes z)\| \leq 2\varepsilon. \qquad \square\end{aligned}$$

4 An existence theorem for $Q_n \otimes C(\mathbb{T})$

The existence theorem for $Q_n \otimes C(\mathbb{T})$ is proved via an approximate existence theorem for $\mathcal{O}_n \otimes C(\mathbb{T})$. From this we obtain an approximate existence theorem for $Q_n \otimes C(\mathbb{T})$. We turn this into an exact existence theorem for $Q_n \otimes C(\mathbb{T})$ using the trick of Connes and Higson to replace the target algebra D with CD/C_0D, in much the same way as above in the proof of Proposition 2.4.1. (In doing this we use the uniqueness theorem of Section 3, which also holds for approximate homomorphisms. More precisely, it turns out to be sufficient to use Lemma 3.1.)

PROPOSITION 4.1. *Let D be a stable, purely infinite, simple C^*-algebra. Then every element of $KK(\mathcal{O}_n \otimes C(\mathbb{T}), CD/C_0D)$ is represented by an injective *-homomorphism $\mathcal{O}_n \otimes C(\mathbb{T}) \to CD/C_0D$.*

Proof. We show that for every pair

$$\alpha_j : K_j(\mathcal{O}_n \otimes C(\mathbb{T})) \to K_j(CD/C_0D),$$

$k = 0, 1$, of group homorphisms and every pair

$$\varepsilon_j \in \operatorname{Ext}(K_j(\mathcal{O}_n \otimes C(\mathbb{T})), K_{1-j}(CD/D_0D)),$$

$j = 0, 1$, of extensions there are injective *-homomorphisms $\varphi, \psi : \mathcal{O}_n \otimes C(\mathbb{T}) \to CD/C_0D$ such that $K_j(\varphi) = K_j(\psi) = \alpha_j$ and $KK(\varphi) - KK(\psi) = (\varepsilon_0, \varepsilon_1)$, when the Ext-groups, via the universal coefficient theorem, are viewed as subgroups of $KK(\mathcal{O}_n \otimes C(\mathbb{T}), CD/C_0D)$. Since the universal coefficient theorem holds for the given pair of C^*-algebras, since the Ext-groups are torsion groups, and since the KK-elements that can be represented by *-homomorphisms form a sub-semigroup, the proposition will follow from the assertion.

Set $\alpha_0([1]) = g_0$ and $\alpha_1([1 \otimes z]) = g_1$. Choose a non-zero projection e in D with $[e] = g_0$. Replace D with eDe, so that D becomes unital and $\alpha_0([1]) = [1]$. Consider representations of the Ext-elements ε_j:

$$0 \to K_{1-j}(CD/C_0D) \xrightarrow{\gamma_j} E_j \xrightarrow{\pi_j} K_j(\mathcal{O}_n \otimes C(\mathbb{T})) \to 0.$$

Lift $[1]$ and $[1 \otimes z]$ to $e_0 \in E_0$ and $e_1 \in E_1$. Since $(n-1)\pi_j(e_j) = 0$ there are elements $h_j \in K_j(CD/C_0D)$ such that $\gamma_j(h_{1-j}) = (n-1)e_j$. We find two unital *-homomorphisms $\varphi, \psi : \mathcal{O}_n \otimes C(\mathbb{T}) \to CD/C_0D$ such that

$$\varphi(1 \otimes z) = \psi(1 \otimes z), \qquad \varphi|B_1 \otimes 1 = \psi|B_1 \otimes 1, \qquad [\varphi(1 \otimes z)] = g_1,$$

$$[\varphi(s_1 \otimes 1)^*\psi(s_1 \otimes 1)] = h_1, \qquad b(\varphi(s_1 \otimes 1)^*\psi(s_1 \otimes 1), \varphi(1 \otimes z)) = h_0.$$

It will then follow from Proposition 2.4.1 that φ and ψ define the given element of the KK-group.

By Theorem 2.2.1 there are commuting unitaries $u, v \in D$ with full spectrum such that $[u] = h_1$, $[v] = g_1$ and $b(u, v) = h_0$. Since D is purely infinite and

$$(n-1)[1_D] = (n-1)\alpha_0([1]) = \alpha_0((n-1)[1]) = 0,$$

there is a unital embedding of $\mathcal{O}_n$ into D, i.e. there are isometries $t_1, \ldots, t_n \in D$ such that $\Sigma\, t_j t_j^* = 1$. Set $\lambda(x) = \Sigma\, t_j x t_j^*$, $x \in D$, and recall from [6] that $K_*(\lambda) = n$ id. Hence

$$[\lambda(v)] = n[v] = ng_1 = \alpha_1(n[1 \otimes z]) = \alpha_1([1 \otimes z]) = g_1 = [v].$$

Theorem 2.3.3 yields a continuous path $\{w_t\}$, $t \in [1, \infty)$, of unitaries in D such that $w_t \lambda(v) w_t^* \to v$. Define *-homomorphisms $\varphi_t, \psi_t : \mathcal{O}_n \to D$, $t \in [1, \infty)$, by

$$\varphi_t(s_j) = w_t t_j, \qquad \psi_t(s_j) = w_t t_j u.$$

Since $\lambda(v) t_j = t_j v$, we obtain that

$$\|\varphi_t(s_j)v - v\varphi_t(s_j)\| \to 0, \qquad \|\psi_t(s_j)v - v\psi_t(s_j)\| \to 0$$

as $t \to \infty$. Notice that $\varphi_t(s_1)^* \psi_t(s_1) = u$ and that $\varphi_t|B_1 = \psi_t|B_1$.

Define now $\varphi, \psi : \mathcal{O}_n \otimes C(\mathbb{T}) \to CD/C_0 D$ by

$$\varphi(a \otimes 1) = \{\varphi_t(a)\} + C_0 D, \qquad \psi(a \otimes 1) = \{\psi_t(a)\} + C_0 D,$$
$$\varphi(1 \otimes z) = \psi(1 \otimes z) = j(v),$$

where $j : D \to CD/C_0 D$ is the canonical inclusion. Since $\varphi(s_1 \otimes 1)^* \psi(s_1 \otimes 1) = j(u)$, it follows from the choice of u and v that φ and ψ are as desired. □

COROLLARY 4.2. *If D is a stable, purely infinite, simple C^*-algebra, then every element of $KK(Q_n \otimes C(\mathbb{T}), CD/C_0 D)$ is represented by an injective *-homomorphism.*

Proof. Given Proposition 4.1 it suffices to find an embedding $\beta : Q_n \otimes C(\mathbb{T}) \to \mathcal{O}_n \otimes C(\mathbb{T})$ which induces an invertible element of $KK(Q_n \otimes C(\mathbb{T}), \mathcal{O}_n \otimes C(\mathbb{T}))$. By the universal coefficient thoerem, [13, Proposition 7.2], this will be the case if (and only if) $K_*(\beta)$ is an isomorphism. Since Q_n belongs to the classifiable class $\mathscr{C}$, [15, Definition 5.5], it follows from that definition or from [15, Propositon 8.3] that there is a *-homorphism $\alpha : Q_n \to \mathcal{O}_n$ such that $K_*(\alpha)$ is an isomorphism. Now, $\beta = \alpha \otimes \mathrm{id}$ is an injective *-homomorphism such that $K_*(\beta)$ is an isomorphism. □

LEMMA 4.3. *Let D be a unital, purely infinite, simple C^*-algebra, let B be a finite dimensional C^*-algebra and let $\delta > 0$. Suppose that $\varphi_t : B \to D$, $t \in [0, 1]$, is a continuous path of *-homomophisms and that $v \in D$ is a unitary with full spectrum such that*

$$\|v\varphi_t(b) - \varphi_t(b)v\| \leq \delta \|b\|$$

for all $b \in B$ *and all* $t \in [0, 1]$. *Then there is a continuous path* $\{w_t\}$, $t \in [0, 1]$, *of unitaries in* D *such that* $w_0 = 1$, $\mathrm{Ad}\, w_t \circ \varphi_t = \varphi_0$, *and*

$$\|w_t v - v w_t\| \leq 5\delta$$

for all $t \in [0, 1]$.

Proof. Standard arguments give continuous paths $\{v_t\}$ and $\{x_t\}$, $t \in [0, 1]$, of unitaries in D such that $\|v_t - v\| \leq 2\delta$, v_t has full spectrum, v_t commutes with $\varphi_t(B)$, $x_0 = 1$ and $\mathrm{Ad}\, x_t \circ \varphi_t = \varphi_0$. Set

$$u_t = \mathrm{Ad}\, x_t(v_t) \in D \cap \varphi_0(B)'.$$

Since $u_0 = v_0$ it follows from Lemma 2.3.4 that there is a continuous path $\{z_t\}$ of unitaries in $D \cap \varphi_0(B)'$ such that $z_0 = 1$ and $\|z_t u_t z_t^* - v_t\| < \delta$ for all $t \in [0, 1]$. Set $z_t x_t = w_t$, and observe that $\mathrm{Ad}\, w_t \circ \varphi_t = \varphi_0$ because z_t commutes with $\varphi_0(B)$. Also,

$$\|w_t v w_t^* - v\| \leq 2\|v - v_t\| + \|w_t v_t w_t^* - v_t\| \leq 4\delta + \|w_t u_t w_t^* - v_t\| \leq 5\delta. \qquad \square$$

THEOREM 4.4. *Every element of* $KK(Q_n \otimes C(\mathbb{T}), D)$ *is represented by an injective *-homomorphism* $Q_n \otimes C(\mathbb{T}) \to D$ *if* D *is a stable, purely infinite, simple* C^**-algebra.*

Proof. We identify the KK-groups $KK(\cdot\,, D)$ and $KK(\cdot\,, CD/C_0D)$ and the K-groups $K_*(D)$ and $K_*(CD/C_0D)$ via the map $j : D \to CD/C_0D$ (see the discussion preceding Proposition 2.4.1). Every element of $KK(Q_n \otimes C(\mathbb{T}), D)$ is in this way by Corollary 4.2 and its proof represented by a continuous path $\{\varphi_t\}$, $t \in [1, \infty)$, of *-homomorphisms $\varphi_t : Q_n \to D$ (which we may assume to be unital upon passing to a corner of D) and a unitary v in D (or in a corner of D) such that $\|\varphi_t(a)v - v\varphi_t(a)\| \to 0$ as $t \to \infty$. The pair $(\{\varphi_t\}, v)$ defines a *-homorphism $\varphi : Q_n \otimes C(\mathbb{T}) \to CD/C_0D$ as described in the proof of Proposition 4.1. It therefore sufficies to find a *-homomorphism $\psi : Q_n \otimes C(\mathbb{T}) \to D$ such that $KK(j \circ \psi) = KK(\varphi)$ when φ is given as above.

Let $\{B_k\}$ be a generating nest of finite dimensional subalgebras of the UHF-subalgebra B of Q_n as in Lemma 3.1. Find an increasing sequence $\{m_k\}$ of positive integers and a decreasing sequence $\{\delta_k\}$ of positive numbers such that for each $k \in \mathbb{N}$ we have that Lemma 3.1 holds with $\varepsilon = 2^{-k}$ when $m = m_k$ and $\delta = \delta_k$. We may assume that $\delta_k \leq \min\{2^{-k}, \frac{2}{3}\delta_0\}$, where δ_0 is as in Section 2.2.

Find $1 \leq t_1 < t_2 < t_3 < \cdots$ such that $t_k \to \infty$ and

$$\|v\varphi_t(s) - \varphi_t(s)v\| \leq \delta_k/2 \quad \text{for } t \geq t_k,$$

$$\|v\varphi_t(b) - \varphi_t(b)v\| \leq \delta_k\,\|b\|/20 \quad \text{for } t \geq t_k \quad \text{and} \quad b \in B_{m_k}.$$

We construct below a sequence $\{w_k\}$ of unitaries in D such that

(i) $\mathrm{Ad}\, w_k \circ \varphi_{t_{k+1}}|B_k = \varphi_{t_k}|B_k$,

(ii) $\|(\mathrm{Ad}\, w_k \circ \varphi_{t_{k+1}})(s) - \varphi_{t_k}(s)\| \le 2^{-k}$,

(iii) $\|\mathrm{Ad}\, w_k(v) - v\| \le 2^{-k+1}$.

Let $k \in \mathbb{N}$ be given, and use Lemma 4.3 to find a path $\{x_t\}$, $t \in [t_k, t_{k+1}]$, of unitaries in D such that $x_{t_k} = 1$ and

$$\mathrm{Ad}\, x_t \circ \varphi_t|B_{m_k} = \varphi_{t_k}|B_{m_k}, \quad \|x_t v - v x_t\| \le \tfrac{1}{4}\delta_k$$

for all $t \in [t_k, t_{k+1}]$. Check that

$$\|(\mathrm{Ad}\, x_t \circ \varphi_t)(s)v - v(\mathrm{Ad}\, x_t \circ \varphi_t)(s)\| \le \delta_k.$$

Set $(\mathrm{Ad}\, x_t \circ \varphi_t)(s)^* \varphi_{t_k}(s) = u_t$, and observe that $\{u_t\}$, $t \in [t_k, t_{k+1}]$, is a continuous path of unitaries in $D \cap \varphi(B_{m-1})'$ with $u_{t_k} = 1$, whence $u_{t_{k+1}} \in U_0(D \cap \varphi(B_{m-1})')$. We also have that $\|u_t v - v u_t\| \le \frac{3}{2}\delta_k \le \delta_0$, which implies that $b(u_{t_{k+1}}, v) = 0$. Lemma 3.1 yields a unitary w in $D \cap \varphi(B_k)'$ such that

$$\|w(\mathrm{Ad}\, x_{t_{k+1}} \circ \varphi_{t_{k+1}})(s)w^* - \varphi_{t_k}(s)\| \le 2^{-k}, \qquad \|wv - vw\| \le 2^{-k}.$$

It follows that the unitary $w_k = w_{t_{k+1}}$ is as desired.

The sequence $\{\mathrm{Ad}\,(w_1 \cdots w_{k-1}) \circ \varphi_{t_k}\}$ converges in the pointwise norm topology to a $*$-homomorphism $\psi_0 : Q_n \to D$, and $\{\mathrm{Ad}\,(w_1 \cdots w_{k-1})(v)\}$ converges to a unitary v_0 in D with full spectrum such that v_0 and $\psi_0(Q_n)$ commute. We therefore have an injective $*$-homomorphism $\psi : Q_n \otimes C(\mathbb{T}) \to D$ such that $\psi(a \otimes 1) = \psi_0(a)$, $a \in Q_n$, and $\psi(1 \otimes z) = v_0$.

Define a $*$-homomorphism $\bar{\gamma} : CD \to l^\infty D$ by $\bar{\gamma}(f) = \{f(t_k)\}$. This reduces to a $*$-homomorphism $\gamma : CD/C_0D \to l^\infty D/c_0 D$. Put

$$w = \{w_1 w_2 \cdots w_{k-1}\} + c_0 D \in l^\infty D/c_0 D,$$

and observe that $\mathrm{Ad}\, w \circ \gamma \circ \varphi = \gamma \circ j \circ \psi$. It follows that $KK(\gamma \circ \varphi) = KK(\gamma \circ j \circ \psi)$, and because $K_*(\gamma)$ is injective, this implies that $KK(\varphi) = KK(j \circ \psi)$. □

5 Classifiable models

We give below a slightly revised definition of the classifiable purely infinite, simple C^*-algebras compared with the original definition ([15, Definition 5.5]). Recall from [15] that $KL(A, B)$ is $KK(A, B)$ divided by the subgroup obtained by taking the closure of $\{0\}$ in a certain natural topology. If $K_*(A)$ is finitely generated, then $KL(A, B) = KK(A, B)$. Recall also from [15] that $H(A, B)$ is the set of full $*$-homomorphisms from $A \otimes \mathcal{K}$ into $B \otimes \mathcal{K}$ (i.e. every non-zero element in $A \otimes \mathcal{K}$ is mapped into a full element of $B \otimes \mathcal{K}$) modulo approximate unitary equivalence, and that there is a natural homomorphism $\kappa : H(A, B) \to KL(A, B)$.

DEFINITION 5.1. (cf. [15, Definition 5.5]) Let $\mathcal{C}$ denote the class of all separable, simple, purely infinite C^*-algebras A for which the map $\kappa : H(A, D) \to KL(A, D)$ is an isomorphism for every purely infinite, simple C^*-algebra D.

This definition differs from the original one by restricting the class of target algebras D (which in [15] were only assumed to have real rank zero and to contain a properly infinite full projection). We note that none of the results about the class $\mathcal{C}$ proved in [15] need to be revised, and that, in particular, two C^*-algebras A and B in $\mathcal{C}$ are isomorphic if they have isomorphic K-theory. It will be convenient also to have the following.

DEFINITION 5.2. Let $\bar{\mathcal{C}}$ denote the class of all separable C^*-algebras A with an approximate unit consisting of projections such that $\kappa : H(A, D) \to KL(A, D)$ is an isomorphism for every purely infinite, simple C^*-algebra D.

We can then reformulate Theorem 3.4 and 4.4 as

THEOREM 5.3. *The C^*-algebras $Q_n \otimes C(\mathbb{T})$ belong to $\bar{\mathcal{C}}$ for every $n \geq 2$. In particular, $\mathcal{O}_{2n} \otimes C(\mathbb{T})$ belongs to $\bar{\mathcal{C}}$ for every n.*

The following is a generalization of [15, Theorem 5.9]. The proof in [15] applies verbatim to the more general statement.

THEOREM 5.4. (cf. [15, Theorem 5.9]) *Let A be the inductive limit of a sequence $A_1 \to A_2 \to A_3 \to \cdots$ with injective maps of C^*-algebras, each of which is a finite direct sum of C^*-algebras in $\bar{\mathcal{C}}$. Then A belongs to $\bar{\mathcal{C}}$. If A in addition is simple and purely infinite, then A belongs to $\mathcal{C}$.*

COROLLARY 5.5. *Suppose that A and B are the inductive limits of sequences $A_1 \to A_2 \to A_3 \to \cdots$ and $B_1 \to B_2 \to B_3 \to \cdots$, with injective maps, where each A_j and B_j is isomorphic to a C^*-algebra of the form*

$$\bigoplus_{i=1}^{r} M_{k_i}(Q_{n_i} \otimes C(\mathbb{T}))$$

(r, k_i and n_i depend on A_j and B_j). Suppose also that A and B are simple. Then A and B belong to $\mathscr{C}$. It follows in particular that A is isomorphic to B if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$ as abelian groups, and either A and B are both non-unital, or they both have a unit and there is an isomorphism $K_0(A) \to K_0(B)$ mapping $[1_A]$ onto $[1_B]$.

Proof. It follows from Theorem 5.3 and 5.4 that A and B belong to $\bar{\mathscr{C}}$, and that they belong to $\mathscr{C}$ if they are simple – which is assumed – and purely infinite. It follows from [15, Proposition 8.5 and Theorem 5.7] that Q_n is isomorphic to $Q_n \otimes M_{n^\infty}$, where M_{n^∞} is the UHF-algebra of type n^∞. It follows in particular that Q_n is approximately divisible (in the sense of [1]), and this implies that the inductive limits A and B are approximately divisible. Since A and B clearly are infinite, [1, Theorem 1.4] yields that A and B are purely infinite. □

Let us conclude with the following description of the classifiable class $\mathscr{C}$ that may justify its name.

THEOREM 5.6. (cf. [15, Theorem 8.2 and Proposition 8.3]) *Let G_0 and G_1 be countable abelian groups and let $g_0 \in G_0$. There is, up to isomorphism, precisely one unital C^*-algebra A_0 in $\mathscr{C}$ such that $(K_0(A_0), [1], K_1(A_0)) \cong (G_0, g_0, G_1)$, and there is precisely one non-unital (and hence necessarily stable) C^*-algebra $\bar{A}_0$ in $\mathscr{C}$ with $(K_0(\bar{A}_0), K_1(\bar{A}_0)) \cong (G_0, G_1)$.*

Let A and $\bar{A}$ be separable, purely infinite, simple C^*-algebras in the bootstrap category $\mathscr{N}$ where the universal coefficient theorem is proved to hold c.f. [13], A with a unit and $\bar{A}$ stable, with $(K_0(A), [1], K_1(A)) \cong (G_0, g_0, G_1)$ and $(K_0(\bar{A}), K_1(\bar{A})) \cong (G_0, G_1)$. Then $\bar{A}$ is isomorphic to $\bar{A}_0$ if and only if

(i) *there is a non-zero *-homomorphism $\varphi : \bar{A} \to \bar{A}_0$ such that $K_*(\varphi)$ is an isomorphism, and*

(ii) *every non-zero *-endomorphism φ of $\bar{A}$ with $KL(\varphi) = KL(\mathrm{id})$ is approximately inner (i.e., approximately unitarily equivalent to the identity).*

Similarly, A is isomorphic to A_0 if and only if

(i) *there is a unital *-homomorphism $\varphi : A \to A_0$ such that $K_*(\varphi)$ is an isomorphism, and*

(ii) *every unital *-endomorphism φ of A with $KL(\varphi) = KL(\mathrm{id})$ is approximately inner.*

Proof. Fix a natural number $n \geq 2$. There is a simple unital C^*-algebra B_n of real rank zero which is the inductive limit of a sequence.

$$C(\mathbb{T}) \to M_{n-1}(C(\mathbb{T})) \to M_{(n-1)^2}(C(\mathbb{T})) \to M_{(n-1)^3}(C(\mathbb{T})) \to \cdots$$

and such that $K_0(B_n) \cong \mathbb{Z}[1/(n-1)]$ and $K_1(B_n) \cong \mathbb{Z}$. (In fact, by [8], B_n is unique, and is a Bunce–Deddens algebra.) It follows from Corollary 5.5 that $Q_n \otimes B_n$ belongs to $\mathscr{C}$. We have that $K_0(Q_n \otimes B_n) = 0$ and $K_1(Q_n \otimes B_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.

Suppose now that G_1 is finitely generated, i.e.,

$$G_1 \cong H_1 \oplus \mathbb{Z}/(n_1 - 1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n_r - 1)\mathbb{Z},$$

where H_1 is torsion free. By [15, Theorem 8.2], there is a unital C^*-algebra A_0 in $\mathscr{C}$ such that $(K_0(A_0), [1], K_1(A_0)) \cong (G_0, g_0, H_1)$. Set

$$A_0 \oplus A_1 \oplus \cdots \oplus A_r = A,$$

where $A_j = Q_{n_j} \otimes B_{n_j}, j = 1, \ldots, r$. Since each A_j belongs to $\mathscr{C}$ by Corollary 5.5 there is a unital endomorphism φ of A such that for each partial homomorphism $\varphi_{ij} : A_j \to A_i$ of φ, the map φ_{ij} is non-zero, $K_*(\varphi_{ii}) = \mathrm{id}$, and $K_*(\varphi_{ij}) = 0$ if $i \neq j$. It follows that $K_*(\varphi) = \mathrm{id}$ and that the inductive limit D of the sequence

$$A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$$

is simple with $K_*(D) \cong K_*(A)$. Theorem 5.3 implies that D belongs to $\mathscr{C}$.

Any arbitrary triple (G_0, g_0, G_1) is the inductive limit of a sequence of triples $(G_0^{(n)}, g_0^{(n)}, G_1^{(n)})$ with $G_1^{(n)}$ finitely generated. The latter triples are, by the preceding argument, the invariants of unital C^*-algebras in $\mathscr{C}$, and so it follows from [15, Theorem 8.2] that there is a unital C^*-algebra A_0 in $\mathscr{C}$ with $(K_0(A_0), [1], K_1(A_0)) \cong (G_0, g_0, G_1)$. The stable C^*-algebra $\bar{A}_0 = A_0 \otimes \mathscr{K}$ also belongs to $\mathscr{C}$, and is therefore as desired. The uniqueness of A_0 and $\bar{A}_0$ follows from [15, Theorem 5.7]. The last part of the theorem is just a reformulation of [15, Proposition 8.3]. □

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