Infinite Coxeter groups virtually surject onto Z

Autor(en): **Gonciulea, Constantin**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **72 (1997)**

PDF erstellt am: **28.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-54588

Comment. Math. Helv. 72 (1997) 257–265
0010-2571/97/020257-9 $ 1.50+0.20/0

Commentarii Mathematici Helvetici

Infinite Coxeter groups virtually surject onto $\mathbb{Z}$

Constantin Gonciulea

Abstract. We prove that any infinite Coxeter group has a subgroup of finite index which homomorphically surjects onto the integers. This implies the known result that infinite Coxeter groups do not have property (T) of Kazhdan.

Mathematics Subject Classification (1991). 20F55.

Keywords. Coxeter group, Coxeter complex.

Introduction

In this paper we prove that an infinite Coxeter group contains a finite index subgroup that admits an epimorphism onto $\mathbb{Z}$. There is a standard way of producing an epimorphism from the fundamental group of a manifold onto $\mathbb{Z}$: we consider a two-sided, non-separating submanifold of codimension one, and assign to each loop the sum of its intersection numbers with this submanifold. The main idea of our proof is to try to transpose what this situation gives at the level of the universal cover, to the context of the Coxeter complex associated to a Coxeter group. The Coxeter complex is not very far from the simplicial analogue of a manifold, and its walls are codimension one subcomplexes that under some conditions give "two-sided", "non-separating" subcomplexes of codimension one in the quotient of the Coxeter complex by a subgroup of finite index of the given Coxeter group. In order to make the presentation as clear and as simple as possible, we will alternate geometric and algebraic points of view.

The question of whether infinite Coxeter groups virtually surject onto $\mathbb{Z}$ was originally posed by Pierre de la Harpe and Alain Valette in the following context: the affirmative answer would imply the known result of [1], that infinite Coxeter groups do not have property (T) of Kazhdan. More details can be found in [6].

Another application is related to the Thurston's conjecture that the fundamental group of an aspherical three-manifold virtually surjects onto $\mathbb{Z}$. This note's result proves this for subgroups of infinite Coxeter groups.

I would like to thank Mike Davis for his tremendous support, Tadeusz Januszkiewicz for many valuable comments, and the referees for some extremely useful corrections and suggestions.

While writing this note, it was brought to my attention that the same result has been proved independently by D. Cooper, D. Long and A. Reid ([5]).

1. Preliminaries on Coxeter groups

In this section we recall some basic definitions and important facts about Coxeter groups and their associated complexes. Most of the material can be found in the excellent references [2], [3], or [4].

A *pre–Coxeter system* (W, S) is a group W generated by a set of involutions S; $m(s,t) \in \{1,2,3,\dots\infty\}$ denotes the order of st in W for $s,t \in S$. Note that $m(s,s) = 1$ and $m(s,t) = m(t,s)$ for any $s,t \in S$.

The set of *reflections* of W is $R = \bigcup_{w \in W} wSw^{-1}$. By a *word* in the generating set S we mean a finite sequence $\mathbf{s} = (s_1, \dots, s_d)$. We will often be less formal and simply say that the expression $s_1 \cdots s_d$ is a word; the *element* $w = s_1 \cdots s_d$ of W that it represents must be distinguished from the word $\mathbf{s}$.

The *length* $\ell(w)$ of an element $w \in W$ is the minimal d such that $w = s_1 \cdots s_d$, with $s_i \in S$ for $1 \le i \le d$.

The word $(s_1, \dots s_d)$ is called *reduced* if the corresponding element $w = s_1 \cdots s_d$ has length $\ell(w) = d$, i.e. it cannot be represented by a shorter word. We will also say in this situation that the given word is a *reduced decomposition* of w or, less formally, that the equation $w = s_1 \cdots s_d$ is a reduced decomposition of w.

We say that the pre–Coxeter system (W, S) is a *Coxeter system*, if W has a presentation of the form:

$$W = \langle S \mid (st)^{m(s,t)} = 1 \text{ for all } s,t \in W \text{ with } m(s,t) < \infty \rangle.$$

This condition can be also understood as a *universal property*, in the following sense: given a group G and a map $f : S \to G$ such that $(f(s)f(t))^{m(s,t)} = 1$ for all $s,t \in S$ with $m(s,t) < \infty$, there exists a homomorphism $g : W \to G$ extending f.

In what follows we will restrict ourselves to Coxeter systems (W, S) with S a finite set.

The *dihedral group* D_m is the Coxeter group with the presentation:

$$D_m = \langle s,t \mid s^2 = t^2 = (st)^m = 1 \rangle$$

This group has order $2m$ and it is easy to enumerate all its elements. Since s and t are involutions, the elements of D_m are obtained as alternating products of s's and t's: 1, s, t, st, ts, sts, tst, $\dots$, with two products of each positive length. But the relation $(st)^m = 1$ gives $\underbrace{stst\dots}_{m \text{ factors}} = \underbrace{tsts\dots}_{m \text{ factors}}$, i.e. the two products of length m give the same element of D_m, which is the element of maximal length, since pre-multiplying or post-multiplying it by either s or t will result in cancellation, giving one of the two elements of length $m-1$. Let us also mention that D_m is isomorphic to the reflection group on $\mathbb{R}^2$ generated by reflections in two lines meeting at the angle π/m.

A Coxeter group is irreducible if it cannot be written as a direct product of two Coxeter groups, or equivalently, if S cannot be partitioned in two subsets S_1 and S_2 such that $m(s_1, s_2) = 2$ for any $(s_1, s_2) \in S_1 \times S_2$.

We will make use of the following facts about Coxeter groups.

Proposition 1.1. *Given $w \in W$ and a non-reduced decomposition $w = s_1 \cdots s_d$, then there are indices $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d$ i.e. the length of a word can be decreased only by deleting pairs of letters.*

In this case the elements s_i and s_j are conjugate by the element represented by the word between them. The above statement is usually called the *Deletion Condition*. See [3], p. 37, or [4] for more details.

Proposition 1.2. *Assume (W, S) is a Coxeter system, and $\mathbf{w}, \mathbf{v}$ are reduced words. Then they are representing the same element of W if and only if $\mathbf{w}$ can be transformed in $\mathbf{v}$ by the application of a finite sequence of operations of the following form:*

Given $s, t \in S$ with $s \neq t$ and $m(s,t) < \infty$, replace an alternating subword $(s, t, \dots)$ of length $m = m(s,t)$ by the alternating word $(t, s, \dots)$ of length m.

This proposition can be found in [3], p. 51.

The following characterization of Coxeter systems, sometimes called the *Hyperplane Condition*, will be crucial in our proof.

Proposition 1.3. *Let (W, S) be a pre–Coxeter system . Then (W, S) is a Coxeter system if and only if there is a map $s \mapsto P_s$ from S to 2^W such that:*
1. $1 \in P_s$ for each $s \in S$;
2. if $w \in W$ and $s, t \in S$ satisfy $w \in P_s$ and $wt \notin P_s$, then $sw = wt$;
3. $P_s \cap sP_s = \emptyset$ for each $s \in S$.
In this case, $P_s = \{w \in W \mid \ell(sw) > \ell(w)\}$.

The reader can find a proof of this proposition in [4], or [2], p. 18.

The next fact is a consequence of the well-known Selberg's Lemma (see [7], p. 326, for example), whose hypotheses are satisfied by a Coxeter group.

Proposition 1.4. *Any finitely generated Coxeter group W has a torsion-free subgroup of finite index.*

By taking the so-called "normal core" of such a subgroup, i.e. the intersection of its conjugates by all elements in W, we obtain a subgroup with the same properties, which in addition is normal in W.

The finite (irreducible) Coxeter groups are completely classified. A list can be found in [2]. What we need to know about it, is that the only finite irreducible Coxeter group with an $m(s,t)$ greater than 5, is a dihedral group.

Now we give a short description of the Coxeter complex associated to a Coxeter system (W,S).

Let P be the poset of all subsets of W, with inclusion reversed. The *Coxeter poset* associated to (W,S) is the sub-poset of P consisting of sets of the form $w\langle T\rangle$ for a proper (possibly empty) subset T of S.

The associated *Coxeter complex* $\Sigma(W,S)$ is defined to be the simplicial complex associated to the Coxeter poset of (W,S). That is, $\Sigma(W,S)$ has simplices which are cosets in W of the form $w\langle T\rangle$ for a proper subset T of S, with face relations opposite to subset inclusion in W. Of course, it is not immediate that this is a simplicial complex, but it turns out to be the case. The maximal simplices, called *chambers*, are of the form $w\langle\emptyset\rangle = \{w\}$.

Since $\Sigma = \Sigma(W,S)$ is constructed as a collection of cosets $w\langle T\rangle$, there is a natural action of W on $\Sigma(W,S)$, by left multiplication. The isotropy group in W of the simplex $w\langle T\rangle$ is $w\langle T\rangle w^{-1}$. The fixed point set of a reflection will be called a *wall*. Thus, a wall is a union of codimension one simplices in Σ.

We will often identify the chambers of Σ with the elements of W, and the walls of Σ with the set of reflections $R = \bigcup_{w\in W} wSw^{-1}$ of W.

For a reflection $r \in R$, we denote by Σ_r its corresponding wall of Σ. If $r = wsw^{-1}$, then $\Sigma_r = w\Sigma_s$. We say that Σ_r is the *translate* of Σ_s by w.

The walls that have codimension one intersection with the chamber C, corresponding to the element $w \in W$, are precisely the walls $w\Sigma_s$, with $s \in S$. The intersections are the *faces* of C. Two chambers $C \neq D$ are *adjacent* if they share a common face. If Σ_r is the unique wall containing that face, then $D = rC$. A sequence $(C_0, C_1, \ldots, C_d)$ of chambers is a *gallery* (of length d, from C_0 to C_d) if the chambers C_{i-1} and C_i are adjacent, for $1 \leq i \leq d$. The sequence $(r_1, \ldots, r_d)$ of reflections defined by $C_i = r_iC_{i-1}$ is called *the reflection sequence* corresponding to the gallery $(C_0, C_1, \ldots, C_d)$. A gallery *crosses* the wall Σ_r if r is contained in the corresponding sequence $(r_1, \ldots, r_d)$. A *minimal gallery* from C to D is a gallery of minimal length. Each wall separates Σ in two connected components called *sides*, or *half-spaces*. A wall Σ_r *separates* chambers C and D, if they are on different sides determined by Σ_r, and this happens if and only if a(ny) gallery from C to D crosses Σ_r an odd number of times. If the gallery is minimal, this number is one.

Two walls, associated to two distinct reflections, determine four half-spaces of $\Sigma(W,S)$, and if any two of them which are not determined by the same reflection have a nonempty intersection (if we want we can call such an intersection a quadrant), then the order of the product of the two reflections is finite.

It is clear that we can identify a word on S with a gallery in Σ starting at $\{1\}$, and vice-versa. The set of decompositions of an element $w \in W$ is in bijective correspondence with the set of galleries from $\{1\}$ to $\{w\}$. If $w = s_1 \cdots s_d$, then the gallery from $\{1\}$ to $\{w\}$ crosses precisely the walls Σ_{r_i}, where $r_i = s_1 \cdots s_{i-1} \cdot s_i \cdot (s_1 \cdots s_{i-1})^{-1} = s_1 \cdots s_{i-1} \cdot s_i \cdot s_{i-1} \cdots s_1$, for $1 \leq i \leq d$.

2. Separability and two-sidedness of a reflection with respect to a subgroup

In this section (W, S) is an arbitrary Coxeter system and G a subgroup of W. For each $s \in S$, consider the wall in $\Sigma(W, S)$ determined by s, and define the map $\theta_s : W \to \mathbb{Z}/2\mathbb{Z}$, by $\theta_s(w) =$ the number of times (mod 2) a gallery from 1 to w crosses G-translates of Σ_s. Also, define $W_s^+ = \theta_s^{-1}(0)$, and $W_s^- = \theta_s^{-1}(1)$. We will not complicate the notations by emphasizing the dependence of θ_s on the fixed subgroup G of W.

Remark 2.1. It is easy to see that in general $\theta_s : W \to \mathbb{Z}/2\mathbb{Z}$ is not a group homomorphism, but its restriction to G is. This is a consequence of the following lemma, also used later:

Lemma 2.2. *If $g \in G$ and $w \in W$, then $\theta_s(gw) = \theta_s(g) + \theta_s(w)$. Equivalently, if $g \in G \cap W_s^{\varepsilon}$ and $w \in W_s^{\delta}$, then $gw \in W_s^{\varepsilon\delta}$, for any $\varepsilon, \delta \in \{+, -\}$.*

Proof. Just note that a gallery from 1 to w crosses the wall $\gamma\Sigma_s$, for a $\gamma \in G$, if and only if its translate by g, which is a gallery from g to gw, crosses the wall $g\gamma\Sigma_s$. $\square$

Definition 2.3. We say that $s \in S$ is *G-separating* if $\theta_s\big|_G = 0$ or, equivalently $G \subset W_s^+$.

Recall the notation $P_s = \{w \in W \mid \ell(sw) > \ell(w)\}$.

Definition 2.4. We say that $s \in S$ is *two-sided with respect to the subgroup* G of W if $C_G(s) \subset P_s$, where $C_G(s)$ denotes the centralizer of s in G.

In the Coxeter complex $\Sigma(W, S)$ associated to the Coxeter system (W, S) we denote by Σ_+ the half-space of $\Sigma(W, S)$ determined by Σ_s which contains 1, and by Σ_- the other one. The chambers in Σ_+ correspond to the elements in P_s, and the chambers in Σ_- to those in sP_s. The fact that s is two-sided with respect to G means that all the chambers corresponding to the centralizers of s in G are in Σ_+. Notice that the centralizers of s in W come in pairs $\{z, zs\}$, and the wall between z and zs is determined by the reflection $zsz^{-1} = s$, i.e. it is exactly Σ_s. Hence, all centralizers of s in W line up along Σ_s.

Proposition 2.5. *Let (W,S) be a Coxeter system, G a subgroup of W, and suppose that some $s \in S$ is two-sided with respect to G. Then the homomorphism $\theta_s : G \to \mathbb{Z}/2\mathbb{Z}$ can be lifted to a homomorphism $\phi_s : G \to \mathbb{Z}$. If, in addition, s is not G-separating, then ϕ_s is non-zero.*

Proof. First we assign signs to the half-spaces determined by the G-translates of Σ_s as follows: $g\Sigma_+$ is positive and $g\Sigma_-$ is negative, for $g \in G$. In other words, a side of a wall is positive if it contains (the) chambers along that wall which correspond to elements in G. These are well-defined. Indeed, if we take two chambers along such a wall, corresponding to the elements g and g' of G, then $g\Sigma_s = g'\Sigma_s$, which implies $gsg^{-1} = g's(g')^{-1}$, i.e. $g^{-1}g' \in C_G(s) \subset P_s$, and therefore the chambers corresponding to g and g' are on the same side of the wall $g\Sigma_s = g'\Sigma_s$.

Next, we assign a crossing number to each crossing of a G-translate of Σ_s by a gallery starting at 1. Such a crossing will be assigned $+1$ if the gallery crosses from the positive side to the negative one, and -1 otherwise. Equivalently, the crossing is assigned a $+1$ if $1 \in W$ is on the positive side of that translate.

And now, for each $g \in G$, define $\phi_s(g)$ to be the sum of the crossing numbers associated to a gallery joining 1 and g.

Let us prove that $\phi_s(g)$ does not depend on the choice of such a gallery. First, Proposition 1.1 tells us that if a gallery is not minimal, then it crosses at least one wall twice, and a shorter gallery is obtained by reflecting a part of the gallery with respect to that wall, hence eliminating two crossings of it. Whether this wall is a G-translate of Σ_s or not, it is clear that the sum of the crossing numbers remains unchanged. Therefore, it is enough to prove that two minimal galleries representing g give the same value for $\phi_s(g)$, i.e. this value does not change when we apply the operation described in Proposition 1.2 to a minimal gallery. Consider the minimal galleries $\mathcal{G}_1 = Artrt\cdots B$ and $\mathcal{G}_2 = Atrtr\cdots B$, with $r,t \in S$ and the number of factors between A and B equal to $m = m(r,t)$. The subgroup generated by $\{r,t\}$ in W is isomorphic to the dihedral group D_m. In the Coxeter complex $\Sigma(D_m,\{r,t\})$, the two (minimal) galleries of length m starting at 1 and ending at the element of maximal length $w_0 = \underbrace{rtrt\cdots}_{m \text{ factors}} = \underbrace{trtr\cdots}_{m \text{ factors}}$ are crossing the same walls, and all these walls separate 1 and w_0 because of minimality of the galleries in question. This proves that $\mathcal{G}_1$ and $\mathcal{G}_2$ cross the very same walls of $\Sigma(W,S)$ (in particular the same G-translates of Σ_s), and with the same crossing numbers, when defined.

It is clear that ϕ_s is a group homomorphism, and $\phi_s(\text{mod } 2) = \theta_s$ on G. If ϕ_s was zero, then θ_s would be zero on G, and hence s would be G-separating. $\square$

3. A torsion-free normal subgroup gives separability and two-sidedness

Lemma 3.1. *If Γ is a normal subgroup of the Coxeter group W then $sW_s^+ = W_s^-$ for any $s \in S$, where the notation is that of section 2, with respect to the subgroup Γ.*

Proof. Indeed, if $\mathcal{G} = (C_0, \dots, C_d)$ is a gallery of length d from 1 to w, then $\mathcal{G}' = (C_0, s\mathcal{G}) = (C_0, sC_0, \dots, sC_d)$ is a gallery of length $d+1$ from 1 to sw, which has a crossing with Σ_s that $\mathcal{G}$ does not have. Since $\gamma \mapsto s\gamma s$ is an automorphism of Γ, any other Γ-translate of Σ_s crossed by $\mathcal{G}'$ is of the form $s\gamma\Sigma_s = s\gamma s\Sigma_s$, where $\gamma\Sigma_s$ is a Γ-translate of Σ_s crossed by $\mathcal{G}$. We conclude that $\theta_s(sw) = \theta_s(w) + 1 \pmod 2$, and therefore $w \in W_s^+$ if and only if $sw \in W_s^-$. □

Propostion 3.2. *Suppose that the Coxeter system (W, S) is such that $m(s,t)$ is finite for any $s, t \in S$ and Γ is a non-trivial, torsion-free, normal subgroup of W. Then there is an $s \in S$ which is not Γ-separating.*

Proof. We will denote the quotient group W/Γ by $\overline{W}$, and the canonical projection from W onto $\overline{W}$ by $w \mapsto [w]$. Since Γ is torsion free, $s \notin \Gamma$ and $st \notin \Gamma$ for $s, t \in S$. It follows that $[s] \neq [t]$ for any $s, t \in S$, and that $\overline{W}$ is generated by $[S] := \{[s] \mid s \in S\}$, which is a set of involutions in $\overline{W}$ (in fact a copy of S). $(\overline{W}, [S])$ is a pre–Coxeter system, and since Γ is torsion-free and all $m(s,t)$'s are finite, it has the same Coxeter data as (W, S). It cannot be a Coxeter system, since Γ is not trivial. Consequently, the subsets $P_{[s]}$ of $\overline{W}$, defined as the images of W_s^+ through the canonical projection, cannot satisfy all three conditions of Proposition 1.3.

But they obviously satisfy the first condition.

We show now that they also satisfy the second one. Suppose $\overline{w} \in \overline{W}$ and $[s], [t] \in [S]$ are such that $\overline{w} \in P_{[s]}$ and $\overline{w}\,[t] \notin P_{[s]}$. Then there is an element $w \in W_s^+$ with $[w] = \overline{w}$, and $[wt] = \overline{w}\,[t] \notin P_{[s]}$ implies that $wt \in W_s^-$. Since $\{w\}$ and $\{wt\}$ are adjacent chambers across the wall $w\Sigma_t$, this is possible only if $w\Sigma_t$ is a Γ-translate of Σ_s, i.e., if and only if there is a $\gamma \in \Gamma$ with $wtw^{-1} = \gamma s \gamma^{-1}$. But this implies $\overline{w}\,[t]\,(\overline{w})^{-1} = [s]$, i.e. $\overline{w}\,[t] = [s]\,\overline{w}$.

It follows that the third condition does not hold. Then we can find $\overline{w} \in P_{[s]} \cap [s]P_{[s]}$ for some $s \in S$, i.e. $\overline{w} \in P_{[s]}$ and $[s]\overline{w} \in P_{[s]}$. This gives some $w, w' \in W_s^+$ with $[w] = [sw'] = \overline{w}$. But according to Lemma 3.1, we have $sw' \in W_s^-$, and since $sw' = sw'w^{-1} \cdot w$, Lemma 2.2 implies that the element $sw'w^{-1}$ of Γ is in W_s^-, which in turn shows that this s is not Γ-separating. □

Remark 3.3. In geometric terms, under the assumptions of the statement, the quotient Σ/Γ is a thin chamber complex, but not a Coxeter complex, and therefore one of its walls has to be non-separating. For the terminology used here, the reader can consult [3]. Using the ideas in Proposition 3.2 we can prove a slightly more general result: *if Γ is a normal subgroup of W, then the quotient W/Γ is a Coxeter group if and only if every $s \in S$ is Γ-separating.*

Proposition 3.4. *Let (W, S) be a Coxeter system, Γ a torsion-free normal subgroup of W, s an element of S, and Γ_0 the subgroup $\Gamma \cap W_s^+$ of index 1 or 2 in Γ consisting of elements $\gamma \in \Gamma$ such that the galleries joining 1 and γ cross the walls which are Γ-translates of Σ_s an even number of times. Then s is two-sided with respect to Γ_0.*

Proof. We already mentioned that the elements of $C_W(s)$ lie along the wall Σ_s. We want to prove that $C_\Gamma(s) \cap W_s^+ \subset P_s$, i.e., for a $z \in C_\Gamma(s)$, the element in the pair $\{z, zs\}$ which is in W_s^+ is also in P_s. For this it is enough to prove that a minimal gallery from 1 to z crosses no Γ-translate of Σ_s other than Σ_s. Suppose this does not happen, and take $z \in C_\Gamma(s)$ and $\gamma \in \Gamma - C_\Gamma(s)$ such that a gallery from 1 to z crosses $\gamma\Sigma_s$. Then the intersection between any half-space determined by Σ_s and any half-space determined by $\gamma\Sigma_s$ is nonempty, each such intersection containing exactly one element of the set $\{1, s, z, zs\}$. As we pointed out at the end of section 1, it follows from this that the order of the product of the corresponding reflections, s and $\gamma s \gamma^{-1}$, is finite. This product is $s \cdot \gamma s \gamma^{-1} = s\gamma s^{-1} \cdot \gamma^{-1}$, and since Γ is normal in W, this product is a nontrivial element of Γ. But this contradicts the fact that Γ is torsion-free. $\square$

4. The main result

Lemma 4.1. *An infinite Coxeter group W with all $m(s,t)$'s finite contains a finite index subgroup which admits a non-zero homomorphism to $\mathbb{Z}$.*

Proof. Consider a non-trivial, torsion-free, normal and finite index subgroup Γ of W (its existence is assured by Selberg's Lemma). We can also suppose that every $s \in S$ is two-sided with respect to Γ (if not, we can replace Γ by the normal core of the intersection of the subgroups of Γ obtained by applying Proposition 3.4 to each $s \in S$; besides clearly being normal and torsion-free, the subgroup thus obtained has finite index in W and is non-trivial since S is finite). Proposition 3.2 tells us that there is an $s \in S$ which is not Γ-separating. Then the homomorphism $\phi_s : \Gamma \to \mathbb{Z}$ defined in Proposition 2.5 is non-zero. $\square$

Main result. *Any infinite Coxeter group virtually surjects onto $\mathbb{Z}$, i.e. it contains a subgroup of finite index which admits an epimorphism onto $\mathbb{Z}$.*

Proof. Let (W, S) be the infinite Coxeter group. If W is not irreducible, say $W = W_1 \times \cdots \times W_k \times V_1 \times \cdots \times V_l$, with $W_1, \ldots, W_k$ infinite Coxeter groups, and $V_1, \ldots, V_l$ finite Coxeter groups, and if we know that for each $i \in \{1, \ldots, k\}$ there is a finite index subgroup Γ_i of W_i and an epimorphism $f_i : \Gamma_i \to \mathbb{Z}$, then $f_1 \times \cdots \times f_k$ is an epimorphism of $\Gamma_1 \times \cdots \Gamma_k$ (which has finite index in W) onto $\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{k \text{ factors}}$, and the latter group obviously surjects onto $\mathbb{Z}$. Therefore, it is sufficient to consider the case when W is irreducible.

If $W = D_\infty$ we can take $\Gamma = \mathbb{Z}$ to be the desired subgroup.

If not, consider the Coxeter group (W', S) with

$$m'(s,t) = \begin{cases} m(s,t) & \text{if } m(s,t) < \infty \\ 1996 & \text{if } m(s,t) = \infty \end{cases}$$

Then we have an epimorphism $f : W \to W'$ (formally we can use the universal property of W to see this). Let us notice that W' is irreducible, and it is also infinite (the only irreducible finite Coxeter group involving an order greater than 5 is a dihedral group.) But the above lemma tells us that we can find a subgroup of finite index Γ' in W', and a non-zero $g : \Gamma' \to \mathbb{Z}$. Then we can take $\Gamma := f^{-1}(\Gamma')$. It is clear that Γ has finite index as a subgroup of W, and $g \circ f : \Gamma \to \mathbb{Z}$ is a non-zero homomorphism, and consequently an epimorphism onto its image. $\square$

References

[1] M. Bożeiko, T. Januszkiewicz, R.T. Spatzier, Infinite Coxeter groups do not have Kazhdan's property, *J. Operator Theory* **19** (1988), 63–67.

[2] N. Bourbaki, *Groupes et algébres de Lie, Chapitres IV-VI*, Masson, Paris 1981.

[3] K.S. Brown, *Buildings*, Springer-Verlag, New York 1989.

[4] A. M. Cohen, Coxeter groups and three related topics, *Generators and Relations in Groups and Geometries*, ed. by A. Barlotty et al., Kluwer Academic Publishers (1991) 235–278.

[5] D. Cooper, D. Long and A. Reid, *Infinite Coxeter groups are virtually indicable*, Proceedings of the Edinburgh Math. Soc., to appear.

[6] P. de la Harpe and A. Valette, La propriété (T) de Kazhdan pour les groupes localement compacts, *Astérisque* **175**, 1989.

[7] J.G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer-Verlag, New York 1994.

Constantin Gonciulea
Ohio State University
231 W. 18th Avenue
Columbus, OH 43210
USA
e-mail: cgon@math.ohio-state.edu

(Received: July 1, 1996)