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Higher generation subgroup sets and the Σ-invariants of graph groups

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Abstract. We present a general condition, based on the idea of n-generating subgroup sets, which implies that a given character $\chi \in \mathrm{Hom}(G, \ )$ represents a point in the homotopical or homological Σ-invariants of the group G. Let $\mathcal{G}$ be a finite simplicial graph, $\widehat{\mathcal{G}}$ the flag complex induced by $\mathcal{G}$, and $G\mathcal{G}$ the graph group, or 'right angled Artin group', defined by $\mathcal{G}$. We use our result on n-generating subgroup sets to describe the homotopical and homological Σ-invariants of $G\mathcal{G}$ in terms of the topology of subcomplexes of $\widehat{\mathcal{G}}$. In particular, this work determines the finiteness properties of kernels of maps from graph groups to abelian groups. This is the first complete computation of the Σ-invariants for a family of groups whose higher invariants are not determined — either implicitly or explicitly — by Σ^1.

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1. Introduction

For almost two decades there have been two competing notions of finiteness for infinite groups. C.T.C. Wall introduced a geometric measure of 'finiteness' by defining a group G to be $\mathcal{F}_m$ if and only if there is a $K(G,1)$ with finite m-skeleton. Wall's properties $\mathcal{F}_1$ and $\mathcal{F}_2$ are topological reformulations of the two most common finiteness conditions, finite generation and finite presentation. On the other hand, Bieri introduced the FP_m property, where a group G is FP_m if $\mathbb{Z}$, thought of as a trivial $\mathbb{Z}G$-module, admits a projective resolution with finitely generated m-skeleton. More generally, for any commutative ring R, with $0 \neq 1$, G is $\mathrm{FP}_m(R)$ if R thought of as a trivial RG-module admits a projective resolution with finitely generated m-skeleton. (See Chapter VIII of [8] for background on finiteness properties of infinite groups.)

Examples of groups exhibiting one kind of finiteness, but not another, are known. Let G_m be the direct product of m copies of a finitely generated non-abelian free group F. Let $\chi : G_m \to \mathbb{Z}$ be the map where each generator is carried to $1 \in \mathbb{Z}$. Then Stallings ($m = 3$) and Bieri ($m > 3$) have shown that the kernel of χ is $\mathcal{F}_{m-1}$ but not $\mathcal{F}_m$ (or FP_m). (See [25] and [4].)

Using covering spaces it is easy to show that $\mathcal{F}_m \Rightarrow \mathrm{FP}_m$; Hurewicz's Theorem shows that if a group G is $\mathcal{F}_2$ and FP_m, then G is $\mathcal{F}_m$; however, for any $m \geq 2$ Bestvina and Brady have shown that $\mathrm{FP}_m \not\Rightarrow \mathcal{F}_m$ [2]. Their work is a natural outgrowth of the work of Stallings and Bieri in that they discovered their groups by examining kernels of maps from 'graph groups' onto $\mathbb{Z}$.

Given a finite simplicial graph $\mathcal{G}$ the corresponding *graph group*, or 'right-angled Artin group', has generators corresponding to the vertices of $\mathcal{G}$, where two generators commute if and only if they are adjacent in $\mathcal{G}$. The class of graph groups includes all finite direct products of finitely generated free groups. The graph $\mathcal{G}$ is the *defining graph* and the corresponding graph group is denoted $G\mathcal{G}$. For example, if the defining graph $\mathcal{G}$ is the 1-skeleton of an octahedron, then the graph group $G\mathcal{G}$ is the direct product of three copies of F_2. In an abuse of terminology, we use v_i to denote both a vertex of the defining graph and a generator of the corresponding graph group. If Y is a subset of the vertex set of $\mathcal{G}$ and all $v_i, v_j \in Y$ are adjacent in $\mathcal{G}$, then Y is called a *(commuting) clique*.

Our main result is a complete description of the Σ-invariants of graph groups. These Σ-invariants are subsets of the real vector space $\mathrm{Hom}(G, \mathbb{R})$ and were introduced by Bieri, Neumann, Strebel, and Renz. Among other things, the Σ-invariants of a group G determine the FP and $\mathcal{F}$ properties of normal subgroups above the commutator of G. Necessary background on these invariants will be outlined in the next section.

Partial computations of the Σ-invariants of graph groups have already appeared. Let χ map a graph group $G\mathcal{G}$ to the reals. A vertex v is *living* if $\chi(v) \neq 0$ and otherwise it is *dead*. In [17] it was shown that the full subgraph generated by the living vertices, denoted $\mathcal{L}_\chi$, encodes whether or not a character is in $\Sigma^1(G\mathcal{G})$: A character χ represents a point in $\Sigma^1(G\mathcal{G})$ if and only if $\mathcal{L}_\chi$ is a connected and dominating subgraph of $\mathcal{G}$. Recall that a subgraph $\mathcal{L} \subseteq \mathcal{G}$ is *dominating* if each vertex $v \in \mathcal{G} - \mathcal{L}$ is adjacent to some vertex in $\mathcal{L}$. This result was extended by the second author to graph products of groups [20].

The second homotopy invariant $\Sigma^2(G\mathcal{G})$ was computed in [18]. Explicit presentations of kernels of maps $G\mathcal{G} \to \mathbb{Z}$ (in certain special cases) were given in [13]. Also, the structure of all the Σ-invariants was known in the case when $G\mathcal{G}$ is a direct product of finitely generated free groups [19]. The work in [18] leads naturally to the statement of our Main Theorem, and the arguments there have been directly extended by the first and third authors to establish this characterization using the 'Σ^n-criterion'. (See Appendix B of [7] or §4 of [6] for criteria establishing that a map represents a point in $\Sigma^n(G)$ or $\Sigma^n(G, \mathbb{Z})$.) Here we present a shorter proof using the concept of 'n-generating subgroup sets' introduced in [1].

The flag complex $\widehat{\mathcal{G}}$ induced by a simple graph $\mathcal{G}$ is the simplicial complex formed by filling in each complete subgraph of $\widehat{\mathcal{G}}$ by a simplex; its n-skeleton is denoted $\widehat{\mathcal{G}}^{(n)}$. The flag subcomplex induced by the living graph $\mathcal{L}_\chi$ is denoted $\widehat{\mathcal{L}}_\chi$.

The topology of $\widehat{\mathcal{L}}_\chi \subseteq \widehat{\mathcal{G}}$ determines whether or not $[\chi] \in \Sigma^n(G\mathcal{G})$, $\Sigma^n(G\mathcal{G}, \mathbb{Z})$, or $\Sigma^n(G\mathcal{G}, R)$.

Note: We will usually work over an arbitrary commutative ring R with $1 \neq 0$; the reader primarily interested in the FP properties of a group may freely substitute $\mathbb{Z}$ for R throughout this paper. Because we are working over R, it would be more accurate to refer to a complex as being "n-acylic over R" and a subcomplex as being "n-acyclic-dominating over R" in the definitions below. However, this terminology becomes quite awkward. Hence in this paper the term *acyclic* will always implicitly mean *"acyclic over R,"* where R is the commutative ring in which the reader is interested. For the cases in which we need to work with integral homology, we emphasize this by writing $\mathbb{Z}$-*acyclic.*

Similar to being n-connected, a complex is *n-acyclic* if its reduced homology groups (over R), up to and including dimension n, are trivial.

Definition. *A subcomplex L of a simplicial complex K is (-1)-acyclic-dominating if it is non-empty, or equivalently, (-1)-acyclic. For $n \geq 0$, L is an n-acyclic-dominating subcomplex of K, if for any vertex $v \in K - L$, the 'restricted link' $\mathrm{lk}_L(v) = \mathrm{lk}(v) \cap L$ is $(n-1)$-acyclic and an $(n-1)$-acyclic-dominating subcomplex of the 'entire link' $\mathrm{lk}(v)$ of v in K.*

When $L = \widehat{\mathcal{L}}_\chi \subseteq \widehat{\mathcal{G}} = K$ is the living subcomplex induced by a map $\chi : G\mathcal{G} \to \mathbb{R}$, $\mathrm{lk}_L(v)$ is refered to as the 'living link' of v, written as $\mathrm{lk}_{\mathcal{L}}(v)$.

Main Theorem. *Let $\mathcal{G}$ be a simplicial graph, let $\widehat{\mathcal{G}}$ be the induced flag complex based on $\mathcal{G}$, and let $\chi : G\mathcal{G} \to \mathbb{R}$ be a non-zero character. Then:*

- *i) $[\chi] \in \Sigma^n(G\mathcal{G})$ if and only if the subcomplex $\widehat{\mathcal{L}}_\chi$ of $\widehat{\mathcal{G}}$ is $(n-1)$-connected and $(n-1)$-$\mathbb{Z}$-acyclic-dominating.*
- *ii) $[\chi] \in \Sigma^n(G\mathcal{G}, R)$ if and only if the subcomplex $\widehat{\mathcal{L}}_\chi$ of $\widehat{\mathcal{G}}$ is $(n-1)$-acyclic and $(n-1)$-acyclic-dominating.*

By Theorem 2.1, we immediately have the following result.

Corollary A. *Let $\mathcal{G}$ be a simplicial graph, let $\widehat{\mathcal{G}}$ be the induced flag complex based on $\mathcal{G}$, and let $\chi : G\mathcal{G} \to \mathbb{Z}$ be a rational character of the graph group $G\mathcal{G}$. Then the kernel of χ is*

- *i) $\mathcal{F}_n$ if and only if $\widehat{\mathcal{L}}_\chi \subseteq \widehat{\mathcal{G}}$ is $(n-1)$-connected and $(n-1)$-$\mathbb{Z}$-acyclic-dominating;*
- *ii) $\mathrm{FP}_n(R)$ if and only if $\widehat{\mathcal{L}}_\chi \subseteq \widehat{\mathcal{G}}$ is $(n-1)$-acyclic and $(n-1)$-acyclic-dominating.*

Our Main Theorem was announced in [16], which contains additional examples beyond those presented here.

Throughout this paper, $V(K)$ will denote the set of vertices of the simplicial complex K, and, if $\mathcal{V} \subseteq V(K)$, then $K - \mathcal{V}$ is the subcomplex of K formed by removing the vertices in $\mathcal{V}$ as well as the open stars of these vertices.

Recall that graph theorists call a graph $\mathcal{G}$ 'm-connected' if $\mathcal{G} - \{v_1, \ldots, v_k\}$ is connected for any collection of $k < m$ vertices. We will be using a more general (and slightly altered) notion of connectivity in the context of simplicial complexes. A simplicial complex K is *m-n-connected* if for any k vertices $\{v_1, \ldots, v_k\}$ with $0 \leq k \leq m$ and $k < |V(K)|$, the complex $K - \{v_1, \ldots, v_k\}$ is n-connected. For example, the graph theorists' notion of being 'm-connected' is equivalent to our '$(m-1)$-0-connected' property. The *m-n-acyclic* property for simplicial complexes is defined in an analogous fashion. Because we allow the possibility that $k = 0$, m-n-connected [resp. m-n-acyclic] for any m implies n-connected [resp. n-acyclic].

Corollary B'. *A graph group $G\mathcal{G}$ has an abelian quotient of integral rank m with*

i) $\mathcal{F}_n$ kernel if and only if $\widehat{\mathcal{G}}$ is $(m-1)$-$(n-1)$-connected;

ii) $\mathrm{FP}_n(R)$ kernel if and only if $\widehat{\mathcal{G}}$ is $(m-1)$-$(n-1)$-acyclic.

The proof of this Corollary is exactly like the proof of Theorem 6.3 in [17]; we do not include the details here, since the result will follow from Corollary B below.

Let $\xi^m(G)$ denote the space of all normal subgroups N in G with G/N free abelian of integral rank m. Let $\xi^m_n(G)$ be the subspace of $\xi^m(G)$ where N is additionally required to be $\mathcal{F}_n$; similarly $\xi^m_n(G, R)$ is the subspace of $\xi^m(G)$ where N is additionally required to be $\mathrm{FP}_n(R)$.

Corollary B. *For any graph $\mathcal{G}$, and for any choice of m and n:*

i) The space $\xi^m_n(G\mathcal{G})$ is dense in $\xi^m(G\mathcal{G})$ if $\widehat{\mathcal{G}}$ is $(m-1)$-$(n-1)$-connected, and is empty otherwise;

ii) The space $\xi^m_n(G\mathcal{G}, R)$ is dense in $\xi^m(G\mathcal{G})$ if $\widehat{\mathcal{G}}$ is $(m-1)$-$(n-1)$-acyclic, and is empty otherwise.

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2. $\Sigma^n(G)$ and $\Sigma^n(G, R)$

The reader is directed to [5], [6] and [7] for background on the Σ-invariants; in this section we merely establish terminology and quote results relevant to our discussion.

The set of all *characters* of a group G is the complement of the zero map in the real vector space $\mathrm{Hom}(G, \mathbb{R})$. For any character χ let $[\chi] = \{r\chi \mid 0 < r \in \mathbb{R}\}$ be a ray in $\mathrm{Hom}(G, \mathbb{R}) - \{0\}$; the set of all such rays is denoted $S(G)$ and should be thought of as a 'sphere' inside the real vector space $\mathrm{Hom}(G, \mathbb{R})$. Since any character of a graph group $G\mathcal{G}$ must factor through the abelianization of $G\mathcal{G}$, $S(G\mathcal{G}) \simeq S^{|V(\mathcal{G})|-1}$.

Any character χ whose corresponding ray $[\chi]$ intersects an integral point of $\mathrm{Hom}(G,\mathbb{R})$ is a *rational* character. It is easy to check that the image of a rational character is an infinite cyclic group and that the set of rational characters $[\chi]$ is dense in $S(G)$.

Given a group G, there are two sequences of Bieri-Neumann-Strebel-Renz invariants: The homotopical invariants

$$S(G) = \Sigma^0(G) \supseteq \Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots,$$

and the homological invariants

$$S(G) = \Sigma^0(G,R) \supseteq \Sigma^1(G,R) \supseteq \Sigma^2(G,R) \supseteq \cdots.$$

The first invariants in these sequences, $\Sigma^1(G)$ and $\Sigma^1(G,R)$, are the same, and were introduced in a paper by Bieri, Neumann, and Strebel [5]. The higher invariants were introduced by Bieri and Renz [6] who noted that, just as any $\mathcal{F}_n$ group is FP_n, $\Sigma^n(G) \subseteq \Sigma^n(G,\mathbb{Z})$ for any n and G. However, the third Corollary to our Main Theorem indicates that $\Sigma^n(G\mathcal{G})$ can be empty while $\Sigma^n(G\mathcal{G},\mathbb{Z})$ is dense in $S(G\mathcal{G})$.

These Σ-invariants have fairly geometric descriptions which are quite concrete in the case of graph groups. Let $K\mathcal{G}$ denote the finite $K(G\mathcal{G},1)$ constructed in [17], and denote its universal cover by $\widetilde{K\mathcal{G}}$. Since the complex $K\mathcal{G}$ contains a single vertex, there is a one-to-one correspondence between vertices in $\widetilde{K\mathcal{G}}$ and elements in $G\mathcal{G}$. Thus corresponding to any character $\chi : G\mathcal{G} \to \mathbb{R}$ one can define a map $\widetilde{\chi} : \widetilde{K\mathcal{G}} \to \mathbb{R}$; the map $\widetilde{\chi}$ is defined on the vertices of $\widetilde{K\mathcal{G}}$ by $\widetilde{\chi}(v) = \chi(g)$ if $v = b \cdot g$ for some fixed base vertex b, and is extended linearly and $G\mathcal{G}$-equivariantly from the vertices to the entire universal cover. Let $\widetilde{K\mathcal{G}}_\chi^{[a,\infty)}$ denote the maximal subcomplex in $\widetilde{K\mathcal{G}} \cap \widetilde{\chi}^{-1}[a,\infty)$. For any non-negative constant d, the inclusion $\widetilde{K\mathcal{G}}_\chi^{[0,\infty)} \hookrightarrow \widetilde{K\mathcal{G}}_\chi^{[-d,\infty)}$ induces a map between reduced homology groups $\widetilde{H}_i(\widetilde{K\mathcal{G}}_\chi^{[0,\infty)},R) \to \widetilde{H}_i(\widetilde{K\mathcal{G}}_\chi^{[-d,\infty)},R)$ and a map between homotopy groups $\pi_i(\widetilde{K\mathcal{G}}_\chi^{[0,\infty)}) \to \pi_i(\widetilde{K\mathcal{G}}_\chi^{[-d,\infty)})$. A character χ represents a point in $\Sigma^n(G\mathcal{G},R)$ [resp. $\Sigma^n(G\mathcal{G})$] if and only if there exists a non-negative constant d such that the induced map on the reduced homology groups [resp. homotopy groups] is trivial for $i < n$. We remark that in the case of graph groups one can establish that a map is in $\Sigma^n(G\mathcal{G})$ or $\Sigma^n(G\mathcal{G},R)$ with $d = 0$. Since the details using this approach are daunting, we do not establish this condition directly.

The general definition of the homotopical invariant $\Sigma^n(G)$ is similar so we restrict ourselves to a sketch. As $\Sigma^n(G)$ is defined only for $\mathcal{F}_n$ groups, we let K denote a $K(G,1)$-complex with finite n-skeleton. We think of a character $\chi : G \to \mathbb{R}$ as an action of G on the real line, and consider a G-equivariant map $\widetilde{\chi} : \widetilde{K} \to \mathbb{R}$ on the universal cover (such a map always exists). Let $\widetilde{K}_\chi^{[a,\infty)}$ be the maximal subcomplex in $\widetilde{K} \cap \widetilde{\chi}^{-1}([a,\infty))$. Then $[\chi] \in \Sigma^n(G)$ if and only if there is a $d \geq 0$

such that the inclusion-induced maps $\pi_i(\widetilde{K}_\chi^{[0,\infty)}) \to \pi_i(\widetilde{K}_\chi^{[-d,\infty)})$ are trivial for $i < n$.

If G is $\mathcal{F}_n$, the homological invariant $\Sigma^n(G,R)$ can be defined as above, replacing homotopy by reduced homology. However, the definition can be given in a more general context, in the sense that one can associate to each RG-module M purely algebraic invariants $\Sigma_R^n(G,M)$. Given a character $\chi : G \to \mathbb{R}$, the submonoid $\chi^{-1}([0,\infty))$ will be denoted G_χ. Then $\Sigma_R^n(G,M)$ consists of all $[\chi] \in S(G)$ such that M is FP_n over the monoid ring RG_χ, i.e. admits a projective resolution over RG_χ with finitely generated n-skeleton.

When G is $\mathcal{F}_n$ and R is regarded as a trivial RG-module, then the algebraically defined invariant $\Sigma_R^n(G,R)$ coincides with the homological invariant $\Sigma^n(G,R)$ defined above. Therefore we always denote $\Sigma_R^n(G,R)$ by $\Sigma^n(G,R)$.

Given any normal subgroup N, with $A = G/N$ abelian, one can look at the subsphere

$$S(G,N) = \{[\chi \circ \phi] \mid \chi \in \mathrm{Hom}(A,\mathbb{R})\} = \{[\chi] \in S(G) \mid \chi(N) = 0\}$$

corresponding to N, in $S(G)$. One particularly convincing reason to study the Σ-invariants of a group is the following result due to Bieri and Renz, building from work of Bieri, Neumann and Strebel.

Theorem 2.1. *Let N be a normal subgroup of a group G with finitely generated abelian quotient. Then:*

i) N is $\mathcal{F}_n$ if and only if $S(G,N) \subseteq \Sigma^n(G)$;

ii) N is $\mathrm{FP}_n(R)$ if and only if $S(G,N) \subseteq \Sigma^n(G,R)$.

The following is immediate by comments in [17] and the theorem above; we note that such a result is relatively rare for arbitrary groups, and it follows in this case because graph groups have very symmetric presentations.

Corollary 2.1. *Let $G\mathcal{G}$ be a graph group and let $\chi : G\mathcal{G} \to \mathbb{Z}$. Then $[\chi] \in \Sigma^n(G\mathcal{G})$ [resp. $\Sigma^n(G\mathcal{G},R)$] if and only if the kernel of χ is $\mathcal{F}_n$ [resp. $\mathrm{FP}_n(R)$].*

We will also need the following results:

Theorem 2.2. ([5], [6]) *For any group G and any natural number n, $\Sigma^n(G)$, and $\Sigma^n(G,R)$ are open subsets of $S(G)$.*

Theorem 2.3. ([23]) *Let G be a group of type $\mathcal{F}_n$.*

i) If $n = 1$ then $\Sigma^1(G) = \Sigma^1(G,\mathbb{Z}) = \Sigma^1(G,R)$.

ii) If $n \geq 2$ then $\Sigma^n(G) = \Sigma^2(G) \cap \Sigma^n(G,\mathbb{Z})$.

The reader who is puzzled or curious about Σ-invariants should see [7] for a complete introduction; for more widely accessible sources, see [5] and [6].

3. Σ-invariants and generation by subgroups

In this section we prove a sufficient condition for a character of a group G to belong to $\Sigma^n(G)$ if G admits a 'nice' generating set of subgroups in the sense of Abels and Holz [1]. Recall that a set $\mathcal{H}$ of subgroups of a group G is *n-generating (for G)* if the nerve $N(\mathcal{H})$ of the covering of G by all (left) cosets in $\bigcup_{H\in\mathcal{H}} G/H$ is $(n-1)$-connected. If $N(\mathcal{H})$ is n-connected for all n, then $\mathcal{H}$ is *infinitely generating.*

Definition 3.1. *Given a set $\mathcal{H}$ of subgroups of a group G, let $C(\mathcal{H})$ be the simplicial complex with k-simplices the finite, non-empty flags $H_0 \subset H_1 \subset \cdots \subset H_k$ of non-trivial subgroups $H_i \in \mathcal{H}$.*

Example. Let $\mathcal{G}$ be a simplicial graph. Then the collection $\mathcal{A}$ of all free abelian subgroups of $G\mathcal{G}$ based on cliques in $\mathcal{G}$, together with the trivial subgroup $G\emptyset = \{1\}$, forms an infinitely generating set of subgroups. (An analogous statement is true in the more general context of graph *products* of groups [12].)

Moreover, there is a natural action of $G\mathcal{G}$ on a cubical complex where the stabilizers of faces of the fundamental domain correspond to the subgroups in $\mathcal{A}$. We quickly outline this construction; for details, see [14]. Let $D(\mathcal{A})$ be the simplicial complex with k-simplices the non-empty flags $G\Delta_0 \subset G\Delta_1 \subset \cdots \subset G\Delta_k$ where the Δ_i are cliques in $\mathcal{G}$ and Δ_0 *can* be empty. Notice that the complex $D(\mathcal{A})$ is the cone over $C(\mathcal{A})$, and that $C(\mathcal{A})$ is the barycentric subdivision of $\widehat{\mathcal{G}}$. If Δ is a non-empty clique in $\mathcal{G}$, then the barycentric subdivision of Δ naturally embeds in $C(\mathcal{A})$. Corresponding to the subgroup $G\Delta$ there is a 'panel' $P\Delta \subset C(\mathcal{A})$ consisting of the simplices in $C(\mathcal{A})$ whose vertices contain the barycenter of Δ and barycenters of simplices $\Delta' \supset \Delta$. (If $\Delta = \emptyset$ then $P\Delta = D(\mathcal{A})$.) On the other hand, if p is a point in $C(\mathcal{A})$, then there is a corresponding group $G_p = G\Delta$ where $p \in P\Delta$, and $G\Delta$ is maximal with respect to this property. (If $p \in D(\mathcal{A}) - C(\mathcal{A})$ then $G_p = \{1\}$.)

Let $X\mathcal{G}$ be the cubical complex formed by $D(\mathcal{A}) \times G\mathcal{G}$ where $(p, g) \sim (q, h)$ if and only if $p = q$ and $gh^{-1} \in G_p$. There is a natural action of $G\mathcal{G}$ on $X\mathcal{G}$ given by $g \cdot (p, h) = (p, gh)$. This complex can be given a piecewise Euclidean cubical metric structure making it a CAT(0) space. Since simply connected CAT(0) spaces are contractible, it follows that $\mathcal{A}$ is an infinitely generating subgroup set for $G\mathcal{G}$. Notice that the isotropy groups of cells in $X\mathcal{G}$ under the action of $G\mathcal{G}$ are the conjugates of the $G\Delta \subset \mathcal{A}$, and the fundamental domain for the $G\mathcal{G}$-action is $D(\mathcal{A}) \times \{1\}$.

Theorem 3.1. *Assume that $\mathcal{H}$ is a non-empty, finite, intersection-closed, and n-generating set of subgroups of a group G, and that $\chi : G \to \mathbb{R}$ is a non-zero homomorphism with $\chi|_H \neq 0$ for each non-trivial subgroup $H \in \mathcal{H}$.*

i) If either the trivial group does not belong to $\mathcal{H}$ or the simplicial complex $C(\mathcal{H})$ is $(n-1)$-connected, and if $[\chi|_H] \in \Sigma^n(H)$ for all non-trivial $H \in \mathcal{H}$, *then* $[\chi] \in \Sigma^n(G)$.

ii) If either the trivial group does not belong to $\mathcal{H}$ *or the simplicial complex* $C(\mathcal{H})$ *is* $(n-1)$*-acyclic, and if* $[\chi|_H] \in \Sigma^n(H, R)$ *for all non-trivial* $H \in \mathcal{H}$, *then* $[\chi] \in \Sigma^n(G, R)$.

This general result is our key technical tool for establishing that a given character is contained in a given Σ-invariant of a graph group $G\mathcal{G}$. Further applications (for example, to graph products of groups, or to Artin groups) seem possible, and we hope to come back to them later.

The proof of Theorem 3.1 is based upon the following result. The homological version of this Theorem was originally obtained by Schmitt [24]. Proofs can be found in [21] for the homological case, and [22] in the homotopical case.

Theorem 3.2. *Suppose a group* G *acts on a CW-complex* X *by cell-permuting homeomorphisms such that the* n*-skeleton of* X *is finite modulo the action of* G. *Let* $\chi : G \to \mathbb{R}$ *be a character whose restriction to the stabilizer* G_σ *of any* p*-cell* $\sigma \subset X$ *with* $p \leq n$ *is non-zero.*

i) If X *is* $(n-1)$*-connected and if* $[\chi|_{G_\sigma}] \in \Sigma^{n-p}(G_\sigma)$ *for each* p*-cell* σ *with* $p \leq n$ *then* $[\chi] \in \Sigma^n(G)$.

ii) If X *is* $(n-1)$*-acyclic, and if* $[\chi|_{G_\sigma}] \in \Sigma^{n-p}(G_\sigma, R)$ *for each* p*-cell* σ *with* $p \leq n$ *then* $[\chi] \in \Sigma^n(G, R)$.

Proof. (Theorem 3.1.) Let $F = F(\mathcal{H})$ be the flag complex associated with the covering of G by all cosets determined by $\mathcal{H}$. In other words, the k-simplices of F are the finite, non-empty flags $gH_0 \subset gH_1 \subset \cdots \subset gH_k$ with $H_i \in \mathcal{H}$ and $g \in G$. As $\mathcal{H}$ is intersection-closed and n-generating, it follows from [1] that F is $(n-1)$-connected.

By acting on the cosets, the group G also acts on F. The stabilizer of a simplex $gH_0 \subset gH_1 \subset \cdots \subset gH_k$ is gH_0g^{-1} which fixes this simplex pointwise. Moreover, the subcomplex $D = D(\mathcal{H})$ consisting of all flags of the form $H_0 \subset H_1 \subset \cdots \subset H_k$ is a *strong fundamental domain* for the action, in the sense that each simplex in F is equivalent modulo G to a unique simplex in D.

If $\mathcal{H}$ does not contain the trivial subgroup, then the stabilizers of the cells of F are conjugates of the non-trivial groups $H \in \mathcal{H}$. By hypothesis, $[\chi|_H] \in \Sigma^n(H) \subseteq \Sigma^{n-p}(H)$ [resp. $[\chi|_H] \in \Sigma^n(H, R) \subseteq \Sigma^{n-p}(H, R)$] hence the result follows from Theorem 3.2.

If $\{1\} \in \mathcal{H}$, then any cell of the form $\{1\} \subset gH_1 \subset \cdots \subset gH_k$ in F has a trivial isotropy group, hence it might be difficult to apply Theorem 3.2. We circumvent this difficulty by having G act on the subcomplex composed of cells with non-trivial stabilizers. In what follows we identify the vertex $g\{1\} \in F$ with the group element $g \in G$. We denote the full subcomplex of F whose simplices have non-trivial stabilizer by $E = E(\mathcal{H})$. Clearly, G acts on E with $C = C(\mathcal{H})$ as a strong fundamental domain. As in the example above, D is the cone of C, with cone point the vertex $1 \in D$. More generally, if $g \in G$ then the link $\mathrm{lk}(g)$ of the vertex $g \in F$ is gC and the closed star $\mathrm{st}(g)$ is the cone on gC with cone point g.

For a finite subset S of G, let $E(S) = E(\mathcal{H}, S)$ be the full subcomplex of F generated by E together with the vertices in S. We claim that

(a) if C is 1-connected then $\pi_1(E) \cong \pi_1(E(S))$, and
(b) if C is $(n-1)$-acyclic then $\widetilde{\mathrm{H}}_i(E, R) \cong \widetilde{\mathrm{H}}_i(E(S), R)$ for $i < n$.

To see this, let $g \in S$ and put $S' = S \setminus \{g\}$. Then

$$E(S) = E(S') \cup \mathrm{st}(g) \qquad \text{and} \qquad E(S') \cap \mathrm{st}(g) = \mathrm{lk}(g) = gC.$$

To establish (a) simply note that

$$\pi_1(E(S)) \cong \pi_1(E(S')) *_{\pi_1(gC)} \pi_1(\mathrm{st}(g))$$

by Van Kampen's Theorem. However, $\pi_1(gC) = \{1\} = \pi_1(\mathrm{st}(g))$; hence $\pi_1(E(S)) \cong \pi_1(E(S'))$ which by the induction hypothesis is isomorphic to $\pi_1(E)$. A similar argument using the Mayer-Vietoris sequence establishes (b).

The complex F is the union of the subcomplexes $E(S)$ as S ranges over the finite subsets of G. Using the fact that F is $(n-1)$-connected together with the claims above, we see that E is $(n-1)$-connected in situation (i) and $(n-1)$-acyclic in situation (ii). Using the action of G on E, Theorem 3.2 now gives the desired result even when $\{1\} \in \mathcal{H}$. □

4. A special case of the Main Theorem

Here we generalize the Bestvina-Brady result [2] from rational characters to all characters; that is, we prove:

Theorem 4.1. *Let $\mathcal{G}$ be a simplicial graph with induced flag complex $\widehat{\mathcal{G}}$, and let $\chi : G\mathcal{G} \to \mathbb{R}$ be a character such that $\chi(v) \neq 0$ for all vertices $v \in \mathcal{G}$. Then:*
i) $[\chi] \in \Sigma^n(G\mathcal{G})$ if and only if $\widehat{\mathcal{G}}$ is $(n-1)$-connected.
ii) $[\chi] \in \Sigma^n(G\mathcal{G}, R)$ if and only if $\widehat{\mathcal{G}}$ is $(n-1)$-acyclic.

Bestvina and Brady established finiteness properties of kernels of maps $\chi : G\mathcal{G} \to \mathbb{Z}$ where each generator is taken to $1 \in \mathbb{Z}$ [2]. Essentially the same proof works when $0 \neq \chi(v) \in \mathbb{Z}$ for each generator v. That is, Bestvina and Brady's result holds for any rational character χ with $\mathcal{L}_\chi = \mathcal{G}$. In particular:

Theorem 4.2. ([2]) *Assume, in addition to the assumptions above, that the character χ is rational.*
i) If the kernel of χ is $\mathcal{F}_n$ then $\widehat{\mathcal{G}}$ is $(n-1)$-connected.

ii) If the kernel of χ is $\mathrm{FP}_n(R)$ then $\widehat{\mathcal{G}}$ is $(n-1)$-acyclic.

Proof. (Theorem 4.1) First let $\widehat{\mathcal{G}}$ be $(n-1)$-connected [resp. $(n-1)$-acyclic]. Let $\mathcal{A}$ be the set of subgroups of $G\mathcal{G}$ consisting of all $G\Delta$, where Δ is a simplex in $\widehat{\mathcal{G}}$, and the trivial group. As was mentioned in the previous section, $\mathcal{A}$ is infinitely generating for $G\mathcal{G}$ [12].

As $G\Delta$ is a finitely generated free abelian group, we have

$$[\chi|_{G\Delta}] \in \Sigma^n(G\Delta) = \Sigma^n(G\Delta, R) = S(G\Delta)$$

for any n (see Theorem 2.1). Further, the simplicial complex $C(\mathcal{A})$ is isomorphic to the barycentric subdivision of $\widehat{\mathcal{G}}$. It follows from Theorem 3.1 that $[\chi] \in \Sigma^n(G\mathcal{G})$ [resp. $[\chi] \in \Sigma^n(G\mathcal{G}, R)$].

To prove the converse, note that the set of rational characters with $\chi(v) \neq 0$ for all vertices $v \in \mathcal{G}$ is dense in the set of all characters of $G\mathcal{G}$. Assuming that $\widehat{\mathcal{G}}$ is not $(n-1)$-connected [resp. not $(n-1)$-acyclic], the Bestvina-Brady result implies that the kernels of all such characters are not $\mathcal{F}_n$ [resp. $\mathrm{FP}_n(R)$], and hence by Corollary 2.1, all such characters are not in $\Sigma^n(G\mathcal{G})$ [resp. $\Sigma^n(G\mathcal{G}, R)$]. Theorem 2.2 implies that the complement of any given Σ-invariant is closed. The comments above indicate $\Sigma^n(G\mathcal{G})^c$ [resp. $\Sigma^n(G\mathcal{G}, R)^c$] contains a set of characters which is dense in $S(G\mathcal{G})$; hence $\Sigma^n(G\mathcal{G})^c = S(G\mathcal{G})$ [resp. $\Sigma^n(G\mathcal{G}, R)^c = S(G\mathcal{G})$] which implies $\Sigma^n(G\mathcal{G}) = \emptyset$ [resp. $\Sigma^n(G\mathcal{G}, R) = \emptyset$]. □

We highlight the fact established at the end of the proof:

Corollary 4.1. *If $\widehat{\mathcal{G}}$ is not $(n-1)$-connected, then $\Sigma^n(G\mathcal{G}) = \emptyset$; if $\widehat{\mathcal{G}}$ is $(n-1)$-connected, then $\Sigma^n(G\mathcal{G})$ is dense in $S(G\mathcal{G})$. Similarly, if $\widehat{\mathcal{G}}$ is not $(n-1)$-acyclic, then $\Sigma^n(G\mathcal{G}, R) = \emptyset$; if $\widehat{\mathcal{G}}$ is $(n-1)$-acyclic, then $\Sigma^n(G\mathcal{G}, R)$ is dense in $S(G\mathcal{G})$.*

This is a theme we will return to in §8.

5. The invariants of graph groups

Throughout this section we will discuss the links of vertices. If v is a vertex of a simplicial graph $\mathcal{G}$, we let $\mathcal{G}^v$ denote the 1-skeleton of the entire link $\mathrm{lk}(v)$ in $\widehat{\mathcal{G}}$. Thus $\mathcal{G}^v$ is a graph, and $\widehat{\mathcal{G}^v} \cong \mathrm{lk}(v) \subset \widehat{\mathcal{G}}$. As a first step in establishing our Main Theorem we prove:

Theorem 5.1. *Let $\mathcal{G}$ be a simplicial graph, let $\chi : G\mathcal{G} \to \mathbb{R}$ be a character, and let $\mathcal{L} = \mathcal{L}_\chi$ be the living subgraph. If $n \geq 1$, then $[\chi] \in \Sigma^n(G\mathcal{G})$ if and only if:*

i) $[\chi|_{G\mathcal{L}}] \in \Sigma^n(G\mathcal{L})$; and

ii) for each $v \in \mathcal{G} - \mathcal{L}$, $\chi|_{G\mathcal{G}^v} \neq 0$ and $[\chi|_{G\mathcal{G}^v}] \in \Sigma^{n-1}(G\mathcal{G}^v, \mathbb{Z})$.

Similarly, $[\chi] \in \Sigma^n(G\mathcal{G}, R)$ *if and only if:*
i) $[\chi|_{G\mathcal{L}}] \in \Sigma^n(G\mathcal{L}, R)$*; and*
ii) for each $v \in \mathcal{G} - \mathcal{L}$, $\chi|_{G\mathcal{G}^v} \neq 0$ *and* $[\chi|_{G\mathcal{G}^v}] \in \Sigma^{n-1}(G\mathcal{G}^v, R)$.

In the proof we'll need the following two theorems, the first of which follows from Theorem 3.2 and Theorem 9.1 using the action of an HNN-extension on its Bass-Serre tree. Its homological parts (ii) and (iii) are due to Schmitt [24].

Theorem 5.2. *Let* $G = \langle B, t \mid t^{-1}Ct = D\rangle$ *be an HNN-extension with base group* B*, stable letter* t*, and associated subgroups* $C \cong D$*. Suppose that* $\chi : G \to \mathbb{R}$ *is a character such that* $\chi|_C \neq 0$*, and that* $n \geq 1$*.*
i) If $[\chi|_B] \in \Sigma^n(B)$ *and if* $[\chi|_C] \in \Sigma^{n-1}(C)$ *then* $[\chi] \in \Sigma^n(G)$*.*
ii) If $[\chi|_B] \in \Sigma^n(B, R)$ *and if* $[\chi|_C] \in \Sigma^{n-1}(C, R)$ *then* $[\chi] \in \Sigma^n(G, R)$*.*
iii) If $[\chi] \in \Sigma^n(G, R)$ *and if* $[\chi|_B] \in \Sigma^{n-1}(B, R)$ *then* $[\chi|_C] \in \Sigma^{n-1}(C, R)$*.*

Theorem 5.3. (Meinert [21], [22]) *Assume that* $G = N \rtimes H$ *is a semi-direct product, and that* $\chi : G \to \mathbb{R}$ *is a character of* G *such that* $\chi(N) = \{0\}$*. Then:*
i) $[\chi] \in \Sigma^n(G)$ *implies* $[\chi|_H] \in \Sigma^n(H)$*, and*
ii) $[\chi] \in \Sigma^n(G, R)$ *implies* $[\chi|_H] \in \Sigma^n(H, R)$*.*

Proof. (Theorem 5.1) ($\Leftarrow$) We induct on the number of vertices in $\mathcal{G} - \mathcal{L}$. The base case, when $\mathcal{G} = \mathcal{L}$, was established in Theorem 4.1. For the inductive step, choose a vertex $w \in \mathcal{G} - \mathcal{L}$, and let $\mathcal{G}_w$ be the full subgraph on all vertices of $\mathcal{G}$ except w. For all vertices $v \in \mathcal{G}_w - \mathcal{L}$, the vertex sets of $\mathcal{G}^v$ and $\mathcal{G}^v_w$ are either equal or differ by w. As $w \notin \mathcal{L}$, the character χ vanishes on the kernel of the natural split projection $G\mathcal{G}^v \to G\mathcal{G}^v_w$, sending w to the identity (if $w \in \mathcal{G}^v$). By (ii), $[\chi|_{G\mathcal{G}^v}] \in \Sigma^{n-1}(G\mathcal{G}^v, \mathbb{Z})$ [or $\Sigma^{n-1}(G\mathcal{G}^v, R)$]. Applying Theorem 5.3 we find that $[\chi|_{G\mathcal{G}^v_w}] \in \Sigma^{n-1}(G\mathcal{G}^v_w, \mathbb{Z})$ [or $\Sigma^{n-1}(G\mathcal{G}^v_w, R)$] for all vertices $v \in \mathcal{G}_w - \mathcal{L}$. Because $\mathcal{L} = \mathcal{L}_\chi = \mathcal{L}_{\chi|_{G\mathcal{G}_w}} \subseteq \mathcal{G}_w \subset \mathcal{G}$, the induction hypothesis yields $[\chi|_{\mathcal{G}_w}] \in \Sigma^n(G\mathcal{G}_w)$ [or $\Sigma^n(G\mathcal{G}_w, R)$].

To complete the induction step, notice that $G\mathcal{G} = \langle G\mathcal{G}_w, w \mid w^{-1}G\mathcal{G}^w w = G\mathcal{G}^w\rangle$ is an HNN-extension with base $G\mathcal{G}_w$, stable element w, and associated subgroups $G\mathcal{G}^w = G\mathcal{G}^w$. When working with the homological invariants $\Sigma^n(G\mathcal{G}, R)$ the result follows from Theorem 5.2 (ii). To establish the result for the homotopical invariant $\Sigma^2(G\mathcal{G})$ note that $\Sigma^1(G\mathcal{G}) = \Sigma^1(G\mathcal{G}, \mathbb{Z})$, hence the result follows from Theorem 5.2 (i). Because $\Sigma^n(G\mathcal{G}) = \Sigma^2(G\mathcal{G}) \cap \Sigma^n(G\mathcal{G}, \mathbb{Z})$ by Theorem 2.3, in order for $[\chi]$ to be in $\Sigma^n(G\mathcal{G})$, it suffices that $[\chi|_{G\mathcal{L}}]$ belongs to $\Sigma^2(G\mathcal{L}) \cap \Sigma^n(G\mathcal{L}, \mathbb{Z}) = \Sigma^n(G\mathcal{L})$ and that for each $v \in \mathcal{G} - \mathcal{L}$,

$$[\chi|_{G\mathcal{G}^v}] \in \Sigma^1(G\mathcal{G}^v) \cap \Sigma^{n-1}(G\mathcal{G}^v, \mathbb{Z}) = \Sigma^{n-1}(G\mathcal{G}^v, \mathbb{Z}).$$

($\Rightarrow$) Notice that χ factors through the natural split projection $G\mathcal{G} \to G\mathcal{L}$, where all vertices in $\mathcal{G} - \mathcal{L}$ are sent to the identity. If $[\chi] \in \Sigma^n(G\mathcal{G})$ then by Theorem 5.3, $[\chi|_{G\mathcal{L}}] \in \Sigma^n(G\mathcal{L})$. It therefore suffices to show that $[\chi] \in \Sigma^n(G\mathcal{G})$ implies $[\chi|_{G\mathcal{G}^v}] \in$

$\Sigma^{n-1}(G\mathcal{G}^v, \mathbb{Z})$ for all vertices $v \in \mathcal{G} - \mathcal{L}$. Similarly, if $[\chi] \in \Sigma^n(G\mathcal{G}, R)$ then $[\chi|_{G\mathcal{L}}] \in \Sigma^n(G\mathcal{L}, R)$, so to establish the homological case it suffices to show that $[\chi] \in \Sigma^n(G\mathcal{G}, R)$ implies $[\chi|_{G\mathcal{G}^v}] \in \Sigma^{n-1}(G\mathcal{G}^v, R)$ for all vertices $v \in \mathcal{G} - \mathcal{L}$.

It follows from [17] or [20] that $\chi|_{G\mathcal{G}^v} \neq 0$ for all vertices $v \in \mathcal{G} - \mathcal{L}$. So let $v \in \mathcal{G} - \mathcal{L}$, and let $\mathcal{G}_v$ be the full subgraph on all vertices of $\mathcal{G}$ except v. We first establish the homological case. From Theorem 5.3 we infer that $[\chi|_{G\mathcal{G}_v}] \in \Sigma^n(G\mathcal{G}_v, R)$. Again, $G\mathcal{G} = \langle G\mathcal{G}_v, v \mid v^{-1}G\mathcal{G}^v v = G\mathcal{G}^v\rangle$ is an HNN-extension with base $G\mathcal{G}_v$, stable element v, and associated subgroups $G\mathcal{G}^v = G\mathcal{G}^v$. Theorem 5.2 (iii) gives the desired result: $[\chi|_{G\mathcal{G}^v}] \in \Sigma^{n-1}(G\mathcal{G}^v, R)$. On the other hand, assume $[\chi] \in \Sigma^2(G\mathcal{G})$. In this case, since $\Sigma^2(G\mathcal{G}) \subseteq \Sigma^2(G\mathcal{G}, \mathbb{Z})$, we see that Theorem 5.2 (iii) implies $[\chi|_{G\mathcal{G}^v}] \in \Sigma^1(G\mathcal{G}^v, \mathbb{Z})$. Intersecting this description of $\Sigma^2(G\mathcal{G})$ with the description of $\Sigma^n(G\mathcal{G}, \mathbb{Z})$ above completes the proof. $\square$

We restate our Main Theorem in a form that more closely follows our line of proof.

Main Theorem. *Let $\mathcal{G}$ be a simplicial graph, let $\widehat{\mathcal{G}}$ be the induced flag complex based on $\mathcal{G}$, and let $\chi : G\mathcal{G} \to \mathbb{R}$ be a character. If $n \geq 1$ then $[\chi] \in \Sigma^n(G\mathcal{G})$ if and only if:*

i) The subcomplex $\widehat{\mathcal{L}}_\chi$ is $(n-1)$-connected; and

ii) the subcomplex $\widehat{\mathcal{L}}_\chi$ is an $(n-1)$-$\mathbb{Z}$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$.

Similarly, $[\chi] \in \Sigma^n(G\mathcal{G}, R)$ if and only if:

i) The subcomplex $\widehat{\mathcal{L}}_\chi$ is $(n-1)$-acyclic; and

ii) the subcomplex $\widehat{\mathcal{L}}_\chi$ is an $(n-1)$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$.

Proof. We first give the proof in the homological case. Let $\mathcal{L}$ denote $\mathcal{L}_\chi$. Appealing to Theorem 4.1 and Theorem 5.1, we see that the assertion $[\chi] \in \Sigma^n(G\mathcal{G}, R)$ is equivalent to

i) the subcomplex $\widehat{\mathcal{L}}$ is $(n-1)$-acyclic, and

(ii') $\chi|_{G\mathcal{G}^v} \neq 0$, and $[\chi|_{G\mathcal{G}^v}] \in \Sigma^{n-1}(G\mathcal{G}^v, R)$ for all vertices $v \in \mathcal{G} - \mathcal{L}$.

It suffices therefore to show that condition (ii$'$) is equivalent to $\widehat{\mathcal{L}}$ being an $(n-1)$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$. We proceed by induction on n. If $n = 1$ then (ii$'$) is equivalent to the condition that $\chi|_{G\mathcal{G}^v} \neq 0$ for all vertices $v \in \mathcal{G} - \mathcal{L}$, which is the definition of $\mathcal{L}$ being a 0-acyclic-dominating subgraph of $\mathcal{G}$.

Assume our characterization of $\Sigma^*(-, R)$ for graph groups holds through dimension $n-1$ (where $n > 1$): for any character χ of any graph group $G\mathcal{G}$, $[\chi] \in \Sigma^{n-1}(G\mathcal{G}, R)$ if and only if (i) and (ii) hold. In particular, when working with $\Sigma^n(-, R)$, the induction hypothesis implies that condition (ii$'$) holds if and only if for each vertex $v \in \mathcal{G} - \mathcal{L}$, $\widehat{\mathcal{G}^v} \cap \widehat{\mathcal{L}} = \mathrm{lk}_{\mathcal{L}}(v)$ is $(n-2)$-acyclic and is $(n-2)$-acyclic-dominating as a subcomplex of $\widehat{\mathcal{G}^v}$. But this is the definition of $\widehat{\mathcal{L}}$ being an $(n-1)$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$. Thus the proof in the homological setting is complete.

In order to establish the homotopical portion of the Main Theorem we make use of Theorem 5.1 and Theorem 4.1 once again. It follows that $[\chi] \in \Sigma^n(G\mathcal{G})$ if and only if $\widehat{\mathcal{L}}$ is $(n-1)$-connected and for all vertices $v \in \mathcal{G} - \mathcal{L}$, $\chi|_{G\mathcal{G}^v} \neq 0$ and $[\chi|_{G\mathcal{G}^v}] \in \Sigma^{n-1}(G\mathcal{G}^v, \mathbb{Z})$. But the equivalence of this second condition with condition (ii) of the theorem was already established in the homological case. □

6. Some examples

In this section we use our Main Theorem to construct counterexamples to one conjecture, and to establish another. First we construct the counterexamples. If G_1 and G_2 are both of type $\mathcal{F}_n$, then the following formula was conjectured in [19]:

$$\Sigma^n(G_1 \times G_2)^c = \bigcup_{p+q=n} (\pi_1^*\Sigma^p(G_1)^c + \pi_2^*\Sigma^q(G_2)^c)$$

This means that a map $\chi \in \mathrm{Hom}(G_1 \times G_2, \mathbb{R})$ is not in $\Sigma^n(G_1 \times G_2)$ if and only if the restriction of χ to G_1 is not in $\Sigma^p(G_1)$ and the restriction of χ to G_2 is not in $\Sigma^q(G_2)$ for some p and q with $p+q=n$. The conjecture has been proven when $n=1$ or 2 (see [7] and [11]) but it is false for $n>2$.

Let $\mathcal{G}$ be the 1-skeleton of an acyclic flag complex K, which has non-trivial fundamental group. It follows from the Main Theorem, or Corollary 4.1, that $\Sigma^2(G\mathcal{G}) = \emptyset$. Let $\mathcal{G}'$ be the 1-skeleton of the suspension of K; hence $G\mathcal{G}' = G\mathcal{G} \times F_2$. Since $\Sigma^1(F_2) = \emptyset$, if the above conjecture were true, $\Sigma^3(G\mathcal{G}')$ would be empty. However, since K is acyclic, its suspension is contractible, and therefore $\Sigma^n(G\mathcal{G}')$ is dense in $S(G\mathcal{G}')$ for all n.

We see no immediate counterexamples to the homological version of this conjecture, and we hope to pursue this question further.

On the other hand, our Main Theorem implies a conjecture stated in [17], which was previously proven by the second author in a private communication. In [10] Droms showed that chordal graph groups are coherent; this corollary is another example of the 'stability' of subgroups of chordal graph groups. Recall that a graph is *chordal* if every circuit of length greater than three has a chord (an edge connecting 2 nonadjacent vertices in the cycle).

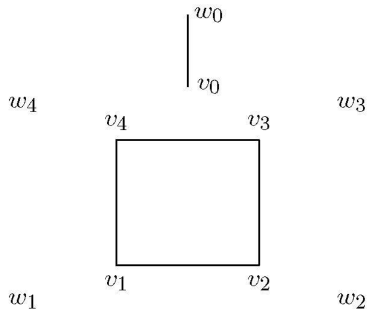
Corollary 6.1. *If $\mathcal{G}$ is a chordal graph, then $\Sigma^1(G\mathcal{G}) = \Sigma^k(G\mathcal{G}) = \Sigma^k(G\mathcal{G}, R)$ for all $k > 1$.*

Proof. We claim that if $\mathcal{L}$ is any connected and dominating full subgraph of a chordal graph $\mathcal{G}$ then, for each vertex $v \in \mathcal{G} - \mathcal{L}$, $\mathcal{L}^v = \mathcal{G}^v \cap \mathcal{L}$ is a connected and dominating subgraph of $\mathcal{G}^v$. Suppose to the contrary that $\mathcal{L}^v$ is not connected. Since $\mathcal{L}$ is connected, any two vertices of $\mathcal{L}^v$ which are not connected within $\mathcal{L}^v$ can be joined by a path in $\mathcal{L}$. Among all those pairs we choose one, say $w_1, w_2 \in \mathcal{L}^v$, where the joining path, say p, is of minimal length (which is at least 2). This implies that, except for its end points, p runs entirely in $\mathcal{L} - \mathcal{L}^v$. Hence the circuit of length ≥ 4, composed of the edge $\{v, w_1\}$, the path p, and the edge $\{w_2, v\}$, has no chord – a contradiction. Suppose now that $\mathcal{L}^v$ is not dominating in $\mathcal{G}^v$. Since $\mathcal{L}$ is dominating, any vertex $w \in \mathcal{G}^v - \mathcal{L}^v$ which is not adjacent to a vertex in $\mathcal{L}^v$ must be adjacent to some vertex $x \in \mathcal{L} - \mathcal{L}^v$. This latter vertex can be joined by a path p in $\mathcal{L}$ with some vertex $\overline{w} \in \mathcal{L}^v$. Among all possible choices of w, x, and p we choose one where the path p is of minimal length. As above we can construct a circuit of length greater then three which has no chord: it is the composition of the edge $\{v, w\}$, the edge $\{w, x\}$, the path p, and the edge $\{\overline{w}, v\}$.

A quick induction argument shows that if $\mathcal{G}$ is a connected chordal graph then $\widehat{\mathcal{G}}$ is contractible. The base case, where $\mathcal{G}$ has one vertex, is trivial. For the induction step, choose a vertex $v \in \mathcal{G}$ such that the dominating subgraph $\mathcal{G} - \{v\}$ is connected. The claim above implies that $\mathcal{G}^v$ is connected. Since $\mathcal{G}^v$ is a full subgraph of $\mathcal{G}$, it is also chordal. Thus the induction hypothesis yields that the link, $\mathrm{lk}(v) = \widehat{\mathcal{G}^v}$, of $v \in \mathcal{G}$ is contractible. By Van Kampen's Theorem and the Mayer-Vietoris Theorem, $\widehat{\mathcal{G}}$ is contractible.

From the claims above one then concludes that the flag complex $\widehat{\mathcal{L}}$ associated with a connected and dominating full subgraph $\mathcal{L}$ of a chordal graph $\mathcal{G}$ is contractible and k-$\mathbb{Z}$-acyclic-dominating for all k. (The proof is left to the reader.) Finally, because $[\chi] \in \Sigma^1(G\mathcal{G})$ if and only if $\widehat{\mathcal{L}}_\chi \subset \widehat{\mathcal{G}}$ is connected and dominating, the discussion above shows that when $[\chi] \in \Sigma^1(G\mathcal{G})$, then $\widehat{\mathcal{L}}_\chi$ is a contractible and k-$\mathbb{Z}$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$ for all k. By our Main Theorem, $[\chi] \in \Sigma^k(G\mathcal{G}) \subseteq \Sigma^k(G\mathcal{G}, R)$ for all $k > 1$. $\square$

The converse of Corollary 6.1 is not true. Consider the following graph $\mathcal{G}$:



Let $\mathcal{G}'$ be the full subgraph generated by the vertices $v_0, \ldots, v_4$. Then $[\chi] \in \Sigma^1(G\mathcal{G})$ if and only if $\mathcal{G}' \subseteq \mathcal{L}_\chi$. In this situation $\widehat{\mathcal{L}}_\chi$ is contractible, and the link of any dead vertex w_i is the corresponding living vertex v_i. By our Main Theorem $\Sigma^1(G\mathcal{G}) = \Sigma^k(G\mathcal{G}) = \Sigma^k(G\mathcal{G}, R)$ for all $k \geq 2$, but the graph $\mathcal{G}$ is not chordal.

7. The simplicial structure

It follows from our Main Theorem that $\Sigma^n(G\mathcal{G})^c$ and $\Sigma^n(G\mathcal{G}, R)^c$ are rational polyhedral for any n. As a matter of fact, these complements are simplicial complexes contained in the 'natural' simplicial decomposition of $S(G\mathcal{G})$. The n-sphere can be described as an iterated join of $(n+1)$ copies of the 0-sphere: the 1-sphere is $S^0 * S^0$, S^2 is then $S^0 * S^1 = S^0 * (S^0 * S^0)$ and so on. For a graph group, $S(G\mathcal{G}) \simeq S^{|V(\mathcal{G})|-1}$, so $S(G\mathcal{G})$ is the $|V(\mathcal{G})|$-fold join of copies of S^0. Each copy of S^0 in $S(G\mathcal{G})$ is simply two vertices in this simplicial decomposition; one of these vertices is associated with a generator $v \in \mathcal{G}$, and the other is associated to v^{-1}. Each simplex in this decomposition is determined by its vertices; hence it can be described by a list of generators, or their inverses, $\{v_1^{\pm 1}, \dots, v_k^{\pm 1}\}$, where v_i and v_i^{-1} are not both present. Thus a character $[\chi] \in S(G\mathcal{G})$ belongs to the *closed* simplex corresponding to $\{v_1^{\varepsilon_1}, \dots, v_k^{\varepsilon_k}\}$ if and only if $\chi(v_i^{\varepsilon_i}) \geq 0$ for $1 \leq i \leq k$ and χ is trivial on all other vertices of $\mathcal{G}$; a character $[\chi] \in S(G\mathcal{G})$ belongs to the *open* simplex corresponding to $\{v_1^{\varepsilon_1}, \dots, v_k^{\varepsilon_k}\}$ if and only if $\chi(v_i^{\varepsilon_i}) > 0$ for $1 \leq i \leq k$ and χ is trivial on all other vertices of $\mathcal{G}$.

The subcomplex $\widehat{\mathcal{L}}_\chi$ is determined by the living vertices in $\mathcal{G}$ corresponding to χ. All other characters χ' in the smallest open simplex of $S(G\mathcal{G})$ containing χ will induce exactly the same living subcomplex. By our Main Theorem, all characters in this open simplex are in $\Sigma^n(G\mathcal{G})$ [resp. $\Sigma^n(G\mathcal{G}, R)$] or they are all not in $\Sigma^n(G\mathcal{G})$ [resp. $\Sigma^n(G\mathcal{G}, R)$]. This discussion essentially establishes the following result.

Proposition 7.1. *An open simplex σ defined by $\{v_1^{\varepsilon_1}, \dots, v_k^{\varepsilon_k}\}$ is contained in $\Sigma^n(G\mathcal{G})$ if and only if the full subcomplex of $\widehat{\mathcal{G}}$ induced by the vertices $\{v_1, \dots, v_k\}$ is an $(n-1)$-connected and $(n-1)$-$\mathbb{Z}$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$. Similarly, σ is contained in $\Sigma^n(G\mathcal{G}, R)$ if and only if the full subcomplex of $\widehat{\mathcal{G}}$ induced by the vertices $\{v_1, \dots, v_k\}$ is an $(n-1)$-acyclic and $(n-1)$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$.*

The following corollary generalizes Proposition 6.4 of [17].

Corollary 7.1. *One can determine if the kernel of a map $\chi : G\mathcal{G} \to \mathbb{Z}^n$ is $\mathcal{F}_n$ or $\mathrm{FP}_n(R)$ by examining the induced maps $\phi \circ \chi : G\mathcal{G} \to \mathbb{Z}$ for finitely many homomorphisms $\phi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$.*

Proof. Recall that, by Theorem 2.1, we need to determine whether or not the $(n-1)$-sphere $S(G\mathcal{G}, \ker(\chi))$ is contained in $\Sigma^n(G\mathcal{G})$ or $\Sigma^n(G\mathcal{G}, R)$. However, by Proposition 7.1, a point $x \in S(G\mathcal{G})$ is contained in $\Sigma^n(G\mathcal{G})$ or $\Sigma^n(G\mathcal{G}, R)$ if and only if the minimal open simplex in $S(G\mathcal{G})$ containing x is contained in $\Sigma^n(G\mathcal{G})$ or $\Sigma^n(G\mathcal{G}, R)$. Thus one simply needs to check a single representative point — and hence a single $\phi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ — in each of the simplices of $S(G\mathcal{G})$ which $S(G\mathcal{G}, \ker(\chi))$ passes through. □

Examples indicating that one can indeed carry out useful computations along these lines are presented in [16].

Notice that a simplex $\sigma \sim \{v_1^{\varepsilon_1}, \dots, v_k^{\varepsilon_k}\}$ determines a great subsphere $\mathcal{S}(\sigma) \subseteq S(G\mathcal{G})$ which is the closure of all simplices corresponding to k-tuples $\{v_1^{\pm 1}, \dots, v_k^{\pm 1}\}$. By Theorem 2.2 the complements of the Σ-invariants are closed in $S(G\mathcal{G})$, so Proposition 7.1 yields the following corollary which will be used in the next section.

Corollary 7.2. *If an open simplex $\sigma \sim \{v_1^{\varepsilon_1}, \dots, v_k^{\varepsilon_k}\}$ is in $\Sigma^n(G\mathcal{G}, R)^c$ [resp. $\Sigma^n(G\mathcal{G})^c$], then $\mathcal{S}(\sigma) \subseteq \Sigma^n(G\mathcal{G}, R)^c$ [resp. $\mathcal{S}(\sigma) \subseteq \Sigma^n(G\mathcal{G})^c$].*

The main result of this section is the following computation of the dimensions of the simplicial complexes determined by the complements of the geometric invariants.

Theorem 7.1. *Let $G\mathcal{G}$ be a graph group based on a simplicial graph $\mathcal{G}$ and let $0 < d < |V(\mathcal{G})|$ be an integer. If the character sphere $S(G\mathcal{G})$ is equipped with its natural simplicial decomposition, then the dimension of the subcomplex $\Sigma^n(G\mathcal{G})^c$ [resp. $\Sigma^n(G\mathcal{G}, R)^c$] is less then d if and only if $\widehat{\mathcal{G}}$ is $(m-1)$-$(n-1)$-connected [resp. $(m-1)$-$(n-1)$-acyclic] where $m = |V(\mathcal{G})| - d$.*

Recall that a simplicial complex K is *m-n-connected* [resp. *m-n-acyclic*] if for any set $\mathcal{V}$ of k vertices of K, where $0 \leq k \leq m$ and $k < |V(K)|$, the full subcomplex $K - \mathcal{V}$ of K generated by all vertices in $V(K) - \mathcal{V}$, is n-connected [resp. n-acyclic].

Lemma 7.1. *If a simplicial complex K is m-n-acyclic then the entire link $\mathrm{lk}(v) \subset K$ of any vertex $v \in K$ is $(m-1)$-$(n-1)$-acyclic. In particular, if K is m-n-connected, then the entire link $\mathrm{lk}(v) \subset K$ of any vertex $v \in K$ is $(m-1)$-$(n-1)$-$\mathbb{Z}$-acyclic.*

Proof. If K is n-connected, then it is n-$\mathbb{Z}$-acyclic, so we only need to prove the assertion in the homological setting. Given a vertex v, let $\mathcal{V}$ be a set of $0 \leq k \leq m-1$ vertices of $\mathrm{lk}(v)$. Then the link of v with respect to the subcomplex $K - \mathcal{V}$ is $\mathrm{lk}_{K-\mathcal{V}}(v) = \mathrm{lk}(v) \cap (K - \mathcal{V}) = \mathrm{lk}(v) - \mathcal{V}$. Putting $\mathcal{V}^+ = \mathcal{V} \cup \{v\}$ yields

$$K - \mathcal{V} = (K - \mathcal{V}^+) \cup \mathrm{st}_{K-\mathcal{V}}(v) \quad \text{and} \quad (K - \mathcal{V}^+) \cap \mathrm{st}_{K-\mathcal{V}}(v) = \mathrm{lk}_{K-\mathcal{V}}(v),$$

where $\mathrm{st}_{K-\mathcal{V}}(v) = \mathrm{st}(v) \cap (K - \mathcal{V}) = \mathrm{st}(v) - \mathcal{V}$ is the closed star of v in the subcomplex $K - \mathcal{V}$. Since $\mathcal{V}$ and $\mathcal{V}^+$ both have less than or equal to m vertices, $K - \mathcal{V}$ and $K - \mathcal{V}^+$ are n-acyclic. Certainly $\mathrm{st}_{K-\mathcal{V}}(v)$ is n-acyclic, hence $\mathrm{lk}_{K-\mathcal{V}}(v) = \mathrm{lk}(v) - \mathcal{V}$ is $(n-1)$-acyclic by the Mayer-Vietoris Theorem. □

Proof. (Theorem 7.1) We only give the proof in the homological case; the homotopical case follows similarly.

If $\widehat{\mathcal{G}}$ is not $(m-1)$-$(n-1)$-acyclic then there is a set $\mathcal{V}$ of $k \leq m-1$ vertices where $\widehat{\mathcal{G}}-\mathcal{V}$ is not $(n-1)$-acyclic. One can easily construct a character $\chi : G\mathcal{G} \to \mathbb{R}$ whose living subcomplex $\widehat{\mathcal{L}_\chi}$ equals $\widehat{\mathcal{G}} - \mathcal{V}$. Since $m = |V(\mathcal{G})| - d$, χ has at least $d+1$ living vertices, and $[\chi] \in \Sigma^n(G\mathcal{G}, R)^c$ by our Main Theorem, so the d-simplex determined by the vertices in $\mathcal{V}$ belongs to $\Sigma^n(G\mathcal{G}, R)^c$.

The proof of the converse is by induction on the number of vertices of $\widehat{\mathcal{G}}$, the case of one vertex being trivial. In the induction step we'll show that any character $\chi : G\mathcal{G} \to \mathbb{R}$ with less then $|V(\mathcal{G})| - d$ dead vertices belongs to $\Sigma^n(G\mathcal{G}, R)$. Notice that $\widehat{\mathcal{L}_\chi}$ is $(n-1)$-acyclic because $\widehat{\mathcal{G}}$ is $(|V(\mathcal{G})| - d - 1)$-$(n-1)$-acyclic. Also notice that $\chi|_{G\mathcal{G}^v}$ is non-zero for each vertex $v \in \mathcal{G} - \mathcal{L}_\chi$ (as in §5, $\mathcal{G}^v$ denotes the 1-skeleton of the entire link $\mathrm{lk}(v) \subset \widehat{\mathcal{G}}$); for otherwise one would get a disconnected complex $\widehat{\mathcal{G}} - V(\mathcal{G}^v)$ by removing $|V(\mathcal{G})| - d - 1$ or fewer (dead) vertices.

By Lemma 7.1 we know that $\mathrm{lk}(v)$ is $(m-2)$-$(n-2)$-acyclic for any vertex $v \in \mathcal{G} - \mathcal{L}_\chi$. We then apply the induction hypothesis; if $m - 2 \geq |V(\mathcal{G}^v)| - 1$ then $\Sigma^{n-1}(G\mathcal{G}^v, R)^c$ is empty; if $m-1 < |V(\mathcal{G}^v)|$ then the dimension of $\Sigma^{n-1}(G\mathcal{G}^v, R)^c$ is less then $|V(\mathcal{G}^v)| - (m-1) = |V(\mathcal{G}^v)| - |V(\mathcal{G})| + d + 1$. Summarizing we find that any character of $G\mathcal{G}^v$ with less than $|V(\mathcal{G})| - d - 1$ dead vertices is in $\Sigma^{n-1}(G\mathcal{G}^v, R)$.

We finish the proof by an appeal to Theorem 5.1. From the discussion above we infer that $[\chi|_{G\mathcal{L}_\chi}] \in \Sigma^n(G\mathcal{L}_\chi, R)$ and that $[\chi|_{G\mathcal{G}^v}] \in \Sigma^{n-1}(G\mathcal{G}^v, R)$ for each vertex $v \in \mathcal{G} - \mathcal{L}_\chi$, hence $[\chi] \in \Sigma^n(G\mathcal{G}, R)$. □

8. The space of kernels

We now turn to the proof of Corollary B from the introduction.

Let $\xi^m(G)$ denote the space of all normal subgroups N in G with G/N free abelian of integral rank m. Let $\xi^m_n(G)$ be the subspace of $\xi^m(G)$ where N is additionally required to be $\mathcal{F}_n$; similarly $\xi^m_n(G, R)$ is the subspace of $\xi^m(G)$ where N is additionally required to be $\mathrm{FP}_n(R)$.

The space $\xi^m(G)$ is topologized as follows. Each normal subgroup N with $G/N \cong \mathbb{Z}^m$ induces a great $(m-1)$-dimensional subsphere $S(G, N)$ of the character sphere $S(G)$. Thus there is an actual distance between these normal subgroups given by taking the Hausdorff distance between these subspheres.

Corollary B. *For any graph $\mathcal{G}$, and for any choice of m and n:*

i) The space $\xi^m_n(G\mathcal{G})$ is dense in $\xi^m(G\mathcal{G})$ if $\widehat{\mathcal{G}}$ is $(m-1)$-$(n-1)$-connected, and is empty otherwise;

ii) The space $\xi^m_n(G\mathcal{G}, R)$ is dense in $\xi^m(G\mathcal{G})$ if $\widehat{\mathcal{G}}$ is $(m-1)$-$(n-1)$-acyclic, and is empty otherwise.

A normal subgroup $N \in \xi^m(G\mathcal{G})$ is in *general position* if for any character $[\chi] \in S(G\mathcal{G}, N)$, $|V(\mathcal{L}_\chi)| > |V(\mathcal{G})| - m$.

Lemma 8.1. *The set of all general position subgroups $N \in \xi^m(G\mathcal{G})$ is dense in $\xi^m(G\mathcal{G})$.*

Proof. Given the kernel $\widetilde{N}$ of any epimorphism $\phi : G\mathcal{G} \to \mathbb{Z}^m$ we will find an $N \in \xi^m(G\mathcal{G})$ which is arbitrarily close to $\widetilde{N} \in \xi^m(G\mathcal{G})$ and is in general position. In the following it will be convenient to endow the various $\mathbb{R}^*$'s with the 1-norm.

First, order the vertices in $\mathcal{G}$ such that $\{\phi(v_1), \dots, \phi(v_m)\}$ is a basis for $\mathbb{R}^m$, and define a homomorphism $\varphi : G\mathcal{G} \to \mathbb{R}^m$ as follows. Start by having $\varphi(v_i) = \phi(v_i)$ for $1 \le i \le m$. Because $\{\phi(v_1), \dots, \phi(v_m)\}$ is a basis for $\mathbb{R}^m$, each $\phi(v_k)$ can be expressed as a linear combination

$$\phi(v_k) = c_{k,1}\phi(v_1) + c_{k,2}\phi(v_2) + \cdots + c_{k,m}\phi(v_m)$$

for $k > m$. Note: Since the image of ϕ is $\mathbb{Z}^m$, each of these coefficients is rational. We define our associated map φ to be

$$\varphi(v_k) = (c_{k,1} + \varepsilon_{k,1})\phi(v_1) + (c_{k,2} + \varepsilon_{k,2})\phi(v_2) + \cdots + (c_{k,m} + \varepsilon_{k,m})\phi(v_m)$$

where the $\varepsilon_{k,i}$ are small, rational, and chosen so that $\varphi(v_k)$ is not a linear combination of any collection of $m-1$ vectors from $\{\varphi(v_1), \dots, \varphi(v_{k-1})\}$. This can be achieved because removing a finite number of $(m-1)$-dimensional linear subspaces from $\mathbb{R}^m$ leaves a dense subset of $\mathbb{R}^m$.

Let N be the kernel of φ. As φ agrees with ϕ on the first m generators, the image of φ, and hence $G\mathcal{G}/N$, is isomorphic to $\mathbb{Z}^m$.

Assume for the moment that there is a character $\chi : G\mathcal{G} \to \mathbb{R}$ that vanishes on N and has at least m dead vertices. Clearly, χ splits as $\psi \circ \varphi$ for some homomorphism $\psi : \mathbb{Z}^m \to \mathbb{R}$. As the image under φ of any m vertices forms a basis of $\mathbb{R}^m$, ψ must be trivial. This contradicts our assumptions that χ is non-zero; hence N is a general position subgroup.

Finally it remains to prove that N can be chosen arbitrarily close to $\widetilde{N}$ in $\xi^m(G\mathcal{G})$. We shall identify $S(G\mathcal{G})$ with the unit sphere in $\mathbb{R}^{|V(\mathcal{G})|}$, the identification being induced by mapping a homomorphism $\chi : G\mathcal{G} \to \mathbb{R}$ onto the vector $(\chi(v_1), \dots, \chi(v_{|V(\mathcal{G})|})) \in \mathbb{R}^{|V(\mathcal{G})|}$. Now note that each χ with $[\chi] \in S(G\mathcal{G}, N)$ [resp. $S(G\mathcal{G}, \widetilde{N})$] factors as $\psi \circ \varphi$ [resp. $\psi \circ \phi$] for some non-zero homomorphism $\psi : \mathbb{Z}^m \to \mathbb{R}$. Consequently, it suffices to show that for each $\psi \in \mathrm{Hom}(\mathbb{Z}^m, \mathbb{R})$ the points on the unit sphere of $\mathbb{R}^{|V(\mathcal{G})|}$ represented by the vectors $(\psi \circ \varphi(v_1), \dots, \psi \circ \varphi(v_{|V(\mathcal{G})|}))$ and $(\psi \circ \phi(v_1), \dots, \psi \circ \phi(v_{|V(\mathcal{G})|}))$ can be chosen to be arbitrarily close. Assuming these vectors are rescaled so that their associated points are actually on the unit sphere, we see that the Hausdorff distance between N and $\widetilde{N}$ is bounded by

$$\sum_{k=1}^{|V(\mathcal{G})|} |\psi \circ \varphi(v_k) - \psi \circ \phi(v_k)| \le \sum_{k=1}^{|V(\mathcal{G})|} \sum_{i=1}^{m} |\varepsilon_{k,i} \cdot \psi \circ \phi(v_i)| \le \sum_{k=1}^{|V(\mathcal{G})|} \sum_{i=1}^{m} |\varepsilon_{k,i}|.$$

But because the $\varepsilon_{k,i}$ can be chosen to be arbitrarily small, the normal subgroup N can be made arbitrarily close to $\widetilde{N}$ in $\xi^m(G\mathcal{G})$. □

Proof. (Corollary B) We only give the proof in the homotopic setting, the homological one is similar. If $\widehat{\mathcal{G}}$ is not $(m-1)$-$(n-1)$-connected then the dimension of $\Sigma^m(G\mathcal{G})^c$ is greater than or equal to $|V(\mathcal{G})|-m$ by Theorem 7.1. However, Corollary 7.2 shows that $\Sigma^m(G\mathcal{G})^c$ not only contains a simplex of dimension $|V(\mathcal{G})|-m$ but, in fact, a great subsphere of this dimension. Because $S(G\mathcal{G})$ is a sphere of dimension $|V(\mathcal{G})|-1$ we see that each great subsphere of dimension $m-1$ must intersect $\Sigma^n(G\mathcal{G})^c$. Thus $\xi^m_n(G\mathcal{G}) = \emptyset$ by Theorem 2.1.

If $\widehat{\mathcal{G}}$ is $(m-1)$-$(n-1)$-connected then the dimension of $\Sigma^n(G\mathcal{G})^c$ is less than $|V(\mathcal{G})|-m$. Consequently any general position subgroup N with $G/N \cong \mathbb{Z}^m$ must be of type $\mathcal{F}_n$. These normal subgroups form a dense subset of $\xi^m(G\mathcal{G})$. □

9. Appendix: On the invariants of groups acting on trees

This appendix contains work of Susanne Schmitt which was used in our proof of Theorem 5.1. As her thesis [24] is not widely accessible, we sketch the proof of her theorem. We have divided these results into several parts which seem to be of independent interest.

Throughout this appendix G will denote a group, M an RG-module, and $\chi : G \to \mathbb{R}$ a non-trivial homomorphism.

Theorem 9.1. (Schmitt [24]) *Suppose that G acts on a tree T such that T is finite modulo the action of G. Suppose further that the restriction $\chi_\sigma : G_\sigma \to \mathbb{R}$ of χ to the stabilizer G_σ of a vertex or an edge σ of T is non-zero.*

i) If $n \geq 1$, if $[\chi_v] \in \Sigma^n_R(G_v, M)$ for all vertices v of T, and if $[\chi_e] \in \Sigma^{n-1}_R(G_e, M)$ for all edges e of T, then $[\chi] \in \Sigma^n_R(G, M)$.

ii) If $n \geq 0$, if $[\chi] \in \Sigma^n_R(G, M)$, and if $[\chi_e] \in \Sigma^n_R(G_e, M)$ for all edges e of T, then $[\chi_v] \in \Sigma^n_R(G_v, M)$ for all vertices v of T.

iii) If $n \geq 1$, if $[\chi] \in \Sigma^n_R(G, M)$, and if $[\chi_v] \in \Sigma^{n-1}_R(G_v, M)$ for all vertices v of T, then $[\chi_e] \in \Sigma^{n-1}_R(G_e, M)$ for all edges e of T.

Remark 9.1. Replacing the short exact sequence of a tree with the cellular chain complex, the proof of (i) generalizes easily to CW-complexes of arbitrary dimension. This provides a proof of Theorem 3.2 (ii).

Lemma 9.1. *Suppose that $H \leq G$ is a subgroup, that N is an RH-module, and that χ restricts to a non-zero character, also denoted by χ, of H.*

i) The monoid ring RG_χ is flat as (left or right) RH_χ-module.

ii) The embeddings $G_\chi \hookrightarrow G$ and $H_\chi \hookrightarrow H$ induce an isomorphism $RG_\chi \otimes_{RH_\chi}$

$N \cong RG \otimes_{RH} N$ *of left* RG_χ*-modules.*

Proof. (i) We only prove that RG_χ is flat as left RH_χ-module, the right hand case follows similarly. Choose a transversal $\mathcal{T}$ for the cosets in $H\backslash G$. Then RG_χ is the direct sum of the RH_χ-submodules $R(Ht \cap G_\chi)$ with $t \in \mathcal{T}$, and it suffices to show that the latter are RH_χ-flat. As $\chi(Ht \cap G_\chi)$ is bounded from below, there is a sequence $(g_i)_{i \geq 1}$ of elements in $Ht \cap G_\chi$ such that $\chi(g_i) \geq \chi(g_{i+1})$ and $\chi(Ht \cap G_\chi)$ is contained in the union of the intervals $[\chi(g_i), \infty)$. Hence $H_\chi g_i \subseteq H_\chi g_{i+1}$, and the sequence $(H_\chi g_i)_{i \geq 1}$ exhausts $Ht \cap G_\chi$. Consequently $R(Ht \cap G_\chi)$ is the ascending union of the free RH_χ-submodules $R(H_\chi g_i)$. It follows that $R(Ht \cap G_\chi)$ is flat over RH_χ.

(ii) To verify this assertion one shows that the inverse φ of the obvious homomorphism $RG_\chi \otimes_{RH_\chi} N \longrightarrow RG \otimes_{RH} N$ can be defined as follows: Given $\lambda \in RG$ and $n \in N$, choose an element $h \in H_\chi$ such that $\lambda h \in RG_\chi$, and put $\varphi(\lambda \otimes n) = \lambda h \otimes h^{-1} n$. □

Theorem 9.2. *Let* $H \leq G$ *be a subgroup, and* N *an* RH*-module. If* $\chi|_H \neq 0$ *then* $[\chi|_H] \in \Sigma_R^n(H, N)$ *if and only if* $[\chi] \in \Sigma_R^n(G, RG \otimes_{RH} N)$.

Proof. By the lemma above, applying the functor $RG_\chi \otimes_{RH_\chi}$ to an RH_χ-free resolution of N with finitely generated n-skeleton produces an RG_χ-free resolution of $RG \otimes_{RH} N$ with finitely generated n-skeleton.

The proof of the converse relies on the Bieri-Eckmann criterion (see [3]): A Λ-module A is of type FP_n if and only if, for any index set $\mathcal{I}$, the natural map $\mathrm{Tor}_k^\Lambda(\prod_\mathcal{I} \Lambda, A) \to \prod_\mathcal{I} \mathrm{Tor}_k^\Lambda(\Lambda, A)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.

So for any index set $\mathcal{I}$ the natural map

$$\mathrm{Tor}_k^{RG_\chi}(\prod_\mathcal{I} RG_\chi, RG \otimes_{RH} N) \longrightarrow \prod_\mathcal{I} \mathrm{Tor}_k^{RG_\chi}(RG_\chi, RG \otimes_{RH} N)$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$. As RG_χ is flat over RH_χ, it follows that $\mathrm{Tor}_k^{RG_\chi}(B, RG \otimes_{RH} N) \cong \mathrm{Tor}_k^{RG_\chi}(B, RG_\chi \otimes_{RH_\chi} N)$ is isomorphic to $\mathrm{Tor}_k^{RH_\chi}(B, N)$ for each RG_χ-module B. Consequently the natural map

$$\mathrm{Tor}_k^{RH_\chi}(\prod_\mathcal{I} RG_\chi, N) \longrightarrow \prod_\mathcal{I} \mathrm{Tor}_k^{RH_\chi}(RG_\chi, N)$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$.

Since the embedding $\iota : RH_\chi \to RG_\chi$ of RH_χ-modules splits, with split projection $\varrho : RG_\chi \to RH_\chi$ say, we have a commutative diagram with monomorphisms

ι_*, $\iota_\sharp$ and epimorphisms ϱ_*, $\varrho_\sharp$:

$$\begin{array}{ccc} \mathrm{Tor}_k^{RH_\chi}(\prod_{\mathcal{I}} RH_\chi, N) & \longrightarrow & \prod_{\mathcal{I}} \mathrm{Tor}_k^{RH_\chi}(RH_\chi, N) \\ \big\downarrow \iota_* \big\uparrow \varrho_* & & \big\downarrow \iota_\sharp \big\uparrow \varrho_\sharp \\ \mathrm{Tor}_k^{RH_\chi}(\prod_{\mathcal{I}} RG_\chi, N) & \longrightarrow & \prod_{\mathcal{I}} \mathrm{Tor}_k^{RH_\chi}(RG_\chi, N). \end{array}$$

From the criterion above one now concludes that N is of type FP_n over RH_χ. □

The following finite index result was first obtained by Bieri and Strebel (1987, unpublished). The presentation here is taken from [24].

Theorem 9.3. *Suppose that $H \leq G$ is a subgroup of finite index, and that χ restricts to a non-zero homomorphism of H. Then $[\chi|_H] \in \Sigma_R^n(H, M)$ if and only if $[\chi] \in \Sigma_R^n(G, M)$.*

Proof. All we have to show is that M is of type FP_n over RG_χ if and only if the module $R(G/H) \otimes_R M$, with diagonal RG_χ-action, is FP_n. This follows from the theorem above together with the fact that $RG \otimes_{RH} M \cong R(G/H) \otimes_R M$ where RG acts on the left hand side of the first module and diagonally on the second.

So assume first that M admits a free resolution $\mathbf{F} \twoheadrightarrow M$ over RG_χ with finitely generated n-skeleton. Tensoring yields an exact chain complex

$$R(G/H) \otimes_R \mathbf{F} \longrightarrow R(G/H) \otimes_R M \longrightarrow 0$$

where RG_χ acts diagonally. Hence it suffices to show that $R(G/H) \otimes_R \mathbf{F}$ has finitely generated n-skeleton over RG_χ. As H has finite index in G, this follows from the observation that $R(G/H) \otimes_R RG_\chi$ with diagonal RG_χ-action is isomorphic to $R(G/H) \otimes_R RG_\chi$ with single RG_χ-action on the right hand side: the isomorphism can be defined by $\lambda \otimes g \mapsto g^{-1}\lambda \otimes g$ for $\lambda \in R(G/H)$ and $g \in G_\chi$.

To prove the converse, we may assume that H is normal in G since every subgroup of finite index, in fact, contains a normal one of finite index. Then we take a free resolution $\mathbf{F}$, with finitely generated modules in all dimensions, of the trivial $R(G/H)$-module R and tensor it, over R, with the RG-module M. As $\mathbf{F}$ splits over R this produces an RG-resolution $\mathbf{E} \twoheadrightarrow M$, the modules E_i being finite direct sums of $R(G/H) \otimes_R M$ with diagonal RG-action. Our assumption now implies that there is an RG_χ-free resolution with finitely generated n-skeleton for each of the modules E_i (see Lemma 9.2 (ii) below). Finally the mapping cone construction yields an RG_χ-free resolution of M with finitely generated n-skeleton. □

Beside these results all we need to prove Theorem 9.1 is the following general observation (see Proposition 1.4 of [3]):

Lemma 9.2. *Suppose that $0 \to M'' \to M' \to M \to 0$ is a short exact sequence of RG-modules.*

i) If $n \geq 1$, if $[\chi] \in \Sigma_R^n(G,M')$, and if $[\chi] \in \Sigma_R^{n-1}(G,M'')$, then $[\chi] \in \Sigma_R^n(G,M)$.

ii) If $n \geq 0$, if $[\chi] \in \Sigma_R^n(G,M)$, and if $[\chi] \in \Sigma_R^n(G,M'')$, then $[\chi] \in \Sigma_R^n(G,M')$.

iii) If $n \geq 1$, if $[\chi] \in \Sigma_R^n(G,M)$, and if $[\chi] \in \Sigma_R^{n-1}(G,M')$, then $[\chi] \in \Sigma_R^{n-1}(G,M'')$.

Proof. (Theorem 9.1) The cellular chain complex $\mathbf{C}(T,R)$ of the tree T with R-coefficients gives an RG-resolution of the trivial module R. A careful study shows that after tensoring this free R-complex with M over R we obtain a short exact sequence

$$0 \longrightarrow \oplus_{e\in\mathcal{E}}(RG \otimes_{RG_e} M_e) \longrightarrow \oplus_{v\in\mathcal{V}}(RG \otimes_{RG_v} M_v) \longrightarrow M \longrightarrow 0$$

of RG-modules (see, e.g., [8]). Here $\mathcal{E}$ is a finite set of representatives for the edges of T, and $\mathcal{V}$ is a finite G-transversal for the vertices. Moreover, for $e \in \mathcal{E}$, M_e is the RG_e-module M with RG_e-action twisted by a homomorphism $\tau_e : G_e \to \{\pm 1\}$ which takes into account whether an element preserves the orientation of the edge e or not. The modules M_v are defined similarly.

We only prove (i); the assertions (ii) and (iii) follow similarly. Given a vertex or an edge $\sigma \in \mathcal{V} \cup \mathcal{E}$, there is a subgroup of finite index in G_σ which maps onto the identity under τ_σ. Using the above theorem, our assumptions now imply that $[\chi_v] \in \Sigma_R^n(G_v, M_v)$ for all $v \in \mathcal{V}$ and $[\chi_e] \in \Sigma_R^{n-1}(G_e, M_e)$ for all $e \in \mathcal{E}$. Referring to Theorem 9.2 along with Lemma 9.2 we see that $[\chi] \in \Sigma_R^n(G,M)$. $\square$

Note added in proof: Bux and Gonzalez have recently written an alternate proof of our Main Theorem, more along the lines of the Bestvina-Brady Morse theory approach [9].

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