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Isotopy and invariants of Albert algebras

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Abstract. Let k be a field with characteristic different from 2 and 3. Let B be a central simple algebra of degree 3 over a quadratic extension K/k, which admits involutions of second kind. In this paper, we prove that if the Albert algebras $J(B, \sigma, u, \mu)$ and $J(B, \tau, v, \nu)$ have same f_3 and g_3 invariants, then they are isotopic. We prove that for a given Albert algebra J, there exists an Albert algebra J' with $f_3(J') = 0$, $f_5(J') = 0$ and $g_3(J') = g_3(J)$. We conclude with a construction of Albert division algebras, which are pure second Tits' constructions.

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Introduction

Let k be a field with characteristic different from 2 and 3. The exceptional central simple Jordan algebras over k are now called *Albert algebras.* There are rational constructions of Albert algebras due to Tits. These are referred to as the *first Tits' construction* and the *second Tits' construction.* In the first construction, one associates to a pair (A, μ), where A is a degree 3 central simple algebra over k and $\mu \in k^*$, an Albert algebra $J(A, \mu)$ over k. For the second construction, one starts with a quadratic extension K/k and a degree 3 central simple algebra B over K with an involution σ of second kind. To any unit $u \in B$ with $\sigma(u) = u$ and $N(u) = \mu\overline{\mu}$ for some $\mu \in K$, one associates an Albert algebra $J(B, \sigma, u, \mu)$ over k (cf. [P-R 1]). There are cohomological invariants attached to these algebras. Let J be an Albert algebra over k. One assigns two mod 2 invariants to J, $f_3(J) \in H^3(k, \mathbb{Z}/2)$, $f_5(J) \in H^5(k, \mathbb{Z}/2)$ and a mod 3 invariant $g_3(J) \in H^3(k, \mathbb{Z}/3)$ (cf. [P-R 1]). Serre asked whether these invariants determine the isomorphism class of J. The question is known to have affirmative answer for the *reduced* Albert algebras (cf. [P-R 1]). It was proved in ([P-S-T]) that if the Albert algebras $J(B, \sigma, u, \mu)$ and $J(B, \sigma, u', \mu')$ have the same invariants f_3 and g_3, then they are isomorphic. In this direction, Petersson and Racine had asked a weaker question ([P-R 1]), namely, if two Albert algebras have same f_3 and g_3, are they isotopic? In this paper, we answer this question for the Albert algebras $J(B, \sigma, u, \mu)$ and

$J(B, \tau, v, \nu)$, in the affirmative (§2). The same authors had asked in another paper ([P-R 2]), whether for a given Albert algebra J, there exists an Albert algebra J' with $f_3(J') = 0$, $f_5(J') = 0$ and $g_3(J') = g_3(J)$. We answer this question in the affirmative in (§3). We end with a construction of pure second Tits' construction Albert division algebras over a field k (§4).

1. Preliminaries

1.1. Isotopes of Jordan algebras

Let J be a Jordan algebra over k with 1. Let $a \in J$ be an invertible element. One defines a new multiplication on J by

$$x_a y = \{xay\},$$

where $\{xyz\}$ is the Jordan triple product in J, given by

$$\{xyz\} = U_{x,z}(y),$$

$$U_{x,z} = R_x R_z + R_z R_x - R_{xz},$$

R_x denoting the homothety on J given by x. The algebra J, with this new multiplication, is a Jordan algebra (cf. [J]), called the $a-$isotope of J. It is denoted by $J^{(a)}$. Two Jordan algebras J_1 and J_2 are *isotopic* if $J_1^{(a)}$ is isomorphic to J_2 for some invertible $a \in J_1$. Isotopy is an equivalence relation on the class of Jordan algebras (cf. [J]).

1.2. Constructions of Albert algebras

In the following, we give a brief review of the Tits' constructions and the Freudenthal's construction of Albert algebras.

Tits' first construction: Let A be a central simple k-algebra of degree 3. Let $\mu \in k^*$. On the k-vector space

$$J(A, \mu) = A_0 \oplus A_1 \oplus A_2, \text{ where } A_i = A \text{ for } i = 0,\ 1,\ 2,$$

we define a multiplication by

$$(a_0, a_1, a_2)(a_0', a_1', a_2')$$
$$= (a_0.a_0' + \widetilde{a_1 a_2'} + \widetilde{a_1' a_2},\ \widetilde{a_0} a_1' + \widetilde{a_0'} a_1 + \mu^{-1} a_2 \times a_2',\ a_2 \widetilde{a_0'} + a_2' \widetilde{a_0} + \mu a_1 \times a_1').$$

Here, for $a, b \in A$,

$$a.b = \frac{1}{2}(ab + ba),\ a \times b = a.b - \frac{1}{2}t(a)b - \frac{1}{2}t(b)a + \frac{1}{2}(t(a)t(b) - t(a.b))$$

and $\widetilde{a} = \frac{1}{2}(t(a) - a)$, t being the reduced trace on A. It is known that $J(A, \mu)$ is an Albert algebra. Further, it is a division algebra if and only if A is a division algebra and μ is not a reduced norm from A.

Tits' second construction: Let K be a quadratic extension of k and let (B, σ) be a central simple K-algebra of degree 3 with an involution σ of the second kind over k. Let $u \in B^*$ be such that $\sigma(u) = u$ and $N(u) = \mu\overline{\mu}$ for some $\mu \in K^*$, bar denoting the nontrivial k-automorphism of K. Let $(B, \sigma)_+$ denote the k-vector space of σ-symmetric elements in B. Let $J(B, \sigma, u, \mu) = (B, \sigma)_+ \oplus B$. We define a multiplication on $J(B, \sigma, u, \mu)$ by

$$(b_0, b)(b_0', b') = (b_0.b_0' + \widetilde{bu\sigma(b')} + \widetilde{b'u\sigma(b)},\ \widetilde{b_0}b' + \widetilde{b_0'}b + \overline{\mu}(\sigma(b) \times \sigma(b'))u^{-1}).$$

Then $J(B, \sigma, u, \mu)$ is known to be an Albert algebra and

$$J(B, \sigma, u, \mu) \otimes_k K \simeq J(B, \mu)$$

over K (cf. [MC]). Further, $J(B, \sigma, u, \mu)$ is a division algebra if and only if B is a division algebra and μ is not a reduced norm from B.

Freudenthal's construction: Let C be a Cayley algebra over k and let $\Gamma = <\gamma_1, \gamma_2, \gamma_3>$ be a diagonal invertible matrix with $\gamma_i \in k$. Let $M_3(C)$ denote the algebra of 3×3 matrices with entries in C. The map $X \mapsto \Gamma^{-1}\overline{X}^t\Gamma$ stabilizes $M_3(C)$, where $\overline{X}$ is the matrix obtained by applying the involution bar on C to the entries of X. Let

$$\mathcal{H}_3(C, \Gamma) = \{X \in M_3(C) | \Gamma^{-1}\overline{X}^t\Gamma = X\}.$$

This is closed under the multiplication $X.Y = \frac{1}{2}(XY + YX)$ and is known to be an Albert algebra (cf. [J]). These are the so called *reduced* Albert algebras.

1.3. Cohomological invariants of Albert algebras

Let k be as before. Let J be an Albert algebra over k. It is a fact that J carries a linear trace form T defined on it (cf. [J]) and this gives rise to a quadratic form Q_J on J given by

$$Q_J(x) = \frac{1}{2}T(x^2).$$

There exists a 3-fold Pfister form ϕ_3 and a 5-fold Pfister form ϕ_5 over k such that

$$Q_J \perp \phi_3 \simeq < 2, 2, 2 > \perp \phi_5$$

over k (cf. [S]). Further, this property characterizes ϕ_3 and ϕ_5 upto isometry. For an n-fold Pfister form $\phi_n = << a_1, a_2, \cdots, a_n >> = < 1, -a_1 > \otimes \cdots \otimes < 1, -a_n >$, one has the Arason invariant $A(\phi_n) \in H^n(k, \mathbb{Z}/2)$ given by

$$A(\phi_n) = (-a_1) \cup (-a_2) \cdots \cup (-a_n),$$

where, for $a \in k^*$, (a) denotes the class of a in $H^1(k, \mathbb{Z}/2)$. The mod 2 invariants for J are defined as

$$f_3(J) = A(\phi_3), \ f_5(J) = A(\phi_5).$$

If $J = \mathcal{H}_3(C, \Gamma)$ then $f_3(J) = A(n_C)$ and $f_5(J) = A(< 1, \gamma_1^{-1}\gamma_2 > \otimes < 1, \gamma_2^{-1}\gamma_3 > \otimes n_C)$, where n_C is the norm on the Cayley algebra C, which is known to be a 3-fold Pfister form. Rost ([R]) attached an invariant mod 3 to J, denoted by $g_3(J)$, which is defined as follows. If $J = J(B, \sigma, u, \mu)$ for some central simple algebra B of degree 3 over a quadratic filed extension K of k, with an involution of second kind, then define

$$g_3(J) = -Cor_{K/k}([B] \cup [\mu]) \in H^3(k, \mathbb{Z}/3),$$

and if $J = J(A, \nu)$ for a central simple algebra A of degree 3 over k, then define

$$g_3(J) = ([A] \cup [\nu]) \in H^3(k, \mathbb{Z}/3).$$

These are independent of the expression of J as a first or a second Tits' construction (cf. [R], [P-R 3]). Rost showed ([R]) that J is a division algebra if and only if $g_3(J) \neq 0$. Further, g_3 is compatible with base change.

2. Classification of Albert algebras upto isotopy

The question, whether the invariants f_3 and g_3 classify a given Albert algebra upto isotopy, itself is an important question. We answer this question in a particular case of second Tits' construction. Namely,

Theorem 2.1. *Let K be a quadratic extension of k and let B denote a central simple algebra of degree 3 over K, which admits involutions of second kind over k. Let $J = J(B, \sigma, u, \mu)$ and $J' = J(B, \sigma', u', \mu')$ be second Tits' construction Albert algebras. Assume that $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$. Then J and J' are isotopic.*

We need the following result. We supply a proof for the sake of completeness.

Theorem 2.2. ([P-R 1]). *Let J_1 and J_2 be two Albert algebras over k which are isotopic. Then $f_3(J_1) = f_3(J_2)$ and $g_3(J_1) = g_3(J_2)$.*

Proof. Since isotopic first Tits' constructions Albert algebras are isomorphic ([P-R 4], 4.9), we may assume that J_1 and J_2 are both second Tits' constructions. Let $J_1 = J(B, \sigma, u, \mu)$ and let K be the centre of B. There is a cubic extension L of k such that $J_1 \otimes_k L$ is reduced. Since J_1 is isotopic to J_2, for some invertible $v \in J_1$ we have,

$$J_1^{(v)} \simeq J_2.$$

Thus

$$(J_1 \otimes_k K)^{(v)} \simeq J_2 \otimes_k K.$$

But $J_1 \otimes_k K \simeq J(B, \mu)$ over K (cf. [MC]), and using the fact that isotopic first Tits' Albert algebras are isomorphic ([P-R 4], 4.9), we get

$$J_1 \otimes_k K \simeq J_2 \otimes_k K$$

over K. This proves $g_3(J_1) = g_3(J_2)$.

To compare the f_3 invariant, we appeal to the following

Theorem 2.3. (cf. [F], 1.9) *Let J be an Albert algebra isotopic to $\mathcal{H}_3(C, \Gamma)$, C a Cayley algebra over k and $\Gamma \in GL_3(k)$ a diagonal matrix. Then there is an isomorphism of J onto $\mathcal{H}_3(C, \Gamma')$ for some $\Gamma' \in GL_3(k)$, a diagonal matrix.*

Now we base change to L to reduce J_1, so that over L we have,

$$(J_1 \otimes_k L)^{(v)} \simeq J_2 \otimes_k L.$$

Let $J_1 \otimes_k L \simeq \mathcal{H}_3(C, \Gamma)$ for some Cayley algebra C over L and Γ defined over L. By the above theorem, $J_2 \otimes_k L \simeq \mathcal{H}_3(C, \Gamma')$ for some Γ'. By a theorem of Serre and Rost (cf. [P-R 5], 1.8), there exits a Cayley algebra $\mathcal{O}$ over k such that $\mathcal{O} \otimes_k L \simeq C$. Now, by the definition of f_3, it follows that $f_3(J_1) = f_3(J_2)$.

For the proof of Theorem 2.1, we need the following

Theorem 2.4. ([P-R 5], 1.5). *Let (B, σ) be as above. Let $v \in B^*$ be such that $\sigma(v) = v$. Let $\sigma' = Int(v)\sigma$. Then the map*

$$(a_o, a) \mapsto (v^{-1}a_0, a)$$

is an isomprphism from $J(B, \sigma', uv^{\#}, N(v)\mu)$ onto $J(B, \sigma, u, \mu)^{(v)}$.

Theorem 2.5. ([P-S-T], 2.8) *Let $J = J(B, \sigma, u, \mu)$ and $J' = J(B, \sigma, u', \mu')$ be Albert algebras arising from Tits' second construction. Assume $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$. Then J is isomorphic to J'.*

Proof of Theorem 2.1. We have $J = J(B, \sigma, u, \mu)$ and $J' = J(B, \sigma', u', \mu')$. Let $v \in (B, \sigma)_+$ be such that $Int(v)\sigma = \sigma'$. Then, by Theorem 2.2,

$$f_3(J) = f_3(J^{(v)}), \ g_3(J) = g_3(J^{(v)}).$$

Now, invoking Theorem 2.4, we have,

$$J^{(v)} \simeq J(B, \sigma', uv^{\#}, N(v)\mu).$$

Thus

$$f_3(J) = f_3(J(B, \sigma', uv^{\#}, N(v)\mu)),\ g_3(J) = g_3(J(B, \sigma', uv^{\#}, N(v)\mu)).$$

Therefore, by Theorem 2.5, we get

$$J(B, \sigma', u', \mu') \simeq J(B, \sigma', uv^{\#}, N(v)\mu) \simeq J(B, \sigma, u, \mu)^{(v)}.$$

Hence J and J' are isotopic.

3. Albert algebras with trivial mod 2 invariants

In this section, we construct, for a given Albert algebra J, an Albert algebra J' with $f_3(J') = 0,\ f_5(J') = 0$ and $g_3(J') = g_3(J)$. We begin with reviewing some results on involutions of second kind. Let K/k be a quadratic extension. Let (B, σ) be a central simple algebra of degree 3 over K with an involution σ of second kind over k. The restriction Q_σ of the trace quadratic form $x \mapsto T(x^2)$ to $(B, \sigma)_+$, the k-space of σ-symmetric elements in B, is a quadratic form with values in k. It is shown in ([H-K-R-T]) that Q_σ is an invariant of σ. The decomposition of Q_σ is given by the following

Theorem 3.1. ([H-K-R-T], 4) *Let $K = k(\sqrt{\alpha})$. Then there exist $b, c \in k^*$ such that*

$$Q_\sigma \simeq <1,1,1> \perp <2><<\alpha>><-b,-c,bc> .$$

In the same paper, it is shown that the Arason invariant of the 3-fold Pfister form $<<\alpha, b, c>>$ determines the isomorphism class of σ. More precisely,

Proposition 3.2. ([H-K-R-T], 15) *The following statements are equivalent for (B, σ), with B as above.*

(1) $\sigma \simeq \sigma'$.

(2) $A(<<\alpha, b, c>>) = A(<<\alpha, b', c'>>)$. *Where b', c' are the elements of k^*, corresponding to the decomposition of $Q_{\sigma'}$.*

The invariant $A(<<\alpha, b, c>>) \in H^3(k, \mathbb{Z}/2)$ is also denoted by $f_3(B, \sigma)$. An involution σ on B is called *distinguished* if $f_3(B, \sigma) = 0$. The following result from ([H-K-R-T]) will be needed.

Proposition 3.3. ([H-K-R-T], 17) *On any central simple algebra B of degree 3 over K, with an involution of second kind, there exists a distinguished involution.*

We now come to the promised construction. We note first that we need only consider the case of second Tits' construction Albert algebras.

Theorem 3.4. *Let $J = J(B, \sigma, u, \mu)$ be an Albert algebra, B a degree 3 central simple algebra over a quadratic extension K/k and with an involution of second kind. There exists an Albert algebra J' with $f_3(J') = 0$, $f_5(J') = 0$ and $g_3(J') = g_3(J)$.*

Proof. By ([K-M-R-T], 39.2), we may assume that $N(u) = 1 = \mu\overline{\mu}$, bar denoting the nontrivial k-automorphism of K. Let σ' be a distinguished involution on B (3.3). Set $J' = J(B, \sigma', 1, \mu)$. Then by ([K-M-R-T], 40.2), we have $f_3(J') = f_3(B, \sigma') = 0$ and $f_5(J')$, being a multiple of $f_3(J')$, is zero as well. Further, $g_3(J) = -Cor_{K/k}([B] \cup [\mu]) = g_3(J')$. This completes the proof.

4. Pure second Tits' construction Albert algebras.

In this brief section, we exihibit how one can construct pure second Tits' construction Albert division algebras. We recall (cf. [P-R 6]) that an Albert algebra J over k is called a *pure second Tits' construction* if it does not arise from Tits' first construction.

Let B be a central division algebra of degree 3 over a quadratic extension K/k. Assume that B admits involutions of second kind. Assume further that σ is an involution on B which is not distinguished. In the terms of the invariant mod 3 associated to (B, σ), this means that $f_3(B, \sigma) \neq 0$. Let $\mu \in K^*$ be such that $\mu\overline{\mu} = 1$ and μ is not a reduced norm from B. Set $J = J(B, \sigma, 1, \mu)$. Then J is an Albert division algebra over k. Further, by ([K-M-R-T], 40.2), $f_3(J) = f_3(B, \sigma) \neq 0$. Thus by ([K-M-R-T], 40.5), J is a pure second Tits' construction. We record this as

Theorem 4.1. *Let (B, σ) be a central division algebra of degree 3 over K, with an involution σ of second kind. Assume that $f_3(B, \sigma) \neq 0$. Let $\mu \in K^*$ be such that μ is not a reduced norm from B and $\mu\overline{\mu} = 1$. Then the Albert algebra $J(B, \sigma, 1, \mu)$ is a pure second Tits' construction division algebra.*

Remarks.

(1) The construction of J' in the proof of Theorem 3.4 yields a division algebra if J is division.

(2) The Albert algebra J' as above, must be a first Tits' construction due to the fact that Albert algebras of first Tits' type are precisely those with the f_3 invariant zero ([K-M-R-T], 40.5).

(3) As a consequence of Remark 2 and Theorem 3.4, we see that $g_3(J)$ is always decomposable, i.e., is a product of H^1-classes, since this is the case when J is a first Tits' construction (cf. also [K-M-R-T, 40.9]).

(4) In light of the fact that the Albert algebra $J(B, \sigma, u, \mu)$ has the f_5 invariant zero if and only if σ is distinguished ([K-M-R-T], 40.7), we note that the invariant

f_5 is sensitive to isotopy, in contrast with f_3 and g_3 (2.2). For example, if $\sigma' = Int(v)\sigma$ is distinguished, then $f_5(J(B,\sigma,u,\mu)^{(v)}) = 0$. Whereas by (2.2), the f_3 and g_3 for this isotope are the same as for $J(B,\sigma,u,\mu)$.

(5) The Albert algebra $J(B,\sigma,u,\mu)$ can be pure even when σ is distinguished, as the above remark shows.

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