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Abstract. For a closed hypersurface in a space form, this work provides some sharp upper bounds for its first positive Laplacian eigenvalue. These bounds are extrinsic as they rely on the mean curvatures and center(s) of gravity of the hypersurface.

1. Notations, introduction and the result

Let $(\mathbb{M} = \mathbb{M}_{n+1}(\kappa), g = \langle \cdot, \cdot \rangle)$ be a space form, that is a connected simply connected Riemannian manifold of dimension $n + 1 \geq 2$, with constant sectional curvature $\kappa \in \mathbb{R}$, d its Riemannian distance, $d\tilde{\nu}$ its volume element, $\widetilde{\nabla}$ its Levi-Civita connection, $\widetilde{\nabla}^2$ its Hessian, $\widetilde{\Delta} = -\text{trace } \widetilde{\nabla}^2$ its Laplacian, exp its exponential map and $U_p\mathbb{M}$ the unit sphere of the tangent space $T_p\mathbb{M}$ of $\mathbb{M}$ at a point p. If M is a closed (compact without boundary) connected hypersurface of $\mathbb{M}$, we endow M with the induced metric, also denoted by $\langle \cdot, \cdot \rangle$. The induced volume element, connection, Hessian and Laplacian are denoted by $d\nu$, ∇, ∇^2 and Δ respectively. We recall that its mean curvatures are the functions $(H_i)_{0 \leq i \leq n}$ defined by $\prod_{i=1}^{n}(1 + Xk_i) = \sum_{i=0}^{n} \binom{n}{i} H_i X^i$ where $(k_i)_{1 \leq i \leq n}$ are the principal curvatures of M. By the generalized Jordan theorem, M is orientable and divides $\mathbb{M}$ into two connected components, one of which, say Ω, is relatively compact and has M as its oriented boundary. We introduce the function sn_κ solution of the differential equation $\ddot{y} + \kappa y = 0$ with the initial conditions $(y(0), \dot{y}(0)) = (0, 1))$ and its primitive h_κ which vanishes at 0. A *center of gravity* of M is a critical point of the smooth function $\mathcal{E} : \mathbb{M} \to \mathbb{R} : p \mapsto \int_M h_\kappa \circ d_p \cdot d\nu$. This definition differs slightly from the one commonly used. The introduction of h_κ has an immediate utility: the distance function d_p may be non smooth at some points but, thanks to h_κ, $\mathcal{E}$ is nevertheless smooth on the whole manifold $\mathbb{M}$. Center(s) of gravity has several applications: for example, we recall that it can be used to prove that any compact

group of isometries in an Hadamard manifold has a fixed point. In this paper, we will use center(s) of gravity to provide some sharp upper bounds for the first positive Laplacian eigenvalue involving the mean curvatures of M. The method used here is to apply the min-max principle to a collection of appropriate functions and goes back to Bleecker and Weiner [2]. Their work, with the generalization given by Reilly [3], dealt with Euclidean submanifolds. The author was naturally led to expect similar results for space form submanifolds. This attempt has been possible by the introduction of center(s) of gravity. The proof will show that the function h_κ will make possible the use of generalized Hsiung-Minkowski formulae. For sake of simplicity, this work concerns only hypersurfaces but for higher codimension, similar formulae can be derived. Our result is the following one (the case $\kappa = 0$ is the one studied by the authors quoted above):

Theorem. *Let M be a closed connected hypersurface of $\mathbb{M}$, λ_1 the first positive eigenvalue of the Laplacian of M and c a center of gravity of M.*

i) If κ is non-negative, then we have the following inequalities:

$$\lambda_1 \int_M \mathrm{sn}_\kappa^2 \circ d_c \cdot d\nu \leq n\,VolM \tag{1}$$

$$\lambda_1 \left(\int_M H_i\,\dot{\mathrm{sn}}_\kappa \circ d_c \cdot d\nu \right)^2 \leq n\,VolM \int_M H_{i+1}^2 \cdot d\nu \tag{2}$$

$$\lambda_1 (n+1)^2 \left(\int_\Omega \dot{\mathrm{sn}}_\kappa \circ d_c \cdot d\tilde{\nu} \right)^2 \leq n\,Vol^2 M \tag{3}$$

for any $i \in [\![0, n-1]\!]$ where $VolM$ denotes the volume of M. Equality occurs in one of these three inequalities if and only if M is a geodesic sphere centered at c. In this case, $\lambda_1 = n/\mathrm{sn}_\kappa^2 r$, r being the radius of this geodesic sphere.

ii) If κ is negative, then

$$\lambda_1 \int_M \mathrm{sn}_\kappa^2 \circ d_c \cdot d\nu \leq n\,VolM - \kappa \int_M \mathrm{sn}_\kappa^2 \circ d_c \cdot d\nu \tag{4}$$

For negative κ, inequality (4) is non sharp unfortunately (see the final remark) and this means that the min-max principle has to be applied to "better" functions. Before going further, the problem of the existence of a center of gravity has to be studied: while for positive κ, two centers of gravity, at least, exist by the compactness of $\mathbb{M}$, the question for non-positive κ is solved by the following result:

Proposition 1. *The function $h_\kappa \circ d_p$ satisfies:*

$$\widetilde{\nabla}^2(h_\kappa \circ d_p) = (\dot{\mathrm{sn}}_\kappa \circ d_p) \cdot g$$

If κ is non-positive, M admits a unique center of gravity.

2. Proof

2.1. Preliminaries

We first prove proposition 1: the claim is trivial for $\kappa = 0$. For non-zero κ, it is well-known that if we equip the space $\mathbb{R}^{n+2}$ with the pseudo-metric $\langle x, y \rangle = \varepsilon x_0 y_0 + \sum_{i=1}^{n+1} x_i y_i$ where $\varepsilon = \text{sign}\,\kappa$ and its associated Levi-Civita pseudo-Riemannian connection D, then, a model space for $\mathbb{M}$ is the sphere $\mathbb{S}^{n+1}(1/\sqrt{\kappa}) = \{x \in \mathbb{R}^{n+2}/\langle x, x \rangle = 1/\kappa\}$ for positive κ and the upper hyperboloid $\mathbb{H}^{n+1}(-1/\sqrt{-\kappa}) = \{x \in \mathbb{R}^{n+2}/\langle x, x \rangle = 1/\kappa$ and $x_0 > 0\}$ for negative κ, with the induced metric, which is Riemannian. The announced formula follows then from two elementary facts: the distance of two points x and y of $\mathbb{M}$ is $d(x, y) = \text{sn}_\kappa^{-1}(\kappa\langle x, y \rangle)$ which implies that $h_\kappa \circ d_p$ is the restriction to $\mathbb{M}$ of the linear form $-\langle p, \cdot \rangle + 1/\kappa$ on $\mathbb{R}^{n+2}$. On a second hand, $(D_X Y)_q = (\widetilde{\nabla}_X Y)_q - \kappa\langle X, Y \rangle q$ for any point $q \in \mathbb{M}$ and vector fields X, Y on $\mathbb{M}$. We deduce immediatly that if γ is a 1-time speed geodesic of $\mathbb{M}$, then $(\mathcal{E} \circ \gamma)''(t) = \int_M \widetilde{\nabla}^2 (h_\kappa \circ d_q)_{\gamma(t)} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \cdot d\nu(q) = \int_M (\text{sn}_\kappa \circ d_q)(\gamma(t)) \cdot d\nu(q) > 0$, which shows the strict convexity of $\mathcal{E}$ and proves the assertion. $\qquad \square$

As c is a critical point of $\mathcal{E}$, we note that $0 = \langle \widetilde{\nabla}\mathcal{E}(c), u \rangle = \int_M \langle \widetilde{\nabla}(h_\kappa \circ d_q)(c), u \rangle \cdot d\nu(q)$ for any $u \in U_c\mathbb{M}$ and the key point is the following:

Lemma. *Let u and v be unit vectors tangent to $\mathbb{M}$ at c and q respectively, q being arbitrary in $\mathbb{M}$. Then the smooth function $F_{c,u} : \mathbb{M} \to \mathbb{R} : q \mapsto \langle \widetilde{\nabla}(h_\kappa \circ d_q)(c), u \rangle$ satisfies:*

$$\int_{U_c\mathbb{M}} F_{c,u}^2(q) \cdot du = \omega_{n+1} \cdot \text{sn}_\kappa^2(d_c(q)) \tag{5}$$

$$\int_{U_c\mathbb{M}} \langle \widetilde{\nabla} F_{c,u}(q), v \rangle^2 \cdot du = \omega_{n+1} \left\{ 1 - \langle v, \widetilde{\nabla} d_c(q) \rangle^2 \kappa\, \text{sn}_\kappa^2(d_c(q)) \right\} \tag{6}$$

where du is the canonical measure of $U_c\mathbb{M}$ and ω_{n+1} the volume of the unit ball of $\mathbb{R}^{n+1}$.

Formula (5) follows from the classical trick on quadratic forms:

Proposition 2. *Let $\mathfrak{B} : V \times V \to \mathbb{R}$ be a bilinear form on a real $(n+1)$-dimensional inner product space V and (S, du) the unit sphere of V endowed with its canonical measure du. Then*

$$\int_S \mathfrak{B}(u, u) \cdot du = \omega_{n+1} \cdot \text{trace } \mathfrak{B}$$

To prove (6), we introduce for convenience the function $\mu_\kappa(t) = \text{sn}_\kappa t / t$ and we set

$\ell = d(c, q)$. Then we can write $F_{c,u}(q) = -\mu_\kappa(\ell)\langle \exp_c^{-1} q, u\rangle$ and

$$\langle \widetilde{\nabla} F_{c,u}(q), v\rangle = -\dot{\mu}_\kappa(\ell)\langle \widetilde{\nabla} d_c(q), v\rangle\langle \exp_c^{-1} q, u\rangle - \mu_\kappa(\ell)\langle \widetilde{\nabla} J(0), u\rangle$$

where $J : [0, 1] \to T\mathbb{M}$ is the unique Jacobi vector field along the geodesic $\gamma :$ $[0, 1] \to \mathbb{M} : t \mapsto \exp_c(t\exp_c^{-1} q)$ in $\mathbb{M}$ joining c to q with the initial conditions $(J(0), J(1)) = (0, v)$. Let us write $v = v^T + v^\perp$ where $v^T = \langle v, \widetilde{\nabla} d_c(q)\rangle\widetilde{\nabla} d_c(q) = \langle v, \dot{\gamma}(1)\rangle\dot{\gamma}(1)/\ell^2$ is the tangential component of v relatively to the geodesic speed vector $\dot{\gamma}(1)$. Solving the differential equation satisfied by J, one easily obtains that

$$J(t) = \frac{\langle v, \widetilde{\nabla} d_c(q)\rangle}{\ell} t\dot{\gamma}(t) + \frac{\mathrm{sn}_\kappa(t\ell)}{\mathrm{sn}_\kappa \ell} P^\perp(t)$$

where $P^\perp$ is the unique parallel vector field along γ with $P^\perp(1) = v^\perp$. This implies that

$$\widetilde{\nabla} J(0) = \frac{\langle v, \widetilde{\nabla} d_c(q)\rangle}{\ell}\dot{\gamma}(0) + \frac{1}{\mu_\kappa(\ell)} P^\perp(0)$$

So,

$$\langle \widetilde{\nabla} F_{c,u}(q), v\rangle = -\left\langle \left\{\dot{\mu}_\kappa(\ell) + \frac{\mu_\kappa(\ell)}{\ell}\right\}\langle v, \widetilde{\nabla} d_c(q)\rangle\dot{\gamma}(0) + P^\perp(0), u\right\rangle$$

$$= -\left\langle \frac{\mathrm{sn}_\kappa \ell}{\ell}\langle v, \widetilde{\nabla} d_c(q)\rangle\dot{\gamma}(0) + P^\perp(0), u\right\rangle$$

As $P^\perp$ is parallel along the geodesic γ and perpendicular to the geodesic at $t = 1$, $P^\perp$ is of constant norm $|v^\perp|$ and always perpendicular to the geodesic. By proposition 2 and the straightforward relation $\mathrm{sn}_\kappa^2 t + \kappa\,\mathrm{sn}_\kappa^2 t = 1$, one obtains that

$$\frac{1}{\omega_{n+1}}\int_{U_c\mathbb{M}}\langle \widetilde{\nabla} F_{c,u}(q), v\rangle^2 \cdot du = |v^T|^2\mathrm{sn}_\kappa^2\ell + |v^\perp|^2 = 1 - |v^T|^2\kappa\,\mathrm{sn}_\kappa^2\ell \qquad \square$$

2.2. Proof of the theorem

Let η be the outward smooth unit vector field normal to M, that is the one pointing in the opposite direction to D. The previous lemma, the Green formula which by proposition 1 is: $\int_M \langle \widetilde{\nabla}(h_\kappa \circ d_c), \eta\rangle \cdot d\nu = \int_\Omega \widetilde{\Delta}(h_\kappa \circ d_c) \cdot d\tilde{\nu} = (n+1)\int_\Omega \mathrm{sn}_\kappa \circ d_c \cdot d\tilde{\nu}$ and the generalized Hsiung-Minkowski formulae [1]: $\int_M \{H_i\,\mathrm{sn}_\kappa \circ d_c + H_{i+1}\langle \widetilde{\nabla}(h_\kappa \circ d_c), \eta\rangle\} \cdot d\nu = 0$ for any $i \in [\![0, n-1]\!]$ show formulae (2) and (3) as easy consequences of (1). Formulae (5) et (6) prove (1): indeed, if we denote by $f_{c,u}$ the restriction of $F_{c,u}$ to M, then $|\nabla f_{c,u}|^2 = |\widetilde{\nabla} F_{c,u}|^2 - \langle \widetilde{\nabla} F_{c,u}, \eta\rangle^2$ and we deduce immediatly from (6) that

$$\frac{1}{\omega_{n+1}}\int_{U_c\mathbb{M}}|\nabla f_{c,u}|^2 \cdot du = n - \left(1 - \langle \eta, \widetilde{\nabla} d_c\rangle^2\right)\kappa\,\mathrm{sn}_\kappa^2 \circ d_c \qquad (7)$$

The definition of c imply that the integrals of the functions $(f_{c,u})_{u \in U_c \mathbb{M}}$ on M all vanish. By the Rayleigh min-max principle, the first positive Laplacian eigenvalue of M satisfy $\lambda_1 \int_M f_{c,u}^2 \cdot d\nu \leq \int_M |\nabla f_{c,u}|^2 \cdot d\nu$ for any vector $u \in U_c \mathbb{M}$ and equality is achieved if and only if $f_{c,u}$ is an eigenfunction, that is $\Delta f_{c,u} = \lambda_1 f_{c,u}$. Integrating this last inequality on $U_c \mathbb{M}$, using Fubini theorem, (5) and (7), we obtain

$$\lambda_1 \int_M \mathrm{sn}_\kappa^2 \circ d_c \cdot d\nu \leq \int_M \left\{ n - \left(1 - \langle \eta, \widetilde{\nabla} d_c \rangle^2 \right) \kappa \, \mathrm{sn}_\kappa^2 \circ d_c \right\} \cdot d\nu$$

If κ is negative, inequality (4) is immediate. For non-negative κ, the right term is bounded from above by $n \mathrm{Vol}\, M$ and we arrive to the announced inequality (1).

If κ is zero and equality holds, then $\Delta f_{c,u} = \lambda_1 f_{c,u}$ for any vector u. In order to be self-contained, we recall the classical argument [4] which concludes the proof: it is an easy result that $\Delta f_{c,u} = n H_1 \langle \eta, u \rangle$ where H_1 is the 1-mean curvature of M. This implies that $n H_1 \eta = -\lambda_1 \widetilde{\nabla}(h_\kappa \circ d_c)$. We choose a local orthonormal basis $(X_i)_{1 \leq i \leq n}$ of principal vectors in TM associated to the principal curvatures $(k_i)_{1 \leq i \leq n}$ and we differentiate this last equation by X_i. Using Weingarten equation and proposition 1, we obtain $n(X_i H_1)\eta - n H_1 k_i X_i = -\lambda_1 X_i$. Taking its scalar product with X_i and summing the relations obtained for all i, we deduce that $\lambda_1 = n H_1^2$. This shows that M is of constant 1-mean curvature and by this way, included in a geodesic sphere centered at c with radius $\sqrt{n/\lambda_1}$. As this geodesic sphere is also n-dimensional and connected, we deduce by the compactness of M that M coincides with this geodesic sphere.

If κ is positive and equality holds, this means that $\eta = \pm \widetilde{\nabla} d_c$. Therefore, the gradient of $(h_\kappa \circ d_c)_{|M}$ satisfy $\langle \nabla((h_\kappa \circ d_c)_{|M}), \xi \rangle = \langle \widetilde{\nabla}(h_\kappa \circ d_c), \xi \rangle = (\mathrm{sn}_\kappa \circ d_c)\langle \widetilde{\nabla} d_c, \xi \rangle = 0$ for any vector field ξ on M. As M is connected, the function $h_\kappa \circ d_c$ is then constant on M, i.e. M is included in a geodesic sphere centered at c. As this geodesic sphere is also n-dimensional and connected, we deduce by the compactness of M that M coincides with this geodesic sphere.

For negative κ, we note at last that equality cannot hold in (4): indeed, equality would imply that $\langle \eta, \widetilde{\nabla} d_c \rangle = 0$ on M and this is in contradiction with the first generalized Hsiung-Minkowski formulae which says that $0 = \int_M \{ \mathrm{sn}_\kappa \circ d_c + H_1 \langle \widetilde{\nabla}(h_\kappa \circ d_c), \eta \rangle \} \cdot d\nu = \int_M \{ \mathrm{sn}_\kappa \circ d_c + H_1 \cdot \mathrm{sn}_\kappa \circ d_c \langle \widetilde{\nabla} d_c, \eta \rangle \} \cdot d\nu$. $\qquad \square$

Acknowledgements

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