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Real structures on minimal ruled surfaces

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Abstract. In this paper, we give a complete description of the deformation classes of real structures on minimal ruled surfaces. In particular, we show that these classes are determined by the topology of the real structure, which means, using the terminology of [5], that real minimal ruled surfaces are quasi-simple. As an intermediate result, we obtain the classification, up to conjugation, of real structures on decomposable ruled surfaces.

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0. Introduction

Let X be a smooth compact complex surface. A *real structure* on X is an antiholomorphic involution $c_X : X \to X$. The *real part* of (X, c_X) is by definition the fixed point set of c_X. If X admits a holomorphic submersion on a smooth compact complex irreducible curve B whose fibers have genus zero, then it is called a *minimal ruled surface*. These surfaces are all algebraic, minimal – with the exception of the blown-up projective plane – and of Kodaira dimension $-\infty$ (see [2]). Real minimal ruled surfaces are one of the few examples of real algebraic surfaces of special type whose classification under real deformation is not known, see the recent results [5], [4], [3] and the survey [6] for detailed history and references. The purpose of this paper is to fill this gap. Since all the ruled surfaces considered in this paper will be minimal, from now on we will call them "ruled" rather than "minimal ruled".

Rational surfaces are well known (see [5]), so we can restrict ourselves to non-rational ruled surfaces. The ruling $p : X \to B$ is then unique and any real structure c_X on X is fibered over a real structure c_B on B in the sense that $c_B \circ p = p \circ c_X$. The topology of the real part of X as well as the topology of the real curve (B, c_B) provide us with a topological invariant under real deformation which we call *the topological type* of the surface. This invariant is encoded by a quintuple of integers: the number of tori and Klein bottles of $\mathbb{R}X$, the genus of B, the number of components of $\mathbb{R}B$ and the type of (B, c_B) (see §3.2). The main result of this paper is the following (see theorem 3.7 and proposition 3.4):

Theorem 0.1. *Two real (minimal) non-rational ruled surfaces are in the same real deformation class if and only if they have the same topological type and homeomorphic quotients. Moreover, any allowable quintuple of integers is realized as the topological type of a real non-rational ruled surface.*

Remember that in the case of rational ruled surfaces, the analogous result is (see [5] or [6]):

Theorem 0.2. *There are four deformation classes of real structures on rational ruled surfaces, one for which the real part is a torus, one for which the real part is a sphere and two for which the real part is empty. These two latter have non-homeomorphic quotients.* □

Note that as soon as the bases of the surfaces have non-empty real parts, the condition on the quotients in theorem 0.1 can be removed. A quintuple of integers is called *allowable* when it satisfies the few obvious conditions satisfied by topological types of real non-rational ruled surfaces, see §3.2 for a precise definition. Remember that any compact complex surface lying in the deformation class of a non-rational ruled surface is itself a non-rational ruled surface (see, for example, [1]). A definition of real deformation classes can be given as follows. Equip the Poincaré's disk $\Delta \subset \mathbb{C}$ with the complex conjugation *conj*. A *real deformation* of surfaces is a proper holomorphic submersion $\pi : Y \to \Delta$ where Y is a complex manifold of dimension 3 equipped with a real structure c_Y and π satisfies $\pi \circ c_Y = conj \circ \pi$. Then, when $t \in]-1, 1[\in \Delta$, the fibers $Y_t = \pi^{-1}(t)$ are invariant under c_Y and hence are compact real surfaces. Two real surfaces X' and X'' are said to be *in the same deformation class* if there exists a chain $X' = X_0, \dots, X_k = X''$ of compact real surfaces such that for every $i \in \{0, \dots, k-1\}$, the surfaces X_i and X_{i+1} are isomorphic to some real fibers of a real deformation.

Remember that every ruled surface is the projectivization $P(E)$ of a rank two complex vector bundle E over B (see [2]). Moreover $P(E)$ and $P(E')$ are isomorphic if and only if $E' = E \otimes L$ where L is a complex line bundle over B. A ruled surface is said to be *decomposable* if E is decomposable, that is if E is the direct sum of two complex line bundles. The paper is organized as follows. In the first section, we give constructions of some particular real structures on decomposable ruled surfaces. In the second section we obtain a classification, up to conjugation, of real structures on decomposable ruled surfaces (see theorem 2.3). This result, of independent interest, plays a crucial rôle in the proof of theorem 0.1. In this section is also given a result independent of real algebraic geometry, which concerns the lifting of automorphisms of the ruled surface X to automorphisms of the rank two vector bundle E, see proposition 2.1. Finally, the third section is devoted to the proof of theorem 0.1. This gives a complete description of the deformation classes of real structures on ruled surfaces. In particular, it shows that these classes are determined by the topology of the real structure, which means, using the terminology of [5], that real ruled surfaces are quasi-simple.

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1. Construction of some particular real structures

1.1. Meromorphic functions and real structures

Let B be a smooth compact complex irreducible curve. Denote by $\mathrm{Pic}(B)$ the group of complex line bundles over B. This group is identified with the group of divisors modulo principal ones. Let $\phi : B \to B$ be a holomorphic or anti-holomorphic automorphism, and let $D = \sum_{i=1}^k n_i p_i$, $n_i \in \mathbb{Z}$, $p_i \in B$, be a divisor on B. Then we denote by $\phi^*(D)$ the divisor $\sum_{i=1}^k n_i \phi^{-1}(p_i)$ and by $\phi(D)$ the divisor $\sum_{i=1}^k n_i \phi(p_i)$. The morphism on the quotient $\mathrm{Pic}(B)$ of the group of divisors induced by ϕ^* will also be denoted by ϕ^*. We denote by L_0 the trivial line bundle over B and by L^* the line bundle dual to L, so that $L \otimes L^* = L_0$.

Suppose from now on that B is equipped with a *real structure* c_B, that is an anti-holomorphic involution c_B.

Lemma 1.1. *Let $L \in \mathrm{Pic}(B)$ be a line bundle such that $c_B^*(L) = L$. Then, for every divisor D associated to L, there exists a meromorphic function f_D on B such that* $\mathrm{div}(f_D) = c_B(D) - D$ *and* $f_D \overline{f_D \circ c_B} = \pm 1$.

Proof. By assumption, D and $c_B(D)$ are linearly equivalent. As a consequence, there exists a meromorphic function f such that $\mathrm{div}(f) = c_B(D) - D$. Then, $h = \overline{f \circ c_B}$ is a meromorphic function on B satisfying $\mathrm{div}(h) = D - c_B(D)$. So fh is a holomorphic function on B. This means that there exists a constant $\lambda \in \mathbb{C}^*$ such that $f\overline{f \circ c_B} = \lambda$.

But for all $x \in B$,

$$\lambda = (f\overline{f \circ c_B})(c_B(x)) = f \circ c_B(x)\overline{f(x)} = \overline{f(x)\overline{f \circ c_B(x)}} = \overline{\lambda}.$$

Thus $\lambda \in \mathbb{R}^*$, and we define $f_D = \frac{1}{\sqrt{|\lambda|}} f$. □

Remark 1.2. As soon as $\mathbb{R}B$ is non-empty, $f_D\overline{f_D \circ c_B} = +1$, since for every $x \in \mathbb{R}B$ we have $f_D\overline{f_D \circ c_B}(x) = |f(x)|^2 \geq 0$. Nevertheless, when $\mathbb{R}B = \emptyset$, there always exists a divisor D on B, of degree congruent to $g(B) - 1 \mod (2)$ where $g(B)$ is the genus of B, such that $f_D\overline{f_D \circ c_B} = -1$ (see [7], proposition 2.2).

Note also that the function f_D given by lemma 1.1 is not unique, since for every constant $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, the function λf_D has the same properties.

Lemma 1.3. *Let $L \in \operatorname{Pic}(B)$ be a line bundle such that $c_B^*(L) = L^*$. Then, for every divisor D associated to L, there exists a meromorphic function f_D on B such that* $\operatorname{div}(f_D) = D + c_B(D)$ *and* $f_D = \overline{f_D \circ c_B}$.

Proof. By assumption, $c_B(D)$ and $-D$ are linearly equivalent. As a consequence, there exists on B a meromorphic function f such that $\operatorname{div}(f) = D + c_B(D)$. Then, $h = \overline{f \circ c_B}$ is a meromorphic function on B satisfying $\operatorname{div}(h) = c_B(D) + D = \operatorname{div}(f)$. Thus there exists a constant $\lambda \in \mathbb{C}^*$ such that $h = \lambda f$. But,

$$\lambda = \frac{h \circ c_B}{f \circ c_B} = \frac{\overline{f}}{f \circ c_B} = \overline{\left(\frac{f}{\overline{f \circ c_B}}\right)} = \frac{1}{\overline{\lambda}}.$$

Hence there exists $\theta \in \mathbb{R}$ such that $\lambda = \exp(2i\theta)$, and we define $f_D = \exp(i\theta) f$. □

Remark 1.4. The function f_D given by lemma 1.3 is not unique: for every $\lambda \in \mathbb{R}^*$, the function λf_D has the same properties. Note that every zero or pole of f_D on $\mathbb{R}B$ has even order, so that the sign of f_D is constant on every component of $\mathbb{R}B$.

1.2. Some particular real structures

Let $D = \sum_{i=1}^{k} n_i p_i$ be a divisor on B, where $p_i \in B$ and $n_i \in \mathbb{Z}$ ($i \in \{1, \ldots, k\}$). We can assume that the set $\{p_i \,|\, 1 \leq i \leq k\}$ is invariant under c_B (add some points with zero coefficients to D if necessary). Denote by $U_0 = B \setminus \{p_i \,|\, 1 \leq i \leq k\}$ and for every $i \in \{1, \ldots, k\}$, choose a holomorphic chart (U_{p_i}, ϕ_{p_i}) such that $U_{p_i} \cap U_{p_j} = \emptyset$ if $i \neq j$, $c_B(U_{p_i}) = U_{c_B(p_i)}$ and $\phi_{p_i} : U_{p_i} \to \Delta = \{z \in \mathbb{C} \,|\, |z| < 1\}$ is a biholomorphism. Require in addition that $\phi_{p_i}(p_i) = 0 \in \Delta$ and $\phi_{c_B(p_i)} \circ c_B \circ \phi_{p_i}^{-1}(z) = \overline{z}$ for all $z \in \Delta$ and $i \in \{1, \ldots, k\}$ (such charts always exist, see [12]). Such an atlas is called *compatible* with the divisor D and the group $< c_B >$.

For every $i \in \{1, \ldots, k\}$, denote by ψ_i the morphism:

$$\begin{aligned} (U_{p_i} \setminus p_i) \times \mathbb{C} &\to U_0 \times \mathbb{C} \\ (x, z) &\mapsto (x, \phi_{p_i}(x)^{-n_i} z). \end{aligned}$$

The morphisms ψ_i allow to glue together the trivializations $U_{p_i} \times \mathbb{C}$, $i \in \{0, \ldots, k\}$, in order to define the line bundle L associated to D. Such trivializations are called *compatible* with the divisor D and the group $< c_B >$.

Let L (resp. X) be a line bundle (resp. a ruled surface) over B. The real structure c_L on L (resp. c_X on X) is said to be *fibered over* c_B, or that it *lifts* c_B, if $p \circ c_L = c_B \circ p$ (resp. $p \circ c_X = c_B \circ p$) where p is the projection $L \to B$ (resp. $X \to B$).

Lemma 1.5. *There exists a real structure on $L \in \operatorname{Pic}(B)$ which lifts c_B if and only if $c_B^*(L) = L$ and for every couple (D, f_D) given by lemma 1.1,* $f_D \overline{f_D \circ c_B} = +1$.

Proof. $\Longrightarrow$: Let s be a meromorphic section of L and $D = \text{div}(s)$. Let c_L be a real structure on L and $\tilde{s} = c_L \circ s \circ c_B$. Then $\tilde{s}$ is another meromorphic section of L. This implies that $\text{div}(\tilde{s})$ and $\text{div}(s)$ are linearly equivalent. Since $\text{div}(\tilde{s}) = c_B(\text{div}(s))$, we deduce that $c_B^*(L) = L$. Moreover, $\tilde{s} = fs$ where f is a meromorphic function on B satisfying $\text{div}(f) = c_B(D) - D$. Since $s = c_L \circ \tilde{s} \circ c_B = c_L \circ (fs) \circ c_B = (\overline{f \circ c_B})\tilde{s} = (\overline{f \circ c_B}) fs$, we have $(\overline{f \circ c_B}) f = +1$. Changing the section s, the same is obtained for any couple (D, f_D) given by lemma 1.1.

$\Longleftarrow$: Let L be a line bundle such that $c_B^*(L) = L$ and (D, f_D) a couple given by lemma 1.1 such that $f_D \overline{f_D \circ c_B} = +1$. Denote $D = \sum_{i=1}^k n_i p_i$ and let $U_0 = B \setminus \{p_i \,|\, 1 \leq i \leq k\}$ and (U_{p_i}, ϕ_{p_i}), $i \in \{1, \dots, k\}$, be an atlas compatible with the divisor D and the group $< c_B >$.

The maps

$$\begin{aligned} U_0 \times \mathbb{C} &\to U_0 \times \mathbb{C} \\ (x, z) &\mapsto (c_B(x), f_D \circ c_B(x) \overline{z}), \end{aligned}$$

and for every $i \in \{1, \dots, k\}$,

$$\begin{aligned} U_{p_i} \times \mathbb{C} &\to U_{c_B(p_i)} \times \mathbb{C} \\ (x, z) &\mapsto (c_B(x), f_D \circ c_B(x) \overline{\phi_{p_i}(x)}^{n_{c_B(p_i)} - n_{p_i}} \overline{z}) \end{aligned}$$

glue together to form an antiholomorphic map c_L on L. This map lifts c_B and is an involution, hence the result. $\square$

Proposition 1.6. *Let $L \in \text{Pic}(B)$ be a line bundle such that $c_B^*(L) = L^*$. Then to every couple (D, f_D) given by lemma 1.3 is associated a real structure c_{f_D} on the ruled surface $X = P(L \oplus L_0)$ which lifts c_B. The real part of (X, c_{f_D}) is orientable and consists of t^+ tori, where t^+ is the number of components of $\mathbb{R}B$ on which f_D is non-negative (see remark 1.4).*

Remark 1.7. For the sake of simplicity, when there will not be any ambiguity on the choice of the function f_D, we will denote by c_X^+ (resp. c_X^-) the real structure c_{f_D} (resp. c_{-f_D}). The real part of (X, c_{-f_D}) consists of t^- tori, where t^- is the number of components of $\mathbb{R}B$ on which $f_D \leq 0$. Obviously, $t^+ + t^- = \mu(\mathbb{R}B)$, where $\mu(\mathbb{R}B)$ is the number of components of $\mathbb{R}B$. Thus, when $\mu(\mathbb{R}B)$ is odd, the real structures c_X^+ and c_X^- on X cannot be conjugated, since the numbers of components of their real parts do not have the same parity. Nevertheless, these two real structures may sometimes be conjugated. This situation will be studied in the next section, proposition 2.6.

Proof. Let (D, f_D) be a couple given by lemma 1.3, so that $f_D = \overline{f_D \circ c_B}$ and $\text{div}(f_D) = D + c_B(D)$. Let $p_i \in B$ and $n_i \in \mathbb{Z}$, $i \in \{1, \dots, k\}$, be such that $D = \sum_{i=1}^k n_i p_i$. We can assume that the set $\{p_i \,|\, 1 \leq i \leq k\}$ is invariant under c_B. Let $U_0 = B \setminus \{p_i \,|\, 1 \leq i \leq k\}$ and (U_{p_i}, ϕ_{p_i}), $i \in \{1, \dots, k\}$, be an atlas compatible with the divisor D and the group $< c_B >$.

The morphisms:

$$\begin{aligned}(U_{p_i} \setminus p_i) \times \mathbb{C}P^1 &\to U_0 \times \mathbb{C}P^1\\ (x,(z_1:z_0)) &\mapsto (x,(\phi_{p_i}(x)^{-n_i} z_1 : z_0))\end{aligned}$$

($i \in \{1,\dots,k\}$) allow to glue together the trivializations $U_{p_i} \times \mathbb{C}P^1$, $i \in \{0,\dots,k\}$, in order to define the ruled surface X.

Now, the maps

$$\begin{aligned}U_0 \times \mathbb{C}P^1 &\to U_0 \times \mathbb{C}P^1\\ (x,(z_1:z_0)) &\mapsto (c_B(x),(\overline{z_0} : f_D \circ c_B(x)\overline{z_1})),\end{aligned}$$

and for every $i \in \{1,\dots,k\}$,

$$\begin{aligned}U_{p_i} \times \mathbb{C}P^1 &\to U_{c_B(p_i)} \times \mathbb{C}P^1\\ (x,(z_1:z_0)) &\mapsto (c_B(x),(\overline{z_0} : f_D \circ c_B(x)\overline{\phi_{p_i}(x)}^{-n_{c_B(p_i)}-n_{p_i}}\overline{z_1})\end{aligned}$$

glue together to form an antiholomorphic map c_{f_D} on X. This map lifts c_B and is an involution. The first part of proposition 1.6 is proved.

Now, the fixed point set of c_{f_D} in $U_0 \times \mathbb{C}P^1$ is:

$$\{(x,(\theta : \sqrt{f_D(x)})) \in U_0 \times \mathbb{C}P^1 \,|\, x \in \mathbb{R}B, f_D(x) \geq 0 \text{ and } \theta \in \mathbb{C}, |\theta| = 1\}.$$

The connected components of this fixed point set are then tori or cylinders depending on whether the corresponding component of $\mathbb{R}B$ is completely included in U_0 or not. Similarly, the fixed point set of c_{f_D} in $U_{p_i} \times \mathbb{C}P^1$ is:

$$\{(x,(\theta_i : \sqrt{f_D(x)x_i^{-2n_i}})) \in U_{p_i} \times \mathbb{C}P^1 \,|\, x \in \mathbb{R}B, f_D(x) \geq 0 \text{ and } \theta_i \in \mathbb{C}, |\theta_i| = 1\},$$

where $x_i = \phi_{p_i}(x)$. This fixed point set is a cylinder if $p_i \in \mathbb{R}B$ and is empty otherwise.

The gluing maps between these cylinders are given by $\theta = -\theta_i$ if $x_i = \phi_{p_i}(x) < 0$ and by $\theta = \theta_i$ if $x_i = \phi_{p_i}(x) > 0$. Since both id and $-id$ preserve the orientation of the circle $U^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$, the results of these gluings are always tori. Thus, the real part of (X, c_{f_D}) consists only of tori and the number of such tori is the number of components of $\mathbb{R}B$ on which $f_D \geq 0$, that is t^+. □

2. Conjugacy classes of real structures on decomposable ruled surfaces

2.1. Lifting of automorphisms of X

I could not find the following proposition in the literature, so I give it here.

Proposition 2.1. *Let L be a complex line bundle over B and X be the ruled surface $P(E)$, where $E = L \oplus L_0$.*

If $L \neq L^*$ or if $L = L_0$, then every automorphism of X fibered over the identity of B lifts to an automorphism of E. If $L = L^*$ and $L \neq L_0$, then the automorphisms of X fibered over the identity of B which lift to automorphisms of E form an index two subgroup of the group of automorphisms of X fibered over the identity. In that case, the automorphisms of X which do not lift are of the form

$$\phi_\lambda = \begin{bmatrix} 0 & \lambda s \\ s & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{C}^*$ and s is a non-zero meromorphic section of L.

Remark 2.2. The automorphims ϕ_λ introduced in proposition 2.1 are holomorphic involutions of X.

Proof. Denote by $\mathcal{O}_B^*$ the sheaf of holomorphic functions on B which do not vanish and by $\mathcal{A}ut(E)$ (resp. $\mathcal{A}ut(X)$) the sheaf of automorphisms of E (resp. of X) fibered over the identity. These sheaves satisfy the exact sequence:

$$1 \to \mathcal{O}_B^* \to \mathcal{A}ut(E) \to \mathcal{A}ut(X) \to 1.$$

We deduce the following long exact sequence:

$$\begin{aligned} 1 \to H^0(B, \mathcal{O}_B^*) \to H^0(B, \mathcal{A}ut(E)) \to H^0(B, \mathcal{A}ut(X)) \\ \to H^1(B, \mathcal{O}_B^*) \to H^1(B, \mathcal{A}ut(E)) \end{aligned}$$

We are searching for the image of the morphism $H^0(B, \mathcal{A}ut(E)) \to H^0(B, \mathcal{A}ut(X))$. To compute this image, let us study the kernel of the map $i_* : H^1(B, \mathcal{O}_B^*) \to H^1(B, \mathcal{A}ut(E))$.

Remember that the group $H^1(B, \mathcal{O}_B^*)$ is isomorphic to $\mathrm{Pic}(B)$. Such an isomorphism can be defined as follows: fix a divisor $\sum_{j=1}^t r_j q_j$, where for $j \in \{1, \dots, t\}$, $r_j \in \mathbb{Z}$ and $q_j \in B$. Denote by $U_0 = B \setminus \{q_j \mid 1 \leq j \leq t\}$ and for every $j \in \{1, \dots, t\}$, choose a holomorphic chart (U_{q_j}, ϕ_{q_j}) of B such that $U_{q_j} \cap U_{q_{j'}} = \emptyset$ if $j \neq j'$, $\phi_{q_j} : U_{q_j} \to \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ is a biholomorphism and $\phi_{q_j}(q_j) = 0 \in \Delta$. Denote by $\mathcal{U}$ the covering of B defined by $U_0, \dots, U_t$ and consider the following sections of $\mathcal{O}_B^*$ ($j \in \{1, \dots, t\}$):

$$\begin{aligned} l_{0j}^1 : U_0 \cap U_j &\to \mathbb{C}^* \\ x &\mapsto \phi_{q_j}(x)^{r_j} = x_j^{r_j}, \end{aligned}$$

where by definition $x_j = \phi_{q_j}(x) \in \Delta$. These sections define a 1-cocycle of B with coefficient in $\mathcal{O}_B^*$ and we denote with the same letter l^1 its cohomology class in $H^1(\mathcal{U}, \mathcal{O}_B^*)$ and in $H^1(B, \mathcal{O}_B^*)$. This construction defines an isomorphism between $\mathrm{Pic}(B)$ and $H^1(B, \mathcal{O}_B^*)$.

So let $l^1 \in H^1(B, \mathcal{O}_B^*)$ be associated to the divisor $\sum_{j=1}^t r_j q_j$. Then $m^1 = i_*(l^1)$ is the cohomology class of the 1-cocycle with coefficient in $\mathcal{A}ut(E)$ defined by the following sections ($j \in \{1, \ldots, t\}$):

$$\begin{aligned} m^1_{0j} : U_0 \cap U_j &\longrightarrow \mathcal{A}ut(E) \\ x &\longmapsto \begin{bmatrix} x_j^{r_j} & 0 \\ 0 & x_j^{r_j} \end{bmatrix}. \end{aligned}$$

Suppose that $m^1 = 0 \in H^1(B, \mathcal{A}ut(E))$. Then $\sum_{j=1}^t r_j q_j$ is of degree zero, since $0 = \det(m^1) = 2l^1 \in H^1(B, \mathcal{O}_B^*)$. Moreover, since the map $H^1(\mathcal{U}, \mathcal{A}ut(E)) \to H^1(B, \mathcal{A}ut(E))$ is injective (see [10], lemma 3.11, p. 294), m^1 is the coboundary of a 0-cochain given in the covering $\mathcal{U}$ by the following sections ($j \in \{0, \ldots, t\}$):

$$\begin{aligned} m^0_j : U_j &\longrightarrow \mathcal{A}ut(E) \\ x &\longmapsto \begin{bmatrix} a_j(x) & c_j(x) \\ b_j(x) & d_j(x) \end{bmatrix}, \end{aligned}$$

where a_j, d_j are 0-cochains with coefficients in $\mathcal{O}_B$, c_j is a 0-cochain with coefficient in $\mathcal{O}_B(L)$, d_j is a 0-cochain with coefficient in $\mathcal{O}_B(L^*)$ and $a_j d_j - b_j c_j$ does not vanish. Then, the equality $m^1 = \delta m^0$ can be written:

$$\forall j \in \{1, \ldots, t\}, \quad m^1_{0j} = m^0_0 (m^0_j)^{-1},$$

which rewrites as $m^0_0 = x_j^{r_j} m^0_j$ ($j \in \{1, \ldots, t\}$). Hence, we deduce that for $j \in \{1, \ldots, t\}$, $a_0 = x_j^{r_j} a_j$, $d_0 = x_j^{r_j} d_j$, $b_0 = x_j^{r_j} b_j$ and $c_0 = x_j^{r_j} c_j$. As soon as a_0 (resp. d_0) is non-zero, this implies that a_0 (resp. d_0) is a meromorphic function over B satisfying $\operatorname{div}(a_0) \geq \sum_{j=1}^t r_j q_j$ (resp. $\operatorname{div}(d_0) \geq \sum_{j=1}^t r_j q_j$). Since these two divisors are of degree zero, they are equal. So $\sum_{j=1}^t r_j q_j$ is a principal divisor and $l^1 = 0$. When $a_0 = d_0 = 0$, we deduce that b_0 (resp. c_0) is a meromorphic section of L^* (resp. of L) satisfying $\operatorname{div}(b_0) \geq \sum_{j=1}^t r_j q_j$ (resp. $\operatorname{div}(c_0) \geq \sum_{j=1}^t r_j q_j$). Since $\deg(L) = -\deg(L^*)$, these divisors are equal. We then deduce that $L = L^*$ and that this line bundle is associated to the divisor $\sum_{j=1}^t r_j q_j$.

In conclusion, when $L \neq L^*$, the morphism i_* is injective and when $L = L^*$, $L \neq L_0$, the kernel of i_* is included into the subgroup of $H^1(B, \mathcal{O}_B^*) = \operatorname{Pic}(B)$ generated by L, which is of order two. In that case, it is not difficult to check that the kernel of i_* is exactly this subgroup of order two. Indeed, with the preceding notations, it suffices to let a_0 and d_0 be equal to 0 and let b_0 and c_0 be equal to a same meromorphic section of L. This constructs a 0-cochain m^0 such that $\delta m^0 = i_*(L)$. The first part of the proposition is proved.

To check the second part of the proposition, note that when $L = L^* \neq L_0$, $H^0(B, L) = H^0(B, L^*) = 0$, so that the automorphisms of $E = L \oplus L_0$ fibered over the identity of B are of the form

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

where $a, d \in \mathbb{C}^*$. The automorphisms of X fibered over the identity which lift to

E are then of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \quad (\lambda \in \mathbb{C}^*).$$

It follows that the automorphisms ϕ_λ do not lift to automorphims of E and that they are the only ones with this property. $\square$

2.2. The conjugation theorem

Denote by c_{L_0} the real structure on L_0 defined by:

$$\begin{aligned} B \times \mathbb{C} &\to B \times \mathbb{C} \\ (x, z) &\mapsto (c_B(x), \overline{z}). \end{aligned}$$

This real structure lifts c_B.

Theorem 2.3. *Let L be a line bundle over a smooth compact complex irreducible curve B equipped with a real structure c_B and let $X = P(L \oplus L_0)$ be the associated decomposable ruled surface.*

1. *Suppose that $L \neq L^*$ and that there exists a real structure c_L on L which lifts c_B. Then there exists, up to conjugation by a biholomorphism of X, one and only one real structure on X which lifts c_B. It is the real structure induced by $c_L \oplus c_{L_0}$.*
2. *Suppose that $c_B^*(L) = L^*$. If $L \neq L^*$, then every real structure on X which lifts c_B is conjugated to one of the two structures c_X^+ or c_X^- given by proposition 1.6. The same result occurs when $L = L_0$ or when $L = L^*$ and there is no real structure on L which lifts c_B.*
3. *Suppose that $c_B^*(L) = L = L^*$, that $L \neq L_0$ and that there exists a real structure c_L on L which lifts c_B. Then every real structure on X which lifts c_B is conjugated to the real structure $c_L \oplus c_{L_0}$, or to one of the two structures c_X^+ or c_X^- given by proposition 1.6.*

In any other case, X does not admit real structures fibered over c_B.

Remark 2.4. It follows from lemma 1.5 and remark 1.2 that when $\mathbb{R}B \neq \emptyset$, there exists a real structure on L which lifts c_B if and only if $c_B^*(L) = L$.

Note that in the third case, the real structures c_X^+ and c_X^- are not conjugated to $c_L \oplus c_{L_0}$, since they are exchanging the two disjoint holomorphic sections of zero square of X and $c_L \oplus c_{L_0}$ does not. Note also that when $X = B \times \mathbb{C}P^1$, or when $\mu(\mathbb{R}B)$ is odd, the real structures c_X^+ and c_X^- on X are not conjugated (see remark 1.7). Nevertheless, these two real structures may sometimes be conjugated, see proposition 2.6.

Proposition 2.5. *Let L be a line bundle over (B, c_B) and let $X = P(L \oplus L_0)$. Then there exists a real structure on X which lifts c_B if and only if there exists a real structure on L which lifts c_B or $c_B^*(L) = L^*$.*

Proof. $\Longrightarrow$: To begin with, suppose that $\deg(L) \neq 0$. Then, without loss of generality, we can assume that $d = \deg(L) > 0$. The holomorphic section e of X associated to L satisfy $e \circ e = -d < 0$, since its normal bundle is L^*. Any other section $\tilde{e}$ of X is homologous to $e + kv$, where $k \in \mathbb{Z}$ and v is the integer homology class of a fiber. When $\tilde{e} \neq e$, we have $\tilde{e} \circ e \geq 0$, which means that $k \geq d$. Then $\tilde{e} \circ \tilde{e} \geq d$ and this proves that e is the only holomorphic section of X with negative square. Thus this section is invariant under the real structure of X, and so is its normal bundle. This implies that there exists a real structure on L^* which lifts c_B. Using duality, there exists one on L which lifts c_B.

Suppose now that $\deg(L) = 0$. If L is the trivial bundle, then $X = B \times \mathbb{C}P^1$ and nothing has to be proved. Otherwise, the sections of X associated to L and L_0 are the only ones with zero squares. Indeed, a third holomorphic section with zero square should be disjoint from them and these three sections would give a trivialization of X. This would contradict the assumption that $X \neq B \times \mathbb{C}P^1$. As a consequence, we deduce the following alternative: either the real structure c_X preserves these two sections, or it exchanges them. In the first case, c_X preserves the normal bundles and we conclude as before. In the second case, c_X exchanges the normal bundles and so defines a morphism $\hat{c}_X : L^* \to L$, fibered over c_B. Let s be a meromorphic section of L^*, so that $\operatorname{div}(s) = -D$ where D is a divisor associated to L. Then $\hat{c}_X \circ s \circ c_B$ is a meromorphic section of L and $\operatorname{div}(\hat{c}_X \circ s \circ c_B) = c_B^*(\operatorname{div}(s)) = -c_B^*(D)$. Hence $c_B^*(L) = L^*$.

$\Longleftarrow$: If there exists a real structure on L which lifts c_B, then taking the direct sum with c_{L_0} we get a real structure on $L \oplus L_0$ which lifts c_B. This structure induces on $X = P(L \oplus L_0)$ a real structure which lifts c_B. If $c_B^*(L) = L^*$, the result follows from proposition 1.6. □

Proof of theorem 2.3. When $X = B \times \mathbb{C}P^1$, the second part of theorem 2.3 is clear. Indeed, in this case every real structure on X which lifts c_B is the direct sum of c_B and a real structure on $\mathbb{C}P^1$. Moreover, the group of automorphisms of X fibered over the identity is then equal to $\{id\} \times Aut(\mathbb{C}P^1)$. So the second part of theorem 2.3 follows from the well known fact that, up to conjugation, there are two real structures on $\mathbb{C}P^1$. Thus, from now on, we can assume that $L \neq L_0$. It follows from proposition 2.5 that if there exists a real structure on X which lifts c_B, then either there exists a real structure c_L on L which lifts c_B, or $c_B^*(L) = L^*$. This already proves the last line of theorem 2.3. We will show the theorem in three steps.

In the first step, we will prove that if there exists a real structure c_L on L which lifts c_B, then every real structure on X of the form $c_X \circ \phi$, where c_X is the real structure of X induced by $c_L \oplus c_{L_0}$ and ϕ is an automorphism of X fibered over the identity of B which lifts to an automorphism of $E = L \oplus L_0$, is conjugated to c_X. In the second step, we will prove that if $c_B^*(L) = L^*$, then every real structure on X of the form $c_X^+ \circ \phi$, where ϕ is an automorphism of X fibered over the identity of B which lifts to an automorphism of $E = L \oplus L_0$, is conjugated either to c_X^+ or to c_X^-. Finally, in the third step, we will prove that if $c_B^*(L) = L^* = L$, then every real structure on X of the form $c_X^+ \circ \phi$, where ϕ is an automorphism of X fibered over the identity of B which does not lift to an automorphism of $E = L \oplus L_0$, is conjugated to a real structure of the form $c_L \oplus c_{L_0}$, where c_L is a real structure on L which lifts c_B. Furthermore, this conjugation is given by an automorphism of X fibered over the identity of B which lifts to an automorphism of $E = L \oplus L_0$. In particular, when there is no real structure on L which lifts c_B, every antiholomorphic map of the form $c_X^+ \circ \phi$, where ϕ is an automorphism of X fibered over the identity of B which does not lift to an automorphism of $E = L \oplus L_0$, is not an involution. The theorem follows from these three steps and proposition 2.1.

First step: Suppose that there exists a real structure c_L on L which lifts c_B and let c_X be the real structure of X induced by $c_L \oplus c_{L_0}$. Let $\tilde{c}_X$ be another real structure on X which is of the form $c_X \circ \phi$, where ϕ is an automorphism of X fibered over the identity of B which lifts to an automorphism of $E = L \oplus L_0$. The aim of this first step is to prove that c_X and $\tilde{c}_X$ are conjugated.

Let Φ be an automorphism of $E = L \oplus L_0$ which lifts ϕ. Then $\Phi \in End(E) = E \otimes E^* = L \oplus L^* \oplus L_0 \oplus L_0$. Thus there exist $a, d \in \mathbb{C}^*$, $b \in H^0(B, L^*)$ and $c \in H^0(B, L)$ such that

$$\Phi = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

By assumption, the line bundle L is not trivial, so that either L or L^* has no non-zero holomorphic section. Without loss of generality, we can assume that it is L, so that $c = 0$ and

$$\Phi = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}.$$

By assumption, $\tilde{c}_X^2 = id$, which implies that $c_X \circ \phi \circ c_X = \phi^{-1}$. So there exists $\lambda \in \mathbb{C}^*$ such that $c_E \circ \Phi \circ c_E = \lambda \Phi^{-1}$. But

$$\Phi^{-1} = \frac{1}{ad} \begin{bmatrix} d & 0 \\ -b & a \end{bmatrix},$$

and

$$c_E \circ \Phi \circ c_E = \begin{bmatrix} \overline{a} & 0 \\ c_{L_0} \circ b \circ c_L & \overline{d} \end{bmatrix}.$$

Put $\tilde{\lambda} = \frac{1}{ad}\lambda$, we have $\tilde{\lambda} d = \overline{a}$, $\tilde{\lambda} a = \overline{d}$ and $-\tilde{\lambda} b = c_{L_0} \circ b \circ c_L$. The two first conditions imply that $|\tilde{\lambda}| = 1$. Thus there exists $\theta \in \mathbb{R}$ such that $\tilde{\lambda} = \exp(2i\theta)$. So the previous conditions can be rewritten as $\exp(i\theta)d = \overline{\exp(i\theta)a}$, $\exp(i\theta)a = \overline{\exp(i\theta)d}$ and $-\exp(i\theta)b = c_{L_0} \circ (\exp(i\theta)b) \circ c_L$. Hence we can assume that

$$\Phi = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix},$$

where $d = \overline{a}$ and $b = -c_{L_0} \circ b \circ c_L$ (replace Φ by $\exp(i\theta)\Phi$ which also lifts ϕ).

Now, denote by Ψ the automorphism of E defined by

$$\Psi = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}b & \overline{a} \end{bmatrix}.$$

Then

$$\Psi^{-1} = \frac{1}{\overline{a}} \begin{bmatrix} \overline{a} & 0 \\ -\frac{1}{2}b & 1 \end{bmatrix}, \text{ and}$$

$$\begin{aligned} \Psi^{-1} \circ c_E \circ \Psi &= \frac{1}{\overline{a}} \begin{bmatrix} \overline{a}c_L & 0 \\ -\frac{1}{2}b \circ c_L + \frac{1}{2}c_{L_0} \circ b & ac_{L_0} \end{bmatrix} \\ &= \frac{1}{\overline{a}} \begin{bmatrix} \overline{a}c_L & 0 \\ c_{L_0} \circ b & ac_{L_0} \end{bmatrix} \\ &= \frac{1}{\overline{a}} c_E \circ \Phi. \end{aligned}$$

(For the second equality, we used the relation $-b \circ c_L = c_{L_0} \circ b$.) Denote by ψ the automorphism of X induced by Ψ, we then deduce that $\psi^{-1} \circ c_X \circ \psi = \tilde{c}_X$, which was the aim of this first step.

Second step: Suppose that $c_B^*(L) = L^*$ and fix a real structure c_X^+ on X given by proposition 1.6 (see remark 1.7). Let $\tilde{c}_X$ be another real structure on X which is of the form $c_X^+ \circ \phi$, where ϕ is an automorphism of X fibered over the identity of B which lifts to an automorphism of $E = L \oplus L_0$. The aim of this second step is to prove that $\tilde{c}_X$ is conjugated either to c_X^+ or to c_X^-. Let Φ be an automorphism of $E = L \oplus L_0$ which lifts ϕ. Since $\deg(L) = 0$ and since L is not trivial, we know that $H^0(B, L) = H^0(B, L^*) = 0$. As a consequence, there exists $a, d \in \mathbb{C}^*$ such that

$$\Phi = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Since $\tilde{c}_X^2 = id$, we know that $\frac{a}{d} \in \mathbb{R}^*$ and we can assume that $a = 1$, $d \in \mathbb{R}^*$ (replace Φ by $\frac{1}{a}\Phi$). Let ψ be the automorphism of X defined by

$$\psi = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix},$$

where $\delta = \frac{1}{\sqrt{|d|}}$. Then ψ conjugates $\tilde{c}_X$ to one of the two real structures c_X^+ or c_X^-.

Third step: Suppose that $c_B^*(L) = L^*$ and fix a real structure c_X^+ on X given by proposition 1.6. Let $\tilde{c}_X$ be another real structure on X which is of the form $c_X^+ \circ \phi$, where ϕ is an automorphism of X fibered over the identity of B which does not lift to an automorphism of $E = L \oplus L_0$. The aim of this third step is to prove that $\tilde{c}_X$ is conjugated to a real structure of the form $c_L \oplus c_{L_0}$ where c_L is a real structure on L which lifts c_B. Note that the automorphism ϕ and the involution c_X^+ both exchange the sections of X associated to L and L_0. Thus $\tilde{c}_X$ preserves these two sections. As a consequence, it preserves also the normal bundles of these sections and so induces a real structure on the line bundle L which lifts c_B. Consider then the real structure $c_L \oplus c_{L_0}$ on X, it follows from the first and the second steps that it is conjugated to $\tilde{c}_X$ by an automorphism of X which lifts to an automorphism of E. □

2.3. When are c_X^+ and c_X^- conjugated?

In this subsection, a sufficient condition for c_X^+ and c_X^- to be conjugated is given (see proposition 2.6). One important example where this occurs is given by corollary 2.8.

Proposition 2.6. *Let L be a line bundle over (B, c_B) such that $c_B^*(L) = L^*$ and let $X = P(L \oplus L_0)$ be the associated ruled surface. Let (D, f_D) be a couple given by lemma 1.3 and c_{f_D}, c_{-f_D} be the associated real structures of X (see proposition 1.6). Suppose that there exists $\varphi \in Aut(B)$ of finite order such that $\varphi \circ c_B = c_B \circ \varphi$ and:*

a. either $\varphi^*(L) = L$ and there exists a meromorphic function g on B such that $\mathrm{div}(g) = \varphi(D) - D$ and $(f_D \circ \varphi)(g \circ \varphi)\overline{g \circ c_B \circ \varphi} = -f_D$,

b. or $\varphi^*(L) = L^*$ and there exists a meromorphic function h on B such that $\mathrm{div}(h) = \varphi(D) + D$ and $(h \circ \varphi)\overline{h \circ c_B \circ \varphi} = -f_D f_D \circ \varphi$.

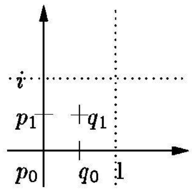
Then, the real structures c_{f_D} and c_{-f_D} are conjugated in X.

Remark 2.7. When $\mathbb{R}B \neq \emptyset$, the conditions a and b can be replaced by $\varphi^*(L) \in \{L, L^*\}$ and there exists $x \in \mathbb{R}B$ such that $(f_D f_D \circ \varphi)(x) < 0$. Indeed, it is not difficult to check that in the situation a, there always exists a meromorphic function g on B such that $\mathrm{div}(g) = \varphi(D) - D$ and $(f_D \circ \varphi)(g \circ \varphi)\overline{g \circ c_B \circ \varphi} = \epsilon f_D$ where $\epsilon = \pm 1$. Similarly, in the situation b, there always exists a meromorphic function h on B such that $\mathrm{div}(h) = \varphi(D) + D$ and $(h \circ \varphi)\overline{h \circ c_B \circ \varphi} = \epsilon f_D f_D \circ \varphi$, where $\epsilon = \pm 1$. Hence, conditions a or b are equivalent to require that $\epsilon = -1$, which is equivalent, when $\mathbb{R}B \neq \emptyset$, to require that there exists $x \in \mathbb{R}B$ such that $(f_D f_D \circ \varphi)(x) < 0$.

Note that when $g(B) \geq 2$, the conditions given by proposition 2.6 are in fact necessary and sufficient for c_{f_D} and c_{-f_D} to be conjugated, but this will not be needed in what follows.

Corollary 2.8. *Let $g \geq 1$ be an odd integer. Then there exists a smooth compact irreducible real algebraic curve (B, c_B) of genus g and empty real part together with a complex line bundle L over B satisfying $c_B^*(L) = L^*$, such that the real structures c_X^+ and c_X^- on $X = P(L \oplus L_0)$ are conjugated.*

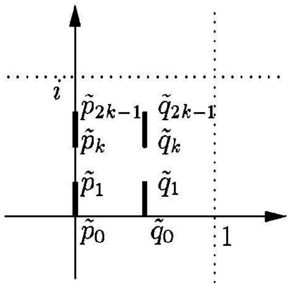
Proof. Let us consider first the case $g = 1$. Let B be the elliptic curve $\mathbb{C}/\mathbb{Z}[i]$ equipped with the real structure $c_B(z) = \overline{z} + \frac{1}{2}$, so that $\mathbb{R}B = \emptyset$. Let $p_0 = 0$, $q_0 = \frac{1}{2}$, $p_1 = \frac{i}{2}$ and $q_1 = \frac{1}{2} + \frac{i}{2}$.



Let $D = p_1 - p_0$ and denote by L the associated complex line bundle over B. Then $c_B^*(L) = L = L^*$. Denote by φ the involutive automorphism of B defined by $\varphi(z) = z + \frac{1}{2}$. Then $\varphi \circ c_B = c_B \circ \varphi$ and $\varphi^*(L) = L$. We will prove that φ satisfies condition a of proposition 2.6.

For this, let f be a meromorphic function on B given by lemma 1.3, such that $\overline{f \circ c_B} = f$ and $\mathrm{div}(f) = D + c_B(D)$. Then $f \circ \varphi = f$. Indeed, there exists a holomorphic section s of the line bundle L such that $\mathrm{div}(s) = D$ and $s \otimes (s \circ \varphi) = f$. Thus $f \circ \varphi = (s \circ \varphi) \otimes s = s \otimes (s \circ \varphi) = f$. Now let g be a meromorphic function on B such that $\mathrm{div}(g) = \varphi(D) - D = q_1 - p_1 - q_0 + p_0$ and $g(\overline{g \circ c_B}) = -1$. Such a function is given by lemma 1.1 and [7], proposition 2.2, since D belong to the nontrivial component of the real part of $(\mathrm{Jac}(B), c_B)$. Then $(f \circ \varphi)(g \circ \varphi)\overline{g \circ c_B \circ \varphi} = -f$, so that the condition a of proposition 2.6 is satisfied. We deduce that the real structures c_X^+ and c_X^- on $X = P(L \oplus L_0)$ defined by f and $-f$ (see proposition 1.6) are conjugated.

Now, let us consider the case $g = 2k+1$, $k \geq 1$. For $j \in \{0, \dots, 2k-1\}$, denote by $\tilde{p}_j = \frac{j}{2k} i \in B$ and $\tilde{q}_j = \frac{1}{2} + \frac{j}{2k} i \in B$ (so that $p_1 = \tilde{p}_k$ and $q_1 = \tilde{q}_k$). Denote by B_k the double covering of B ramified over the $4k$ points $\tilde{p}_j$, $\tilde{q}_j$, $j \in \{0, \dots, 2k-1\}$. This covering can be chosen so that its characteristic class in $H^1(B \setminus \{\tilde{p}_j, \tilde{q}_j \,|\, j \in \{0, \dots, 2k-1\}\}; \mathbb{Z}/2\mathbb{Z})$ is Poincaré dual to the sum of the $2k$ segments $\{(0,t) \,|\, t \in]\frac{2j}{2k}, \frac{2j+1}{2k}[, j \in \{0, \dots, k-1\}\}$ and $\{(\frac{1}{2}, t) \,|\, t \in]\frac{2j}{2k}, \frac{2j+1}{2k}[, j \in \{0, \dots, k-1\}\}$.



Denote by $\pi_k : B_k \to B$ the projection associated to the covering. The automorphism φ of B lifts to an automorphism φ_k of B_k such that $\varphi \circ \pi_k = \pi_k \circ \varphi_k$. Similarly, the real structure c_B lifts to a real structure c_{B_k} on B_k such that $c_B \circ \pi_k = \pi_k \circ c_{B_k}$ and $\mathbb{R}B_k = \emptyset$. Denote by $L_k = \pi_k^*(L)$. This bundle satisfies $c_{B_k}^*(L_k) = L_k = L_k^* = \varphi_k^*(L_k)$. Finally, denote by $f_k = f \circ \pi_k$ and $g_k = g \circ \pi_k$. Then $f_k = \overline{f_k \circ c_{B_k}}$ and $g_k \overline{g_k \circ c_{B_k}} = -1$. Moreover, $\mathrm{div}(f_k) =$

$c^*_{B_k}(D_k) + D_k$, where $D_k = \pi^*_k(D) = 2p_1 - 2p_0$, and $\mathrm{div}(g_k) = \varphi^*_k(D_k) - D_k$. We have, $(f_k \circ \varphi_k)(g_k \circ \varphi_k)\overline{g_k \circ c_{B_k} \circ \varphi_k} = -f_k$, so that the condition a of proposition 2.6 is satisfied. We deduce that the real structures $c^+_{X_k}$ and $c^-_{X_k}$ on $X_k = P(L_k \oplus L_0)$ defined by f_k and $-f_k$ (see proposition 1.6) are conjugated. $\square$

Proof of proposition 2.6. Denote $D = \sum_{i=1}^k n_i p_i$, where $p_i \in B$ and $n_i \in \mathbb{Z}$, $i \in \{1, \dots, k\}$. We can assume that the set $\{p_i \,|\, 1 \leq i \leq k\}$ is invariant under φ (add some points with zero coefficients to D if necessary). Denote by $U_0 = B \setminus \{p_i \,|\, 1 \leq i \leq k\}$ and for every $i \in \{1, \dots, k\}$, choose some holomorphic chart (U_{p_i}, ϕ_{p_i}) such that $U_{p_i} \cap U_{p_j} = \emptyset$ if $i \neq j$, $\varphi(U_{p_i}) = U_{\varphi(p_i)}$ and $\phi_{p_i} : U_{p_i} \to \Delta = \{z \in \mathbb{C} \,|\, |z| < 1\}$ is a biholomorphism. We require in addition that $\phi_{p_i}(p_i) = 0$ and

$$\begin{aligned} \phi_{\varphi(p_i)} \circ \varphi \circ \phi^{-1}_{p_i} : \Delta &\to \Delta \\ x &\mapsto \exp(\tfrac{2i\pi}{m_i})x \text{ if } p_i \text{ is a fixed point of order } m_i \text{ of } \varphi. \end{aligned}$$

(We put $m_i = 1$ if $\varphi(p_i) \neq p_i$. This atlas and these trivializations are compatible with D and the group $< \varphi >$. It always exists, see [12].)

For every $i \in \{1, \dots, k\}$, denote by ψ_i the morphism:

$$\begin{aligned} (U_{p_i} \setminus p_i) \times \mathbb{C}P^1 &\to U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) &\mapsto (x, (\phi_{p_i}(x)^{-n_i} z_1 : z_0)). \end{aligned}$$

The morphisms ψ_i allow to glue together the trivializations $U_{p_i} \times \mathbb{C}P^1$, $i \in \{0, \dots, k\}$, in order to define the ruled surface X.

Now suppose we are in the case a. Let g be the meromorphic function on B such that $\mathrm{div}(g) = \varphi(D) - D$ and $(f_D \circ \varphi)(g \circ \varphi)\overline{g \circ c_B \circ \varphi} = -f_D$. Consider the maps:

$$\begin{aligned} U_0 \times \mathbb{C}P^1 &\to U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) &\mapsto (\varphi(x), (g \circ \varphi(x) z_1 : z_0)), \end{aligned}$$

and for every $i \in \{1, \dots, k\}$,

$$\begin{aligned} U_{p_i} \times \mathbb{C}P^1 &\to U_{p_j} \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) &\mapsto (\varphi(x), (g \circ \varphi(x) \phi_{p_i}(x)^{n_j - n_i} \exp(\tfrac{2i\pi}{m_i}) z_1 : z_0)), \end{aligned}$$

where p_j denotes the point $\varphi(p_i)$. These maps glue together to form an element $\Phi_g \in Aut(X)$ fibered over φ.

The map Φ_g^{-1} is given by:

$$\begin{aligned} U_0 \times \mathbb{C}P^1 &\to U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) &\mapsto (\varphi^{-1}(x), (z_1 : g(x) z_0)). \end{aligned}$$

And the map c^-_X is given by:

$$\begin{aligned} U_0 \times \mathbb{C}P^1 &\to U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) &\mapsto (c_B(x), (\overline{z_0} : -f_D \circ c_B(x) \overline{z_1})). \end{aligned}$$

Thus $\Phi_g^{-1} \circ c^-_X \circ \Phi_g$ is given in this trivialization by:

$$\begin{aligned} U_0 \times \mathbb{C}P^1 &\to U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) &\mapsto (c_B(x), (\overline{z_0} : -(f_D \circ c_B \circ \varphi)(\overline{g \circ \varphi})(g \circ c_B \circ \varphi)(x) \overline{z_1})). \end{aligned}$$

Since $(f_D \circ \varphi)(g \circ \varphi)\overline{g \circ c_B \circ \varphi} = -f_D$, we conclude that $\Phi_g^{-1} \circ c_X^- \circ \Phi_g = c_X^+$.

Suppose now we are in the case b. Let h be the meromorphic function on B such that $\mathrm{div}(h) = \varphi(D) + D$ and $(h \circ \varphi)\overline{h \circ c_B \circ \varphi} = -f_D f_D \circ \varphi$. Consider then the maps:

$$\begin{array}{rcl} U_0 \times \mathbb{C}P^1 & \to & U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) & \mapsto & (\varphi(x), (z_0 : h \circ \varphi(x) z_1)) \end{array}$$

and for all $i \in \{1, \dots, k\}$,

$$\begin{array}{rcl} U_{p_i} \times \mathbb{C}P^1 & \to & U_{p_j} \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) & \mapsto & (\varphi(x), (z_0 : h \circ \varphi(x) \phi_{p_i}(x)^{-n_i - n_j} \exp(-\frac{2i\pi}{m_i}) z_1)), \end{array}$$

where p_j denotes the point $\varphi(p_i)$. These maps glue together to form an element $\Phi_h \in Aut(X)$ fibered over φ.

The map Φ_h^{-1} is given by:

$$\begin{array}{rcl} U_0 \times \mathbb{C}P^1 & \to & U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) & \mapsto & (\varphi^{-1}(x), (z_0 : h(x) z_1)). \end{array}$$

And the map c_X^- is given by:

$$\begin{array}{rcl} U_0 \times \mathbb{C}P^1 & \to & U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) & \mapsto & (c_B(x), (\overline{z_0} : -f_D \circ c_B(x) \overline{z_1})). \end{array}$$

Thus $\Phi_h^{-1} \circ c_X^- \circ \Phi_h$ is given in this trivialization by:

$$\begin{array}{rcl} U_0 \times \mathbb{C}P^1 & \to & U_0 \times \mathbb{C}P^1 \\ (x, (z_1 : z_0)) & \mapsto & (c_B(x), (-f_D \circ c_B \circ \varphi(x) \overline{z_0} : \overline{h \circ \varphi(x)} (h \circ c_B \circ \varphi)(x) \overline{z_1})). \end{array}$$

Since $(h \circ \varphi)\overline{h \circ c_B \circ \varphi} = -f_D f_D \circ \varphi$, we conclude that $\Phi_h^{-1} \circ c_X^- \circ \Phi_h = c_X^+$. $\square$

3. Deformation classes of real structures on ruled surfaces

3.1. The real part of $(\mathrm{Jac}(B), -c_B^*)$

Remember the following well known result (see, for instance, [7], propositions 3.2 and 3.3):

Proposition 3.1. *Let (B, c_B) be a smooth compact irreducible real algebraic curve. The Jacobian $\mathrm{Jac}(B)$ of B is equipped with the real structure $-c_B^*$. Then if $\mathbb{R}B \neq \emptyset$, the real part of $(\mathrm{Jac}(B), -c_B^*)$ has $2^{\mu(\mathbb{R}B)-1}$ connected components, where $\mu(\mathbb{R}B)$ is the number of components of $\mathbb{R}B$. If $\mathbb{R}B = \emptyset$, the real part of $(\mathrm{Jac}(B), -c_B^*)$ is connected if $g(B)$ is even and consists of two connected components otherwise.* $\square$

(Note that multiplication of c_B^* by -1 does not change the topology of the real part of $\mathrm{Jac}(B)$.)

Let L be a complex line bundle over B such that $c_B^*(L) = L^*$, that is an element of the real part of $(\mathrm{Jac}(B), -c_B^*)$, where $\mathrm{Jac}(B)$ is identified with the part of $\mathrm{Pic}(B)$ of degree zero. Let (D, f_D) be a couple given by lemma 1.3. The function f_D is real and of constant sign on every component of $\mathbb{R}B$, thus it induces a partition of $\mathbb{R}B$ in two elements $\mathbb{R}B \cap \overline{f_D^{-1}(\mathbb{R}_+^*)}$ and $\mathbb{R}B \cap \overline{f_D^{-1}(\mathbb{R}_-^*)}$. It follows from theorem 2.3 that this partition only depends on the bundle L and not on the choice of (D, f_D), since it corresponds to the projections on $\mathbb{R}B$ of the real parts of $(P(L \oplus L_0), c_X^+)$ and $(P(L \oplus L_0), c_X^-)$. For the same reason, this partition actually only depends on the connected component of the real part of $(\mathrm{Jac}(B), -c_B^*)$ and hence is an invariant associated to these components. Note that when $\mathbb{R}B \neq \emptyset$ has $\mu(\mathbb{R}B)$ components, the number of partitions of $\mathbb{R}B$ in two elements is $2^{\mu(\mathbb{R}B)-1}$.

Lemma 3.2. *When $\mathbb{R}B \neq \emptyset$, the partitions associated to the real components of $(\mathrm{Jac}(B), -c_B^*)$ establish a bijection between the set of these components and the set of partitions of $\mathbb{R}B$ in two elements.*

Proof. Let L and L' be two complex line bundles which belong to $\mathbb{R}\,\mathrm{Jac}(B)$ and such that their associated partitions of $\mathbb{R}B$ are the same. We will prove that they belong to the same component of $\mathbb{R}\,\mathrm{Jac}(B)$. The result follows, since the "partition map" is then injective and hence bijective for cardinality reasons.

Let D (resp. D') be a divisor associated to L (resp. L'). Let f_D (resp. $f_{D'}$) be a non-zero meromorphic function on B such that $\overline{f_D \circ c_B} = f_D$ (resp. $\overline{f_{D'} \circ c_B} = f_{D'}$) and $\mathrm{div}(f_D) = D + c_B(D)$ (resp. $\mathrm{div}(f_{D'}) = D' + c_B(D')$). It follows from lemma 1.3 that such meromorphic functions exist. Since the partitions of L and L' are the same, we can assume that f_D and $f_{D'}$ have the same signs on every components of $\mathbb{R}B$ (replace $f_{D'}$ by $-f_{D'}$ otherwise). For every $t \in [0,1]$, let $g_t = (1-t)f_D + tf_{D'}$. Then $g_0 = f_D$, $g_1 = f_{D'}$ and for every $t \in [0,1]$, $\overline{g_t \circ c_B} = g_t$. Moreover, for every $t \in [0,1]$, g_t is non-zero and of constant sign on each component of $\mathbb{R}B$. Thus every real zero and real pole of g_t is of even order. This implies that there exists a continuous path $(D_t)_{t\in[0,1]}$ of divisors such that $D_0 = D$ and for every $t \in [0,1]$, $\mathrm{div}(g_t) = D_t + c_B(D_t)$. In particular, L and L_1 are in the same component of $\mathbb{R}\,\mathrm{Jac}(B)$, where L_1 is the complex line bundle associated to D_1. It suffices then to prove that L_1 and L' lie in the same component of $\mathbb{R}\,\mathrm{Jac}(B)$.

Now $D_1 + c_B(D_1) = D' + c_B(D') = \mathrm{div}(g_1)$. So the divisor $E = D_1 - D'$ satisfy $c_B(E) = -E$. Thus there exist $k \in \mathbb{N}$ and $p^1, \dots, p^k \in B$ such that $E = \sum_{i=1}^k n_i(p^i - c_B(p^i))$. For every $i \in \{1, \dots, k\}$, choose a continuous path $(p^i_\tau)_{\tau\in[0,1]}$ such that $p^i_0 = p^i$ and $p^i_1 \in \mathbb{R}B$. For every $\tau \in [0,1]$, let $E_\tau = \sum_{i=1}^k n_i(p^i_\tau - c_B(p^i_\tau))$. Then $E_0 = E$, $E_1 = 0$ and for every $\tau \in [0,1]$, $c_B(E_\tau) = -E_\tau$. The path $F_\tau = D' + E_\tau$ is a continuous path of divisors such that $F_0 = D_1$, $F_1 = D'$ and for every $\tau \in [0,1]$, $F_\tau + c_B(F_\tau) = \mathrm{div}(g_1)$. This implies that the bundles L_1 and L' belong to the same component of $\mathbb{R}\,\mathrm{Jac}(B)$, hence the result. □

Remark 3.3. Actually, the surjectivity of the "partition map" follows from a theorem of Witt (see [17] or [16], p. 101–102).

3.2. The topological type of a real ruled surface

Remember that to every smooth compact irreducible real algebraic curve (B, c_B) is associated a triple (g, μ, ϵ), called the *topological type* of (B, c_B), where g is the genus of B, μ is the number of connected components of $\mathbb{R}B$ and $\epsilon = 1$ (resp. $\epsilon = 0$) if B is dividing (resp. if B is non-dividing). Two smooth compact irreducible real algebraic curves are in the same deformation class if and only if they have the same topological type (see [11]). Moreover, there exists a smooth compact irreducible real algebraic curve of topological type (g, μ, ϵ) if and only if $\epsilon = 0$ and $0 \leq \mu \leq g$ or $\epsilon = 1$, $1 \leq \mu \leq g + 1$ and $\mu = g + 1 \mod (2)$.

With the exception of the ellipsoid, that is $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the real structure $(x, y) \mapsto (\overline{y}, \overline{x})$, for every real structure c_X on a ruled surface $p : X \to B$, there exists a real structure c_B on the base B such that $p \circ c_X = c_B \circ p$. In particular, the connected components of $\mathbb{R}X$ are tori or Klein bottles. Note also that in the case of $\mathbb{C}P^1 \times \mathbb{C}P^1$, the ruling given by the projection p is not unique, whereas it is for any other ruled surface. Since real structures on rational ruled surfaces are well known (see theorem 3.6), we will assume from now on that **the genus of the base is non-zero**. So let (X, c_X) be a real non-rational ruled surface of base (B, c_B). The *topological type* of (X, c_X) is by definition the quintuple (t, k, g, μ, ϵ), where (g, μ, ϵ) is the topological type of (B, c_B), k is the number of Klein bottles of $\mathbb{R}X$ and t the number of tori of $\mathbb{R}X$. Obviously $t, k \geq 0$ and $t + k \leq \mu$. A quintuple (t, k, g, μ, ϵ) is called *allowable* if $t, k \geq 0$, $t + k \leq \mu$, $g \geq 1$ and either $\epsilon = 0$ and $0 \leq \mu \leq g$ or $\epsilon = 1$, $1 \leq \mu \leq g + 1$ and $\mu = g + 1 \mod (2)$.

Proposition 3.4. *There exists a real ruled surface of topological type (t, k, g, μ, ϵ) if and only if the quintuple (t, k, g, μ, ϵ) is allowable.*

Proof. If (t, k, g, μ, ϵ) is the topological type of a real ruled surface, then the quintuple (t, k, g, μ, ϵ) is clearly allowable. Now, let (t, k, g, μ, ϵ) be an allowable quintuple. It is well known (see [11] for instance) that there exists a smooth compact connected real algebraic curve (B, c_B) whose topological type is (g, μ, ϵ). If $\mu = 0$, the ruled surface $(B \times \mathbb{C}P^1, c_B \times conj)$, where $conj$ is a real structure on $\mathbb{C}P^1$, is of topological type $(0, 0, g, 0, 0)$. If $\mu \neq 0$, choose a partition $\mathcal{P}$ of $\mathbb{R}B$ in two elements such that one of them contains $t + k$ components of $\mathbb{R}B$ and the other one $\mu - t - k$. It follows from lemma 3.2 that there exists a line bundle L over B such that $c_B^*(L) = L^*$ and the partition associated to L is $\mathcal{P}$. Thus, it follows from proposition 1.6 that there exists a real structure c_X^+ on the ruled surface $X = P(L \oplus L_0)$ such that the real part of X consists of $t + k$ tori. Choose k of these tori and make an elementary transformation on each of them, that is the composition of the blowing up at one point and the blowing down of the strict transform of the fiber passing through this point. The result is still a real ruled surface of base (B, c_B) and the real part of this ruled surface consists of t tori and k Klein bottles, hence the result. □

3.3. The deformation theorem

Let $\Delta \subset \mathbb{C}$ be the Poincaré's disk equipped with the complex conjugation $conj$. A *real deformation* of surfaces is a proper holomorphic submersion $\pi : Y \longrightarrow \Delta$ where (Y, c_Y) is a real analytic manifold of dimension 3 and π satisfies $\pi \circ c_Y = conj \circ \pi$. When $t \in]-1, 1[\in \Delta$, the fibers $Y_t = \pi^{-1}(t)$ are invariant under c_Y and are then compact real analytic surfaces. Two real analytic surfaces X' and X'' are said to be *in the same deformation class* if there exists a chain $X' = X_0, \dots, X_k = X''$ of compact real analytic surfaces such that for every $i \in \{0, \dots, k-1\}$, the surfaces X_i and X_{i+1} are isomorphic to some real fibers of a real deformation.

Proposition 3.5. *The topological type of a real non-rational ruled surface is invariant under deformation.*

Proof. Let $(X, c_X) \longrightarrow (B, c_B)$ be a real ruled surface of topological type (t, k, g, μ, ϵ) with $g \geq 1$. Let $\pi : Y \longrightarrow \Delta$ be a real deformation of surfaces such that $(Y_0, c_Y|_{Y_0}) = (X, c_X)$. Then every fiber of π is a ruled surface with base of genus g (see [1] for instance). Now since the deformation is trivial from the differentiable point of view, the topology of the real part and the topology of the involution on the base are invariant under deformation, hence the result. □

For the sake of completeness, let us recall the following well known result, see [5] or [6]:

Theorem 3.6. *There are four deformation classes of real structures on rational ruled surfaces, one for which the real part is a torus, one for which the real part is a sphere and two for which the real part is empty. These two latter have non-homeomorphic quotients.* □

Remember that the real structure for which the real part is a sphere is very special. It only exists on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and is fibered over no real structure on the base $\mathbb{C}P^1$. This comes from the existence of two rulings on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the involution $(x, y) \mapsto (y, x)$ reversing them. This is the main reason why we do not include the case of rational ruled surfaces in theorem 3.7.

Theorem 3.7. *Two real non-rational ruled surfaces are in the same deformation class if and only if they have the same topological type (t, k, g, μ, ϵ), except when $\mu = 0$. There are two deformation classes of real non-rational ruled surfaces of topological type $(0, 0, g, 0, 0)$. For one such class of ruled surfaces (X, c_X), the* *quotient $X' = X/c_X$ is spin, for the other one it is not.*

This theorem 3.7 is a reformulation of the theorem 0.1 mentioned in the introduction. Using the terminology introduced in [5], it means that real ruled surfaces are quasi-simple. The definition of the topological type of a real ruled surface is given in §3.2. Note that every allowable quintuple is the topological type of a real ruled surface (see proposition 3.4).

Remark 3.8. If $X = P(E)$ is a real non-rational ruled surface of topological type (t, k, g, μ, ϵ) with $t + k < \mu$ and $k \neq 0$, then X is not decomposable, whereas any other topological type is realized by a decomposable real ruled surface. Remember also that the deformation classes of complex ruled surfaces are described by the genus of the base and by whether the surface is spin or not (see [14], theorem 5). Then, real structures for which k is even only exist on spin ruled surfaces and real structures for which k is odd only exist on non-spin ruled surfaces.

Let us sketch the proof of theorem 3.7.

Let (X, c_X) be a real ruled and non-decomposable surface with base (B, c_B). If X admits a real holomorphic section, then we will prove that (X, c_X) is in the same deformation class that a real decomposable ruled surface (see proposition 3.9). If X does not admit a real holomorphic section, then we will prove that there exists a complex line bundle $L \in \mathrm{Pic}(B)$ satisfying $c_B^*(L) = L^*$, such that (X, c_X) is in the same deformation class that the surface obtained from $(P(L \oplus L_0), c_X^{\pm})$ after at most one elementary transformation on each component of its real part (see proposition 3.10).

After these two steps, it is possible to reduce the study of deformation classes of real structures on ruled surfaces to the study of deformation classes of real structures on decomposable ruled surfaces. It suffices then to check the theorem 3.7 for decomposable real ruled surfaces.

Proposition 3.9. *Let (X, c_X) be a real ruled surface of base (B, c_B) which admits a real holomorphic section. Then there exists a real deformation $\pi : Y \longrightarrow \Delta$ such that for every $t \in \mathbb{R}^* \cap \Delta$, $(Y_t, c_Y|_{Y_t})$ is isomorphic to (X, c_X) and such that $(Y_0, c_Y|_{Y_0})$ is isomorphic to $(P(L \oplus L_0), c_L \oplus c_{L_0})$ where $L \in \mathrm{Pic}(B)$ and c_L is a real structure on L which lifts c_B.*

Proof. Let E be a rank two complex vector bundle over B such that $X = P(E)$. The real holomorphic section of X is given by a complex sub-line bundle M of E. Denote by N the quotient line bundle E/M so that the bundle E is an extension of N by M. Let $\mu \in H^1(B, M \otimes N^*)$ be the extension class of this bundle and let μ^1 be a 1-cocycle with coefficients in the sheaf $\mathcal{O}_B(M \otimes N^*)$, defined on a covering $\mathcal{U} = (U_i)_{i \in I}$ of B, realizing the cohomology class $\mu \in H^1(B, M \otimes N^*)$. The bundle

E is then obtained as the gluing of the bundles $(M \oplus N)|_{U_i}$ by the gluing maps:

$$\begin{aligned} (M \oplus N)|_{U_i \cap U_j} &\to (M \oplus N)|_{U_j \cap U_i} \\ (m,n) &\mapsto \begin{bmatrix} 1 & \mu_{ij} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = (m + \mu_{ij} n, n). \end{aligned}$$

We can assume that for every open set U_i of $\mathcal{U}$, there exists $\bar{\imath} \in I$ such that $U_{\bar{\imath}} = c_B(U_i)$ (add these open sets to $\mathcal{U}$ if not). We can also assume that there exists $J \subset I$ such that the open sets $(U_i)_{i \in J}$ cover B and such that the real structure $c_X : X|_{U_i} \to X|_{U_{\bar{\imath}}}$ lifts to an antiholomorphic map $E|_{U_i} \to E|_{U_{\bar{\imath}}}$ (take a refinement of $\mathcal{U}$ if not). Since by hypothesis the section of X associated to M is real, these antiholomorphic maps are of the form:

$$\begin{aligned} (M \oplus N)|_{U_i} &\to (M \oplus N)|_{U_{\bar{\imath}}} \\ (x,(m,n)) &\mapsto (c_B(x), \begin{bmatrix} a_i & b_i \\ 0 & d_i \end{bmatrix} \begin{pmatrix} m \\ n \end{pmatrix}), \end{aligned}$$

where a_i (resp. b_i, resp. d_i) is an antiholomorphic morphism $M|_{U_i} \to M|_{U_{\bar{\imath}}}$ (resp. $N|_{U_i} \to M|_{U_{\bar{\imath}}}$, resp. $N|_{U_i} \to N|_{U_{\bar{\imath}}}$) which lifts c_B. Since c_X is an involution, we have for every $i \in J$, $a_{\bar{\imath}} \circ a_i = d_{\bar{\imath}} \circ d_i \in \mathcal{O}_B^*|_{U_i}$ and $a_{\bar{\imath}} \circ b_i + b_{\bar{\imath}} \circ d_i = 0 \in \mathcal{O}_B(N^* \otimes M)|_{U_i}$. Moreover, for $i, j \in J$ such that $U_i \cap U_j \neq \emptyset$, the gluing conditions are the following: $a_i = \lambda a_j$, $d_i = \lambda d_j$ and $b_i + \mu_{\bar{\imath}\bar{\jmath}} \circ d_i = \lambda(a_j \circ \mu_{ij} + b_j)$ where $\lambda \in \mathcal{O}_B^*|_{U_i \cap U_j}$.

Now let Y be the complex analytic manifold of dimension three defined as the gluing of the charts $\mathbb{C} \times P(M \oplus N)|_{U_i}$, $i \in J$, with change of charts given by the maps:

$$\begin{aligned} \mathbb{C} \times P(M \oplus N)|_{U_i} &\to \mathbb{C} \times P(M \oplus N)|_{U_j} \\ (t, x, (m:n)) &\mapsto (t, x, \begin{bmatrix} 1 & t\mu_{ij} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} m \\ n \end{pmatrix}) = (t, x, (m + t\mu_{ij} n : n)). \end{aligned}$$

The projection on the first coordinate defines a holomorphic submersion $\pi : Y \to \mathbb{C}$. The surface $\pi^{-1}(0)$ is isomorphic to the decomposable ruled surface $P(M \oplus N)$, whereas, as soon as $t \in \mathbb{C}^*$, the fiber $Y_t = \pi^{-1}(t)$ is isomorphic to the ruled surface $X = P(E)$. Such an isomorphism $\psi_t : Y_t \to X$ is given in the charts $P(M \oplus N)|_{U_i}$, $i \in J$, by:

$$\begin{aligned} P(M \oplus N)|_{U_i} &\to P(M \oplus N)|_{U_j} \\ (x, (m:n)) &\mapsto (x, (m : tn)). \end{aligned}$$

Denote by c_Y the real structure on Y defined on charts $\mathbb{C} \times P(M \oplus N)|_{U_i}$ by:

$$\begin{aligned} \mathbb{C} \times P(M \oplus N)|_{U_i} &\to \mathbb{C} \times P(M \oplus N)|_{U_{\bar{\imath}}} \\ (t, x, (m:n)) &\mapsto \left(\bar{t}, c_B(x), \begin{bmatrix} a_i & \bar{t} b_i \\ 0 & d_i \end{bmatrix} \begin{pmatrix} m \\ n \end{pmatrix}\right). \end{aligned}$$

This real structure satisfies $\pi \circ c_Y = conj \circ \pi$ where $conj$ is the complex conjugation on $\mathbb{C}$. Moreover, when $t \in \mathbb{R}^*$, ϕ_t gives an isomorphism between the real ruled surfaces $(Y_t, c_Y|_{Y_t})$ and (X, c_X). Hence, the restriction of $\pi : Y \to \mathbb{C}$ over $\Delta \subset \mathbb{C}$ is a real deformation which satisfies proposition 3.9. $\square$

Proposition 3.10. *Let (X, c_X) be a real ruled surface of base (B, c_B), which does not admit any real holomorphic section. Then, there exists $L \in \mathrm{Pic}(B)$ satisfying $c_B^*(L) = L^*$ and a ruled surface $(X', c_{X'})$ obtained from $(P(L \oplus L_0), c_X^{\pm})$ after at most one elementary transformation on each of its real components, such that (X, c_X) and $(X', c_{X'})$ are in the same deformation class.*

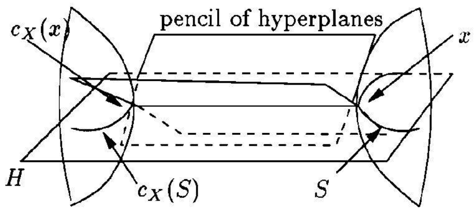
Remember that an *elementary transformation* on the ruled surface X is by definition the composition of a blowing up of X at one point and the blowing down of the strict transform of the fiber passing through this point.

Lemma 3.11. *Let $X = P(L \oplus L_0)$ be a decomposable ruled surface of base B. Let $s : B \to X$ be the section defined by L and D be a divisor associated to L. Then the ruled surface obtained from X after an elementary transformation at the point $s(x)$, $x \in B$, is the surface $P(L(x) \oplus L_0)$ where $L(x)$ is the complex line bundle associated to the divisor $D + x$.* $\square$

Lemma 3.12. *Let (X, c_X) be a real ruled surface of base (B, c_B), which does not admit any real holomorphic section. Then X has a very ample holomorphic section S which is transversal to its image under c_X.*

Proof. Let us first construct a very ample section on X. Let E be a rank two complex vector bundle over B such that $X = P(E)$, and let A be an ample line bundle over B. Then by definition, for sufficiently large n, the bundle $E^* \otimes A^n$ is generated by its global sections. Choosing N such global sections, it provides a surjective morphism of bundles $B \times \mathbb{C}^N \to E^* \otimes A^n$. This induces an injective morphism between the dual bundles $E \otimes (A^*)^n \to B \times \mathbb{C}^N$ and thus an embedding $X \to B \times \mathbb{C}P^{N-1}$. Fixing an embedding $B \to \mathbb{C}P^3$, we deduce an embedding $X \to \mathbb{C}P^3 \times \mathbb{C}P^{N-1}$. Finally, combining this with Segre embedding, we obtain an embedding $X \to \mathbb{C}P^{4N-1}$ associated to a very ample linear system of sections on X.

Now, let us prove that in this linear system, there exists a smooth section S transversal to $c_X(S)$. From Bertini's theorem (see [8], theorem 8.18) there exists, in this linear system, a smooth section S associated to a hyperplane H of $\mathbb{C}P^{4N-1}$ transversal to X. By hypothesis, S cannot be real, so that the intersection $c_X(S) \cap S$ consists of a finite number of points. We will prove that after a small perturbation of H, this intersection can be assumed transversal. Indeed, let $x \in c_X(S) \cap S$. If $x \in \mathbb{R}X$, the intersection of H with T_xX is a line, which is the tangent of S at X. The section S is transverse to $c_X(S)$ at x if and only if this line is not fixed by the differential $d_x c_X$. Since the fixed point set of this involution is of half dimension, the intersection of S and $c_X(S)$ at x can be made transversal after a small perturbation of H, keeping the intersection point x. Now, if $x \notin \mathbb{R}X$, then since the section S is smooth, the points x and $c_X(x)$ belong to two different fibers of X and in particular to non-real ones. Suppose that the line $D_x \subset \mathbb{C}P^{4N-1}$ joining them is transversal to both the planes T_xX and $T_{c_X(x)}X$. Then there exists a pencil $\mathcal{P}$ of hyperplanes of $\mathbb{C}P^{4N-1}$ containing H and parametrized both by the lines of $T_xX \subset \mathbb{C}P^{4N-1}$ and the lines of $T_{c_X(x)}X \subset \mathbb{C}P^{4N-1}$. This means that each line of T_xX passing through x, and similarly each line of $T_{c_X(x)}$ passing through $c_X(x)$, is contained in one and only one hyperplane of $\mathcal{P}$. Also, $\mathcal{P}$ contains no other hyperplane.



This pencil $\mathcal{P}$ thus provides us with a holomorphic identification between the projective lines $P(T_xX)$ and $P(T_{c_X(x)}X)$. Under this identification, the differential d_xc_X reads as an anti-holomorphic involution of T_xX and once more, the section S is transversal to $c_X(S)$ at x if and only its tangent line is not fixed by this involution d_xc_X. This can always be guaranteed after a small perturbation of H. Since small perturbations do not perturb the transversality of transversal points, this process strictly increases the number of transversal points between S and $c_X(S)$ and so gives the result after a finite number of steps. It thus only remains to prove that the line D_x can indeed be assumed transverse to both the planes T_xX and $T_{c_X(x)}X$, after a small perturbation of H if necessary.

For this, note that the embedding $B \to \mathbb{C}P^3$ can be chosen real. The set of points of B whose tangent is not a real line of $\mathbb{C}P^3$ is a dense open subset $U \subset B$ (for the usual topology, not the Zariski's one), invariant under c_B. The set U is in fact the complementary of the real part of the dual curve. Let $x \in X$ be a point such that $y = p(x) \in U$ where p is the projection $X \to B$. Since the line joining y to $c_B(y)$ is real, it is not tangent to B at y and $c_B(y)$. Let H_1 be a hyperplane of $\mathbb{C}P^3$ passing through y and $c_B(y)$ and transverse to B. Then $H_1 \times \mathbb{C}P^{N-1}$ is transverse to X in $\mathbb{C}P^3 \times \mathbb{C}P^{N-1}$. Let H_2 be a hyperplane of $\mathbb{C}P^{N-1}$ such that $\mathbb{C}P^3 \times H_2$ does contain neither x nor $c_X(x)$. Then the divisor $(H_1 \times \mathbb{C}P^{N-1}) + (\mathbb{C}P^3 \times H_2)$ is associated to a hyperplane H_0 of $\mathbb{C}P^{4N-1}$, which contains both x and $c_X(x)$ and which is transverse to X at these points. Then H_0 contains the line D_x and since by construction it also contains the fibers through x and $c_X(x)$, its transversality with X at x and $c_X(x)$ implies the one of D_x. Hence for any point x belonging to the open set $p^{-1}(U)$ of X, the line D_x is transverse to X at x and $c_X(x)$. Since it is not hard to observe that any non-real intersection point of S and $c_X(S)$ can be moved to $p^{-1}(U)$ after a small perturbation of H, this completes the proof of lemma 3.12. $\square$

Remark 3.13. To prove the transversality part of lemma 3.12, the following simpler argument has been communicated to me by V. Kharlamov. First notice that once you have two very ample line bundles and a generic couple of sections of these bundles, the zero sets of these sections intersect transversally. Now take L_1 to be the line bundle associated to the constructed very ample section S and L_2 to be the bundle associated to $c_X(S)$. On the space of couples of holomorphic sections of these bundles, we have the following real structure: $(s_1, s_2) \mapsto (\overline{s_2 \circ c_X}, \overline{s_1 \circ c_X})$. We are exactly interested in couples which belong to the real locus of this real structure. The result follows from the standard fact that there are generic points on this (non-singular) real locus.

Proof of proposition 3.10. Let $S \subset X$ be a very ample holomorphic smooth section, transverse to its image under c_X. Such a section is given by lemma 3.12. The set $c_X(S) \cap S$ is finite and invariant under c_X. Denote by X_1 the ruled surface obtained from X after an elementary transformation on every point of this set. Since it is invariant under c_X, the real structure c_X induces a real structure c_{X_1} on X_1. Moreover, the strict transform S_1 of S satisfies $c_{X_1}(S_1) \cap S_1 = \emptyset$. Thus X_1 is a decomposable ruled surface, and c_{X_1} exchanges the two holomorphic sections S_1 and $c_X(S_1)$. The inverse of an elementary transformation is still an elementary transformation, so we deduce that (X, c_X) is obtained from the real decomposable ruled surface (X_1, c_{X_1}) after performing elementary transformations on points $\{x_1, \dots, x_k, y_1, \dots, y_l, \overline{y}_1, \dots, \overline{y}_l\}$ where $c_{X_1}(x_i) = x_i$ and $c_{X_1}(y_j) = \overline{y}_j$. Note that all the points $\{x_1, \dots, x_k, y_1, \dots, y_l, \overline{y}_1, \dots, \overline{y}_l\}$ belong to different fibers of X_1. It remains to see that this number of points can be reduced to one at most for each component of $\mathbb{R}X_1$, changing the decomposable real ruled surface X_1 if necessary.

For every $j \in \{1, \dots, l\}$, choose a piecewise analytic path $y_j(t)$, $t \in [0,1]$, such that $y_j(0) = y_j$, $y_j(1) \in S_1$ and $p(y_j(t))$ is constant, which means that $y_j(t)$ stays in a same fiber of X_1. Let $\overline{y}_j(t) = c_{X_1}(y_j(t))$ and denote by X_2 the ruled surface obtained from X_1 after elementary transformations in the points $y_1(1), \dots, y_l(1), \overline{y}_1(1), \dots, \overline{y}_l(1)$. The real structure c_{X_1} induces a real structure c_{X_2} on X_2. The surface (X_2, c_{X_2}) is in the same deformation class that (X_1, c_{X_1}). Moreover, X_2 is also a decomposable ruled surface. Indeed, the strict transform S_2 of S_1 is a holomorphic section of X_2 satisfying $c_{X_2}(S_2) \cap S_2 = \emptyset$. Thus (X, c_X) is in the same deformation class that the surface obtained from the real decomposable ruled surface (X_2, c_{X_2}) after performing elementary transformations on the strict transforms of the points $x_1, \dots, x_k \in \mathbb{R}X_1$, still denoted by $x_1, \dots, x_k \in \mathbb{R}X_2$. Now for each pair of points x_1, x_2 lying in a same connected component of $\mathbb{R}X_2$, we can make the elementary transformation on the point x_2. Then, the image of the fiber passing through x_2 is a real point x_2' in the new surface X_2' obtained. So we can choose an analytic path from x_1 to x_2' in the real part of X_2' and we deduce that the surface obtained from X_2 after making the elementary transformations on the points x_1, x_2 is in the same deformation class that the one obtained from X_2' after an elementary transformation on x_2', which is X_2 itself. Hence each pair of points lying in a same connected component of $\mathbb{R}X_2$ can be removed and so (X, c_X) is in the same deformation class that the surface obtained from the real decomposable ruled surface (X_2, c_{X_2}) after performing at most one elementary transformation on each of its real components. Since c_{X_2} exchanges two disjoint holomorphic sections of X_2, it follows from theorem 2.3 that (X_2, c_{X_2}) is of the form $(P(L \oplus L_0), c_X^+)$ where $L \in \text{Pic}(B)$ and $c_B^*(L) = L^*$. □

Lemma 3.14. *Let $g \geq 1$ be an odd integer and (B, c_B) be a smooth compact irreducible real algebraic curve of genus g and empty real part. Let L be a complex line bundle over B satisfying $c_B^*(L) = L^*$. Then the real ruled surfaces $(P(L \oplus L_0), c_X^+)$ and $(P(L \oplus L_0), c_X^-)$ are in the same deformation class.*

(In lemma 3.14, the real structures c_X^+ and c_X^- on $X = P(L \oplus L_0)$ are those given by proposition 1.6.)

Proof. Without changing the deformation class of $X = P(L \oplus L_0)$, we can assume that the base of this surface is the real algebraic curve (B, c_B) given by corollary 2.8. Then, if L belong to the same real component of $(\text{Jac}(B), -c_B^*)$ that the bundle given by corollary 2.8, we can assume, without changing the deformation class of $X = P(L \oplus L_0)$, that L is exactly this bundle. In that case, the result comes from corollary 2.8.

Let $X = P(L \oplus L_0)$ be the ruled surface given by corollary 2.8, and $\Phi : X \to X$ be the automorphism conjugating c_X^+ and c_X^-. Let x_1 be a point on the section of X associated to L and $y_1 = c_X^+(x_1) = c_X^-(x_1)$. Let $x_2 = \Phi(x_1)$ and $y_2 = \Phi(y_1) = c_X^+(x_2) = c_X^-(x_2)$. Denote by Y_1 (resp. Y_2) the ruled surface obtained from X after one elementary transformation on the points x_1 and y_1 (resp. x_2 and y_2). Then the real structures c_X^+ and c_X^- lift to the real structures $c_{Y_1}^{\pm}$ (resp. $c_{Y_2}^{\pm}$) on Y_1 (resp. Y_2), and Φ lifts to a biholomorphism $\Psi : Y_1 \to Y_2$ such that $c_{Y_1}^+ = \Psi^{-1} \circ c_{Y_2}^- \circ \Psi$ and $c_{Y_1}^- = \Psi^{-1} \circ c_{Y_2}^+ \circ \Psi$. But the real ruled surface $(Y_1, c_{Y_1}^-)$ is in the same deformation class that $(Y_2, c_{Y_2}^-)$. Indeed, it suffices to choose an analytic path x_t linking x_1 to x_2 in the section of X associated to L and to consider the surfaces $(Y_t, c_{Y_t}^-)$ obtained from (X, c_X^-) after an elementary transformation on the points x_t and $c_X^-(x_t)$.

Hence the real ruled surfaces $(Y_1, c_{Y_1}^-)$ and $(Y_1, c_{Y_1}^+)$ are in the same deformation class. To conclude, it remains to see that they do not come from the same connected component of $(\text{Jac}(B), -c_B^*)$ that $(X, c_X^{\pm})$. This follows from the fact that the quotients $Y_1/c_{Y_1}^{\pm}$ and $X/c_X^{\pm}$ are not homeomorphic. Indeed, these two quotients are sphere bundles over the non-orientable surface $B' = B/c_B$. But $Y_1/c_{Y_1}^{\pm}$ is obtained from $X/c_X^{\pm}$ after one elementary transformation in one point. Thus one of these two quotient is spin, and one is not. Hence the result. □

Proof of theorem 3.7. Let (X_1, c_{X_1}) and (X_2, c_{X_2}) be two real non-rational ruled surfaces of bases (B_1, c_{B_1}) and (B_2, c_{B_2}) respectively, which have the same topological type (t, k, g, μ, ϵ). We have to prove that they are in the same deformation class, as soon as $\mu \neq 0$.

Let us first consider the case of decomposable ruled surfaces, that is let us assume that X_1 and X_2 are decomposable. If $t + k < \mu$, it follows from theorem 2.3 that $X_1 = P(L_1 \oplus L_0)$ (resp. $X_2 = P(L_2 \oplus L_0)$), where $L_1 \in \mathrm{Pic}(B_1)$ (resp. $L_2 \in \mathrm{Pic}(B_2)$) and $c^*_{B_1}(L_1) = L_1^*$ (resp. $c^*_{B_2}(L_1) = L_2^*$). Moreover, it follows from proposition 1.6 that in this case $k = 0$. The partition $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) in two elements of $\mathbb{R}B_1$ (resp. $\mathbb{R}B_2$) associated to L_1 (resp. L_2) consists of one element containing t components of $\mathbb{R}B_1$ (resp. $\mathbb{R}B_2$) and one element containing $\mu - t$ components of $\mathbb{R}B_1$ (resp. $\mathbb{R}B_2$) (see §3.1 for the definition of the partition). Since (B_1, c_{B_1}) and (B_2, c_{B_2}) have same topological type (g, μ, ϵ), there exists a piecewise analytic path of smooth real algebraic curves connecting them (see [11]). Moreover, this path can be chosen such that the t components of $\mathbb{R}B_2$, which form an element of the partition $\mathcal{P}_2$, deform into the t components of $\mathbb{R}B_1$ which form an element of the partition $\mathcal{P}_1$. This follows from the presentation in [11] of a real algebraic curve as the gluing of a Riemann surface with boundary with its conjugate, the gluing maps being either identity or antipodal. Thus (X_2, c_{X_2}) is in the same deformation class that a ruled surface $(\widetilde{X}_2, c_{\widetilde{X}_2})$ of base (B_1, c_{B_1}). Moreover, $\widetilde{X}_2 = P(\widetilde{L}_2 \oplus L_0)$ where $\widetilde{L}_2 \in \mathrm{Pic}(B_1)$, $c^*_{B_1}(\widetilde{L}_2) = \widetilde{L}_2^*$ and the partitions associated to $\widetilde{L}_2$ and L_1 are the same. From lemma 3.2, it follows that $\widetilde{L}_2$ and L_1 are in the same component of the real part of $(\mathrm{Jac}(B_1), -c^*_{B_1})$ and hence the surfaces $(\widetilde{X}_2, c_{\widetilde{X}_2})$ and (X_1, c_{X_1}) are in the same deformation class.

If $t + k = \mu$, it follows from theorem 2.3 that $X_1 = P(L_1 \oplus L_0)$ (resp. $X_2 = P(L_2 \oplus L_0)$), where $L_1 \in \mathrm{Pic}(B_1)$ (resp. $L_2 \in \mathrm{Pic}(B_2)$) and either $c^*_{B_1}(L_1) = L_1^*$ (resp. $c^*_{B_2}(L_1) = L_2^*$), or $c^*_{B_1}(L_1) = L_1$ (resp. $c^*_{B_2}(L_1) = L_2$). In the first case, L_1 (resp L_2) is in the same component of the real part of $(\mathrm{Jac}(B_1), -c^*_{B_1})$ (resp. $(\mathrm{Jac}(B_2), -c^*_{B_2})$) that L_0, since $t + k = \mu$. Thus (X_1, c_{X_1}) (resp. (X_2, c_{X_2})) is in the same deformation class that $(B_1 \times \mathbb{C}P^1, c^{\pm}_X)$ (resp. $(B_2 \times \mathbb{C}P^1, c^{\pm}_X)$). Moreover, when $\mu \neq 0$, only one of the two real structures $c^{\pm}_X$, say c^+_X, satisfies $t + k = \mu$. In the second case, denote by $D_+ - D_-$ a divisor associated to L_1, where D_+, D_- are positive divisors and invariant under c_{B_1}. Then $X_1 = P(L_{D_+} \oplus L_{D_-})$ and $c_{X_1} = c_{L_{D_+}} \oplus c_{L_{D_-}}$. Thus, it follows from lemma 3.11 that (X_1, c_{X_1}) is obtained from $(B_1 \times \mathbb{C}P^1, c_{L_0} \oplus c_{L_0})$ after performing elementary transformations on the points of the section associated to L_{D_+} (resp. L_{D_-}) over the locus of $D_+ \in B_1$ (resp. $D_- \in B_1$). Without changing the deformation class of the surface, we can assume that the elementary transformations are only done on real points of $(B_1 \times \mathbb{C}P^1, c_{L_0} \oplus c_{L_0})$ with at most one on each of its real components. Indeed, the extra real points can be removed as in proposition 3.10 and every couple of conjugated imaginary points can be moved to real points following a standard deformation: embed the disk $(\Delta, conj)$ in a real section of X, and for every $t \in \Delta$, denote by Y_t the surface obtained from X after an elementary transformation on the points t and $-t$ in Δ (we still denote by Δ its image in X by the chosen embedding).

The dimension 3 complex manifold Y obtained gets two real structures, one which lifts $conj$ in Δ and one which lifts $-conj$. This thus define two real deformations of ruled surfaces and shows that the real ruled surfaces obtained from X after making elementary transformations on the points $\pm\frac{1}{2} \in \Delta$ or $\pm\frac{i}{2} \in \Delta$ are in the same deformation class. Hence, without changing the deformation class of the surface (X_1, c_{X_1}), we can assume that the elementary transformations are done only on real points of $(B_1 \times \mathbb{C}P^1, c_{L_0} \oplus c_{L_0})$ with at most one on each of its real components. The total number of such elementary transformations is then k since the topological type of (X_1, c_{X_1}) is (t, k, g, μ, ϵ). If X_1 and X_2 are two such surfaces, there exists a piecewise analytic path of smooth real algebraic curves connecting (B_1, c_{B_1}) and (B_2, c_{B_2}), such that the k components of $\mathbb{R}B_2$ over which are done the elementary transformations deform on the k components of $\mathbb{R}B_1$ over which are done the elementary transformations. Hence in both cases, (X_1, c_{X_1}) and (X_2, c_{X_2}) are in the same deformation class. Since the real structures c_X^+ and $c_{L_0} \oplus c_{L_0}$ are conjugated on $B_1 \times \mathbb{C}P^1$, which follows from theorem 2.3 for instance, we deduce that the real decomposable ruled surfaces (X_1, c_{X_1}) and (X_2, c_{X_2}) are in the same deformation class if and only if they have the same topological type (t, k, g, μ, ϵ), except when $\mu = 0$. In that case, if g is even, it follows from proposition 3.1 that the same method as before leads to the fact that (X_1, c_{X_1}) and (X_2, c_{X_2}) are in the same deformation class that $(B \times \mathbb{C}P^1, c_X^+)$ or $(B \times \mathbb{C}P^1, c_X^-)$. But the quotient $(B \times \mathbb{C}P^1)/c_X^+$ is spin and $(B \times \mathbb{C}P^1)/c_X^-$ is not, so the surfaces $(B \times \mathbb{C}P^1, c_X^+)$ and $(B \times \mathbb{C}P^1, c_X^-)$ are not in the same deformation class. If g is odd, it follows from proposition 3.1 that the same method as before leads to the fact that (X_1, c_{X_1}) and (X_2, c_{X_2}) are in the same deformation class that $(P(L \oplus L_0), c_X^{\pm})$, where L belongs to one of the two components of the real part of $(Jac(B), -c_B^*)$. But it follows from lemma 3.14 that $(P(L \oplus L_0), c_X^+)$ and $(P(L \oplus L_0), c_X^-)$ are in a same deformation class. The result follows from the fact that $(B \times \mathbb{C}P^1)/c_X^{\pm}$ is spin and $P(L \oplus L_0)/c_X^{\pm}$ is not when L is not in the same component of the real part of $(Jac(B), -c_B^*)$ that L_0.

Now let us prove the theorem in the general case, which means that we no more assume that X_1 and X_2 are decomposable. From propositions 3.9 and 3.10, it follows that these surfaces are either in the same deformation class that some real decomposable ruled surfaces, or in the same deformation class that some ruled surface obtained from a decomposable one of the form $(P(L \oplus L_0), c_X^{\pm})$ after at most one elementary transformation on each of its real components. In this second case, we can assume that L does not belong to the same component of the real part of $(Jac(B), -c_B^*)$ that L_0 (otherwise the surface can be deformed to a decomposable ruled surface). Since the topological types of these surfaces are different from those realized by decomposable ruled surfaces, we can assume that either X_1 and X_2 are both decomposable, or that they are both from this second class. In the first case, the theorem follows from what we have already done. Let us assume we are in the second case. Then there exists $L_1 \in \mathrm{Pic}(B_1)$ (resp. $L_2 \in \mathrm{Pic}(B_2)$) such that $c_{B_1}^*(L_1) = L_1^*$ (resp. $c_{B_2}^*(L_2) = L_2^*$) and (X_1, c_{X_1})

is obtained from $(P(L_1 \oplus L_0), c_X^+)$ after making k elementary transformations in k disjoint real components. The surfaces $(P(L_1 \oplus L_0), c_X^+)$ and $(P(L_2 \oplus L_0), c_X^+)$ have same topological type $(t+k, 0, g, \mu, \epsilon)$, with $\mu > 0$. Thus they are in the same deformation class. Moreover, in the same way as before, this deformation can be chosen so that the k marked real components of $(P(L_2 \oplus L_0), c_X^+)$ deform to the k marked real components of $(P(L_1 \oplus L_0), c_X^+)$. It follows that (X_1, c_{X_1}) and (X_2, c_{X_2}) are in the same deformation class. $\square$

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