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Autor(en): **Bodin, Arnaud**

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Invariance of Milnor numbers and topology of complex polynomials

Arnaud Bodin

Abstract. We give a global version of Lê–Ramanujam μ-constant theorem for polynomials. Let (f_t), $t \in [0,1]$, be a family of polynomials of n complex variables with isolated singularities, whose coefficients are polynomials in t. We consider the case where some numerical invariants are constant (the affine Milnor number $\mu(t)$, the Milnor number at infinity $\lambda(t)$, the number of critical values, the number of affine critical values, the number of critical values at infinity). Let $n = 2$, we also suppose the degree of the f_t is a constant, then the polynomials f_0 and f_1 are topologically equivalent. For $n > 3$ we suppose that critical values at infinity depend continuously on t, then we prove that the geometric monodromy representations of the f_t are all equivalent.

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1. Introduction

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map, $n \geqslant 2$. By a result of Thom [Th] there is a finite minimal set of complex numbers $\mathcal{B}$, the *critical values*, such that $f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ is a fibration.

1.1. Affine singularities

We suppose that *affine singularities are isolated i.e.* that the set $\{x \in \mathbb{C}^n \mid \operatorname{grad}_f x = 0\}$ is a finite set. Let μ_c be the sum of the local Milnor numbers at the points of $f^{-1}(c)$. Let

$$\mathcal{B}_{\mathit{aff}} = \{c \mid \mu_c > 0\} \quad \text{and} \quad \mu = \sum_{c \in \mathbb{C}} \mu_c$$

be the *affine critical values* and the *affine Milnor number*.

1.2. Singularities at infinity

See [Br]. Let d be the degree of $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, let $f = f^d + f^{d-1} + \cdots + f^0$ where f^j is homogeneous of degree j. Let $\bar{f}(x, x_0)$ (with $x = (x_1, \ldots, x_n)$) be the homogenization of f with the new variable x_0: $\bar{f}(x, x_0) = f^d(x) + f^{d-1}(x)x_0 + \ldots + f^0(x)x_0^d$. Let

$$X = \Big\{((x : x_0), c) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x, x_0) - cx_0^d = 0\Big\}.$$

Let $\mathcal{H}_\infty$ be the hyperplane at infinity of $\mathbb{P}^n$ defined by $(x_0 = 0)$. The singular locus of X has the form $\Sigma \times \mathbb{C}$ where

$$\Sigma = \left\{(x : 0) \mid \frac{\partial f^d}{\partial x_1} = \cdots = \frac{\partial f^d}{\partial x_n} = f^{d-1} = 0\right\} \subset \mathcal{H}_\infty.$$

We suppose that f has *isolated singularities at infinity* that is to say that Σ is finite. This is always true for $n = 2$. For $n \geqslant 2$ such polynomials have been studied by S. Broughton [Br] and by A. Parusiński [Pa]. For a point $(x : 0) \in \mathcal{H}_\infty$, assume, for example, that $x = (x_1, \ldots, x_{n-1}, 1)$. Set $\check{x} = (x_1, \ldots, x_{n-1})$ and

$$F_c(\check{x}, x_0) = \bar{f}(x_1, \ldots, x_{n-1}, 1) - cx_0^d.$$

Let $\mu_{\check{x}}(F_c)$ be the local Milnor number of F_c at the point $(\check{x}, 0)$. If $(x : 0) \in \Sigma$ then $\mu_{\check{x}}(F_c) > 0$. For a generic s, $\mu_{\check{x}}(F_s) = \nu_{\check{x}}$, and for finitely many c, $\mu_{\check{x}}(F_c) > \nu_{\check{x}}$. We set $\lambda_{c,\check{x}} = \mu_{\check{x}}(F_c) - \nu_{\check{x}}$, $\lambda_c = \sum_{(x:0)\in\Sigma} \lambda_{c,\check{x}}$. Let

$$\mathcal{B}_\infty = \big\{c \in \mathbb{C} \mid \lambda_c > 0\big\} \quad \text{and} \quad \lambda = \sum_{c\in\mathbb{C}} \lambda_c$$

be the *critical values at infinity* and the *Milnor number at infinity.* We can now describe the set of critical values $\mathcal{B}$ as follows (see [HL] and [Pa]):

$$\mathcal{B} = \mathcal{B}_{aff} \cup \mathcal{B}_\infty.$$

Moreover, by [HL] and [ST], for $s \notin \mathcal{B}$, $f^{-1}(s)$ has the homotopy type of a wedge of $\lambda + \mu$ spheres of real dimension $n - 1$.

1.3. Statement of the results

Theorem 1. *Let $(f_t)_{t\in[0,1]}$ be a family of complex polynomials from $\mathbb{C}^n$ to $\mathbb{C}$ whose coefficients are polynomials in t. We suppose that affine singularities and singularities at infinity are isolated. Let suppose that the integers $\mu(t)$, $\lambda(t)$, $\#\mathcal{B}(t)$, $\#\mathcal{B}_{aff}(t)$, $\#\mathcal{B}_\infty(t)$ do not depend on $t \in [0, 1]$. Moreover let us suppose that critical values at infinity $\mathcal{B}_\infty(t)$ depend continuously on t. Then the fibrations $f_0 : f_0^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_1 : f_1^{-1}(\mathbb{C} \setminus \mathcal{B}(1)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(1)$ are fiber homotopy equivalent, and for $n \neq 3$ are differentiably isomorphic.*

Remark 1. As a consequence for $n \neq 3$ and $* \notin \mathcal{B}(0) \cup \mathcal{B}(1)$ the monodromy representations

$$\pi_1(\mathbb{C} \setminus \mathcal{B}(0), *) \longrightarrow \mathrm{Diff}(f_0^{-1}(*)) \text{ and}$$
$$\pi_1(\mathbb{C} \setminus \mathcal{B}(1), *) \longrightarrow \mathrm{Diff}(f_1^{-1}(*))$$

are equivalent (where $\mathrm{Diff}(f_t^{-1}(*))$ denotes the diffeomorphisms of $f_t^{-1}(*)$ modulo diffeomorphisms isotopic to identity).

Remark 2. The restriction $n \neq 3$, as in [LR], is due to the use of the h-cobordism theorem. The proof is based on the articles of H. V. Hà – T. S. Pham [HP] and of D. T. Lê – C. P. Ramanujam [LR].

Remark 3. This result extends a theorem of H. V. Hà and T. S. Pham [HP] which deals only with monodromy at infinity (which corresponds to a loop around the whole set $\mathcal{B}(t)$) for $n = 2$. For $n \neq 3$, the fact that the monodromies at infinity are diffeomorphic is proved in [HZ] (for M-tame polynomials, with affine Milnor number constant) and in [Ti] (for generic fibers with homotopy type equivalent to a fixed number of $(n-1)$-spheres, with the hypothesis that $\mathcal{B}(t)$ is included in a compact set for all t).

Lemma 2. *Under the hypotheses of the previous theorem (except the hypothesis of continuity of the critical values), and one of the following conditions:*

- $n = 2$, *and* $\deg f_t$ *does not depend on* t;
- $\deg f_t$, *and* $\Sigma(t)$ *do not depend on* t, *and for all* $(x : 0) \in \Sigma(t)$, $\nu_{\check{x}}(t)$ *is independent of* t;

we have that $\mathcal{B}_\infty(t)$ *depends continuously on* t, i.e. *if* $c(\tau) \in \mathcal{B}_\infty(\tau)$ *then for all* t *near* τ *there exists* $c(t)$ *near* $c(\tau)$ *such that* $c(t) \in \mathcal{B}_\infty(t)$.

Under the hypothesis that there is no singularity at infinity we can prove the stronger result:

Theorem 3. *Let* $(f_t)_{t\in[0,1]}$ *be a family of complex polynomials whose coefficients are polynomials in* t. *Suppose that* $\mu(t)$, $\#\mathcal{B}_{\mathrm{aff}}(t)$ *do not depend on* $t \in [0,1]$. *Moreover suppose that* $n \neq 3$ *and for all* $t \in [0,1]$ *we have* $\mathcal{B}_\infty(t) = \varnothing$. *Then the polynomials* f_0 *and* f_1 *are topologically equivalent, that is to say, there exist homeomorphisms* Φ *and* Ψ *such that*

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\Phi"] \arrow[d, "f_0"'] & \mathbb{C}^n \arrow[d, "f_1"] \\ \mathbb{C} \arrow[r, "\Psi"'] & \mathbb{C}. \end{tikzcd}$$

For the proof we glue the former study with the version of the μ-constant theorem of D. T. Lê and C. P. Ramanujam stated by J. G. Timourian [Tm]: a

μ-constant deformation of germs of isolated hypersurface singularity is a product family.

For polynomials in two variables we can prove the following theorem which is a global version of Lê–Ramanujam–Timourian theorem:

Theorem 4. *Let $n = 2$. Let $(f_t)_{t\in[0,1]}$ be a family of complex polynomials whose coefficients are polynomials in t. Suppose that the integers $\mu(t)$, $\lambda(t)$, $\#\mathcal{B}(t)$, $\#\mathcal{B}_{aff}(t)$, $\#\mathcal{B}_\infty(t)$, $\deg f_t$ do not depend on $t \in [0,1]$. Then the polynomials f_0 and f_1 are topologically equivalent.*

It uses a result of L. Fourrier [Fo] that gives a necessary and sufficient condition for polynomials to be topologically equivalent outside sufficiently large compact sets of $\mathbb{C}^2$.

Remark 4. In theorems 3 and 4 not only f_0 and f_1 are topologically equivalent but we can prove that it is a topologically trivial family.

This work was initiated by an advice of D. T. Lê concerning the article [Bo]: "It is easier to find conditions for polynomials to be equivalent than find all polynomials that respect a given condition."

We will denote $B_R = \{x \in \mathbb{C}^n \mid \|x\| \leqslant R\}$, $S_R = \partial B_R = \{x \in \mathbb{C}^n \mid \|x\| = R\}$ and $D_r(c) = \{s \in \mathbb{C} \mid |s - c| \leqslant r\}$.

2. Fibrations

In this paragraph we give some properties for a complex polynomial of n variables. The two first lemmas are consequences of transversality properties. There are direct generalizations of lemmas of [HP]. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial with isolated affine singularities and with isolated singularities at infinity. Let choose $r > 0$ such that $\mathcal{B}$ is contained in the interior of $D_r(0)$. For each fiber $f^{-1}(c)$ there is a finite number of real numbers $R > 0$ such that $f^{-1}(c)$ has non-transversal intersection with the sphere S_R (see [M3], Corollaries 2.8 and 2.9). So, for a sufficiently large number $R(c)$, the intersection $f^{-1}(c)$ with S_R is transversal for all $R \geqslant R(c)$. Let R_1 be greater than the maximum of the $R(c)$ with $c \in \mathcal{B}$, we also choose $R_1 \gg r$. We choose a small ε, $0 < \varepsilon \ll 1$ such that for all values c in the bifurcation set $\mathcal{B}$ of f and for all $s \in D_\varepsilon(c)$ the intersection $f^{-1}(s) \cap S_{R_1}$ is transversal, this is possible by continuity of the transversality. We denote

$$K = D_r(0) \setminus \bigcup_{c\in\mathcal{B}} \mathring{D}_\varepsilon(c).$$

Lemma 5. *There exists $R_0 \gg 1$ such that for all $R \geqslant R_0$ and for all s in K, $f^{-1}(s)$ intersects S_R transversally.*

Proof. We have to adapt the beginning of the proof of [HP]. If the assertion is false then we have a sequence (x_k) of points of $\mathbb{C}^n$ such that $f(x_k) \in K$ and $\|x_k\| \to +\infty$ as $k \to +\infty$ and such that there exist complex numbers λ_k with $\operatorname{grad}_f x_k = \lambda_k x_k$, where the gradient is Milnor gradient: $\operatorname{grad}_f = \left(\overline{\frac{\partial f}{\partial x_1}}, \dots, \overline{\frac{\partial f}{\partial x_n}}\right)$. Since K is a compact set we can suppose (after extracting a sub-sequence, if necessary) that $f(x_k) \to c \in K$ as $k \to +\infty$. Then by the Curve Selection Lemma of [NZ] there exists a real analytic curve $x :]0, \varepsilon[\longrightarrow \mathbb{C}^n$ such that $x(\tau) = a\tau^\beta + a_1\tau^{\beta+1} + \cdots$ with $\beta < 0$, $a \in \mathbb{R}^{2n} \setminus \{0\}$ and $\operatorname{grad}_f x(\tau) = \lambda(\tau)x(\tau)$. Then $f(x(\tau)) = c + c_1\tau^\rho + \cdots$ with $\rho > 0$. So $f(x(\tau)) \to c$ as $\tau \to 0$. Then we can redo the calculus of [HP]:

$$\frac{df(x(\tau))}{d\tau} = \langle \frac{dx}{d\tau}, \operatorname{grad}_f x(\tau) \rangle = \bar{\lambda}(\tau) \langle \frac{dx}{d\tau}, x(\tau) \rangle$$

it implies

$$|\lambda(\tau)| \leqslant 2 \frac{\left| \frac{df(x(\tau))}{d\tau} \right|}{\frac{d\|x(\tau)\|^2}{d\tau}}.$$

As $\|x(\tau)\| = b_1\tau^\beta + \cdots$ with $b_1 \in \mathbb{R}_+^*$ and $\beta < 0$ we have, for small enough τ, $|\lambda(\tau)| \leqslant \gamma \frac{\tau^{\rho-1}}{\tau^{2\beta-1}} = \gamma\tau^{\rho-2\beta}$ where γ is a constant. We end the proof be using the characterization of critical value at infinity in [Pa]:

$$\|x(\tau)\|^{1-1/N} \|\operatorname{grad}_f x(\tau)\| = \|x(\tau)\|^{1-1/N} |\lambda(\tau)| \, \|x(\tau)\| \leqslant \gamma\tau^{\rho-\beta/N}$$

As $\rho > 0$ and $\beta < 0$, for all $N > 0$ we have that $\|x(\tau)\|^{1-1/N} \|\operatorname{grad}_f x(\tau)\| \to 0$ as $\tau \to 0$. It implies that the value c (the limit of $f(x(\tau))$ as $\tau \to 0$) is in $\mathcal{B}_\infty$. But as $c \in K$ it is impossible. □

Lemma 5 enables us to get the following result: because of the transversality we can find a vector field tangent to the fibers of f and pointing out the spheres S_R. Integration of such a vector field gives the next lemma (see [HP] Paragraph 2.2 or [Ti] Lemma 1.8).

Lemma 6. *The fibrations $f : f^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$ and $f : f^{-1}(K) \longrightarrow K$ are differentiably isomorphic.*

As $\mathring{K}$ is diffeomorphic to $\mathbb{C} \setminus \mathcal{B}$ we have the following fact:

Lemma 7. *The fibrations $f : f^{-1}(\mathring{K}) \longrightarrow \mathring{K}$ and $f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ are differentiably isomorphic.*

The following lemma is adapted from [LR]. For completeness we give the proof.

Lemma 8. *Let R, R' with $R \geqslant R'$ be real numbers such that the intersections $f^{-1}(K) \cap S_R$ and $f^{-1}(K) \cap S_{R'}$ are transversal. Let us suppose that $f : f^{-1}(K) \cap B_{R'} \longrightarrow K$ and $f : f^{-1}(K) \cap B_R \longrightarrow K$ are fibrations with fibers homotopic to*

a wedge of ν $(n-1)$-dimensional spheres. Then the fibrations are fiber homotopy equivalent. And for $n \neq 3$ the fibrations are differentiably equivalent.

Proof. The first part is a consequence of a result of A. Dold [Do, Th. 6.3]. The first fibration is contained in the second. By the result of Dold we only have to prove that if $* \in \partial D_r$ then the inclusion of $F' = f^{-1}(*) \cap B_{R'}$ in $F = f^{-1}(*) \cap B_R$ is a homotopy equivalence. To see this we choose a generic x_0 in $\mathbb{C}^n$ near the origin such that the real function $x \mapsto \|x - x_0\|$ has only non-degenerate critical points of index less than n (see [M1, §7]). Then F is obtained from F' by attaching cells of index less than n.

For $n = 2$ the fibers are homotopic to a wedge of ν circles, then the inclusion of F' in F is a homotopy equivalence. For $n > 2$ the fibers F, F' are simply connected and the morphism $H_i(F') \longrightarrow H_i(F)$ induced by inclusion is an isomorphism. For $i \neq n-1$ this is trivial since F and F' have the homotopy type of a wedge of $(n-1)$-dimensional spheres, and for $i = n-1$ the exact sequence of the pair (F, F') is

$$0 \longrightarrow H_{n-1}(F) \longrightarrow H_{n-1}(F') \longrightarrow H_{n-1}(F, F')$$

with $H_n(F, F') = 0$, $H_{n-1}(F)$ and $H_{n-1}(F')$ free of rank ν, and $H_{n-1}(F, F')$ torsion-free. Then the inclusion of F' in F is a homotopy equivalence.

The second part is based on the h-cobordism theorem. Let $X = f^{-1}(K) \cap B_R \setminus \mathring{B}_{R'}$, then as f has no affine critical point in X (because there is no critical value in K) and f is transversal to $f^{-1}(K) \cap S_R$ and to $f^{-1}(K) \cap S_{R'}$ then, by Ehresmann theorem, $f : X \longrightarrow K$ is a fibration. We denote $F \setminus \mathring{F}'$ by F^*. We get an isomorphism $H_i(\partial F') \longrightarrow H_i(F^*)$ for all i because $H_i(F^*, \partial F') = H_i(F, F') = 0$. For $n = 2$ it implies that F^* is diffeomorphic to a product $[0,1] \times \partial F'$.

For $n > 3$ we will use the h-cobordism theorem applied to F^* to prove this. We have $\partial F^* = \partial F' \cup \partial F$; $\partial F'$ and ∂F are simply connected: if we look at the function $x \mapsto -\|x - x_0\|$ on $f^{-1}(*)$ for a generic x_0, then $F = f^{-1}(*) \cap B_R$ and $F' = f^{-1}(*) \cap B_{R'}$ are obtained by gluing cells of index more or equal to $n-1$. So their boundary is simply connected. For a similar reason F^* is simply connected. As we have isomorphisms $H_i(\partial F') \longrightarrow H_i(F^*)$ and both spaces are simply connected then by Hurewicz–Whitehead theorem the inclusion of $\partial F'$ in F^* is a homotopy equivalence.

Now F^*, $\partial F'$, ∂F are simply connected, the inclusion of $\partial F'$ in F^* is a homotopy equivalence and F^* has real dimension $2n - 2 \geqslant 6$. So by the h-cobordism theorem, [M2], F^* is diffeomorphic to the product $[0,1] \times \partial F'$. Then the fibration $f : X \longrightarrow K$ is differentiably equivalent to the fibration $f : [0,1] \times (f^{-1}(K) \cap S_{R'}) \longrightarrow K$; so the fibrations $f : f^{-1}(K) \cap B_{R'} \longrightarrow K$ and $f : f^{-1}(K) \cap B_R \longrightarrow K$ are differentiably equivalent. $\square$

3. Family of polynomials

Let $(f_t)_{t\in[0,1]}$ be a family of polynomials that verify hypotheses of theorem 1.

Lemma 9 ([HP]). *There exists $R \gg 1$ such that for all $t \in [0,1]$ the affine critical points of f_t are in $\mathring{B}_R$.*

Proof. It is enough to prove it on $[0,\tau]$ with $\tau > 0$. We choose $R \gg 1$ such that all the affine critical points of f_0 are in $\mathring{B}_R$. We denote

$$\phi_t = \frac{\operatorname{grad}_{f_t}}{\|\operatorname{grad}_{f_t}\|} : S_R \longrightarrow S_1.$$

Then $\deg \phi_0 = \mu(0)$. For all $x \in S_R$, $\operatorname{grad}_{f_0} x \neq 0$, and by continuity there exists $\tau > 0$ such that for all $t \in [0,\tau]$ and all $x \in S_R$, $\operatorname{grad}_{f_t} x \neq 0$. Then the maps ϕ_t are homotopic (the homotopy is $\phi : S_R \times [0,\tau] \longrightarrow S_1$ with $\phi(x,t) = \phi_t(x)$). And then $\mu(0) = \deg \phi_0 = \deg \phi_t \leqslant \mu(t)$. If there exists a family $x(t) \in \mathbb{C}^n$ of affine critical points of ϕ_t such that $\|x(t)\| \to +\infty$ as $t \to 0$, then for a sufficiently small t, $x(t) \notin B_R$ and then $\mu(t) > \deg \phi_t$. It contradicts the hypothesis $\mu(0) = \mu(t)$. $\square$

Lemma 10. *There exists $r \gg 1$ such that the subset $\big\{(c,t) \in D_r(0) \times [0,1] \mid c \in \mathcal{B}(t)\big\}$ is a braid of $D_r(0) \times [0,1]$.*

It enables us to choose $* \in \partial D_r(0)$ which is a regular value for all f_t, $t \in [0,1]$. In other words if we enumerate $\mathcal{B}(0)$ as $\{c_1(0),\dots,c_m(0)\}$ then there are continuous functions $c_i : [0,1] \longrightarrow D_r(0)$ such that for $i \neq j$, $c_i(t) \neq c_j(t)$. This enables us to identify $\pi_1(\mathbb{C} \setminus \mathcal{B}(0),*)$ and $\pi_1(\mathbb{C} \setminus \mathcal{B}(1),*)$ by means of the previous braid.

Proof. Let τ be in $[0,1]$ and $c(\tau)$ be a critical value of f_τ, then for all t near τ there exists a critical value $c(t)$ of f_t. It is a hypothesis for critical values at infinity and this fact is well-known for affine critical values as the coefficients of f_t are smooth functions of t, see for example [Br, Prop. 2.1].

Moreover there can not exist critical values that escape at infinity *i.e.* a $\tau \in [0,1]$ such that $|c(t)| \to +\infty$ as $t \to \tau$. For affine critical values it is a consequence of lemma 9 (or we can make the same proof as we now will perform for the critical values at infinity). For $\mathcal{B}_\infty(t)$ let us suppose that there are critical values that escape at infinity. By continuity of the critical values at infinity with respect to t we can suppose that there is a continuous function $c_0(t)$ on $]0,\tau]$ ($\tau > 0$) with $c_0(t) \in \mathcal{B}_\infty(t)$ and $|c(t)| \to +\infty$ as $t \to 0$. By continuity of the critical values at infinity, if $\mathcal{B}_\infty(0) = \{c_1(0),\dots,c_p(0)\}$ there exist continuous functions $c_i(t)$ on $[0,\tau]$ such that $c_i(t) \in \mathcal{B}_\infty(t)$ for all $i = 1,\dots,p$. And for a sufficiently small $t > 0$, $c_0(t) \neq c_i(t)$ ($i = 1,\dots,p$) then $\#\mathcal{B}_\infty(0) < \#\mathcal{B}_\infty(t)$ which contradicts the constancy of $\#\mathcal{B}_\infty(t)$.

Finally there can not exist ramification points: suppose that there is a τ such that $c_i(\tau) = c_j(\tau)$ (and $c_i(t), c_j(t)$ are not equal in a neighborhood of τ). Then if

$c_i(\tau) \in \mathcal{B}_{aff}(\tau) \setminus \mathcal{B}_\infty(\tau)$ (resp. $\mathcal{B}_\infty(\tau) \setminus \mathcal{B}_{aff}(\tau)$, $\mathcal{B}_\infty(\tau) \cap \mathcal{B}_{aff}(\tau)$) there is a jump in $\#\mathcal{B}_{aff}(t)$ (resp. $\#\mathcal{B}_\infty(t)$, $\#\mathcal{B}(t)$) near τ which is impossible by assumption. □

Let $R_0, K, D_r(0), D_\varepsilon(c)$ be the objects of section 2 for the polynomial $f = f_0$. Moreover we suppose that R_0 is greater than the R obtained in lemma 9.

Lemma 11. *There exists $\tau \in]0,1]$ such that for all $t \in [0,\tau]$ we have the properties:*
- *$c_i(t) \in D_\varepsilon(c_i(0))$, $i = 1, \dots, m$;*
- *for all $s \in K$, $f_t^{-1}(s)$ intersects S_{R_0} transversally.*

Proof. The first point is just the continuity of the critical values $c_i(t)$. The second point is the continuity of transversality: if the property is false then there exist sequences $t_k \to 0$, $x_k \in S_{R_0}$ and $\lambda_k \in \mathbb{C}$ such that $\operatorname{grad}_{f_{t_k}} x_k = \lambda_k x_k$. We can suppose that (x_k) converges (after extraction of a sub-sequence, if necessary). Then $x_k \to x \in S_{R_0}$, $\operatorname{grad}_{f_{t_k}} x_k \to \operatorname{grad}_{f_0} x$, and $\lambda_k = \langle \operatorname{grad}_{f_{t_k}} x_k \,|\, x_k\rangle / \|x_k\|^2 = \langle \operatorname{grad}_{f_{t_k}} x_k \mid x_k\rangle / {R_0}^2$ converges towards $\lambda \in \mathbb{C}$. Then $\operatorname{grad}_{f_0} x = \lambda x$ and the intersection is non-transversal. □

Lemma 12. *The fibrations $f_0 \colon f_0^{-1}(K) \cap B_{R_0} \longrightarrow K$ and $f_\tau \colon f_\tau^{-1}(K) \cap B_{R_0} \longrightarrow K$ are differentiably isomorphic.*

Proof. Let
$$F : \mathbb{C}^n \times [0,1] \longrightarrow \mathbb{C} \times [0,1], \qquad (x,t) \mapsto (f_t(x), t).$$
We want to prove that the fibrations
$$F_0 : \Sigma_0 = F^{-1}(K \times \{0\}) \cap (B_{R_0} \times \{0\}) \longrightarrow K, \qquad (x,0) \mapsto f_0(x)$$
and
$$F_\tau : \Sigma_\tau = F^{-1}(K \times \{\tau\}) \cap (B_{R_0} \times \{\tau\}) \longrightarrow K, \qquad (x,\tau) \mapsto f_\tau(x)$$
are differentiably isomorphic. Let denote $[0,\tau]$ by I. Then F has maximal rank on $F^{-1}(K \times I) \cap (\mathring{B}_{R_0} \times I)$ and on the boundary $F^{-1}(K \times I) \cap (S_{R_0} \times I)$. By Ehresmann theorem $F : F^{-1}(K \times I) \cap (B_{R_0} \times I) \longrightarrow K \times I$ is a fibration.

As in [HP] we build a vector field that gives us a diffeomorphism between the two fibrations F_0 and F_τ. Moreover it provides a control of the diffeomorphism near S_{R_0} that we will need later. Let $0 < \eta \ll 1$ be a real number. We build a vector field v_1:
- which is defined on $F^{-1}(K \times I) \cap (\cup_{R_0 - 2\eta < R < R_0} S_R \times I)$,
- such that $d_z F.v_1(z) = (0,1)$ for all z,
- and such that $v_1(z)$ is tangent to $S_R \times I$ for $z \in S_R \times I$, $R_0 - 2\eta < R < R_0$.

This is possible because F is a fibration on $F^{-1}(K \times I) \cap (B_{R_0} \times I)$. On the set $F^{-1}(K \times I) \cap (\mathring{B}_{R_0 - \eta} \times I)$ we build a second vector field v_2 such that $d_z F.v_2(z) = (0,1)$.

By gluing these vector fields v_1 and v_2 by a partition of unity and by integrating the corresponding vector field we obtain integral curves

$$p_z : [0,1] \longrightarrow F^{-1}(K \times I) \cap B_{R_0} \times I$$

such that $p_z(0) = z \in \Sigma_0$ and $p_z(\tau) \in \Sigma_\tau$. It induces a diffeomorphism $\Phi : \Sigma_0 \longrightarrow \Sigma_\tau$ such that $F_0 = F_\tau \circ \Phi$; that makes the fibrations isomorphic. □

Proof of theorem 1. It is sufficient to prove the theorem for a family (f_t) parameterized by t in an interval $[0,\tau]$ for a small $\tau > 0$. We choose τ as in lemma 11. By lemma 7, $f_0 : f^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_0 : f_0^{-1}(\mathring{K}) \longrightarrow \mathring{K}$ are differentiably isomorphic fibrations. Then by lemma 6, the fibration $f_0 : f_0^{-1}(K) \longrightarrow K$ is differentiably isomorphic to $f_0 : f_0^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$ which is, by lemma 12 differentiably isomorphic to $f_\tau : f_\tau^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$.

By continuity of transversality (lemma 11) $f_\tau^{-1}(K)$ has transversal intersection with S_{R_0}. Lemma 5 applied to f_τ gives us a large real number R, such that $f_\tau^{-1}(K)$ intersects S_R transversally, R may be much more greater than R_0. The fibration $f_\tau : f_\tau^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$ is fiber homotopy equivalent to $f_\tau : f_\tau^{-1}(K) \cap \mathring{B}_R \longrightarrow K$: it is the first part of lemma 8 because the fiber $f_\tau^{-1}(*) \cap \mathring{B}_{R_0}$ is homotopic to a wedge of $\mu(0) + \lambda(0)$ spheres and the fiber $f_\tau^{-1}(*) \cap \mathring{B}_R$ is homotopic to a wedge of $\mu(\tau) + \lambda(\tau)$ spheres; as $\mu(0) + \lambda(0) = \mu(\tau) + \lambda(\tau)$ we get the desired conclusion. Moreover for $n \neq 3$ by the second part of lemma 8 the fibrations are differentiably isomorphic.

By applying lemmas 6 and 7 to f_τ, the fibration $f_\tau : f_\tau^{-1}(\mathring{K}) \cap \mathring{B}_R \longrightarrow \mathring{K}$ is differentiably isomorphic to $f_\tau : f_\tau^{-1}(\mathbb{C} \setminus \mathcal{B}(\tau)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(\tau)$. As a conclusion the fibrations $f_0 : f_0^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_\tau : f_\tau^{-1}(\mathbb{C} \setminus \mathcal{B}(\tau)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(\tau)$ are fiber homotopy equivalent, and for $n \neq 3$ are differentiably isomorphic □

4. Around affine singularities

We now work with $t \in [0,1]$. We suppose in this paragraph that the critical values $\mathcal{B}(t)$ depend analytically on $t \in [0,1]$. This enables us to construct a diffeomorphism χ such that:

- $\chi : \mathbb{C} \times [0,1] \longrightarrow \mathbb{C} \times [0,1]$,
- $\chi(x,t) = (\chi_t(x), t)$,
- $\chi_0 = \mathrm{id}$,
- $\chi_t(\mathcal{B}(t)) = \mathcal{B}(0)$.

We denote χ_1 by Ψ, so that $\Psi : \mathbb{C} \longrightarrow \mathbb{C}$ verifies $\Psi(\mathcal{B}(1)) = \mathcal{B}(0)$. Moreover we can suppose that χ_t is equal to id on $\mathbb{C} \setminus D_r(0)$ because all the critical values are in $D_r(0)$. Finally χ defines a vector field w of $\mathbb{C} \times [0,1]$ by $\frac{\partial \chi}{\partial t}$.

We need a non-splitting result of the affine singularities, this principle has been proved by C. Has Bey ([HB], $n = 2$) and by F. Lazzeri ([La], for all n).

Lemma 13. *Let $x(\tau)$ be an affine singular point of f_τ and let U_τ be an open neighborhood of $x(\tau)$ in $\mathbb{C}^n$ such that $x(\tau)$ is the only affine singular point of f_τ in U_τ. Suppose that for all t closed to τ, the restriction of f_t to U_τ has only one critical value. Then for all t sufficiently closed to τ, there is one, and only one, affine singular point of f_t contained in U_τ.*

This lemma is a local lemma; it enables to enumerate the singularities: if we denote the affine singular points of f_0 by $\{x_i(0)\}_{i\in J}$ then there are continuous functions $x_i : [0,1] \longrightarrow \mathbb{C}^n$ such that $\{x_i(t)\}_{i\in J}$ is the set of affine singularities of f_t. Let us notice that there can be two distinct singular points of f_t with the same critical value.

We suppose

- that (f_t) verifies the hypotheses of theorem 1,
- that $n \neq 3$,
- and $\mathcal{B}(t)$ depends analytically on t.

This and lemma 13 imply that for all $t \in [0,1]$ the local Milnor number of f_t at $x(t)$ is equal to the local Milnor number of f_0 at $x(0)$. The improved version of Lê–Ramanujam theorem by J. G. Timourian [Tm] for a family of germs with constant local Milnor number proves that (f_t) is locally a product family.

Theorem 14 (Lê–Ramanujam–Timourian). *Let $x(t)$ be a singular point of f_t. There exist U_t, V_t neighborhoods of $x(t)$, $f_t(x(t))$ respectively and a homeomorphism Ω^{in} such that if $U = \bigcup_{t\in[0,1]} U_t \times \{t\}$ and $V = \bigcup_{t\in[0,1]} V_t \times \{t\}$ the following diagram commutes:*

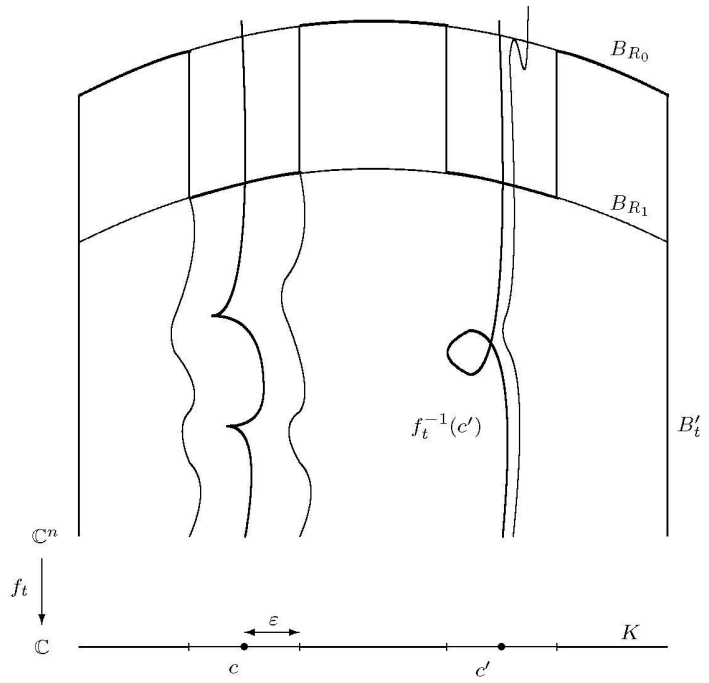
$$\begin{tikzcd} U \arrow[r, "\Omega^{\text{in}}"] \arrow[d, "F"'] & U_0 \times [0,1] \arrow[d, "f_0\times\mathrm{id}"] \\ V \arrow[r, "\chi"'] & V_0 \times [0,1]. \end{tikzcd}$$

In particular it proves that the polynomials f_0 and f_1 are locally topologically equivalent: we get a homeomorphism Φ_{in} such that the following diagram commutes:

$$\begin{tikzcd} U_1 \arrow[r, "\Phi_{\text{in}}"] \arrow[d, "f_1"'] & U_0 \arrow[d, "f_0"] \\ V_1 \arrow[r, "\Psi"'] & V_0. \end{tikzcd}$$

By lemma 9 we know that for all $t \in [0,1]$, $\mathcal{B}(t) \subset D_r(0)$. Now we redefine the radius R_0 and R_1 of section 2. By continuity of transversality and compactness of $[0,1]$ we choose R_1 such that

$$\forall c \in \mathcal{B}(0) \quad \forall R \geqslant R_1 \quad f_0^{-1}(c) \pitchfork S_R \quad \text{and} \quad \forall t \in [0,1] \quad \forall c \in \mathcal{B}(t) \quad f_t^{-1}(c) \pitchfork S_{R_1}.$$



For a sufficiently small ε we denote

$$K(0) = D_r(0) \setminus \bigcup_{c \in \mathcal{B}_\infty(0)} \mathring{D}_\varepsilon(c), \qquad K(t) = \chi_t^{-1}(K(0))$$

and we choose $R_0 \geqslant R_1$ such that

$$\forall s \in K(0) \quad \forall R \geqslant R_0 \quad f_0^{-1}(s) \pitchfork S_R \ \text{ and } \ \forall t \in [0,1] \quad \forall s \in K(t) \quad f_t^{-1}(s) \pitchfork S_{R_0}.$$

We denote

$$B'_t = \big(f_t^{-1}(D_r(0)) \cap B_{R_1}\big) \cup \big(f_t^{-1}(K(t)) \cap B_{R_0}\big), \quad t \in [0,1].$$

Lemma 15. *There exists a homeomorphism Φ such that we have a commutative diagram:*

$$\begin{tikzcd} B'_1 \arrow[r, "\Phi"] \arrow[d, "f_1"'] & B'_0 \arrow[d, "f_0"] \\ D_r(0) \arrow[r, "\Psi"'] & D_r(0). \end{tikzcd}$$

Proof. We denote by U'_t a neighborhood of $x(t)$ such that $\bar{U'_t} \subset U_t$. We denote by $\mathcal{U}_t$ (resp. $\mathcal{U}'_t$), the union (on the affine singular points of f_t) of the U_t (resp. U'_t). We set

$$B''_t = B'_t \setminus \mathcal{U}'_t, \quad t \in [0,1].$$

We can extend the homeomorphism Φ of lemma 12 to $\Phi_{\text{out}} : B''_1 \longrightarrow B''_0$. We just have to extend the vector field of lemma 12 to a new vector field denoted by v' such that

- v' is tangent to $\partial \mathcal{U}'_t$,
- v' is tangent to $S_{R_1} \times [0,1]$ on $F^{-1}(D_r(0) \setminus K(t) \times \{t\})$ for all $t \in [0,1]$,
- v' is tangent to $S_{R_0} \times [0,1]$ on $F^{-1}(K(t) \times \{t\})$ for all t.
- $d_z F.v'(z) = w(F(z))$ for all $z \in \bigcup_{t\in[0,1]} B''_t \times \{t\}$, which means that Φ_{out} respects the fibrations (w is defined by $\frac{\partial \chi}{\partial t}$).

If we set $B'' = \bigcup_{t\in[0,1]} B''_t \times \{t\}$ the integration of v' gives Ω^{out} and Φ_{out} such that:

$$\begin{tikzcd} B'' \arrow[r, "\Omega^{\text{out}}"] \arrow[d, "F"'] & B''_0 \times [0,1] \arrow[d, "f_0 \times \mathrm{id}"] \\ D_r(0) \times [0,1] \arrow[r, "\chi"'] & D_r(0) \times [0,1], \end{tikzcd} \qquad \begin{tikzcd} B''_1 \arrow[r, "\Phi_{\text{out}}"] \arrow[d, "f_1"'] & B''_0 \arrow[d, "f_0"] \\ D_r(0) \arrow[r, "\Psi"'] & D_r(0). \end{tikzcd}$$

We now explain how to glue Φ_{in} and Φ_{out} together. We can suppose that there exist spheres S_t centered at the singularities $x(t)$ such that if $S = \bigcup_{t\in[0,1]} S_t \times \{t\}$ then we have $\Omega^{\text{in}} : S \longrightarrow S_0 \times [0,1]$ and $\Omega^{\text{out}} : S \longrightarrow S_0 \times [0,1]$. It defines $\Omega^{\text{in}}_t : S_t \longrightarrow S_0$ and $\Omega^{\text{out}}_t : S_t \longrightarrow S_0$. On S_1 we have $\Omega^{\text{in}}_1 = \Phi_{\text{in}}$ and $\Omega^{\text{out}}_1 = \Phi_{\text{out}}$.

Now we define

$$\Theta_t : S_1 \longrightarrow S_0, \quad \Theta_t = \Omega^{\text{in}}_t \circ (\Omega^{\text{out}}_t)^{-1} \circ \Phi_{\text{out}}.$$

Then $\Theta_0 = \Phi_{\text{out}}$ and $\Theta_1 = \Phi_{\text{in}}$. On a set homeomorphic to $S \times [0,1]$ included in $\bigcup_{t\in[0,1]} U_t \setminus U'_t$ we glue Φ_{in} to Φ_{out}, moreover this gluing respects the fibrations f_0 and f_1. We end by doing this construction for all affine singular points. □

Proof of theorem 3. In the hypotheses of this theorem we supposed that there is no critical value at infinity. In order to apply the results of this section we have to prove that affine critical values are analytic functions of t. Let $c(0) \in \mathcal{B}_{\mathcal{aff}}(0)$, by lemma 10 it defines a continuous function $c : [0,1] \longrightarrow \mathbb{C}$. The set $\mathcal{C} = \{(c(t), t) \mid t \in [0,1]\}$ is a real algebraic subset of $\mathbb{C} \times [0,1]$ as all affine critical points are contained in B_{R_0} (lemma 9). In fact there is a polynomial $P \in \mathbb{C}[x,t]$ such that $\mathcal{C}$ is equal to $(P = 0) \cap (\mathbb{C} \times [0,1])$. Because the set of critical values is a braid of $\mathbb{C} \times [0,1]$ (lemma 10) then $c : [0,1] \longrightarrow \mathbb{C}$ is an analytic function.

If we suppose that $\mathcal{B}_\infty(t) = \varnothing$ for all $t \in [0,1]$ then by lemma 6 we can extend $\Phi : B'_1 \longrightarrow B'_0$ to $\Phi : f_1^{-1}(D_r(0)) \longrightarrow f_0^{-1}(D_r(0))$. And as $\mathcal{B}(t) \subset D_r(0)$ by a lemma similar to lemma 7 we can extend the homeomorphism to the whole space. □

Remark. We can improve the end of the proof of lemma 15 in order to get a trivialization of the whole family, that is to say $(f_t)_{t\in[0,1]}$ is topologically a product family. For each $t \in [0,1]$ we thicken the sphere S_t in a set $S_t \times [0,1]$. We parameterize this interval $[0,1]$ by s. Let

$$\Lambda : S \times [0,1] \longrightarrow S_0 \times [0,1] \times [0,1], \quad \Lambda(x,t,s) \mapsto (\Lambda_{t,s}(x), t, s)$$

where $\Lambda_{t,s}$ is a map defined by

$$\Lambda_{t,s} : S_t \longrightarrow S_0, \quad \Lambda_{t,s} = \Omega^{\text{in}}_{s\times t} \circ (\Omega^{\text{out}}_{s\times t})^{-1} \circ \Omega^{\text{out}}_t.$$

By fixing $s = 0$ the map Λ can be identified with Ω^{out} and for $s = 1$ it can be identified with Ω^{in}. So we are able to glue together the trivializations in order to get a homeomorphism Ω with a commutative diagram:

$$\begin{tikzcd} B' \arrow[r, "\Omega"] \arrow[d, "F"'] & B'_0 \times [0,1] \arrow[d, "f_0\times\mathrm{id}"] \\ D_r(0) \times [0,1] \arrow[r, "\chi"'] & D_r(0) \times [0,1], \end{tikzcd}$$

where $B' = \bigcup_{t\in[0,1]} B'_t \times \{t\}$. Now if $\mathcal{B}_\infty(t)$ is empty for all $t \in [0,1]$, then we can extend Ω in order to get:

$$\begin{tikzcd} \mathbb{C}^n \times [0,1] \arrow[r, "\Omega"] \arrow[d, "F"'] & \mathbb{C}^n \times [0,1] \arrow[d, "f_0\times\mathrm{id}"] \\ \mathbb{C} \times [0,1] \arrow[r, "\chi"'] & \mathbb{C} \times [0,1]. \end{tikzcd}$$

5. Polynomials in two variables

We set $n = 2$. We recall a result of L. Fourrier [Fo]. Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ with set of critical values at infinity $\mathcal{B}_\infty$. Let $* \notin \mathcal{B}$ and $Z = f^{-1}(*) \cup \bigcup_{c\in\mathcal{B}_\infty} f^{-1}(c)$. The *total link of* f is $L_f = Z \cap S_R$ for a sufficiently large R.

To f we associate a resolution $\phi : \Sigma \longrightarrow \mathbb{P}^1$,

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r] \arrow[d, "f"'] & \mathbb{P}^2 \arrow[d, "\tilde{f}"'] & \Sigma \arrow[l, "\pi"'] \arrow[dl, "\phi"] \\ \mathbb{C} \arrow[r] & \mathbb{P}^1 & \end{tikzcd}$$

where $\tilde{f}$ is the map coming from the homogenization of f; π is the minimal blow-up of some points on the line at infinity $\mathcal{L}_\infty$ of $\mathbb{P}^2$ in order to obtain a well-defined morphism $\phi : \Sigma \longrightarrow \mathbb{P}^1$. The components of the divisor $\pi^{-1}(\mathcal{L}_\infty)$ on which ϕ is surjective are the *dicritical components*. For each dicritical component D we have

a branched covering $\phi : D \longrightarrow \mathbb{P}^1$. If the union of dicritical components is D_{dic} we then have the restriction $\phi_{\mathrm{dic}} : D_{\mathrm{dic}} \longrightarrow \mathbb{P}^1$ of ϕ. The 0-*monodromy representation* is the representation

$$\pi_1(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathrm{Aut}\left(\phi_{\mathrm{dic}}^{-1}(*)\right).$$

The set $\phi_{\mathrm{dic}}^{-1}(*)$ is in bijection with the boundary components of $f^{-1}(*)$. Then the 0-monodromy representation can be seen as the action of $\pi_1(\mathbb{C} \setminus \mathcal{B})$ on the boundary components of $f^{-1}(*)$.

Theorem 16 (Fourrier). *Let f, g be complex polynomials in two variables with equivalent 0-monodromy representations and equivalent total links. Then there exist compact sets C, C' of $\mathbb{C}^2$ and homeomorphisms Φ_∞ and Ψ_∞ that make the diagram commute:*

$$\begin{tikzcd} \mathbb{C}^2 \setminus C \arrow[r, "\Phi_\infty"] \arrow[d, "f"'] & \mathbb{C}^2 \setminus C' \arrow[d, "g"] \\ \mathbb{C} \arrow[r, "\Psi_\infty"'] & \mathbb{C}. \end{tikzcd}$$

Let $f_t : \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that the coefficients of this family are algebraic in t. We suppose that the integers $\mu(t)$, $\lambda(t)$, $\#\mathcal{B}(t)$, $\#\mathcal{B}_{aff}(t)$, $\#\mathcal{B}_\infty(t)$ do not depend on $t \in [0,1]$. We also suppose the $\deg f_t$ does not depend on t. For our family (f_t), by theorem 1 we know that the geometric monodromy representations are all equivalent, then they act similarly on the boundary components of $f_t^{-1}(*)$. It implies that all the 0-monodromy representations of (f_t) are equivalent. Moreover if we suppose that for any $t, t' \in [0,1]$ the total links L_{f_t} and $L_{f_{t'}}$ are equivalent, then by theorem 16 the polynomials f_t and $f_{t'}$ are topologically equivalent out of some compact sets of $\mathbb{C}^2$. We need a result a bit stronger which can be proved by similar arguments than in [Fo] and we will omit the proof:

Lemma 17. *Let $(f_t)_{t\in[0,1]}$ be a polynomial family such that the coefficients are algebraic functions of t. We suppose that the 0-monodromy representations and the total links are all equivalent. Then there exist compact sets $C(t)$ of $\mathbb{C}^2$ and a homeomorphism Ω^∞ such that if $\mathcal{C} = \bigcup_{t\in[0,1]} C(t) \times \{t\}$ we have a commutative diagram:*

$$\begin{tikzcd} \mathbb{C}^2 \times [0,1] \setminus \mathcal{C} \arrow[r, "\Omega^\infty"] \arrow[d, "F"'] & \left(\mathbb{C}^2 \setminus C(0)\right) \times [0,1] \arrow[d, "f_0 \times \mathrm{id}"] \\ \mathbb{C} \times [0,1] \arrow[r, "\chi"'] & \mathbb{C} \times [0,1]. \end{tikzcd}$$

We now prove a strong version of the continuity of critical values.

Lemma 18. *The critical values are analytic functions of t. Moreover for $c(t) \in \mathcal{B}(t)$, the integers $\mu_{c(t)}$ and $\lambda_{c(t)}$ do not depend on $t \in [0,1]$.*

Proof. For affine critical values, refer to the proof of theorem 3. The constancy of $\mu_{c(t)}$ is a consequence of lemma 9 and lemma 13. For critical values at infinity we need a result of [Ha] and [HP] that enables to calculate critical values and Milnor numbers at infinity. As $\deg f_t$ is constant we can suppose that this degree is $\deg_y f_t$. Let denote $\Delta(x,s,t)$ the discriminant $\mathrm{Disc}_y(f_t(x,y)-s)$ with respect to y. We write

$$\Delta(x,s,t) = q_1(s,t)x^{k(t)} + q_2(s,t)x^{k(t)-1} + \cdots$$

First of all Δ has constant degree $k(t)$ in x because $k(t) = \mu(t) + \lambda(t) + \deg f_t - 1$ (see [HP]). Secondly by [Ha] we have

$$\mathcal{B}_\infty(t) = \{s \mid q_1(s,t) = 0\}$$

then we see that critical values at infinity depend continuously on t and that critical values at infinity are a real algebraic subset of $\mathbb{C} \times [0,1]$. For the analyticity we end as in the proof of theorem 3. Finally, for a fixed t, we have that $\lambda_c = k(t) - \deg_x \Delta(x,c,t)$. In other words $q_i(c,t)$ is zero for $i = 1,\ldots,\lambda_c$ and non-zero for $i = \lambda_c + 1$. For $c(t) \in \mathcal{B}_\infty(t)$ we now prove that $\lambda_{c(t)}$ is constant. The former formula proves that $\lambda_{c(t)}$ is constant except for finitely many $\tau \in [0,1]$ for which $\lambda_{c(\tau)} \geqslant \lambda_{c(t)}$. But if $\lambda_{c(\tau)} > \lambda_{c(t)}$ then $\lambda(\tau) = \sum_{c\in\mathcal{B}_\infty(\tau)} \lambda_c > \sum_{c\in\mathcal{B}_\infty(t)} \lambda_c = \lambda(t)$ which contradicts the hypotheses. □

To apply lemma 17 we need to prove:

Lemma 19. *For any $t, t' \in [0,1]$ the total links L_{f_t} and $L_{f'_t}$ are equivalent.*

Proof. The problem is similar to the one of [LR] and to lemma 8. Let $c(t) \in \mathcal{B}_\infty(t) \cup \{*\}$. As in lemma 15 we have $R_1 \gg 1$ such that $f_0^{-1}(c(0)) \cap S_{R_1}$ is the link at infinity of $f_0^{-1}(c(0))$. Moreover by lemma 15 we know that the link at infinity $f_0^{-1}(c(0)) \cap S_{R_1}$ is equivalent to the link $f_1^{-1}(c(1)) \cap S_{R_1}$. But $f_1^{-1}(c(1)) \cap S_{R_1}$ is not necessarily the link at infinity for $f_1^{-1}(c(1))$.

We now prove this fact; let denote $c = c(1)$. Let $R_2 \geqslant R_1$ such that for all $R \geqslant R_2$, $f_1^{-1}(c) \pitchfork S_R$, then $f_1^{-1}(c) \cap S_{R_2}$ is the link at infinity of $f_1^{-1}(c)$. We choose η, $0 < \eta \ll 1$ such that $f_1^{-1}(D_\eta(c))$ has transversal intersection with S_{R_1} and S_{R_2} and such that $f_1^{-1}(\partial D_\eta(c))$ has transversal intersection with all S_R, $R \in [R_1, R_2]$. Notice that η is much smaller than the ε of the former paragraphs and that $f_1^{-1}(s) \cap S_{R_2}$ is *not* the link at infinity of $f_1^{-1}(s)$ for $s \in \partial D_\eta(c)$. We fix R_0 smaller than R_1 such that $f_1^{-1}(D_\eta(c))$ has transversal intersection with S_{R_0}. We denote $f_1^{-1}(D_\eta(c)) \cap B_{R_i} \setminus \mathring{B}_{R_0}$ by $\mathcal{A}_i$, $i = 1,2$.

The proof is now similar to the one of lemma 8. Let A_1 and A_2 be connected components of $\mathcal{A}_1$ and $\mathcal{A}_2$ with $A_1 \subset A_2$. By Ehresmann theorem, we have fibrations $f_1 : A_1 \longrightarrow D_\eta(c)$, $f_1 : A_2 \longrightarrow D_\eta(c)$. From one hand $f_1^{-1}(c) \cap B_{R_1}$ is

diffeomorphic to $f_0^{-1}(c(0)) \cap B_{R_1}$. So by Suzuki formula (see [HL]) $f_1^{-1}(c) \cap B_{R_1}$ has Euler characteristic $1-\mu-\lambda+\mu_{c(0)}+\lambda_{c(0)}$. From the other hand $f_1^{-1}(c) \cap B_{R_2}$ has Euler characteristic $1-\mu-\lambda+\mu_{c(1)}+\lambda_{c(1)}$ by Suzuki formula. By lemma 18 we have that $\mu_{c(0)}+\lambda_{c(0)} = \mu_{c(1)}+\lambda_{c(1)}$, with $c = c(1)$. So the fiber $f_1^{-1}(c) \cap B_{R_1}$ and $f_1^{-1}(c) \cap B_{R_2}$ have the same Euler characteristic. As the number of connected components of $f_1^{-1}(c) \cap B_R$ is constant for $R \in [R_1, R_2]$ we have that $f_1^{-1}(c) \cap B_{R_1}$ and $f_1^{-1}(c) \cap B_{R_2}$ are homotopic. It implies that the fibrations $f_1 : A_1 \longrightarrow D_\eta(c)$ and $f_1 : A_2 \longrightarrow D_\eta(c)$ are fiber homotopy equivalent, and even more are diffeomorphic.

It provides a diffeomorphism $\Xi : A_1 \cap S_{R_1} = A_2 \cap S_{R_1} \longrightarrow A_2 \cap S_{R_2}$ and we can suppose that $\Xi(f_1^{-1}(c) \cap A_1 \cap S_{R_1})$ is equal to $f_1^{-1}(c) \cap A_2 \cap S_{R_2}$. By doing this for all connected components of $\mathcal{A}_1$, $\mathcal{A}_2$, for all values $c \in \mathcal{B}_\infty(1) \cup \{*\}$ and by extending Ξ to the whole spheres we get a diffeomorphism $\Xi : S_{R_1} \longrightarrow S_{R_2}$ such that $\Xi(f_1^{-1}(c) \cap S_{R_1}) = f_1^{-1}(c) \cap S_{R_2}$ for all $c \in \mathcal{B}_\infty(1) \cup \{*\}$. Then the total links for f_0 and f_1 are equivalent. □

Proof of theorem 4. By lemma 17 we have a trivialization $\Omega^\infty : \mathbb{C}^2 \times [0,1] \setminus \mathcal{C} \longrightarrow (\mathbb{C}^2 \setminus C(0)) \times [0,1]$. We can choose the R_1 (before lemma 15) such that $\mathring{C}(t) \subset B_{R_1}$. And then the proof of lemma 15 gives us an $\Omega^{\text{out}} : \bigcup_{t\in[0,1]} B''(t) \times \{t\} \longrightarrow B''(0) \times [0,1]$. By gluing Ω^{out} and Ω^∞ as in the proof of lemma 15, we obtain $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that:

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\Phi"] \arrow[d, "f_1"'] & \mathbb{C}^2 \arrow[d, "f_0"] \\ \mathbb{C} \arrow[r, "\Psi"'] & \mathbb{C}. \end{tikzcd}$$

Then f_0 and f_1 are topologically equivalent. □

Remark. As in the remark after the proof of theorem 3, we can glue Ω^{out} and Ω^∞ in order to get a topologically product family.

6. Continuity of the critical values at infinity

We now give a proof of the second part of lemma 2 in the introduction. The first part has been proven in lemma 18.

Lemma 20. *Let $(f_t)_{t\in[0,1]}$ be a family of polynomials such that the coefficients are polynomials in t. We suppose that:*

- *the total affine Milnor number $\mu(t)$ is constant;*
- *the degree $\deg f_t$ is constant;*
- *the set of critical points at infinity $\Sigma(t)$ is finite and does not vary: $\Sigma(t) = \Sigma$;*
- *for all $(x:0) \in \Sigma$, the generic Milnor number $\nu_{\bar{x}}(t)$ is independent of t.*

Then the critical values at infinity depend continuously on t, i.e. *if $c(t_0) \in \mathcal{B}_\infty(t_0)$ then for all t near t_0 there exists $c(t)$ near $c(t_0)$ such that $c(t) \in \mathcal{B}_\infty(t)$.*

Let f be a polynomial. For $x \in \mathbb{C}^n$ we have $(x : 1)$ in $\mathbb{P}^n$ and if $x_n \neq 0$ we divide x by x_n to obtain local coordinates at infinity $(\check{x}', x_0)$. The following lemma explains the link between the critical points of f and those of F_c. It uses Euler relation for the homogeneous polynomial part of f of degree d.

Lemma 21.

- *F_c has a critical point $(\check{x}', x_0)$ with $x_0 \neq 0$ of critical value 0 if and only if f has a critical point x with critical value c.*
- *F_c has a critical point $(\check{x}', 0)$ of critical value 0 if and only if $(x : 0) \in \Sigma$.*

Proof of lemma 20. We suppose that critical values at infinity are *not* continuous functions of t. Then there exists (t_0, c_0) such that $c_0 \in \mathcal{B}_\infty(t_0)$ and for all (t, c) in a neighborhood of (t_0, c_0), we have $c \notin \mathcal{B}_\infty(t)$. Let P be the point of irregularity at infinity for (t_0, c_0). Then $\mu_P(F_{t_0,c_0}) > \mu_P(F_{t_0,c})$ $(c \neq c_0)$ by definition of $c_0 \in \mathcal{B}_\infty(t_0)$ and by semi-continuity of the local Milnor number at P we have $\nu_P(t_0) = \mu_P(F_{t_0,c}) \geqslant \mu_P(F_{t,c}) = \nu_P(t)$, $(t, c) \neq (t_0, c_0)$.

We consider t as a complex parameter. By continuity of the critical points and by conservation of the Milnor number for $(t, c) \neq (t_0, c_0)$ we have critical points $M(t, c)$ near P of $F_{t,c}$ that are not equal to P. This fact uses that $\deg f_t$ is a constant, in order to prove that $F_{t,c}$ depends continuously on t.

Let denote by V' the algebraic variety of $\mathbb{C}^3 \times \mathbb{C}^n$ defined by $(t, c, s, x) \in V'$ if and only if $F_{t,c}$ has a critical point x with critical value s (the equations are $\operatorname{grad} F_{t,c}(x) = 0, F_{t,c}(x) = s$). If $\mu_P(F_{t,c}) > 0$ for a generic (t, c) then $\{(t, c, 0, P) \mid (t, c) \in \mathbb{C}^2\}$ is a subvariety of V'. We define V to be the closure of V' minus this subvariety. Then for a generic (t, c), $(t, c, 0, P) \notin V$. We call $\pi : \mathbb{C}^3 \times \mathbb{C}^n \longrightarrow \mathbb{C}^3$ the projection on the first factor. We set $W = \pi(V)$. Then W is locally an algebraic variety around $(t_0, c_0, 0)$. For each (t, c) there is a non-zero finite number of values s such that $(t, c, s) \in W$. So W is locally an equi-dimensional variety of codimension 1. Then it is a germ of hypersurface of $\mathbb{C}^3$. Let $R(t, c, s)$ be the polynomial that defines W locally. We set $Q(t, c) = R(t, c, 0)$. As $Q(t_0, c_0) = 0$ then in all neighborhoods of (t_0, c_0) there exists $(t, c) \neq (t_0, c_0)$ such that $Q(t, c) = 0$. Moreover there are solutions for t a real number near t_0 and we now suppose that t is a real parameter.

Then for $(t, c) \neq (t_0, c_0)$ we have that: $Q(t, c) = 0$ if and only if $F_{t,c}$ has a critical point $M(t, c) \neq P$ with critical value 0. The point $M(t, c)$ is not equal to P because for $t \neq t_0$, $(t, c, 0, P) \notin V$: it uses that $c \notin \mathcal{B}_\infty(t)$ for $t \neq t_0$, and that $\nu_P(t) = \nu_P(t_0)$. Let us notice that $M(t, c) \to P$ as $(t, c) \to (t_0, c_0)$.

We end the proof be studying the different cases:

- if we have $M(t, c)$ in $\mathcal{H}_\infty$ (of equation $(x_0 = 0)$) then $M(t, c) \in \Sigma$ which provides a contradiction because then it is equal to P;

- if we have points $M(t,c)$, not in $\mathcal{H}_\infty$, with $t \neq t_0$ then there are affine critical points $M'(t,c)$ of f_t (lemma 21), and as $M(t,c)$ tends towards P (as (t,c) tends towards (t_0,c_0)) we have that $M'(t,c)$ escapes at infinity. It contradicts the fact that the critical points of f_t are bounded (lemma 9).
- if we have points $M(t_0,c)$, not in $\mathcal{H}_\infty$, then there are infinitely many affine critical points for f_{t_0}, which is impossible since the singularities of f_{t_0} are isolated. □

7. Examples

Example 1. Let $f_t = x(x^2y + tx + 1)$. Then $\mathcal{B}_{aff}(t) = \varnothing$, $\mathcal{B}_\infty(t) = \{0\}$, $\lambda(t) = 1$ and $\deg f_t = 4$. Then by theorem 4, f_0 and f_1 are topologically equivalent. These are examples of polynomials that are topologically but not algebraically equivalent, see [Bo].

Example 2. Let $f_t = (x+t)(xy+1)$. Then f_0 and f_1 are not topologically equivalent. One has $\mathcal{B}_\infty(t) = \varnothing$, $\mathcal{B}_{aff}(t) = \{0,t\}$ for $t \neq 0$, but $\mathcal{B}_\infty(0) = \{0\}$, $\mathcal{B}_{aff}(0) = \varnothing$. In fact the two affine critical points for f_t "escape at infinity" as t tends towards 0.

Example 3. Let $f_t = x\big(x(y+tx^2)+1\big)$. Then f_0 is topologically equivalent to f_1. We have for all $t \in [0,1]$, $\mathcal{B}_{aff}(t) = \varnothing$, $\mathcal{B}_\infty(t) = \{0\}$, and $\lambda(t) = 1$, but $\deg f_t = 4$ for $t \neq 0$ while $\deg f_0 = 3$.

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Arnaud Bodin
Centre de Recerca Matemàtica
Apartat 50
08193 Bellatera
Spain
e-mail: abodin@crm.es

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