Resonance category

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Resonance category

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Abstract. The main purpose of this paper is to introduce a new category, which we call a resonance category, whose combinatorics reflect that of canonical stratifications of n-fold symmetric smash products. The study of the stratifications can then be abstracted to the study of functors satisfying certain sets of axioms, which we name resonance functors.

One frequently studied stratification is that of the set of all polynomials of degree n, defined by fixing the allowed multiplicities of roots. We apply our abstract combinatorial framework, in particular the notion of direct product of relative resonances, to study the Arnold problem of computing the algebro-topological invariants of these strata.

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1. Introduction

Complicated combinatorial problems often arise when one studies the homological properties of strata in some topological space with a given natural stratification. In this paper, we study the symmetric smash products stratified by point multiplicities.

More specifically, let X be a pointed topological space (we refer to the base point as a point at infinity), and denote

$$X^{(n)} = \underbrace{X \wedge X \wedge \cdots \wedge X}_{n} / \mathcal{S}_n,$$

where $\wedge$ is the smash product of pointed spaces, $\mathcal{S}_n$ is the symmetric group, and the action of $\mathcal{S}_n$ on the n-fold smash product of X is the permutation action. In other words, $X^{(n)}$ is the set of all unordered collections of n points on X with the collections having at least one of the points at infinity identified, to form a new infinity point. $X^{(n)}$ is naturally stratified by point coincidences, and the strata are indexed by the

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number partitions of n. Note that we consider the closed strata, so, for example, the stratum indexed $(\underbrace{1, 1, \ldots, 1}_{n})$ is the whole space $X^{(n)}$.

The main open stratum, that is the complement of the closed stratum $(2, 1, \ldots, 1)$, is a frequently studied object. It was suggested by Arnold in a much more general context, see for example [2], that in situations of this kind one should study the problem for all closed strata. The main argument in support of this point of view is that there is usually no natural stratification on the main open stratum, while there is one on its complement, also known as discriminant. Having a natural stratification allows one to apply such computational techniques as spectral sequences in a canonical way. Once some information has been obtained about the closed strata, one can try to find out something about the open stratum by means of some kind of duality.

If one specifies $X = \mathbb{S}^1$, resp. $X = \mathbb{S}^2$, one obtains as strata the spaces of all monic real hyperbolic, resp. monic complex, polynomials of degree n with specified root multiplicities. These spaces naturally appear in singularity theory, [1]. Homological invariants of several of these strata were in particular computed by Arnold, Shapiro, Sundaram, Welker, Vassiliev, and the author, see [2], [4], [5], [8], [9]. These are the special cases which have inspired this general study.

Here we take a different, more abstract look at this set of problems. More specifically, the idea is to introduce a new canonical combinatorial object, independent of topology of particular X, where the combinatorial aspects of these stratifications would be fully reflected. This object is a certain category, which we name the *resonance category*. It was suggested to the author by B. Shapiro, [7], to use the term *resonance* as a generic reference to a certain type of linear relations among parts of a number partition.

Having this canonically defined category at hand, one then can, for each specific topological space X, view the natural stratification of $X^{(n)}$ as a certain functor from the resonance category to $\mathbf{Top}^*$. These functors satisfy a system of axioms, which we take as a definition of *resonance functors*. The combinatorial structures in the resonance category will then project to the corresponding structures in each specific $X^{(n)}$. This opens the door to develop the general combinatorial theory of the resonance category, and then prove facts valid for all resonance functors satisfying some further conditions, such as for example acyclicity of certain spaces.

The main combinatorial structure inside the resonance category, which we study, is that of *relative resonances* and their direct products. Intuitively, a relative resonance encodes the combinatorial type of a stratum with a union of some substrata shrunk to form the new infinity point. These spaces appear naturally if we are trying to compute the homology groups of our strata by means of long exact sequences, or, more generally, spectral sequences.

Our idea is that the combinatorial knowledge of which relative resonances are *reducible* (that is, are direct products of other relative resonances) serves as a guidance

for which long exact sequences one is to consider for the actual homology computations. This way, the Arnold problem of computing the algebraic invariants of the strata splits into two parts: the combinatorial one, embodied by various structures in the resonance category such as the relative resonances, and the topological one, reflecting the specific properties of X.

Our notions of sequential and strongly sequential resonances are intended to capture the combinatorial structure of those resonances, which are particularly compatible with the spectral sequence computations. This, in turn, leads to the natural notion of complexity of resonances.

As mentioned above, to illustrate a possible appearance of this abstract framework we choose to use a class of topological spaces which come in particular from the singularity theory, and whose topological properties have been studied: spaces of polynomials (real or complex) with prescribed root multiplicities. In particular, in case of strata $(k^m, 1^t)$, which were studied in [2], [4] for the complex case, and in [5], [8] for the real case, we demonstrate how the inherent combinatorial structure of the resonance category makes this particular resonance especially reducible.

The paper is organized as follows:

Section 2. We introduce the notion of resonance category and describe the structure of its set of morphisms.
Section 3. We introduce the notions of relative resonances, direct products of relative resonances, and resonance functors.
Section 4. We formulate the problem of Arnold and Shapiro which motivated this research as that concerning a specific resonance functor. Then we analyze the combinatorial structure of resonances (a^k, b^l), which leads to the complete determination of the homotopy types of the corresponding strata for $X = S^1$.
Section 5. We analyze the combinatorial structure of the sequential and strongly sequential resonances. For $X = S^1$ this leads to the complete computation of homotopy types of the strata corresponding to resonances $(a^k, b^l, 1^m)$ such that $a - bl \leq m$. Next, we consider division chain resonances, which constitute a vast generalization of the case $(a^k, 1^l)$. We prove that in this case the strata always have a homotopy type of a bouquet of spheres. We describe a combinatorial model to enumerate these spheres as paths in a certain weighted directed graph, with dimensions of the spheres being given by the total weights of the paths.
Section 6. We introduce the notion of a complexity of a resonance and give a series of examples of resonances having arbitrarily high complexity.

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2. Resonance Category

2.1. Resonances and their symbolic notation. For every positive integer n, let $\{-1, 0, 1\}^n$ denote the set of all points in $\mathbb{R}^n$ with coordinates in the set $\{-1, 0, 1\}$. We say that a subset $S \subseteq \{-1, 0, 1\}^n$ is *span-closed* if $\operatorname{span}(S) \cap \{-1, 0, 1\}^n = S$, where $\operatorname{span}(S)$ is the linear subspace spanned by the origin and points in S. Of course the origin lies in every span-closed set. For $x = (x_1, \dots, x_n) \in \{-1, 0, 1\}^n$, we use the notations $\operatorname{Plus}(x) = \{i \in [n] \mid x_i = 1\}$ and $\operatorname{Minus}(x) = \{i \in [n] \mid x_i = -1\}$.

Definition 2.1. (1) A subset $S \subseteq \{-1, 0, 1\}^n$ is called an *n-cut* if it is span-closed and for every $x \in S \setminus \{\text{origin}\}$ we have $\operatorname{Plus}(x) \neq \emptyset$ and $\operatorname{Minus}(x) \neq \emptyset$. We denote the set of all n-cuts by $\mathcal{R}_n$.

(2) $\mathcal{S}_n$ acts on $\{-1, 0, 1\}^n$ by permuting coordinates, which in turn induces $\mathcal{S}_n$-action on $\mathcal{R}_n$. The *n-resonances* are defined to be the orbits of the latter $\mathcal{S}_n$-action. We let $[S]$ denote the n-resonance represented by the n-cut S.

The resonance consisting of origin only is called *trivial.*

Example 2.2 (n-resonances for small values of n). (1) There are no nontrivial 1-resonances.

(2) There is one nontrivial 2-resonance: $[\{(0, 0), (1, -1), (-1, 1)\}]$.

(3) There are four nontrivial 3-resonances:

$$[\{(0, 0, 0), (1, -1, 0), (-1, 1, 0)\}],$$

$$[\{(0, 0, 0), (1, -1, 0), (-1, 1, 0), (1, 0, -1), (-1, 0, 1), (0, 1, -1), (0, -1, 1)\}],$$

$$[\{(0, 0, 0), (1, -1, -1), (-1, 1, 1)\}],$$

$$[\{(0, 0, 0), (1, -1, -1), (-1, 1, 1), (0, 1, -1), (0, -1, 1)\}].$$

(4) Here is an example of a nontrivial 6-resonance:

$$[\{(0,0,0,0,0,0), \pm(1,1,0,-1,-1,0), \pm(0,1,1,0,-1,-1), \pm(1,0,-1,-1,0,1)\}].$$

Symbolic notation. To describe an n-resonance, rather than to list all of the elements of one of its representatives, it is more convenient to use the following symbolic notation: we write a sequence of n linear expressions in some number (between 1 and n) of parameters, the order in which the expressions are written is inessential.

Here is how to get from such a symbolic expression to the n-resonance: choose an order on the n linear expressions and observe that now they parameterize some linear subspace of $\mathbb{R}^n$, which we denote by A. The n-resonance is now the orbit of $A^\perp \cap \{-1, 0, 1\}^n$.

Reversely, to go from an n-resonance to a symbolic expression: choose a representative n-cut S, the symbolic expression can now be obtained as a linear parameterization of $\mathrm{span}(S)^{\perp}$.

For example the 6 nontrivial resonances listed in Example 2.2 are (in the same order):

$$(a,a),\quad (a,a,b),\quad (a,a,a),\quad (a+b,a,b),$$
$$(2a,a,a),\quad (a+b,b+c,a+d,b+d,c+d,2d).$$

2.2. Acting on cuts with ordered set partitions. We say that π is an *ordered set partition* of $[n]$ with m parts (sometimes called *blocks*) when $\pi = (\pi_1,\dots,\pi_m)$, $\pi_i \neq \emptyset$, $[n] = \bigcup_{i=1}^m \pi_i$, and $\pi_i \cap \pi_j = \emptyset$, for $i \neq j$. If the order of the parts is not specified, then π is just called a *set partition*. We denote the set of all partitions, resp. ordered partitions, of a set A by $\mathrm{P}(A)$, resp. $\mathrm{OP}(A)$. For $\mathrm{P}([n])$, resp. $\mathrm{OP}([n])$, we use the shorthand notations $\mathrm{P}(n)$, resp. $\mathrm{OP}(n)$. Furthermore, for every set A, we let $\mathrm{un}\colon \mathrm{OP}(A) \to \mathrm{P}(A)$ be the map which takes the ordered partition to the associated unordered partition.

Definition 2.3. Given $\pi = (\pi_1,\dots,\pi_k)$ an ordered set partition of $[m]$ with k parts, and $\nu = (\nu_1,\dots,\nu_m)$ an ordered set partition of $[n]$ with m parts, their *composition* $\pi \circ \nu$ is an ordered set partition of $[n]$ with k parts, defined by $\pi \circ \nu = (\mu_1,\dots,\mu_k)$, $\mu_i = \bigcup_{j\in\pi_i} \nu_j$, for $i = 1,\dots,k$.

Analogously, we can define $\pi \circ \nu$ for an ordered set partition ν and a set partition π, in which case $\pi \circ \nu$ is a set partition without any specified order on the blocks.

In particular, when $m = n$, and $|\pi_i| = 1$, for $i = 1,\dots,n$, we can identify $\pi = (\pi_1,\dots,\pi_n)$ with the corresponding permutation of $[n]$. The composition of two such ordered set partitions corresponds to the multiplication of corresponding permutations, and we denote the ordered set partition $(\{1\},\dots,\{n\})$ by id_n, or just id.

Definition 2.4. For $A \subseteq B$, let $p_{B,A}\colon \mathrm{P}(B) \to \mathrm{P}(A)$ denote map induced by the restriction from B to A. For two disjoint set A and B, and $\Pi \subseteq \mathrm{P}(A)$, $\Lambda \subseteq \mathrm{P}(B)$, we define $\Pi \times \Lambda = \{\pi \in \mathrm{P}(A \cup B) \mid p_{A\cup B,A}(\pi) \in \Pi,\ p_{A\cup B,B}(\pi) \in \Lambda\}$.

The following definition provides the combinatorial constructions necessary to describe the morphisms of the resonance category, as well as to define the relative resonances.

Definition 2.5. Assume S is an n-cut. For an ordered set partition of $[n]$, denoted $\pi = (\pi_1,\dots,\pi_m)$, we define $\pi S \in \mathcal{R}_m$ to be the set of all m-tuples $(t_1,\dots,t_m) \in \{-1,0,1\}^m$, for which there exists $(s_1,\dots,s_n) \in S$ such that for all $j \in [m]$, and $i \in \pi_j$, we have $s_i = t_j$.

Clearly id $S = S$, and one can see that $(\pi \circ \nu)S = \pi(\nu S)$.

Verification of $(\pi \circ \nu)S = \pi(\nu S)$. By definition we have

$$(\pi \circ \nu)S = \{(t_1, \dots, t_k) \mid \exists (s_1, \dots, s_n) \in S \text{ s.t. } \forall j \in [k], i \in \mu_j : s_i = t_j\},$$

$$\nu S = \{(x_1, \dots, x_m) \mid \exists (s_1, \dots, s_n) \in S \text{ s.t. } \forall q \in [m], i \in \nu_q : s_i = x_q\},$$

$$\pi(\nu S) = \{(t_1, \dots, t_k) \mid \exists (x_1, \dots, x_m) \in \nu S \text{ s.t. } \forall j \in [k], q \in \pi_j, i \in \nu_q : s_i = t_j\}.$$

The identity $(\pi \circ \nu)S = \pi(\nu S)$ follows now from the equality $\mu_j = \bigcup_{q \in \pi_j} \nu_q$.

There are many different ways to formulate Definition 2.5. We chose the ad hoc combinatorial language, but it is also possible to put it in the linear-algebraic terms. An ordered set partition of $[n]$, $\pi = (\pi_1, \dots, \pi_m)$, defines an inclusion map $\phi\colon \mathbb{R}^m \to \mathbb{R}^n$ by $\phi(e_i) = \sum_{j \in \pi_i} \tilde{e}_j$, where $\{e_1, \dots, e_m\}$, resp. $\{\tilde{e}_1, \dots, \tilde{e}_n\}$, is the standard orthonormal basis of $\mathbb{R}^m$, resp. $\mathbb{R}^n$. Given $S \in \mathcal{R}_n$, πS can then be defined as $\phi^{-1}(\operatorname{Im} \phi \cap S)$.

2.3. The definition of the resonance category and the terminology for its morphisms

Definition 2.6. The *resonance category*, denoted $\mathcal{R}$, is defined as follows:

(1) The set of objects is the set of all n-cuts, for all positive integers n, $\mathcal{O}(\mathcal{R}) = \bigcup_{n=1}^{\infty} \mathcal{R}_n$.

(2) The set of morphisms is indexed by triples (S, T, π), where $S \in \mathcal{R}_m$, $T \in \mathcal{R}_n$, and π is an ordered set partition of $[n]$ with m parts such that $S \subseteq \pi T$. For the reasons which will become clear later we denote the morphism indexed with (S, T, π) by $S \twoheadrightarrow \pi T \overset{\pi}{\hookrightarrow} T$.

As the notation suggests, the initial object of the morphism $S \twoheadrightarrow \pi T \overset{\pi}{\hookrightarrow} T$ is S and terminal object is T. The composition rule is defined by

$$(S \twoheadrightarrow \pi T \overset{\pi}{\hookrightarrow} T) \circ (T \twoheadrightarrow \nu Q \overset{\nu}{\hookrightarrow} Q) = S \twoheadrightarrow \pi\nu Q \overset{\pi\nu}{\hookrightarrow} Q,$$

where $S \in \mathcal{R}_k$, $T \in \mathcal{R}_m$, $Q \in \mathcal{R}_n$, π is an ordered set partition of $[m]$ with k parts, and ν is an ordered set partition of $[n]$ with m parts.

An alert reader will notice that the resonances themselves did not appear explicitly in the definition of the resonance category. In fact, it is not difficult to notice that resonances are isomorphism classes of objects of $\mathcal{R}$. Let us now look at the set of morphisms of $\mathcal{R}$ in some more detail.

(1) For $S \in \mathcal{R}_n$, the identity morphism of S is $S \twoheadrightarrow S \overset{\mathrm{id}}{\hookrightarrow} S$.

(2) Let us introduce short hand notations: $S \twoheadrightarrow T$ for $S \twoheadrightarrow T \overset{\mathrm{id}}{\hookrightarrow} T$, and $\pi T \overset{\pi}{\hookrightarrow} T$ for $\pi T \twoheadrightarrow \pi T \overset{\pi}{\hookrightarrow} T$. Then we have

$$S \twoheadrightarrow \pi T \overset{\pi}{\hookrightarrow} T = (S \twoheadrightarrow \pi T) \circ (\pi T \overset{\pi}{\hookrightarrow} T).$$

Note also that $S \twoheadrightarrow S = S \overset{\mathrm{id}}{\hookrightarrow} S$.

(3) The associativity of the composition rule can be derived from the commutation relation

$$(\pi S \overset{\pi}{\hookrightarrow} S) \circ (S \twoheadrightarrow T) = (\pi S \twoheadrightarrow \pi T) \circ (\pi T \overset{\pi}{\hookrightarrow} T)$$

as follows:

$$\begin{aligned}
&(S \twoheadrightarrow \pi T \hookrightarrow T) \circ (T \twoheadrightarrow \nu Q \hookrightarrow Q) \circ (Q \twoheadrightarrow \rho X \hookrightarrow X) \\
&= (S \twoheadrightarrow \pi T) \circ (\pi T \hookrightarrow T) \circ (T \twoheadrightarrow \nu Q) \circ (\nu Q \hookrightarrow Q) \circ (Q \twoheadrightarrow \rho X) \circ (\rho X \hookrightarrow X) \\
&= (S \twoheadrightarrow \pi T) \circ (\pi T \twoheadrightarrow \pi \nu Q) \circ (\pi \nu Q \twoheadrightarrow \pi \nu \rho X) \\
&\qquad \circ (\pi \nu \rho X \hookrightarrow \nu \rho X) \circ (\nu \rho X \hookrightarrow \rho X) \circ (\rho X \hookrightarrow X).
\end{aligned}$$

(4) We shall use the following names: morphisms $S \twoheadrightarrow T$ are called *gluings* (or n-gluings, if it is specified that $S, T \in \mathcal{R}_n$); morphisms $\pi T \overset{\pi}{\hookrightarrow} T$ are called *inclusions* (or (n, m)-inclusions, if it is specified that $T \in \mathcal{R}_n$, $\pi T \in \mathcal{R}_m$), the inclusions are called *symmetries* if π is a permutation. As observed above, the symmetries are the only isomorphisms in $\mathcal{R}$. Here are two examples of inclusions:

$$\{(0,0), (1,-1), (-1,1)\} \xhookrightarrow{(\{1\},\{2,3\})} \{(0,0,0), (1,-1,-1), (-1,1,1)\},$$

$$\{(0,0), (1,-1), (-1,1)\} \xhookrightarrow{(\{1\},\{2,3\})} \{(0,0,0), \pm(1,-1,-1), \pm(0,1,-1)\}.$$

3. Relative resonances, direct products, and resonance functors

3.1. Relative Resonances. Let $A(n)$ denote the set of all collections of non-empty multisubsets of $[n]$, and let $\mathrm{P}(n) \subseteq A(n)$ be the set of all partitions of $[n]$. For every $S \in \mathcal{R}_n$ let us define a closure operation on $A(n)$, resp. on $\mathrm{P}(n)$.

Definition 3.1. Let $\mathcal{A} \in A(n)$. We define $\mathcal{A} \Downarrow S \subseteq A(n)$ to be the minimal set satisfying the following conditions:

(1) $\mathcal{A} \in \mathcal{A} \Downarrow S$;
(2) if $\{B_1, B_2, \ldots, B_m\} \in \mathcal{A} \Downarrow S$, then $\{B_1 \cup B_2, B_3, \ldots, B_m\} \in \mathcal{A} \Downarrow S$;
(3) if $\{B_1, B_2, \ldots, B_m\} \in \mathcal{A} \Downarrow S$, and there exists $x \in S$ such that $\mathrm{Plus}(x) \subseteq B_1$, then $\{(B_1 \setminus \mathrm{Plus}(x)) \cup \mathrm{Minus}(x), B_2, \ldots, B_m\} \in \mathcal{A} \Downarrow S$.

For $\pi \in \mathrm{P}(n)$, we define $\pi \downarrow S \subseteq \mathrm{P}(n)$ as $\pi \downarrow S = (\pi \Downarrow S) \cap \mathrm{P}(n)$. For a set $\Pi \subseteq \mathrm{P}(n)$ we define $\Pi \downarrow S = \bigcup_{\pi \in \Pi} \pi \downarrow S$. We say that Π is *S-closed* if $\Pi \downarrow S = \Pi$.

The idea behind this definition comes from the context of the standard stratification of the n-fold symmetric product. Given a stratum X indexed by a number partition of n with m parts, let us fix some order on the parts. A substratum Y is obtained by choosing some partition π of $[m]$ and summing the numbers within the blocks of π. Since the order of the parts of the number partition indexing X is fixed, X gives rise to a unique m-cut S. The set $\pi \downarrow S$ describes all partitions ν of $[m]$ such that if the numbers within the blocks of ν are summed then the obtained stratum Z satisfies $Z \subseteq Y$. In particular, if Y is shrunk to a point, then so is Z. The two following examples illustrate how the different parts of Definition 3.1 might be needed.

Example 1. *The equivalences of type* (2) *from the Definition* 3.1 *are needed.* Let the stratum X be indexed by $(3, 2, 1, 1, 1)$ (fix this order of the parts), and let $\pi = \{1\}\{23\}\{4\}\{5\}$. Then, the stratum Y is indexed by $(3, 3, 1, 1)$. Clearly, the stratum Z, which is indexed by $(3, 3, 2)$, lies inside Y, hence $\{1\}\{2\}\{345\} \in \pi \downarrow S$, where S is the cut corresponding to $(3, 2, 1, 1, 1)$. However, if one starts from the partition π and uses equivalences of type (3) from Definition 3.1, the only other partitions one can obtain are $\{1\}\{24\}\{3\}\{5\}$, and $\{1\}\{25\}\{3\}\{4\}$. None of them refines $\{1\}\{2\}\{345\}$, hence it would not be enough in Definition 3.1 to just take the partitions which can be obtained via the equivalences of type (3) and then take $\pi \downarrow S$ to be the set of all the partitions which are refined by these.

Example 2. *It is necessary to view the equivalence relation on the larger set* $A(n)$. This time, let the stratum X be indexed by $(a + b, b + c, a + d, b + d, c + d, 2d)$ (fix this order of the parts, and assume as usual that there are no linear relations on the parts other than those induced by the algebraic identities on the variables a, b, c, and d). Furthermore, let $\pi = \{16\}\{23\}\{45\}$. Then the stratum Y is indexed by $(a + b + 2d, a + b + c + d, b + c + 2d)$. Clearly, we have $\{34\}\{15\}\{26\} \in \pi \downarrow S$, where S is the cut corresponding to $(a + b, b + c, a + d, b + d, c + d, 2d)$.

A natural idea for Definition 3.1 could have been to define the equivalence relation directly on the set $\mathrm{P}(n)$ and use "swaps" instead of the equivalences of type (3), i.e., to replace the condition (3) by the following one:

> *If* $\{B_1, B_2, \ldots, B_m\} \in \mathcal{A} \downarrow S$, *and there exists* $x \in S$, *such that* $\mathrm{Plus}(x) \subseteq B_1$, *and* $\mathrm{Minus}(x) \subseteq B_2$, *then* $\{(B_1 \setminus \mathrm{Plus}(x)) \cup \mathrm{Minus}(x), (B_2 \setminus \mathrm{Minus}(x)) \cup \mathrm{Plus}(x), B_3, \ldots, B_m\} \in \mathcal{A} \downarrow S$.

However, this would not have been sufficient as this example shows, since no swaps would be possible on $\pi = \{16\}\{23\}\{45\}$.

Definition 3.2. Let S be an n-cut, $\Pi \subseteq \mathrm{P}(n)$ an S-closed set of partitions. We define

$$S \setminus \Pi = S \setminus \{s \in S \mid (\mathrm{Minus}(s), \mathrm{Plus}(s), \text{singletons}) \in \Pi\},$$

where $(\mathrm{Minus}(s), \mathrm{Plus}(s), \text{singletons})$ is the partition which has only two nonsingleton blocks: $\mathrm{Minus}(s)$ and $\mathrm{Plus}(s)$.

In the next definition we give a combinatorial analog of viewing a stratum relative to a substratum.

Definition 3.3. (1) A *relative n-cut* is a pair (S, Π), where $S \subseteq \{-1, 0, 1\}^n$, $\Pi \subseteq \mathrm{P}(n)$, such that the following two conditions are satisfied:

- $(\mathrm{span}\, S) \setminus \Pi = S$;
- Π is $(\mathrm{span}\, S)$-closed.

(2) The permutation $\mathcal{S}_n$-action on $\{-1, 0, 1\}^n$ induces an $\mathcal{S}_n$-action on the relative n-cuts by $(S, \Pi) \overset{\sigma}{\mapsto} (\sigma S, \Pi\sigma^{-1})$, for $\sigma \in \mathcal{S}_n$. The *relative n-resonances* are defined to be the orbits of this $\mathcal{S}_n$-action. We let $[S, \Pi]$ denote the relative n-resonance represented by the relative n-cut (S, Π).

When $S \in \mathcal{R}_n$ and $\Pi \subseteq \mathrm{P}(n)$, Π is S-closed, it is convenient to use the notation $Q(S, \Pi)$ to denote the relative cut $(S \setminus \Pi, \Pi)$. Clearly we have $(S, \Pi) = Q(\mathrm{span} S, \Pi)$. Analogously, $[Q(S, \Pi)]$ denotes the relative resonance $[S \setminus \Pi, \Pi]$. We use these two notations interchangeably depending on which one is more natural in the current context.

The special case of the particular importance for our computations in the later sections is that of $Q(S, \pi \downarrow S)$, where π is a partition of $[n]$ with m parts. In this case, we call $(S \setminus (\pi \downarrow S), \pi \downarrow S)$ the relative (n, m)-cut associated to S and π.

By Definition 3.3, the relative cut $(S, \Pi) = ((\mathrm{span}\, S) \setminus \Pi, \Pi)$ consists of two parts. We intuitively think of $(\mathrm{span}\, S) \setminus \Pi$ as the set of all resonances which survive the shrinking of the strata associated to the elements of Π, so it is natural to call them *surviving elements*. We also think of Π as the set of all partitions whose associated strata are shrunk to the infinity point, so, accordingly, we call them *partitions at in□nity*.

3.2. Direct products of relative resonances

Definition 3.4. For relative resonances (S, Π) and (T, Λ) we define

$$(S, \Pi) \times (T, \Lambda) = (S \times T, (\Pi \times \mathrm{P}(m)) \cup (\mathrm{P}(n) \times \Lambda)).$$

Clearly the orbit $[(S, \Pi) \times (T, \Lambda)]$ does not depend on the choice of representatives of the orbits $[S, \Pi]$ and $[T, \Lambda]$, so we may define $[S, \Pi] \times [T, \Lambda]$ to be $[(S, \Pi) \times (T, \Lambda)]$.

The following special cases are of particular importance for our computation:

(1) *A direct product of two resonances.* For an m-cut S, and an n-cut T, we have

$$S \times T = \{(x_1, \dots, x_m, y_1, \dots, y_n) \mid (x_1, \dots, x_m) \in S, (y_1, \dots, y_n) \in T\} \in \mathcal{R}_{m+n},$$

and $[S] \times [T] = [S \times T]$.

(2) *A direct product of a relative resonance and a resonance.* For $S \in \mathcal{R}_n$, $\Pi \subseteq \mathrm{P}(n)$ an S-closed set of partitions, and $T \in \mathcal{R}_k$, we have $Q(S, \Pi) \times T = Q(S \times T, \widetilde{\Pi})$, where $\widetilde{\Pi} = \Pi \times \mathrm{P}(\{n+1, n+2, \dots, n+k\})$, and $[Q(S, \Pi)] \times [T] = [Q(S, \Pi) \times T]$.

Example.

$$\begin{aligned} &Q(\{(0,0,0), \pm(1,-1,-1), \pm(0,1,-1)\}, \{1\}\{23\}) \\ &\qquad = \{(0)\} \times Q(\{(0,0), \pm(1,-1)\}, \{12\}). \end{aligned}$$

Remark 3.5. One can define a category, called *relative resonance category*, whose set of objects is the set of all relative n-cuts. A new structure which it has in comparison to $\mathcal{R}$ is provided by "shrinking morphisms": $(S, \Pi) \rightsquigarrow (T, \Lambda)$, for $S, T \subseteq \{-1, 0, 1\}^n$, $\mathrm{P}(n) \supseteq \Lambda \supseteq \Pi$, such that $(\operatorname{span} S) \setminus \Lambda = T$. They correspond to shrinking strata to infinity.

3.3. Resonance Functors. Given a functor $\mathcal{F} : \mathcal{R} \longrightarrow \mathbf{Top}^*$, we introduce the following notation:

$$\mathcal{F}(Q(S, \Pi)) = \mathcal{F}(S) \Big/ \bigcup_{\mathrm{un}(\pi) \in \Pi} \operatorname{Im} \mathcal{F}(\pi S \overset{\pi}{\hookrightarrow} S).$$

Definition 3.6. A functor $\mathcal{F} : \mathcal{R} \longrightarrow \mathbf{Top}^*$ is called a *resonance functor* if it satisfies the following axioms:

(A1) *Inclusion axiom.*
If $S \in \mathcal{R}_n$, and $\pi \in \mathrm{OP}(n)$, then $\mathcal{F}(\pi S \overset{\pi}{\hookrightarrow} S)$ is an inclusion map, and $\mathcal{F}(S) \big/ \operatorname{Im} \mathcal{F}(\pi S \overset{\pi}{\hookrightarrow} S) \simeq \mathcal{F}(Q(S, \pi \downarrow S))$.

(A2) *Relative resonance axiom.*
If, for some $S, T \in \mathcal{R}_n$, and $\Pi, \Lambda \subseteq \mathrm{P}(n)$, $[Q(S, \Pi)] = [Q(T, \Lambda)]$, then $\mathcal{F}(Q(S, \Pi)) \simeq \mathcal{F}(Q(T, \Lambda))$.

(A2) *Direct product axiom.*
For two relative n-cuts (S, Π) and (T, Λ) we have

$$\mathcal{F}(S, \Pi) \times \mathcal{F}(T, \Lambda) \simeq \mathcal{F}((S, \Pi) \times (T, \Lambda)).$$

Given $S \in \mathcal{R}_n$, and $\pi \in \mathrm{OP}(n)$, let $i_{S,\pi}$ denote the inclusion map $\mathcal{F}(\pi S \overset{\pi}{\hookrightarrow} S)$. There is a canonical homology long exact sequence associated to the triple

$$\mathcal{F}(\pi S) \xrightarrow{i_{S,\pi}} \mathcal{F}(S) \xrightarrow{p} \mathcal{F}(Q(S, \pi \downarrow S)), \tag{3.1}$$

namely

$$\begin{aligned} \cdots \xrightarrow{\partial_*} \widetilde{H}_n(\mathcal{F}(\pi S)) \xrightarrow{(i_{S,\pi})_*} \widetilde{H}_n(\mathcal{F}(S)) \xrightarrow{p_*} \widetilde{H}_n(\mathcal{F}(Q(S, \pi \downarrow S))) \\ \xrightarrow{\partial_*} \widetilde{H}_{n-1}(\mathcal{F}(\pi S)) \xrightarrow{(i_{S,\pi})_*} \cdots \end{aligned} \tag{3.2}$$

We call (3.1), resp. (3.2), the *standard triple*, resp. the *standard long exact sequence* associated to the morphism $\pi S \overset{\pi}{\hookrightarrow} S$ and the functor $\mathcal{F}$ (usually $\mathcal{F}$ is fixed, so its mentioning is omitted).

4. First applications

4.1. Resonance compatible compactifications. As mentioned in the introduction we shall now look at the natural strata of the spaces $X^{(n)}$. The strata are defined by point coincidences and are indexed by number partitions of n. Let Σ_λ^X denote the stratum indexed by λ.

Let λ be a number partition of n and let $\tilde{\lambda}$ be λ with some fixed order on the parts. Then $\tilde{\lambda}$ can be thought of as a vector with positive integer coordinates in $\mathbb{R}^n$. Let $S_{\tilde{\lambda}}$ be the set $\{x \in \{-1, 0, 1\}^n \mid \langle x, \tilde{\lambda}\rangle = 0\}$. Obviously, $S_{\tilde{\lambda}}$ is an n-cut and the n-resonance S_λ, which it defines, does not depend on the choice of $\tilde{\lambda}$, but only on the number partition λ.

The crucial topological observation is that if ν is another partition of n such that $S_\lambda = S_\nu$, then the spaces Σ_λ^X and Σ_ν^X are homeomorphic. This is precisely the fact which leads one to introduce resonances and the surrounding combinatorial framework and to forget about the number partitions themselves.

This allows us to introduce a functor $\mathcal{F}$ mapping $S_{\tilde{\lambda}}$ to Σ_λ^X; the morphisms map accordingly. Clearly, $\mathcal{F}(1^l) = X^{(l)}$. One can observe in this example the justification for the names which we chose for the morphisms of $\mathcal{R}$: "inclusions" and "gluings". Furthermore, it is easy, in this case, to verify the axioms of Definition 3.6, and hence to conclude that $\mathcal{F}$ is a resonance functor. The only nontrivial point is the verification of the second part of (A1), which we do in the next proposition.

Proposition 4.1. *Let S be an n-cut and $\pi \in \mathrm{OP}(n)$. Then $\mathcal{F}(\nu S) \subseteq \mathcal{F}(\pi S)$ if and only if* $\mathrm{un}(\nu) \in \mathrm{un}(\pi) \downarrow S$.

Proof. It is obvious that all the steps of the definition of $\mathrm{un}(\pi) \downarrow S$ which change the partition preserve the property $\mathcal{F}(\nu S) \subseteq \mathcal{F}(\pi S)$, hence the *if* direction follows.

Assume now $\mathcal{F}(\nu S) \subseteq \mathcal{F}(\pi S)$. This means that there exists $\tau \in \mathrm{OP}(m)$, where m is the number of parts of π, such that $\mathcal{F}(\tau\pi S) = \mathcal{F}(\nu S)$. By definition, $\tau \circ \pi \in \mathrm{un}(\pi) \downarrow S$. Now, we can reach $\mathrm{un}(\nu)$ from $\mathrm{un}(\tau \circ \pi)$ by moves of type (3) from the definition of the relative resonances.

Indeed, if $\mathcal{F}(\tau\pi S) = \mathcal{F}(\nu S) = \Sigma_\lambda^X$, then the sizes of the resulting blocks after gluing along $\tau \circ \pi$ and along ν are the same. For every block b of λ we can go, by means of moves of type (3), from the block of $\mathrm{un}(\tau \circ \pi)$ which glues to b to the block of $\mathrm{un}(\nu)$ which glues to b. Since we can do it for any block of λ, we can go from $\mathrm{un}(\tau \circ \pi)$ to $\mathrm{un}(\nu)$, and hence $\mathrm{un}(\nu) \in \mathrm{un}(\pi) \downarrow S$. □

In the context of this stratification the following central question arises.

The Main Problem (Arnold, Shapiro [7]). Describe an algorithm which, for a given resonance λ, would compute the Betti numbers of $\Sigma_\lambda^{S^1}$, or $\Sigma_\lambda^{S^2}$.

The case of the strata $\Sigma_\lambda^{S^1}$ is simpler, essentially because of the following elementary, but important property of smash products: if X and Y are pointed spaces and X is contractible, then $X \wedge Y$ is also contractible.

In the subsequent subsections we shall look at a few interesting special cases, and also will be able to say a few things about the general problem.

4.2. Resonances $(a^k, 1^l)$. Let a, k, l be positive integers such that $a \geq 2$. Let S be the $(l+k)$-cut consisting of all the elements of $\{-1, 0, 1\}^{l+k}$, which are orthogonal to the vector $(\underbrace{1, \ldots, 1}_{l}, \underbrace{a, \ldots, a}_{k})$. Clearly, the $(l+k)$-resonance $[S]$ is equal to $(a^k, 1^l)$. The case $l < a$ is not very interesting, since then $(a^k, 1^l) = (1^k) \times (1^l)$. Therefore we may assume that $l \geq a$.

We would like to understand the topological properties of the space $\mathcal{F}(a^k, 1^l)$. In general, this is rather hard. However, as the following theorem shows, it is possible under some additional conditions on $\mathcal{F}$.

Theorem 4.2. *Let $\mathcal{F} : \mathcal{R} \longrightarrow \mathbf{Top}^*$ be a resonance functor such that $\mathcal{F}(1^l)$ is contractible for $l \geq 2$. Let $l = am + \epsilon$, where $0 \leq \epsilon \leq a - 1$.*

(a) *If $k \neq 1$, or $\epsilon \geq 2$, then $\mathcal{F}(a^k, 1^l)$ is contractible.*
(b) *If $k = 1$, and $\epsilon \in \{0, 1\}$, then*

$$\mathcal{F}(a^k, 1^l) \simeq \mathrm{susp}^m(\mathcal{F}(1)^{m+\epsilon+1}), \tag{4.1}$$

where $\mathcal{F}(1)^{m+\epsilon+1}$ denotes the $(m+\epsilon+1)$-fold smash product.

Since for the resonance functor $\mathcal{F}$ described in the subsection 4.1 we have $\mathcal{F}(1^l) = X^{(l)}$, we have the following corollary.

Corollary 4.3. *If $X^{(l)}$ is contractible for $l \geq 2$, then*

(a) *If $k \neq 1$, or $\epsilon \geq 2$ (again $l = am + \epsilon$), then $\Sigma^X_{(a^k,1^l)}$ is contractible.*

(b) *If $k = 1$, and $\epsilon \in \{0, 1\}$, then $\Sigma^X_{(a^k,1^l)} \simeq \mathrm{susp}^m(X^{m+\epsilon+1})$, where $X^{m+\epsilon+1}$ denotes the $(m + \epsilon + 1)$-fold smash product.*

Note. Clearly, $(S^1)^{(l)}$ is contractible for $l \geq 2$, so the Corollary 4.3 is valid. In this situation, the case $k > 1$ was proved in [5], and the case $k = 1$ in [3], [8].

Before we proceed with proving Theorem 4.2 we need a crucial lemma. Let $\pi \in \mathrm{P}(k + l)$ be $(\{1, \ldots, a\}, \{a + 1\}, \{a + 2\}, \ldots, \{k + l\})$. It is immediate that $[\tilde{\pi} S] = (a^{k+1}, 1^{l-a})$, if $\mathrm{un}(\tilde{\pi}) = \pi$.

Lemma 4.4. *Let S be as above, $T \in \mathcal{R}_l$ such that $[T] = (1^l)$, and let ν be the partition $(\{1, \ldots, a\}, \{a + 1\}, \{a + 2\}, \ldots, \{l\})$, then we have*

$$[Q(S, \pi \downarrow S)] = [Q(T, \nu \downarrow T)] \times (1^k). \tag{4.2}$$

Note. Lemma 4.4 is a special case of Lemma 4.6, however we choose to include a separate proof for it for two reasons: firstly, it is the first, still not too technical example of investigating the combinatorial structure of the resonance category, which is a new object; secondly, the particular case of $(a^k, 1^l)$ resonances was a subject of substantial previous attention.

Proof of Lemma 4.4. Recall that by the definition of the direct product,

$$[Q(T, \nu \downarrow T)] \times (1^k) = [Q(T \times U, (\nu \downarrow T) \times \mathrm{P}(\{l + 1, \ldots, l + k\}))],$$

where $U \in \mathcal{R}_k$ and $[U] = (1^k)$. Clearly, $(\nu \downarrow T) \times \mathrm{P}(\{l + 1, \ldots, l + k\}) = \pi \downarrow S$, hence we just need to show that $S \setminus (\pi \downarrow S) = (T \times U) \setminus ((\nu \downarrow T) \times \mathrm{P}(\{l + 1, \ldots, l + k\}))$. Note that $(T \times U) \setminus ((\nu \downarrow T) \times \mathrm{P}(\{l + 1, \ldots, l + k\})) = (T \setminus (\nu \downarrow T)) \times U$. Furthermore,

$$S = \left\{(x_1, \ldots, x_{l+k}) \in \{-1, 0, 1\}^{l+k} \mid \sum\nolimits_{j=l+1}^{l+k} x_j + a \sum\nolimits_{i=1}^{l} x_i = 0\right\},$$

and the set which we need to remove from S to get $S \setminus (\pi \downarrow S)$ is

$$\left\{(x_1, \ldots, x_{l+k}) \in \{-1, 0, 1\}^{l+k} \mid \sum\nolimits_{j=l+1}^{l+k} x_j + a \sum\nolimits_{i=1}^{l} x_i = 0, \right.$$
$$\left. \max(|\,\mathrm{Plus}(x_1, \ldots, x_l)|,\ |\,\mathrm{Minus}(x_1, \ldots, x_l)|) \geq a\right\}.$$

Therefore, by the definition of the relative resonances, we have

$$S \setminus (\pi \downarrow S) = \Big\{(x_1, \ldots, x_{l+k}) \in \{-1, 0, 1\}^{l+k} \mid \sum\nolimits_{i=1}^{l} x_i = 0, \\ \sum\nolimits_{j=l+1}^{l+k} x_j = 0, \ |\operatorname{Plus}(x_1, \ldots, x_l)| < a\Big\}.$$

On the other hand, $(1^k) = [\{(y_1, \ldots, y_k) \in \{-1, 0, 1\}^k \mid \sum_{j=1}^{k} y_j = 0\}]$, and

$$T \setminus (\nu \downarrow T) = \Big\{(z_1, \ldots, z_l) \in \{-1, 0, 1\}^l \mid \sum\nolimits_{i=1}^{l} z_i = 0, \ |\operatorname{Plus}(z_1, \ldots, z_l)| < a\Big\},$$

which proves (4.2). □

Proof of Theorem 4.2. (a) We use induction on l. The case $l < a$ can be taken as an induction base, since then $(a^k, 1^l) = (1^k) \times (1^l)$, hence, by the axiom (A3), $\mathcal{F}(a^k, 1^l) = \mathcal{F}(1^k) \wedge \mathcal{F}(1^l)$, which is contractible, since $\mathcal{F}(1^k)$ is. Thus we assume that $l \geq a$, and $\mathcal{F}(a^k, 1^{l'})$ is contractible for all $l' < l$.

Let S and π be as in Lemma 4.4. The standard triple associated to the morphism $\pi S \overset{\pi}{\hookrightarrow} S$ is $\mathcal{F}(a^{k+1}, 1^{l-a}) \hookrightarrow \mathcal{F}(a^k, 1^l) \to \mathcal{F}(a^k, 1^l)/\mathcal{F}(a^{k+1}, 1^{l-a})$. Since, by the induction assumption, $\mathcal{F}(a^{k+1}, 1^{l-a})$ is contractible, we conclude that $\mathcal{F}(a^k, 1^l) \simeq \mathcal{F}(a^k, 1^l)/\mathcal{F}(a^{k+1}, 1^{l-a})$.

Basically by the definition, we have

$$\mathcal{F}(a^k, 1^l)/\mathcal{F}(a^{k+1}, 1^{l-a}) = \mathcal{F}(Q(S, \pi \downarrow S)).$$

On the other hand, we have proved in Lemma 4.4 that $[Q(S, \pi \downarrow S)] = Q(T, \nu \downarrow T) \times (1^k)$, where T and ν are described in the formulation of that lemma. By axioms (A2) and (A3) we get that $\mathcal{F}(Q(S, \pi \downarrow S)) \simeq \mathcal{F}(Q(T, \nu \downarrow T)) \wedge \mathcal{F}(1^k)$, which is contractible, since $\mathcal{F}(1^k)$ is. Therefore, $\mathcal{F}(a^k, 1^l)$ is also contractible.

(b) The argument is very similar to (a). We again assume $l \geq a$, which implies $l \geq 2$. By the using the same ordered set partition π as in (a), we get that $\mathcal{F}(a, 1^l) \simeq \mathcal{F}(a, 1^l)/\mathcal{F}(a^2, 1^{l-a})$. Further, by Lemma 4.4 and the axioms (A2) and (A3) we conclude that $\mathcal{F}(a, 1^l) \simeq \mathcal{F}(1) \wedge (\mathcal{F}(1^l)/\mathcal{F}(a, 1^{l-a}))$. Since $\mathcal{F}(1^l)$ is contractible, we get

$$\mathcal{F}(a, 1^l) \simeq \mathcal{F}(1) \wedge \operatorname{susp} \mathcal{F}(a, 1^{l-a}). \tag{4.3}$$

Since $\mathcal{F}(a) = \mathcal{F}(1)$, $\mathcal{F}(a, 1) = \mathcal{F}(1) \wedge \mathcal{F}(1)$, and $\mathcal{F}(a, 1^l)$ is contractible if $2 \leq l < a$, we obtain (4.1) by the repeated usage of (4.3). □

4.3. Resonances (a^k, b^l). The algebraic invariants of these strata have not been computed before, not even in the case $X = S^1$, and $\mathcal{F}$ - the standard resonance functor associated to the stratification of $X^{(n)}$.

We would like to apply a technique similar to the one used in the subsection 4.2. A problem is that, once one starts to "glue" a's, one cannot get b's in the same way as one could in the previous section from 1's. Thus, we are forced to consider a more general case of resonances, namely (g^m, a^k, b^l), where g is the least common multiple of a and b. Assume $g = a \cdot \bar{a} = b \cdot \bar{b}$, and $b > a \geq 2$. Analogously with the Theorem 4.2 we have the following result.

Theorem 4.5. *Let $\mathcal{F}$ be as in Theorem 4.2. Let furthermore $k = x \cdot \bar{a} + \epsilon_1$, $l = y \cdot \bar{b} + \epsilon_2$, where $0 \leq \epsilon_1 < \bar{a}$, $0 \leq \epsilon_2 < \bar{b}$. Then*

$$\mathcal{F}(g^m, a^k, b^l) \simeq \begin{cases} \operatorname{susp}^{x+y+m-1}(\mathcal{F}(1)^{x+y+m+\epsilon_1+\epsilon_2}), & \text{if } m, \epsilon_1, \epsilon_2 \in \{0, 1\}; \\ \text{point}, & \text{otherwise.} \end{cases} \tag{4.4}$$

Just as in Subsection 4.2 (Corollary 4.3), Theorem 4.5 is true if one replaces $\mathcal{F}(\lambda)$ with $\Sigma_\lambda^{S^1}$.

The proof of Theorem 4.5 follows the same general scheme as that of Theorem 4.2, but the technical details are more numerous. Again there is a crucial combinatorial lemma.

Let S be an $(m + k + l)$-cut consisting of all the elements of $\{-1, 0, 1\}^{m+k+l}$ which are orthogonal to the vector $(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{l}, \underbrace{g, \ldots, g}_{m})$. Assume $k \geq \bar{a}$, and let an unordered set partition π be equal to $(\{1, \ldots, \bar{a}\}, \{\bar{a}+1\}, \ldots, \{k+l+m\})$. We see that $[S] = (g^m, a^k, b^l)$, and $[\tilde{\pi} S] = (g^{m+1}, a^{k-\bar{a}}, b^l)$, if $\pi = \operatorname{un}(\tilde{\pi})$.

Lemma 4.6. *Let $T \in \mathcal{R}_k$ such that $[T] = (1^k)$, and $\nu = (\{1, \ldots, \bar{a}\}, \{\bar{a} + 1\}, \ldots, \{k\})$, then*

$$[Q(S, \pi \downarrow S)] = [Q(T, \nu \downarrow T)] \times (\bar{b}^m, 1^l). \tag{4.5}$$

Proof. Again, it is easy to see that the sets of the partitions at infinity on both sides of (4.5) coincide. Indeed,

$$[Q(T, \nu)] \times (\bar{b}^m, 1^l) = [Q(T \times U, (\nu \downarrow T) \times \mathrm{P}(\{k + 1, \ldots, k + m + l\}))],$$

where $U \in \mathcal{R}_{m+l}$ such that $[U] = (\bar{b}^m, 1^l)$, and $(\nu \downarrow T) \times \mathrm{P}(\{k+1, \ldots, k+m+l\}) = \pi \downarrow S$. Also, we again have the equality

$$(T \times U) \setminus ((\nu \downarrow T) \times \mathrm{P}(\{k + 1, \ldots, k + m + l\})) = T \setminus (\nu \downarrow T) \times U,$$

which greatly helps to prove that the sets if the surviving elements on the two sides of (4.5) coincide.

By the definition

$$S = \Big\{(x_1, \ldots, x_{k+l+m}) \in \{-1, 0, 1\}^{k+l+m} \;\Big|\; a \sum\nolimits_{i=1}^{k} x_i + b \sum\nolimits_{i=k+1}^{k+l} x_i + g \sum\nolimits_{i=k+l+1}^{k+l+m} x_i = 0\Big\},$$

and, again, the set which we have to remove from S to get $S \setminus (\pi \downarrow S)$ is

$$\Big\{(x_1, \dots, x_{k+l+m}) \in \{-1, 0, 1\}^{k+l+m} \;\Big|\; a \sum\nolimits_{i=1}^{k} x_i + b \sum\nolimits_{i=k+1}^{k+l} x_i + g \sum\nolimits_{i=k+l+1}^{k+l+m} x_i = 0,\ \max(|\operatorname{Plus}(x_1, \dots, x_k)|, |\operatorname{Minus}(x_1, \dots, x_k)|) \geq \bar{a}\Big\}.$$

By the definition of the relative resonances and some elementary number theory we conclude that

$$S \setminus (\pi \downarrow S) = \Big\{(x_1, \dots, x_{k+l+m}) \in \{-1, 0, 1\}^{k+l+m} \;\Big|\; |\operatorname{Plus}(x_1, \dots, x_k)| < \bar{a},\ \sum\nolimits_{i=1}^{k} x_i = 0,\ b \sum\nolimits_{i=k+1}^{k+l} x_i + g \sum\nolimits_{i=k+l+1}^{k+l+m} x_i = 0\Big\}.$$

The number theory argument which we need is that if $ax + by + \operatorname{lcm}(a, b)z = 0$, then $\bar{a} \mid x$, where $\bar{a} \cdot a = \operatorname{lcm}(a, b)$. This can be seen by, for example, noticing that if $ax + by + \operatorname{lcm}(a, b)z = 0$, then $b \mid ax$, but since also $a \mid ax$, we have $\operatorname{lcm}(a, b) \mid ax$, hence $\bar{a} \mid x$.

The equation (4.5) follows now from the earlier observations together with the equalities

$$T \setminus (\nu \downarrow T) = \Big\{(x_1, \dots, x_k) \in \{-1, 0, 1\}^k \;\Big|\; |\operatorname{Plus}(x_1, \dots, x_k)| < \bar{a},\ \sum\nolimits_{i=1}^{k} x_i = 0\Big\},$$

and

$$(\bar{b}^m, 1^l) = \Big\{\Big(y_1, \dots, y_{m+l}) \in \{-1, 0, 1\}^{m+l} \;\Big|\; \sum\nolimits_{i=1}^{l} y_i + \bar{b} \sum\nolimits_{i=l+1}^{l+m} x_i = 0\Big\}\Big]. \quad \square$$

Proof of Theorem 4.5. The cases $k < \bar{a}$ and $l < \bar{b}$ are easily reduced to Theorem 4.2. Assume therefore that $k \geq \bar{a}$ and $l \geq \bar{b}$. Recall also that $b > a \geq 2$, and hence $\bar{a} \geq 2$.

Let S and π be as in the formulation of Lemma 4.6. The standard triple associated to the morphism $\pi S \overset{\pi}{\hookrightarrow} S$ is

$$\mathcal{F}(g^{m+1}, a^{k-\bar{a}}, b^l) \hookrightarrow \mathcal{F}(g^m, a^k, b^l) \to \mathcal{F}(g^m, a^k, b^l)/\mathcal{F}(g^{m+1}, a^{k-\bar{a}}, b^l). \tag{4.6}$$

We break the rest of the proof into 3 cases.

Case $m \geq 2$. Again, we prove that $\mathcal{F}(g^m, a^k, b^l)$ is contractible by induction on k. This is clear if $k < \bar{a}$. If $k \geq \bar{a}$, it follows from (4.6) that $\mathcal{F}(g^m, a^k, b^l) \simeq \mathcal{F}(g^m, a^k, b^l)/\mathcal{F}(g^{m+1}, a^{k-\bar{a}}, b^l) = \mathcal{F}(Q(S, \pi \downarrow S))$. By Lemma 4.6 we conclude that $\mathcal{F}(g^m, a^k, b^l) \simeq \mathcal{F}(Q(T, \nu \downarrow T)) \wedge \mathcal{F}(\bar{b}^m, 1^l)$. By Theorem 4.2, $\mathcal{F}(\bar{b}^m, 1^l)$ is contractible, hence so is $\mathcal{F}(g^m, a^k, b^l)$.

Case $m = 0$. By Lemma 4.6 we get that

$$\mathcal{F}(Q(S, \pi \downarrow S)) \simeq \mathcal{F}(Q(T, \nu \downarrow T)) \wedge \mathcal{F}(1^l).$$

Since $l \geq 2$, we have that $\mathcal{F}(1^l)$ is contractible, hence so is $\mathcal{F}(Q(S, \pi \downarrow S)) = \mathcal{F}(a^k, b^l)/\mathcal{F}(g, a^{k-\bar{a}}, b^l)$. Therefore, by (4.6) $\mathcal{F}(a^k, b^l) \simeq \mathcal{F}(g, a^{k-\bar{a}}, b^l)$.

Case $m = 1$. Since $\mathcal{F}(g^2, a^{k-\bar{a}}, b^l)$ is contractible, we conclude by (4.6) that $\mathcal{F}(g, a^k, b^l) \simeq \mathcal{F}(g, a^k, b^l)/\mathcal{F}(g^2, a^{k-\bar{a}}, b^l) = \mathcal{F}(Q(S, \pi \downarrow S))$. By Lemma 4.6, and the properties of the resonance functors, we have

$$\begin{aligned} \mathcal{F}(g, a^k, b^l) &\simeq \mathcal{F}(\bar{b}, 1^l) \wedge (\mathcal{F}(1^k)/\mathcal{F}(\bar{a}, 1^{k-\bar{a}})) \\ &\simeq \mathcal{F}(\bar{b}, 1^l) \wedge \operatorname{susp}(\mathcal{F}(\bar{a}, 1^{k-\bar{a}})). \end{aligned} \tag{4.7}$$

By the repeated usage of (4.7) we obtain (4.4). □

5. Sequential resonances

5.1. The structure theory of strata associated to sequential resonances

Definition 5.1. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \leq \cdots \leq \lambda_n$, be a number partition. We call λ *sequential* if, whenever $\sum_{i \in I} \lambda_i = \sum_{j \in J} \lambda_j$, and $q \in I$ such that $q = \max(I \cup J)$, then there exists $\widetilde{J} \subseteq J$, such that $\lambda_q = \sum_{j \in \widetilde{J}} \lambda_j$.

Correspondingly, we call a resonance S sequential, if it can be associated to a sequential partition.

Note that the set of sequential partitions is closed under removing blocks.

Examples of sequential partitions. (1) All partitions whose blocks are equal to powers of some number.

(2) $(a^k, b^l, 1^m)$ such that $a > bl$; more generally $(a_1^{k_1}, \ldots, a_t^{k_t}, 1^m)$ such that $a_i > \sum_{j=i+1}^{t} a_j k_j$, for all $i \in [t]$.

Through the rest of this subsection, we let λ be as in Definition 5.1. For such λ we use the following additional notations:

- $mm(\lambda) = |\{i \in [n] \mid \lambda_i = \lambda_n\}|$. In other words

$$\lambda_{n-mm(\lambda)} \neq \lambda_{n-mm(\lambda)+1} = \cdots = \lambda_n.$$

- $I(\lambda) \subseteq [n]$ is the lexicographically maximal set (see below the convention that we use to order lexicographically), such that $|I(\lambda)| \geq 2$, and $\lambda_n = \sum_{i \in I(\lambda)} \lambda_i$. Note that it may happen that $I(\lambda)$ does not exist, in which case $\mathcal{F}(\lambda) \simeq \mathcal{F}(\lambda_1, \ldots, \lambda_{n-mm(\lambda)}) \wedge \mathcal{F}(1^{mm(\lambda)})$, and can be dealt with by induction.

Let n be a positive integer. We use the following convention for the lexicographic order on $[n]$. For $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_m\}$, $A, B \subseteq [n]$, $a_1 \leq \cdots \leq a_k$, $b_1 \leq \cdots \leq b_m$, we say that A is lexicographically larger than B if either $A \supseteq B$ or there exists $q < \min(k, m)$ such that $a_k = b_m, a_{k-1} = b_{m-1}, \dots, a_{k-q+1} = b_{m-q+1}$, and $a_{k-q} > b_{m-q}$.

Proposition 5.2. *If $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_1 \leq \cdots \leq \lambda_n$, is a sequential partition, then so is $\bar{\lambda} = (\lambda_{j_1}, \dots, \lambda_{j_t}, \sum_{i \in I(\lambda)} \lambda_i)$, where $t = n - |I(\lambda)|$, and $\{j_1, \dots, j_t\} = [n] \setminus I(\lambda)$.*

Proof. Let $\bar{\lambda}_1 = \lambda_{j_1}, \dots, \bar{\lambda}_t = \lambda_{j_t}$, $\bar{\lambda}_{t+1} = \sum_{i \in I(\lambda)} \lambda_i$. We need to check the condition of Definition 5.1 for the identity

$$\sum_{i \in I} \bar{\lambda}_i = \sum_{j \in J} \bar{\lambda}_j. \tag{5.1}$$

If $t+1 \notin I \cup J$, then it follows from the assumption that λ is sequential. Assume $t+1 \in I$. If $\bar{\lambda}_j = \lambda_n$, for some $j \in J$, take $\widetilde{J} = \{j\}$, and we are done. If $\bar{\lambda}_i = \lambda_n$, for some $i \in I \setminus \{t+1\}$, then, since λ is sequential, there exists $\widetilde{J} \subseteq J$ such that $\sum_{j \in \widetilde{J}} \bar{\lambda}_j = \lambda_n = \bar{\lambda}_{t+1}$, and we are done again.

Finally, assume $\bar{\lambda}_i \neq \lambda_n$, for $i \in (I \cup J) \setminus \{t+1\}$. Substituting λ_n instead of $\bar{\lambda}_{t+1}$ into the identity (5.1) is allowed, since λ_n does not appear among $\{\bar{\lambda}_i\}_{i \in (I \cup J) \setminus \{t+1\}}$. This gives us an identity for λ, and again, since λ is sequential, we find the desired set $\widetilde{J} \subseteq J$ such that $\sum_{j \in \widetilde{J}} \bar{\lambda}_j = \bar{\lambda}_{t+1}$. □

Let $S \in \mathcal{R}_n$ be the set of all elements of $\{-1, 0, 1\}^n$, which are orthogonal to the vector $\lambda = (\lambda_1, \dots, \lambda_n)$. Clearly, $[S] = \lambda$. Let $\pi \in \mathrm{P}(n)$ be the partition whose only nonsingleton block is given by $I(\lambda)$. The next lemma expresses the main combinatorial property of sequential partitions.

Lemma 5.3. *Let $\tau \in \mathrm{P}(n)$ be a partition which has only one nonsingleton block B, and assume $\lambda_n = \sum_{i \in B} \lambda_i$. Then $\tau \in \pi \downarrow S$.*

Proof. Assume there exists partitions τ as in the formulation of the lemma such that $\tau \notin \pi \downarrow S$. Choose one so that the block B is lexicographically largest possible. Let $C = B \cap I(\lambda)$. By the definition of $I(\lambda)$, and the choice of B, we have $\sum_{i \in I(\lambda) \setminus C} \lambda_i = \sum_{j \in B \setminus C} \lambda_j$, and $q \in I(\lambda) \setminus C$, where $q = \max((I(\lambda) \cup B) \setminus C)$.

Since partition λ is sequential, there exists $D \subseteq B \setminus C$ such that $\lambda_q = \sum_{j \in D} \lambda_j$. Let $\gamma \in \mathrm{P}(n)$ be the partition whose only nonsingleton block is $G = (B \setminus D) \cup \{q\}$. Clearly, $\sum_{i \in G} \lambda_i = \lambda_n$, and $|G| \geq 2$. By the choice of q, G is lexicographically larger than B, hence $\gamma \in \pi \downarrow S$.

Let furthermore $\tilde{\gamma} \in \mathrm{P}(n)$ be the partition having two nonsingleton blocks: D and G. By Definition 3.1(2) if $\gamma \in \pi \downarrow S$, then $\tilde{\gamma} \in \pi \downarrow S$. By Definition 3.1(3), if $\tilde{\gamma} \in \pi \downarrow S$, then $\tau \in \pi \downarrow S$, which yields a contradiction. □

Let $T \in \mathcal{R}_{n-mm(\lambda)}$ be the set of all elements of $\{-1, 0, 1\}^{n-mm(\lambda)}$, which are orthogonal to the vector $(\lambda_1, \ldots, \lambda_{n-mm(\lambda)})$. Let $\nu \in \mathrm{P}(n - mm(\lambda))$ be the partition whose only nonsingleton block is given by $I(\lambda)$. We are now ready to state the combinatorial result which is crucial for our topological applications.

Lemma 5.4.

$$[Q(S, \pi \downarrow S)] = [Q(T, \nu \downarrow T)] \times (1^{mm(\lambda)}). \tag{5.2}$$

Proof. By definition we must verify that the sets of partitions at infinity and the surviving elements coincide on both sides of the equation (5.2).

Let us start with the partitions at infinity. Filtered through Proposition 4.1, the identity $\pi \downarrow S = (\nu \downarrow T) \times \mathrm{P}(\{n - mm(\lambda) + 1, \ldots, n\})$ becomes essentially tautological. Both sides consist of the partitions $\tau = (\tau_1, \ldots, \tau_k) \in \mathrm{P}(n)$ such that the number partition $\left(\sum_{i\in\tau_1} \lambda_i, \ldots, \sum_{i\in\tau_k} \lambda_i\right)$ can be obtained from the number partition $(\lambda_{j_1}, \ldots, \lambda_{j_t}, \sum_{i\in I(\lambda)} \lambda_i)$, where $\{j_1, \ldots, j_t\} = [n] \setminus I(\lambda)$, by summing parts.

Let us now look at the surviving elements. It is obvious that $S \setminus (\pi \downarrow S) \supseteq (T \setminus (\nu \downarrow T)) \times U$, where $U \in \mathcal{R}_k$ such that $[U] = (1^{mm(\lambda)})$, and we need to show the converse inclusion. Let $x = (x_1, \ldots, x_n) \in S$ such that $\sum_{i=n-mm(\lambda)+1}^{n} x_i \neq 0$ (otherwise $x \in (T \setminus (\nu \downarrow T)) \times U$). We can assume $\sum_{i=n-mm(\lambda)+1}^{n} x_i > 0$. Then, since S is a sequential resonance, there exists $y = (y_1, \ldots, y_n) \in S$ such that

- if $y_i \neq 0$, then $x_i = y_i$;
- $|\mathrm{Plus}(y)| = 1$, and $\mathrm{Plus}(y) \subseteq \{n - mm(\lambda) + 1, \ldots, n\}$.

This, by Lemma 5.3, means that $y \notin S \setminus (\pi \downarrow S)$, which in turn necessitates $x \notin S \setminus (\pi \downarrow S)$. This finishes the proof of the lemma. □

Just as before, this combinatorial fact about the resonances translates into a topological statement, which can be further strengthened by requiring some additional properties from λ.

Definition 5.5. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \leq \cdots \leq \lambda_n$, be a sequential partition, and let $q = \max I(\lambda)$. λ is called *strongly sequential*, if either $I(\lambda)$ does not exist or there exists $J \subseteq I(\lambda) \setminus \{q\}$ such that $\lambda_q = \sum_{i\in J} \lambda_i$ (note that we do not require $|J| \geq 2$).

We are now in a position to prove the main topological structure theorem concerning the sequential resonances.

Theorem 5.6. *Let $\mathcal{F}$ be as in Theorem* 4.2. *Let λ be a sequential partition such that $I(\lambda)$ exists. Then the following holds.*

(1) *If $mm(\lambda) \geq 2$, then $\mathcal{F}(\lambda)$ is contractible.*

(2) *If* $mm(\lambda) = 1$, *then* $\mathcal{F}(\lambda) \simeq \mathcal{F}(Q(T, \nu \downarrow T)) \wedge \mathcal{F}(1)$, *and we have the inclusion triple* $\mathcal{F}(\mu) \overset{i}{\hookrightarrow} \mathcal{F}(\lambda_1, \ldots, \lambda_{n-1}) \to \mathcal{F}(Q(T, \nu \downarrow T))$, *where* $\mu = (\lambda_{j_1}, \ldots, \lambda_{j_t})$, $\{j_1, \ldots, j_t\} = [n] \setminus I(\lambda)$, *and* $\nu \in \mathrm{P}(n - mm(\lambda))$ *is the partition whose only nonsingleton block is given by* $I(\lambda)$. *We set* $\mathcal{F}(\mu)$ *to be a point, if* $I(\lambda)$ *does not exist.*
If moreover λ *is strongly sequential, then the map* i *is homotopic to a trivial map (mapping everything to a point), hence the triple splits and we conclude that*

$$\mathcal{F}(\lambda) \simeq (\mathcal{F}(1) \wedge \mathcal{F}(\lambda_1, \ldots, \lambda_{n-1})) \vee \mathrm{susp}(\mathcal{F}(1) \wedge \mathcal{F}(\mu)). \tag{5.3}$$

Proof. (1) We use induction on $\sum_{i=1}^{n-mm(\lambda)} \lambda_i$. If $I(\lambda)$ does not exist, then λ_n is independent, i.e., $\mathcal{F}(\lambda) \simeq \mathcal{F}(\lambda_1, \ldots, \lambda_{n-mm(\lambda)}) \times \mathcal{F}(1^{mm(\lambda)})$, and hence $\mathcal{F}(\lambda)$ is contractible. Otherwise consider the inclusion triple

$$\mathcal{F}(\bar{\lambda}) \hookrightarrow \mathcal{F}(\lambda) \to \mathcal{F}(\lambda)/\mathcal{F}(\bar{\lambda}) = \mathcal{F}(Q(S, \pi \downarrow S)), \tag{5.4}$$

where $\bar{\lambda} = (\lambda_{j_1}, \ldots, \lambda_{j_t}, \sum_{i \in I(\lambda)} \lambda_i)$, and $\pi \in \mathrm{P}(n)$ is the partition whose only nonsingleton block is given by $I(\lambda)$. By the induction assumption $\mathcal{F}(\bar{\lambda})$ is contractible. On the other hand, by Lemma 5.4, $\mathcal{F}(Q(S, \pi \downarrow S)) \simeq \mathcal{F}(Q(T, \nu \downarrow T)) \wedge \mathcal{F}(1^{mm(\lambda)})$, which is also contractible if $mm(\lambda) \geq 2$.

(2) if $mm(\lambda) = 1$, then we can conclude from (5.4) that $\mathcal{F}(\lambda) \simeq \mathcal{F}(1) \wedge \mathcal{F}(Q(T, \nu \downarrow T))$. Next, consider the inclusion triple

$$\mathcal{F}(\mu) \overset{i}{\hookrightarrow} \mathcal{F}(\lambda_1, \ldots, \lambda_{n-1}) \to \mathcal{F}(Q(T, \nu \downarrow T)). \tag{5.5}$$

If λ is strongly sequential, then there exists $J \subseteq I(\lambda) \setminus \{q\}$ such that $\lambda_q = \sum_{i \in J} \lambda_i$ (here $q = \max I(\lambda)$). The map i factors:

$$\mathcal{F}(\mu) \overset{i_1}{\hookrightarrow} \mathcal{F}\left(\lambda_{p_1}, \ldots, \lambda_{p_{n-1-|J|}}, \sum\nolimits_{i \in I(\lambda)} \lambda_i\right) \overset{i_2}{\hookrightarrow} \mathcal{F}(\lambda_1, \ldots, \lambda_{n-1}), \tag{5.6}$$

where $\{p_1, \ldots, p_{n-1-|J|}\} = [n-1] \setminus J$. Since $(\lambda_{p_1}, \ldots, \lambda_{p_{n-1-|J|}}, \sum_{i \in I(\lambda)} \lambda_i)$ is sequential, and $mm((\lambda_{p_1}, \ldots, \lambda_{p_{n-1-|J|}}, \sum_{i \in I(\lambda)} \lambda_i)) \geq 2$, we can conclude that the middle space in (5.6) is contractible, and hence i in (5.5) is homotopic to a trivial map. This yields the conclusion. □

5.2. Resonances $(a^k, b^l, 1^m)$

Theorem 5.7. *Let* a, b, k, l, m, r *be positive integers such that* $b > 1$, $m \geq r$, *and* $a = bl + r$. *Then*

$$\mathcal{F}(a^k, b^l, 1^m) \simeq \mathrm{susp}(\mathcal{F}(1^k) \wedge \mathcal{F}(a, 1^{m-r})) \vee (\mathcal{F}(1^k) \wedge \mathcal{F}(b^l, 1^m)). \tag{5.7}$$

Note. The restriction $m \geq r$ is unimportant. Indeed, if $m < r$, then $a > bl+m$, hence a is not involved in any resonance other than $a = a$. This implies that $\mathcal{F}(a^k, b^l, 1^m) = \mathcal{F}(1^k) \times \mathcal{F}(b^l, 1^m)$, and we have determined the homotopy type of $\mathcal{F}(a^k, b^l, 1^m)$ by the previous computations.

Proof of Theorem 5.7. Obviously, the condition $a > bl$ guarantees that the partition $(a^k, b^l, 1^m)$ is sequential, hence Theorem 5.6 is valid. It follows that if $k \geq 2$, then $\mathcal{F}(a^k, b^l, 1^m)$ is contractible, hence (5.7) is true.

Furthermore, if $l \geq 2$, or, $l = 1$ and $m \geq b$, then $(a, b^l, 1^m)$ is strongly sequential, hence in this case (5.3) is valid, which in new notations becomes precisely the equation (5.7).

Finally, assume $l = 1$ and $b > m \geq r \geq 1$. Let $a = b + d$. If $\mathcal{F}(a, 1^{m-d})$ or $\mathcal{F}(b, 1^m)$ is contractible, then the map i in the inclusion triple $\mathcal{F}(a, 1^{m-d}) \overset{i}{\hookrightarrow} \mathcal{F}(b, 1^m) \to \mathcal{F}(b, 1^m)/\mathcal{F}(a, 1^{m-d})$ is homotopic to a trivial map, and we again conclude (5.7). If both of these spaces are not contractible then $\mathcal{F}(a, 1^{m-d}) \simeq S^{2y+\epsilon_2+1}$ and $\mathcal{F}(b, 1^m) \simeq S^{2x+\epsilon_1+1}$, where nonnegative integers $x, y, \epsilon_1, \epsilon_2$ are defined by

$$m = bx + \epsilon_1, \quad m - d = (b + d)y + \epsilon_2, \quad \epsilon_1, \epsilon_2 \in \{0, 1\}. \tag{5.8}$$

Let us show that $2x+\epsilon_1 > 2y+\epsilon_2$. If $x > y$, then $2x+\epsilon_1 \geq 2x \geq 2y+2 > 2y+\epsilon_2$. From (5.8) we have that $b(x-y) = d+dy+\epsilon_2-\epsilon_1$. If $x \leq y$, then the left hand side is nonpositive. On the other hand, since $d \geq 1$, the right hand side is nonnegative. Hence, both sides are equal to 0, which implies $x = y$, $d = \epsilon_1 = 1$, $\epsilon_2 = y = 0$. This yields $2x + \epsilon_1 > 2y + \epsilon_2$.

The homotopic triviality of the map i follows now from the fact that the homotopy groups of a sphere are trivial up to the dimension of that sphere, i.e., $\pi_k(S^n) = 0$, for $0 \leq k \leq n - 1$. □

5.3. Division chain resonances. We call the resonance $(b_n^{m_n}, b_{n-1}^{m_{n-1}}, \ldots, b_1^{m_1})$ a *division chain resonance* if $b_i \mid b_{i+1}$, for any $i \in [n-1]$. For convenience, we assume $m_i \geq 1$, for $i \in [n]$, and set $r_i = b_i/b_{i-1}$, for $n \geq i \geq 2$, and $r_1 = b_1$.

Let us see that division chain resonances are strongly sequential. First, we show that $\lambda = (b_n^{m_n}, b_{n-1}^{m_{n-1}}, \ldots, b_1^{m_1})$ is sequential. Assume that

$$\sum_{i \in I} \alpha_i b_i = \sum_{j \in J} \beta_j b_j, \tag{5.9}$$

and there are no equal size parts appearing on both sides. Set $f = \max(I \cup J)$, $g = \min(I \cup J)$. We use induction on $f - g$. If $f = g + 1$ then the condition of sequentiality is obviously satisfied. Otherwise, divide both sides by b_g. The number of parts of size 1 must be divisible by r_{g+1}, hence, in (5.9) all the parts of size b_g can

be replaced by a certain number of parts of size b_{g+1}. By the induction assumption the condition of sequentiality is satisfied for the new relation, hence it follows for (5.9) as well.

Note that it also follows from the previous argument that $I(\lambda)$ must be of the form $\{p, p+1, \dots, n - mm(\lambda) - 1, n - mm(\lambda)\}$, for some p.

It is now easy to see that λ is strongly sequential. Assume $b_n = b_{n-1} + \sum_{i \in I} \alpha_i b_i$, then $(r_n - 1)b_{n-1} = \sum_{i \in I} \alpha_i b_i$. The sequentiality condition is true for the latter relation, hence the strong sequentiality condition is true for the first one.

Thus, Theorem 5.6 applies, and it yields:

(1) if $m_n \geq 2$, then $\mathcal{F}(\lambda)$ is contractible;

(2) if $I(\lambda)$ exists, then

$$\mathcal{F}(b_n, b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1}) \simeq (\mathcal{F}(1) \wedge \mathcal{F}(b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1})) \vee (S^1 \wedge \mathcal{F}(1) \wedge \mathcal{F}(b_n, b_q^{\widetilde{m}_q}, b_{q-1}^{m_{q-1}}, \dots, b_1^{m_1})), \tag{5.10}$$

where $(b_n, b_q^{\widetilde{m}_q}, b_{q-1}^{m_{q-1}}, \dots, b_1^{m_1})$ is obtained from $(b_n, b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1})$ by removing the parts indexed by $I(\lambda)$. We have $\widetilde{m}_q \geq 1$.

(3) If $I(\lambda)$ does not exist, then

$$\mathcal{F}(b_n, b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1}) \simeq \mathcal{F}(1) \wedge \mathcal{F}(b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1}). \tag{5.11}$$

It is immediate from the formulae (5.10) and (5.11) that each topological space $\mathcal{F}(b_n^{m_n}, b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1})$ is homotopy equivalent to a wedge of spaces of the form $\mathcal{F}(1)^\alpha \wedge S^\beta$, where $\mathcal{F}(1)^\alpha$ means an α-fold smash product of $\mathcal{F}(1)$. The natural combinatorial question which arises is how to enumerate these spaces. We shall now construct a combinatorial model: a weighted graph which yields such an enumeration.

For convenience of notations, we set $m_0 = 1$. Γ_λ is a directed weighted graph on the set of vertices $\{0, 1, \dots, n\}$ whose edges and weights are defined by the following rule. For $x, x+d \in \{0, \dots, n\}$, $d \geq 1$, there exists an edge $e(x, x+d)$ (the edge is directed *from x to $x+d$*) if and only if

$$b_{x+d} \mid b_{x+d-1}m_{x+d-1} + b_{x+d-2}m_{x+d-2} + \dots + b_{x+1}m_{x+1} + b_x(m_x - 1).$$

In this case the weight of the edge is defined as

$$w(x, x+d) = (b_{x+d-1}m_{x+d-1} + \dots + b_{x+1}m_{x+1} + b_x(m_x - 1))/b_{x+d}.$$

Note that if $d \geq 2$ and there exists an edge $e(x, x+d)$, then there exists an edge $e(x, x+d-1)$.

We call a directed path in Γ_λ *complete* if it starts in 0 and ends in n. Let γ be a complete path in Γ_λ consisting of t edges, $\gamma = (e(x_0, x_1), \dots, e(x_{t-1}, x_t))$, where $x_0 = 0$, and $x_t = n$. The weight of γ is defined to be the pair $(l(\gamma), w(\gamma))$, where $l(\gamma) = t$, and $w(\gamma) = \sum_{i=1}^{t} w(x_{i-1}, x_i)$.

Theorem 5.8. *Let* $\lambda = (b_n, b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1})$, *then*

$$\mathcal{F}(\lambda) \simeq \bigvee_{\gamma} (\mathcal{F}(1)^{l(\gamma)+w(\gamma)} \wedge S^{w(\gamma)}), \tag{5.12}$$

where the wedge is taken over all complete paths of Γ_λ.

Proof. We use induction on n. The base of the induction is $n = 1$. In this case Γ_λ is a graph with only one edge $e(0, 1)$, $w(0, 1) = 0$. Thus, there is only one complete path. It has weight $(1, 0)$, and $\mathcal{F}(\lambda) \simeq \mathcal{F}(1)$.

Next, we prove the induction step. We break up the proof in three cases.

Case 1. $I(\lambda)$ *does not exist.* By (5.11) we have

$$\mathcal{F}(\lambda) \simeq \mathcal{F}(1) \wedge \mathcal{F}(b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1}). \tag{5.13}$$

On the other hand, $I(\lambda)$ does not exist if and only if $b_n > m_{n-1}b_{n-1} + \cdots + m_1 b_1$. We also know that $n \geq 2$. This implies that there is at most one edge of the type $e(x, n)$, namely $e(n-1, n)$. This edge exists if and only if $m_{n-1} = 1$, in which case $w(n-1, n) = 0$.

If this edge does not exist then there are no complete paths in Γ_λ and, at the same time $\mathcal{F}(b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1})$ is contractible by the previous observations. This agrees with (5.12).

If, on the other hand, this edge does exist, then all complete paths γ must be of the type $\gamma = (\tilde{\gamma}, e(n-1, n))$, where $\tilde{\gamma}$ is a complete path from 0 to $n-1$. Also in this case (5.13) agrees with (5.12).

Case 2. $I(\lambda)$ *exists and* $m_{n-1} \geq 2$. In this case $\mathcal{F}(b_{n-1}^{m_{n-1}}, \dots, b_1^{m_1})$ is contractible, and

$$\mathcal{F}(\lambda) \simeq S^1 \wedge \mathcal{F}(1) \wedge \mathcal{F}(b_n, b_q^{\widetilde{m}_q}, b_{q-1}^{m_{q-1}}, \dots, b_1^{m_1}), \tag{5.14}$$

where $(b_n, b_q^{\widetilde{m}_q}, b_{q-1}^{m_{q-1}}, \dots, b_1^{m_1})$ is as in (5.10).

Let $\tilde{\lambda} = (b_n, b_q^{\widetilde{m}_q}, b_{q-1}^{m_{q-1}}, \dots, b_1^{m_1})$. We can describe the graph $\Gamma_{\tilde{\lambda}}$: it is obtained from Γ_λ by

(1) removing all vertices indexed by $\{q+1, \dots, n-1\}$ and the incident edges;
(2) decreasing the weight of every existing edge $e(x, n)$ by 1;
(3) keeping all the existing edges with the old weights on the set $\{0, \dots, q-1, q\}$.

This operation on Γ_λ is well-defined, since there can be no edges in Γ_Λ of the type $e(x, n)$, for $x \in \{q+1, \dots, n-1\}$, and since the weight of edges $e(x, n)$, for $x \in \{0, \dots, q\}$ must be at least 1, as $\widetilde{m}_q \geq 1$. Furthermore, it is clear from the above combinatorial description of $\Gamma_{\tilde{\lambda}}$ that the set of the complete paths of $\Gamma_{\tilde{\lambda}}$ is the same as that of Γ_λ, and that the weights of the edges in these paths are also the same except

for the edge with the endpoint n, whose weight has been decreased by 1. Thus, (5.14) agrees with (5.12) in this case.

Case 3. *$I(\lambda)$ exists and $m_{n-1} = 1$.* This case is rather similar to the case 2, except that there is an edge $e(n-1, n)$ of weight 0. Thus, $\Gamma_{\tilde{\lambda}}$ bookkeeps all the complete paths of Γ_λ, except for the ones which have this edge $e(n-1, n)$.

However, the first term of the right hand side of (5.10) bookkeeps the paths $(\tilde{\gamma}, e(n-1, n))$, just like in the case 1. Since the set of all complete paths of Γ_λ is the disjoint union of the sets of those paths which contain $e(n-1, n)$, and those which do not, we again get that (5.10) provides the inductive step for (5.12). □

Examples. (1) Let $\lambda = (a, 1^l)$, for $a \geq 2$. Then Γ_λ is a graph on the vertex set $\{0, 1, 2\}$ having either one or two edges:

(a) it has in any case the edge $e(0, 1)$, $w(0, 1) = 0$;
(b) if a divides l, then it has the edge $e(0, 2)$, in which case $w(0, 2) = l/a$;
(c) if a divides $l - 1$, then it has the edge $e(1, 2)$, in which case $w(1, 2) = (l-1)/a$.

Clearly Theorem 5.8 agrees with Theorem 4.2. Indeed, if $\epsilon \notin \{0, 1\}$ (where ϵ is taken from the formulation of Theorem 4.2), then there are no complete paths in Γ_λ. If $\epsilon = 0$, then there is one path $(0, 2)$ of weight $(1, l/a)$; and if $\epsilon = 1$, then there is one path $((0, 1), (1, 2))$ of weight $(2, (l-1)/a)$. Thus, (5.12) and (4.1) are equivalent in this case.

(2) Let $\lambda = (8, 4, 2^3, 1^6)$. Then the graph Γ_λ is

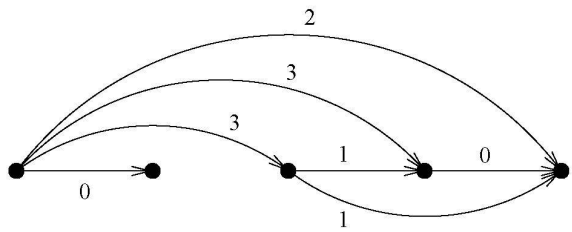

Figure 1

It has 4 directed paths from 0 to 4 and, by Theorem 5.8, we have

$$\mathcal{F}(\lambda) \simeq (\mathcal{F}(1)^3 \wedge S^2) \vee (\mathcal{F}(1)^5 \wedge S^3) \vee (\mathcal{F}(1)^6 \wedge S^4) \vee (\mathcal{F}(1)^7 \wedge S^4),$$

in particular $\Sigma_\lambda^{\mathbb{R}} \simeq S^5 \vee S^8 \vee S^{10} \vee S^{11}$.

6. Remarks on complexity of resonances

The main idea of all our previous computations was to find, for a given n-cut S, a partition $\pi \in \mathrm{P}(n)$ such that $\mathrm{span}(S \setminus (\pi \downarrow S)) \neq S$. Intuitively speaking, shrinking the substratum corresponding to $\tilde{\pi} S$, where $\mathrm{un}(\tilde{\pi}) = \pi$, *essentially* reduces the set of linear identities in S. It is easy to construct examples when such π does not exist, e.g., Example 2.2 (4).

These observations lead us to introduce a formal notion of complexity of a resonance.

Definition 6.1. 1) For $S \in \mathcal{R}_n$, the *complexity* of S is denoted $c(S)$ and is defined by

$$c(S) = \min\{|\Pi| \mid \Pi \subseteq \mathrm{P}(n), \mathrm{span}(S \setminus (\Pi \downarrow S)) \neq S\}. \tag{6.1}$$

2) We define the complexity of an n-resonance to be the complexity of one of its representing cuts. Clearly, it does not depend on the choice of the representative.

Note. The number $c(S)$ would not change if we required the partitions in Π to have one block of size 2, and all other blocks of size 1.

The higher is the complexity of a resonance $[S]$, the less it is likely that one can succeed with analyzing its topological structure using the method of this paper. This is because one would need to take a quotient by a union of $c([S])$ strata and it might be difficult to get a hold on the topology of that union.

We finish by constructing for an arbitrary $n \in \mathbb{N}$, a resonance of complexity n. Let $\lambda_n = (a_1, \ldots, a_n, b_1, \ldots, b_n)$ such that $a_i, b_i \in \mathbb{N}$, $a_i + b_j = a_j + b_i$, for $i, j \in [n]$, and all other linear identities among a_i's and b_i's with coefficients $\pm 1, 0$ are generated by such identities. In other words, the cut S associated to λ is equal to the set

$$\left\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \{-1, 0, 1\}^{2n} \;\middle|\; \sum\nolimits_{i=1}^{n} y_i = 0, \; x_i + y_i = 0, \; \forall i \in [n]\right\}. \tag{6.2}$$

It is not difficult to construct such λ_n directly:

1) Choose $a_1, \ldots, a_n$ such that the only linear identities with coefficients $\pm 1, 0$ on the set $a_1, a_1, a_2, a_2, \ldots, a_n, a_n$ are of the form $a_i = a_i$; in other words, there are no linear identities with coefficients $\pm 2, \pm 1, 0$ on the set $a_1, \ldots, a_n$. One example is provided by the choice $a_1 = 1, a_2 = 3, \ldots, a_n = 3^{n-1}$.

2) Let $b_i = N + a_i$, for $i \in [n]$, where N is sufficiently large. As the proof of Proposition 6.2 will show, it is enough to choose $N > 2\sum_{i=1}^{n} \lambda_i$. This bound is far from sharp, but it is sufficient for our purposes.

Proposition 6.2. *Let S_n be the n-cut associated to the ordered sequence of natural numbers λ_n described above. Then $c(S_n) = n$.*

Proof. First, let us verify that the cut S_n associated to λ_n is equal to the one described in (6.2). Take $(x_1, \dots, x_n, y_1, \dots, y_n) \in S_n$.

Assume first that $\sum_{i=1}^n y_i \neq 0$. Then, $(x_1, \dots, x_n, y_1, \dots, y_n)$ stands for the identity

$$\sum_{i \in I_1} a_i + \sum_{j \in J_1} b_j = \sum_{i \in I_2} a_i + \sum_{j \in J_2} b_j, \tag{6.3}$$

such that $|J_1| \geq |J_2| + 1$. This implies that N is equal to some linear combination of a_i's with coefficients $\pm 2, \pm 1, 0$. This leads to contradiction, since $N > 2 \sum_{i=1}^n \lambda_i$.

Thus, we know that $\sum_{i=1}^n y_i = 0$. Cancelling $N \cdot |J_1|$ out of (6.3) we get an identity with coefficients $\pm 2, \pm 1, 0$ on the set $a_1, \dots, a_n$. By the choice of a_i's, this identity must be trivial, which amounts exactly to saying that $x_i + y_i = 0$, for $i \in [n]$.

Second, it is a trivial observation that $c(S_n) \leq n$. Indeed, let $\pi_i \in \mathrm{P}(n)$ be a partition with only one nonsingleton block $(1, n+i)$, for $i \in [n]$. Then $\mathrm{span}(S_n \setminus (\{\pi_1, \dots, \pi_n\} \downarrow S_n)) \neq S_n$, since for any $(x_1, \dots, x_n, y_1, \dots, y_n) \in S_n \setminus (\{\pi_1, \dots, \pi_n\} \downarrow S_n)$, we have $x_1 = 0$.

Finally, let us see that $c(S_n) > n - 1$. As we have remarked after Definition 6.1, it is enough to consider the case when the partitions of Π have one block of size 2, and the rest are singletons. Let us call the identity $a_i + b_j = a_j + b_i$ *the elementary identity indexed* (i, j).

From the definition of the closure operation $\downarrow$ it is clear that an elementary identity indexed (i, j) is not in $S_n \setminus (\Pi \downarrow S_n)$ if and only if the partition whose only nonsingleton block is $(i, n+j)$ belongs to Π, or the partition whose only nonsingleton block is $(j, n+i)$ belongs to Π. That is because the only reason this identity would not be in $S_n \setminus (\Pi \downarrow S_n)$ would be that one of these two partitions is in $\Pi \downarrow S_n$. But, if such a partition is in $\Pi \downarrow S_n$, then it must be in Π: moves (2) of Definition 3.1 can never produce a partition whose only nonsingleton block has size 2, while the moves (3) of Definition 3.1 may only interchange between partitions $(i, n+j)$ and $(j, n+i)$ in our specific situation. Thus, we can conclude that if $|\Pi| \leq n - 1$, then at most $n - 1$ elementary identities are not in $S_n \setminus (\Pi \downarrow S_n)$.

Next, we note that for any distinct $i, j, k \in [n]$, the elementary identities (i, j) and (j, k) imply the elementary identity (i, k). Let us now think of elementary identities as edges in a complete graph on n vertices, K_n. Then, any set M of elementary identities corresponds to a graph G on n vertices, and the collection of the elementary identities which lie in the span M is encoded by the *transitive closure* of G. It is a well known combinatorial fact that K_n is $(n-1)$-connected, which means that removal of at most $n - 1$ edges from it leaves a connected graph. Hence, if we remove at most $n - 1$ edges from K_n and then take the transitive closure, we get K_n again. Thus, if $|\Pi| \leq n - 1$, all elementary identities lie in $\mathrm{span}(S_n \setminus (\Pi \downarrow S_n))$. Since the elementary

identities generate the whole S_n, we conclude that $S_n = \mathrm{span}(S_n \setminus (\Pi \downarrow S_n))$, hence $c(S_n) > n - 1$. □

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