# Intersection homology and Alexander modules of hypersurface complements

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# Intersection homology and Alexander modules of hypersurface complements

Laurentiu Maxim

**Abstract.** Let $V$ be a degree $d$, reduced hypersurface in $\mathbb{CP}^{n+1}$, $n \geq 1$, and fix a generic hyperplane, $H$. Denote by $\mathcal{U}$ the (affine) hypersurface complement, $\mathbb{CP}^{n+1} - V \cup H$, and let $\mathcal{U}^c$ be the infinite cyclic covering of $\mathcal{U}$ corresponding to the kernel of the total linking number homomorphism. Using intersection homology theory, we give a new construction of the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ of the hypersurface complement and show that, if $i \leq n$, these are torsion over the ring of rational Laurent polynomials. We also obtain obstructions on the associated global polynomials: their zeros are roots of unity of order $d$ and are entirely determined by the local topological information encoded by the link pairs of singular strata of a stratification of the pair $(\mathbb{CP}^{n+1}, V)$. As an application, we give obstructions on the eigenvalues of monodromy operators associated to the Milnor fibre of a projective hypersurface arrangement.

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## 1. Introduction

Intersection homology is the correct theory to extend results from manifolds to singular varieties: e.g., Morse theory, Lefschetz theorems, Hodge decompositions, but especially Poincaré duality, which motivates the theory. It is therefore natural to use it in order to describe topological invariants associated with algebraic varieties.

We will use intersection homology for the study of Alexander modules of hypersurface complements. These are global invariants of the hypersurface, which were introduced and studied by Libgober in a sequence of papers [24], [25], [26], [27] (see also [9]) and can be defined as follows: Let $V$ be a degree $d$, reduced, projective hypersurface in $\mathbb{CP}^{n+1}$, $n \geq 1$; let $H$ be a fixed hyperplane which we call 'the hyperplane at infinity'; set $\mathcal{U} := \mathbb{CP}^{n+1} - V \cup H$. (Alternatively, let $X \subset \mathbb{C}^{n+1}$ be a reduced affine hypersurface and $\mathcal{U} := \mathbb{C}^{n+1} - X$.) Then $H_1(\mathcal{U}) \cong \mathbb{Z}^s$, where $s$

is the number of components of $V$, and one proceeds as in classical knot theory to define Alexander-type invariants of the hypersurface $V$. More precisely, the rational homology groups of any infinite cyclic cover of $\mathcal{U}$ become, under the action of the group of covering transformations, modules over the ring of rational Laurent polynomials, $\Gamma = \mathbb{Q}[t, t^{-1}]$. The modules $H_i(\mathcal{U}^c; \mathbb{Q})$ associated to the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$, defined by the kernel of the total linking number homomorphism, are called *the Alexander modules of the hypersurface complement*. Note that, since $\mathcal{U}$ has the homotopy type of a finite CW complex of dimension $\leq n+1$, the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ are trivial for $i > n+1$ and $H_{n+1}(\mathcal{U}^c; \mathbb{Q})$ is $\Gamma$-free. Thus, of a particular interest are the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ for $i \leq n$.

Libgober showed ([25], [26], [27], [28]) that if $V$ has only isolated singularities (including at infinity), there is essentially only one interesting global invariant of the complement, and that it depends on the "local type and position" of singularities. More precisely, $\tilde{H}_i(\mathcal{U}^c; \mathbb{Z}) = 0$ for $i < n$, and $H_n(\mathcal{U}^c; \mathbb{Q})$ is a torsion $\Gamma$-module. Moreover, if $\delta_n(t)$ denotes the polynomial associated to the torsion module $H_n(\mathcal{U}^c; \mathbb{Q})$, then $\delta_n(t)$ divides (up to a power of $t-1$) the product of the local Alexander polynomials of the algebraic links around the isolated singular points. If $H$ is generic (hence $V$ has no singularities at infinity), then the zeros of $\delta_n(t)$ are roots of unity of order $d$, and $H_n(\mathcal{U}^c; \mathbb{Q})$ is a semi-simple module annihilated by $t^d - 1$.

The aim of this paper is to provide generalizations of these results to the case of hypersurfaces with non-isolated singularities. We will assume that $H$ is generic, i.e., transversal to all strata of a Whitney stratification of $V$. Using intersection homology theory, we will give a new description of the Alexander modules of the hypersurface complement. These will be realized as intersection homology groups of $\mathbb{CP}^{n+1}$, with a certain local coefficient system with stalk $\Gamma := \mathbb{Q}[t, t^{-1}]$, defined on $\mathcal{U}$. Therefore, we will have at our disposal the apparatus of intersection homology and derived categories to study the Alexander modules of the complement.

We now outline our results section by section.

In Section 2, we recall the definitions and main properties of the Alexander modules of the hypersurface complement, $H_i(\mathcal{U}^c; \mathbb{Q})$. We also show that, if $V$ is in general position at infinity, has no codimension one singularities, and is a rational homology manifold, then for $i \leq n$, the modules $H_i(\mathcal{U}^c; \mathbb{Q})$ are torsion and their associated polynomials do not contain factors $t-1$ (see Proposition 2.1).

In Section 3, we realize the Alexander modules of complements to hypersurfaces in general position at infinity as intersection homology modules. Following [4], in Section 3.1 we construct the intersection Alexander modules of the hypersurface $V$. More precisely, by choosing a Whitney stratification $\mathcal{S}$ of $V$ and a generic hyperplane, $H$, we obtain a stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$. We define a local system $\mathcal{L}_H$ on $\mathcal{U}$, with stalk $\Gamma := \mathbb{Q}[t, t^{-1}]$ and action by an element $\alpha \in \pi_1(\mathcal{U})$ determined by multiplication by $t^{\operatorname{lk}(\alpha, V \cup -dH)}$. Then, for any perversity $\bar{p}$, the intersection homology complex $IC^{\bullet}_{\bar{p}} := IC^{\bullet}_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ is defined by using Deligne's axiomatic construction ([2], [15]). The *intersection Alexander modules of the hypersurface* $V$ are then defined as hypercohomology groups of the middle-perversity intersection homology complex: $IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) := \mathbb{H}^{-i}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}})$.

In Section 3.2, we first prove the key technical lemma, which asserts that the restriction to $V \cup H$ of the intersection homology complex $IC^\bullet_{\bar{m}}$ is quasi-isomorphic to the zero complex (see Lemma 3.1). As a corollary, it follows that *the intersection Alexander modules of* $V$ *coincide with the Alexander modules of the hypersurface complement*, i.e., there is an isomorphism of $\Gamma$-modules: $IH_*^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_*(\mathcal{U}^c; \mathbb{Q})$. From now on, we will study the intersection Alexander modules in order to obtain results on the Alexander modules of the complement. Using the superduality isomorphism for the local finite type codimension two embedding $V \cup H \subset \mathbb{CP}^{n+1}$, and the peripheral complex associated with the embedding (see [4]), we show that the $\Gamma$-modules $IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ are *torsion* if $i \leq n$ (see Theorem 3.6). Therefore the classical Alexander modules of the hypersurface complement are torsion in the range $i \leq n$. We denote their associated polynomials by $\delta_i(t)$ and call them *the global Alexander polynomials of the hypersurface*.

Section 4 contains the main theorems of the paper, which provide obstructions on the prime divisors of the polynomials $\delta_i(t)$. Our results are extensions to the case of hypersurfaces with general singularities of the results proven by A. Libgober for hypersurfaces with only isolated singularities ([25], [26], [27]).

The first theorem gives a characterization of the zeros of the global polynomials and generalizes Corollary 4.8 of [25].

**Theorem** (see Theorem 4.1). *If* $V$ *is an* $n$*-dimensional reduced projective hypersurface of degree* $d$*, transversal to the hyperplane at infinity, then the zeros of the global Alexander polynomials* $\delta_i(t)$*,* $i \leq n$*, are roots of unity of order* $d$*.*

The underlying idea of this paper is to use local topological information associated with a singularity to describe some global topological invariants of algebraic varieties. We provide a general divisibility result which restricts the prime factors of the global Alexander polynomial $\delta_i(t)$ to those of the local Alexander polynomials of the link pairs around the singular strata. More precisely, we prove the following result.

**Theorem** (see Theorem 4.2). *Let* $V$ *be a reduced hypersurface in* $\mathbb{CP}^{n+1}$*, which is transversal to the hyperplane at infinity,* $H$*. Fix an arbitrary irreducible component of* $V$*, say* $V_1$*. Let* $\mathcal{S}$ *be a stratification of the pair* $(\mathbb{CP}^{n+1}, V)$*. Then for a fixed integer* $i$ $(1 \leq i \leq n)$*, the prime factors of the global Alexander polynomial* $\delta_i(t)$ *of* $V$ *are among the prime factors of local polynomials* $\xi_r^s(t)$ *associated to the local Alexander modules* $H_r(S^{2n-2s+1} - K^{2n-2s-1}; \Gamma)$ *of link pairs* $(S^{2n-2s+1}, K^{2n-2s-1})$ *of components of strata* $S \in \mathcal{S}$ *such that* $S \subset V_1$*,* $n - i \leq s = \dim S \leq n$*, and* $r$ *is in the range* $2n - 2s - i \leq r \leq n - s$*.*

For hypersurfaces with only isolated singularities, the above theorem can be strengthened to obtain a refinement of Theorem 4.3 of [25].

**Theorem** (see Theorem 4.5). *Let $V$ be a reduced hypersurface in $\mathbb{CP}^{n+1}$, which is transversal to the hyperplane at infinity, $H$, and has only isolated singularities. Fix an irreducible component of $V$, say $V_1$. Then $\delta_n(t)$ divides (up to a power of $(t-1)$) the product $\prod_{p \in V_1 \cap \mathrm{Sing}(V)} \Delta_p(t)$ of the local Alexander polynomials of link pairs of the singular points $p$ of $V$ which are contained in $V_1$.*

We end the section by relating the intersection Alexander modules of $V$ to the modules 'at infinity'. We prove the following extension of Theorem 4.5 of [25].

**Theorem** (see Theorem 4.7). *Let $V$ be a reduced hypersurface of degree $d$ in $\mathbb{CP}^{n+1}$, which is transversal to the hyperplane at infinity, $H$. Let $S_\infty$ be a sphere of sufficiently large radius in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$. Then for all $i < n$,*

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong \mathbb{H}^{-i-1}(S_\infty; IC^\bullet_{\bar{m}}) \cong H_i(\mathcal{U}^c_\infty; \mathbb{Q})$$

*and $IH_n^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ is a quotient of $\mathbb{H}^{-n-1}(S_\infty; IC^\bullet_{\bar{m}}) \cong H_n(\mathcal{U}^c_\infty; \mathbb{Q})$, where $\mathcal{U}^c_\infty$ is the infinite cyclic cover of $S_\infty - (V \cap S_\infty)$ corresponding to the linking number with $V \cap S_\infty$.*

As suggested in [29], we note that the above theorem has as a corollary the semi-simplicity of the Alexander modules of the hypersurface complement (see Proposition 4.9).

In Section 5, we apply the preceding results to the case of a hypersurface $V \subset \mathbb{CP}^{n+1}$, which is the projective cone over a reduced hypersurface $Y \subset \mathbb{CP}^n$. We first note that Theorem 4.2 translates into divisibility results for the characteristic polynomials of the monodromy operators acting on the Milnor fiber $F$ of the projective arrangement defined by $Y$ in $\mathbb{CP}^n$ (see Proposition 5.1), thus generalizing similar results obtained by A. Dimca in the case of isolated singularities ([6], [7]). As a consequence, we obtain obstructions on the eigenvalues of the monodromy operators (see Corollary 5.3), similar to those obtained by Libgober in the case of hyperplane arrangements ([30]), or Dimca in the case of curve arrangements ([7]).

Section 6 deals with examples. We show, by explicit calculations, how to apply the above theorems in obtaining information on the global Alexander polynomials of a hypersurface in general position at infinity.

**Note.** Our overall approach makes use of sheaf theory and the language of derived categories ([2], [4], [15]) and we rely heavily on the material in these references. The necessary background material is also reviewed in the author's thesis (see [32]).

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**Note.** After this paper was written, the Alexander invariants of the total linking number infinite cyclic cover were further studied in [8], where it is shown that there is a natural mixed Hodge structure on the Alexander modules of the complement.

## 2. Alexander modules of hypersurface complements

In this section we recall the definition and main known results on the Alexander modules and polynomials of hypersurface complements. We also consider the special case of hypersurfaces which are rational homology manifolds.

**2.1. Definitions.** Let $X$ be a connected CW complex, and let $\pi_X \colon \pi_1(X) \to \mathbb{Z}$ be an epimorphism. We denote by $X^c$ the $\mathbb{Z}$-cyclic covering associated to the kernel of the morphism $\pi_X$. The group of covering transformations of $X^c$ is infinite cyclic and acts on $X^c$ by a covering homeomorphism $h$. Thus, all the groups $H_*(X^c; A)$, $H^*(X^c; A)$ and $\pi_j(X^c) \otimes A$ for $j > 1$ become in the usual way $\Gamma_A$-modules, where $\Gamma_A = A[t, t^{-1}]$, for any ring $A$. These are called *the Alexander modules of the pair* $(X, \pi_X)$.

If $A$ is a field, then $\Gamma_A$ is a principal ideal domain. Hence any torsion $\Gamma_A$-module $M$ of finite type has a well-defined associated order (see [34]). This is called *the Alexander polynomial of the torsion* $\Gamma_A$*-module* $M$ and denoted by $\delta_M(t)$. We regard the trivial module $(0)$ as a torsion module whose associated polynomial is $\delta(t) = 1$.

With these notations, we have the following simple fact: let $f \colon M \to N$ be an epimorphism of $R$-modules, where $R$ is a PID and $M$ is torsion of finite type. Then $N$ is torsion of finite type and $\delta_N(t)$ divides $\delta_M(t)$.

**2.2. Alexander modules of hypersurface complements.** To fix notations for the rest of the paper, let $V$ be a reduced hypersurface in $\mathbb{CP}^{n+1}$, defined by a degree $d$ homogeneous equation $f = f_1 \dots f_s = 0$, where $f_i$ are the irreducible factors of $f$ and $V_i = \{f_i = 0\}$ the irreducible components of $V$. We will assume that $V$ is *in general position at infinity*, i.e., we choose a generic hyperplane $H$ (transversal to all singular strata in a stratification of $V$) which we call 'the hyperplane at infinity'. Let $\mathcal{U}$ be the (affine) hypersurface complement: $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$. Then $H_1(\mathcal{U}) \cong \mathbb{Z}^s$ ([6], (4.1.3), (4.1.4)), generated by the meridian loops $\gamma_i$ about the non-singular part of each irreducible component $V_i$, $i = 1, \dots, s$. If $\gamma_\infty$ denotes the meridian about the hyperplane at infinity, then there is a relation in $H_1(\mathcal{U})$: $\gamma_\infty + \sum d_i \gamma_i = 0$, where $d_i = \deg(V_i)$. We consider the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$ defined by the kernel of the total linking number homomorphism $\mathrm{lk}\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$, which maps all meridian generators to 1, and thus any loop $\alpha$ to its linking number $\mathrm{lk}(\alpha, V \cup -dH)$ with the divisor $V \cup -dH$ in $\mathbb{CP}^{n+1}$. Note that lk coincides with the homomorphism $\pi_1(\mathcal{U}) \to \pi_1(\mathbb{C}^*)$ induced by the polynomial map defining the affine hypersurface $V_{\mathrm{aff}} := V - V \cap H$ ([6], p. 76–77). The *Alexander modules of the hypersurface complement* are defined as $H_i(\mathcal{U}^c; \mathbb{Q})$, $i \in \mathbb{Z}$.

Since $\mathcal{U}$ has the homotopy type of a finite CW complex of dimension $\leq n+1$ ([6], (1.6.7), (1.6.8)), it follows that all the associated Alexander modules are of finite type over $\Gamma_{\mathbb{Q}}$, but in general not over $\mathbb{Q}$. It also follows that the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ are trivial for $i > n+1$ and $H_{n+1}(\mathcal{U}^c; \mathbb{Q})$ is free over $\Gamma_{\mathbb{Q}}$ ([9]).

Note that if $V$ has no codimension one singularities (e.g. if $V$ is normal), then the fundamental group of $\mathcal{U}$ is infinite cyclic ([25], Lemma 1.5) and $\mathcal{U}^c$ is the universal cover of $\mathcal{U}$. Moreover, if this is the case, then $\pi_i(\mathcal{U}) = 0$ for $1 < i < n-k$, where $k$ is the complex dimension of the singular locus of $V$. In particular, for a *smooth* projective hypersurface $V$, in general position at infinity, we have that $\tilde{H}_i(\mathcal{U}^c; \mathbb{Q}) = 0$ for $i < n+1$.

The next case to consider is that of a hypersurface with only isolated singularities. In this case Libgober showed ([25]) that $\tilde{H}_i(\mathcal{U}^c; \mathbb{Z}) = 0$ for $i < n$, and $H_n(\mathcal{U}^c; \mathbb{Q})$ is a torsion $\Gamma_{\mathbb{Q}}$-module. If we denote by $\delta_n(t)$ the polynomial associated to the torsion module $H_n(\mathcal{U}^c; \mathbb{Q})$, then Theorem 4.3 of [25] asserts that $\delta_n(t)$ divides the product $\prod_{i=1}^{s} \Delta_i(t) \cdot (t-1)^r$ of the Alexander polynomials of link pairs of the singular points of $V$. The factor $(t-1)^r$ can be omitted if $V$ and $V \cap H$ are rational homology manifolds. Moreover, the zeros of $\delta_n(t)$ are roots of unity of order $d = \deg(V)$ and $H_n(\mathcal{U}^c; \mathbb{Q})$ is semi-simple ([25], Corollary 4.8).

**Note.** Libgober's divisibility theorem ([25], Theorem 4.3) holds for hypersurfaces with isolated singularities, *including at infinity* (and $n \geq 1$). However, for non-generic $H$ and for hypersurfaces with more general singularities, the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ $(i \leq n)$ are not torsion in general. Their $\Gamma_{\mathbb{Q}}$ rank is calculated in [9], Theorem 2.10(v). We will show that if $V$ is a reduced hypersurface, in general position at infinity, then the modules $H_i(\mathcal{U}^c; \mathbb{Q})$ are torsion for $i \leq n$.

**2.3. Rational homology manifolds.** Recall that a $n$-dimensional complex variety $V$ is called a *rational homology manifold* if for all points $x \in V$ we have

$$H_i(V, V-x; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & i = 2n, \\ 0, & i \neq 2n. \end{cases}$$

A rational homology manifold of dimension $n$ has pure dimension $n$ as a complex variety. Rational homology manifolds may be thought of as nonsingular for the purposes of rational homology. For example, Poincaré and Lefschetz duality hold for them in rational homology. The Lefschetz hyperplane section theorem also holds.

Examples of rational homology manifolds include complex varieties having rational homology spheres as links of singular strata (see for example [32]).

Note that, if $V$ is a projective hypersurface having rational homology spheres as links of singular strata and $H$ is a generic hyperplane, then $V \cap H$ is a rational homology manifold: indeed, by the transversality assumption, the link in $V \cap H$ of a stratum $S \cap H$ (for $S$ a stratum of $V$) is the same as the link in $V$ of $S$.

As a first example when the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$, $i \leq n$, are torsion, we prove the following result (compare [25], Lemma 1.7, 1.12).

**Proposition 2.1.** *Let $V$ be a degree $d$ projective hypersurface in $\mathbb{CP}^{n+1}$, and let $H$ be a generic hyperplane. Assume that $V$ has no codimension one singularities and is a rational homology manifold. Then for $i \leq n$, $H_i(\mathcal{U}^c; \mathbb{Q})$ is a torsion $\Gamma_{\mathbb{Q}}$-module and $\delta_i(1) \neq 0$, where $\delta_i(t)$ is the associated Alexander polynomial.*

*Proof.* Recall that, under our assumptions, $\mathcal{U}^c$ is the infinite cyclic and universal cover of $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$. We will use Milnor's exact sequence ([25], [9])

$$\cdots \to H_i(\mathcal{U}^c; \mathbb{Q}) \to H_i(\mathcal{U}^c; \mathbb{Q}) \to H_i(\mathcal{U}; \mathbb{Q}) \to H_{i-1}(\mathcal{U}^c; \mathbb{Q}) \to \cdots,$$

where the first morphism is multiplication by $t-1$. We claim that $H_i(\mathcal{U}; \mathbb{Q}) \cong 0$ for $2 \leq i \leq n$, hence the multiplication by $t-1$ in $H_i(\mathcal{U}^c; \mathbb{Q})$ is surjective ($2 \leq i \leq n$). Therefore its cyclic decomposition has neither free summands nor summands of the form $\Gamma_{\mathbb{Q}}/(t-1)^r\Gamma_{\mathbb{Q}}$, with $r \in \mathbb{N}$. On the other hand, $H_1(\mathcal{U}^c; \mathbb{Q}) \cong \pi_1(\mathcal{U}^c) \otimes \mathbb{Q} \cong 0$.

Suppose that $k$ is the dimension of the singular locus of $V$. By our assumptions, $n - k \geq 2$. Let $L \cong \mathbb{CP}^{n-k}$ be a generic linear subspace. Then, by transversality, $L \cap V$ is a non-singular hypersurface in $L$, transversal to the hyperplane at infinity, $L \cap H$. Therefore, by Corollary 1.2 of [25], $L \cap \mathcal{U}$ is homotopy equivalent to $S^1 \vee S^{n-k} \vee \cdots \vee S^{n-k}$. Thus, by Lefschetz hyperplane section theorem (applied $k+1$ times) we obtain $H_i(\mathcal{U}; \mathbb{Q}) \cong H_i(L \cap \mathcal{U}; \mathbb{Q}) = 0$, $2 \leq i \leq n-k-1$.

For $n-k \leq i \leq n$ we have $H_i(\mathcal{U}; \mathbb{Q}) \cong H_{i+1}(\mathbb{CP}^{n+1} - H, \mathbb{CP}^{n+1} - (V \cup H); \mathbb{Q})$, as follows from the exact sequence of the pair $(\mathbb{CP}^{n+1} - H, \mathbb{CP}^{n+1} - (V \cup H))$. Using duality, one can identify this with $H^{2n+1-i}(V \cup H, H; \mathbb{Q})$. And by excision, this group is isomorphic to $H^{2n+1-i}(V, V \cap H; \mathbb{Q})$. Let $u$ and $v$ denote the inclusion of $V - V \cap H$ and respectively $V \cap H$ into $V$. Then the distinguished triangle $u_!u^!\mathbb{Q} \to \mathbb{Q} \to v_*v^*\mathbb{Q} \xrightarrow{[1]}$ (where we regard $\mathbb{Q}$ as a constant sheaf on $V$), upon applying the hypercohomology with compact support functor, yields the isomorphism $H^{2n+1-i}(V, V \cap H; \mathbb{Q}) \cong H_c^{2n+1-i}(V - V \cap H; \mathbb{Q})$ (see [7], Remark 2.4.5.(iii)). By Poincaré duality over $\mathbb{Q}$, the latter is isomorphic to $H_{i-1}(V - V \cap H; \mathbb{Q})$. The Lefschetz theorem on generic hyperplane complements in hypersurfaces ([10], p. 476) implies that $V - V \cap H$ is homotopy equivalent to a wedge of spheres $S^n$. Therefore, $H_{i-1}(V - V \cap H; \mathbb{Q}) \cong 0$ for $0 < i-1 < n$, i.e., for $2 \leq n-k \leq i \leq n$. This finishes the proof of the proposition. $\square$

## 3. Intersection homology and Alexander modules

Using intersection homology theory, we will give a new construction of the Alexander modules of complements of hypersurfaces in general position at infinity. The advantage of the new approach is the use of the powerful language of sheaf theory and derived categories ([15], [2]) in the study of the Alexander invariants associated with singular hypersurfaces. This will alow us to obtain generalizations to classical results known only in the case of hypersurfaces with only isolated singularities. For a quick introduction to derived categories, the reader is advised to consult [31], §1. When dealing with intersection homology, we will always use the indexing conventions of [15].

**3.1. Intersection Alexander modules.** (1) A codimension 2 sub-pseudomanifold $K^n$ of a sphere $S^{n+2}$ is said to be of *finite* (*homological*) *type* if the homology groups $H_i(S^{n+2}-K;\Gamma)$ with local coefficients in $\Gamma := \mathbb{Q}[t,t^{-1}]$ are finite dimensional over $\mathbb{Q}$. Here $\Gamma$ denotes the local system on $S^{n+2}-K$, with stalk $\Gamma$, and it corresponds to the representation $\alpha \mapsto t^{\mathrm{lk}(K,\alpha)}$, $\alpha \in \pi_1(S^{n+2}-K)$, where $\mathrm{lk}(K,\alpha)$ is the linking number of $\alpha$ with $K$ (see [4]).

A sub-pseudomanifold $X$ of a manifold $Y$ is said to be of *finite local type* if the link of each component of a stratification of the pair $(Y,X)$ is of finite type. It is easy to see that the link pairs of components of strata of a sub-pseudomanifold of finite local type also have finite local type ([4]). Algebraic knots are of finite type and of finite local type ([4]).

(2) Let $V$ be a reduced projective hypersurface of degree $d$ in $\mathbb{CP}^{n+1}$ ($n \geq 1$). Choose a Whitney stratification $\mathcal{S}$ of $V$. Recall that there is such a stratification where strata are pure dimensional locally closed algebraic subsets with a finite number of irreducible nonsingular components. Together with the hypersurface complement, $\mathbb{CP}^{n+1}-V$, this yields a stratification of the pair $(\mathbb{CP}^{n+1},V)$, in which $\mathcal{S}$ is the set of singular strata. All links of strata of the pair $(\mathbb{CP}^{n+1},V)$ are algebraic, hence of finite type, so $V \subset \mathbb{CP}^{n+1}$ is of finite local type (see [4], Proposition 2.2). We choose a generic hyperplane $H$ in $\mathbb{CP}^{n+1}$ (i.e., transversal to all strata of $V$) and consider the induced stratification on the pair $(\mathbb{CP}^{n+1}, V\cup H)$, with (open) strata of the form $S-S\cap H$, $S\cap H$ and $H-V\cap H$, for $S\in\mathcal{S}$. We call $H$ 'the hyperplane at infinity' and say that '$V$ is transversal to the hyperplane at infinity'. Following [4], we define a local system $\mathcal{L}_H$ on $\mathbb{CP}^{n+1}-(V\cup H)$, with stalk $\Gamma := \Gamma_{\mathbb{Q}} = \mathbb{Q}[t,t^{-1}]$ and action by an element $\alpha\in\pi_1(\mathbb{CP}^{n+1}-V\cup H)$ determined by multiplication by $t^{\mathrm{lk}(\alpha,V\cup -dH)}$. Here $\mathrm{lk}(\alpha, V\cup -dH)$ is the linking number of $\alpha$ with the divisor $V\cup -dH$ of $\mathbb{CP}^{n+1}$. Then, (using a triangulation of the projective space) $V\cup H$ is a (PL) sub-pseudomanifold of $\mathbb{CP}^{n+1}$ and the intersection complex $IC^{\bullet}_{\bar{p}} := IC^{\bullet}_{\bar{p}}(\mathbb{CP}^{n+1},\mathcal{L}_H)$ is defined for any perversity $\bar{p}$. The middle-perversity intersection homology modules ([15])

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H) := \mathbb{H}^{-i}(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}})$$

will be called the *intersection Alexander modules of the hypersurface* $V$. (Through this paper, $\mathbb{H}^\bullet$ stands for the hypercohomology functor.) Note that these modules are of finite type over $\Gamma$, since $IC^\bullet_{\bar{p}}$ is cohomologically constructible ([2], V.3.12) and $\mathbb{CP}^{n+1}$ is compact (see [2], V.3.4.(a), V.10.13).

It will be useful to describe the links of the pair $(\mathbb{CP}^{n+1}, V \cup H)$ in terms of those of $(\mathbb{CP}^{n+1}, V)$. Because of the transversality assumption, there are stratifications $\{Z_i\}$ of $(\mathbb{CP}^{n+1}, V)$ and $\{Y_i\}$ of $(\mathbb{CP}^{n+1}, V \cup H)$ with $Y_i = Z_i \cup (Z_{i+2} \cap H)$ (here the indices indicate the real dimensions). The link pair of a point $y \in (Y_i - Y_{i-1}) \cap H = (Z_{i+2} - Z_{i+1}) \cap H$ in $(\mathbb{CP}^{n+1}, V \cup H)$ is $(G, F) = (S^1 * G_1, (S^1 * F_1) \cup G_1)$, where $(G_1, F_1)$ is the link pair of $y \in Z_{i+2} - Z_{i+1}$ in $(\mathbb{CP}^{n+1}, V)$. (Here $A * B$ is the join of $A$ and $B$.) Points in $V - V \cap H$ have the same link pairs in $(\mathbb{CP}^{n+1}, V)$ and $(\mathbb{CP}^{n+1}, V \cup H)$. Finally, the link pair at any point in $H - V \cap H$ is $(S^1, \emptyset)$. (For details, see [4]).

By Lemma 2.3.1 of [4], $V \cup H \subset \mathbb{CP}^{n+1}$ is of finite local type. Hence, by Theorem 3.3 of [4], we have the following *superduality isomorphism*:

$$IC^\bullet_{\bar{m}} \cong \mathcal{D} IC^{\bullet \mathrm{op}}_{\bar{l}}[2n+2],$$

where if $\mathcal{A}^\bullet$ is a complex of sheaves, $\mathcal{D}\mathcal{A}^\bullet$ denotes its Verdier dual. (Here $A^{\mathrm{op}}$ is the $\Gamma$-module obtained from the $\Gamma$-module $A$ by composing all module structures with the involution $t \to t^{-1}$.) Recall that the middle and logarithmic perversities are defined as $\bar{m}(s) = [(s-1)/2]$ and $\bar{l}(s) = [(s+1)/2]$. Note $\bar{m}(s) + \bar{l}(s) = s - 1$, i.e., $\bar{m}$ and $\bar{l}$ are superdual perversities.

(3) With the notations from §2, we have an isomorphism of $\Gamma$-modules:

$$H_i(\mathcal{U}; \mathcal{L}_H) \cong H_i(\mathcal{U}^c; \mathbb{Q}),$$

where $\mathcal{L}_H$ is, as above, the local coefficient system on $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$ defined by the representation $\mu \colon \pi_1(\mathcal{U}) \to \mathrm{Aut}(\Gamma) = \Gamma^*$, $\mu(\alpha) = t^{\mathrm{lk}(\alpha, V \cup -dH)}$.

Indeed, $\mathcal{U}^c$ is the covering associated to the kernel of the linking number homomorphism $\mathrm{lk}\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$, $\alpha \mapsto \mathrm{lk}(\alpha, V \cup -dH)$, and note that $\mu$ factors through lk, i.e., $\mu$ is the composition $\pi_1(\mathcal{U}) \xrightarrow{\mathrm{lk}} \mathbb{Z} \to \Gamma^*$, with the second homomorphism mapping 1 to $t$. Thus $\mathrm{Ker}(\mathrm{lk}) \subset \mathrm{Ker}(\mu)$. By definition, $H_*(\mathcal{U}; \mathcal{L}_H)$ is the homology of the chain complex $C_*(\mathcal{U}; \mathcal{L}_H)$ defined by the equivariant tensor product: $C_*(\mathcal{U}; \mathcal{L}_H) := C_*(\mathcal{U}^c) \otimes_{\mathbb{Z}} \Gamma$, where $\mathbb{Z}$ stands for the group of covering transformations of $\mathcal{U}^c$ (see [7], p. 50). Since $\Gamma = \mathbb{Q}[\mathbb{Z}]$, the chain complex $C_*(\mathcal{U}^c) \otimes_{\mathbb{Z}} \Gamma$ is clearly isomorphic to the complex $C_*(\mathcal{U}^c) \otimes \mathbb{Q}$, and the claimed isomorphism follows.

(4) This is also a convenient place to point out the following fact: as $\bar{m}(2) = 0$, the allowable zero- and one-chains ([14]) are those which lie in $\mathbb{CP}^{n+1} - (V \cup H)$. Therefore,

$$IH_0^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_0(\mathbb{CP}^{n+1} - (V \cup H); \mathcal{L}_H) \cong H_0(\mathcal{U}^c; \mathbb{Q}) = \Gamma/(t-1).$$

**3.2. Relation with the classical Alexander modules of the complement.** We are aiming to show that in our setting (i.e., $V$ a degree $d$, reduced, $n$-dimensional projective hypersurface, transversal to the hyperplane at infinity), *the intersection Alexander modules of a hypersurface coincide with the classical Alexander modules of the hypersurface complement.* The key fact will be the following characterization of the support of the intersection homology complex $IC^{\bullet}_{\bar{m}}$.

**Lemma 3.1.** *There is a quasi-isomorphism:*

$$IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)_{|V \cup H} \cong 0.$$

*Proof.* It suffices to show the vanishing of cohomology stalks of the complex $IC^{\bullet}_{\bar{m}}$ at points in strata of $V \cup H$. We will do this in two steps.

*Step* 1. $$IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)_{|H} \cong 0.$$

The link pair of $H - V \cap H$ is $(S^1, \emptyset)$ and this maps to $t^{-d}$ under $\mathcal{L}_H$, therefore the stalk of $IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ at a point in this stratum is zero. Indeed (cf. [2], V.3.15), for $x \in H - V \cap H$:

$$\mathcal{H}^q(IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H))_x \cong \begin{cases} 0, & q > -2n-2, \\ IH^{\bar{m}}_{-q-(2n+1)}(S^1; \Gamma), & q \le -2n-2, \end{cases}$$

and note that $IH^{\bar{m}}_j(S^1; \Gamma) \cong 0$ unless $j = 0$. Here, and in the sequel, $\Gamma$ denotes the stalks and the local systems obtained from $\mathcal{L}_H$ by restricting to various subspaces of $\mathcal{U}$.

Next, consider the link pair $(G, F)$ of a point $x \in S \cap H$, $S \in \mathcal{S}$. Let the real codimension of $S$ in $\mathbb{CP}^{n+1}$ be $2k$. Then the codimension of $S \cap H$ is $2k + 2$ and $\dim(G) = 2k + 1$. The stalk at $x \in S \cap H$ of the intersection homology complex $IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ is given by the local calculation formula ([2], (3.15)):

$$\mathcal{H}^q(IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H))_x \cong \begin{cases} 0, & q > k-2n-2, \\ IH^{\bar{m}}_{-q-(2n-2k+1)}(G; \Gamma), & q \le k-2n-2. \end{cases}$$

We claim that

$$IH^{\bar{m}}_i(G; \Gamma) = 0 \quad \text{for } i \ge k+1.$$

Then, by setting $i = -q-(2n-2k+1)$, we obtain that $IH^{\bar{m}}_{-q-(2n-2k+1)}(G; \Gamma) = 0$ for $q \le k-2n-2$, and therefore

$$\mathcal{H}^q(IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H))_x = 0.$$

In order to prove the claim, we use arguments similar to those used in [4], p. 359–361. Recall that $(G, F)$ is of the form $(S^1 * G_1, (S^1 * F_1) \cup G_1)$, where

$(G_1, F_1)$ is the link pair of $x \in S$ in $(\mathbb{CP}^{n+1}, V)$ (or equivalently, the link of $S \cap H$ in $(H, V \cap H)$). The restriction $\Gamma$ of $\mathcal{L}_H$ to $G - F$ is given by sending $\alpha \in \pi_1(G - F)$ to $t^{\mathrm{lk}(S^1 * F_1 - dG_1, \alpha)} \in \mathrm{Aut}(\Gamma)$. Let $\mathcal{IC} = IC^{\bullet}_{\bar{m}}(G, \Gamma)$. The link of the codimension two stratum $G_1 - G_1 \cap (S^1 * F_1)$ of $G$ is a circle that maps to $t^{-d}$ under $\Gamma$; hence by the stalk cohomology formula ([2], (3.15)), for $y \in G_1 - G_1 \cap (S^1 * F_1)$, we have

$$\mathcal{H}^i(\mathcal{IC})_y \cong \begin{cases} IH^{\bar{m}}_{-i-(\dim G_1+1)}(S^1; \Gamma), & i \le -\dim G, \\ 0, & i > \bar{m}(2) - \dim G = -\dim G. \end{cases}$$

Since $IH^{\bar{m}}_{-i-(\dim G_1+1)}(S^1; \Gamma) \cong H_{-i-(\dim G-1)}(S^1; \Gamma) = 0$ for $-i - (\dim G - 1) \neq 0$, i.e., for $i \neq 1 - \dim G$, we obtain that

$$\mathcal{IC}|_{G_1 - G_1 \cap (S^1 * F_1)} \cong 0.$$

Moreover, $G_1$ is a locally flat submanifold of $G$ and intersects $S^1 * F_1$ transversally. Hence the link pair in $(G, F)$ of a stratum of $G_1 \cap (S^1 * F_1)$ and the restriction of $\Gamma$ will have the same form as links of strata of $V \cap H$ in $(\mathbb{CP}^{n+1}, V \cup H)$. Thus, by induction on dimension we obtain

$$\mathcal{IC}|_{G_1 \cap (S^1 * F_1)} \cong 0.$$

Therefore, $\mathcal{IC}|_{G_1} \cong 0$. Thus, denoting by $i$ and $j$ the inclusions of $G - G_1$ and $G_1$, respectively, the distinguished triangle

$$Ri_! i^* \mathcal{IC} \to \mathcal{IC} \to Rj_* \mathcal{IC}|_{G_1} \to$$

upon applying the compactly supported hypercohomology functor, yields the isomorphisms

$$^c IH^{\bar{m}}_i(G - G_1; \Gamma) \cong IH^{\bar{m}}_i(G; \Gamma)$$

(here $^c IH_*$ denotes intersection homology with compact supports). We have

$$(G - G_1, F - F \cap G_1) \cong (c^\circ G_1 \times S^1, c^\circ F_1 \times S^1),$$

and the local system $\Gamma$ is given on

$$(c^\circ G_1 - c^\circ F_1) \times S^1 \cong (G_1 - F_1) \times \mathbb{R} \times S^1$$

by sending $\alpha \in \pi_1(G_1 - F_1)$ to the multiplication by $t^{\mathrm{lk}(F_1, \alpha)}$, and a generator of $\pi_1(S^1)$ to $t^{-d}$.

We denote by $\mathcal{L}_1$ and $\mathcal{L}_2$ the restrictions of $\Gamma$ to $c^\circ G_1$ and $S^1$ respectively, and note that $IH^{\bar{m}}_b(S^1; \mathcal{L}_2) = 0$ unless $b = 0$, in which case it is isomorphic to $\Gamma/t^d - 1$. Therefore, by the Künneth formula ([17]), we have

$$\begin{aligned} ^c IH^{\bar{m}}_i(c^\circ G_1 \times S^1; \Gamma) \cong \{&^c IH^{\bar{m}}_i(c^\circ G_1; \mathcal{L}_1) \otimes IH^{\bar{m}}_0(S^1; \mathcal{L}_2)\} \\ &\oplus \{^c IH^{\bar{m}}_{i-1}(c^\circ G_1; \mathcal{L}_1) * IH^{\bar{m}}_0(S^1; \mathcal{L}_2)\}. \end{aligned}$$

Lastly, the formula for the compactly supported intersection homology of a cone yields ([2], [21], [11]) ${}^cIH_i^{\bar{m}}(c^\circ G_1; \mathcal{L}_1) = 0$ for $i \geq \dim G_1 - \bar{m}(\dim G_1 + 1) = k$ (as well as ${}^cIH_i^{\bar{m}}(c^\circ G_1; \mathcal{L}_1) = IH_i^{\bar{m}}(G_1; \mathcal{L}_1)$, for $i < k$) and, consequently, ${}^cIH_{i-1}^{\bar{m}}(c^\circ G_1; \mathcal{L}_1) = 0$ for $i \geq k+1$.

Altogether,

$$IH_i^{\bar{m}}(G; \Gamma) \cong {}^cIH_i^{\bar{m}}(G - G_1; \Gamma) \cong {}^cIH_i^{\bar{m}}(c^\circ G_1 \times S^1; \Gamma) = 0 \quad \text{for } i \geq k+1,$$

as claimed.

*Step* 2.
$$IC_{\bar{m}}^\bullet(\mathbb{CP}^{n+1}, \mathcal{L}_H)_{|V} \cong 0.$$

It suffices to show the vanishing of stalks of the complex $IC_{\bar{m}}^\bullet$ at points in strata of the form $S - S \cap H$ of the affine part, $V_{\mathrm{aff}}$, of $V$. Note that, assuming $S$ connected, the link pair of $S - S \cap H$ in $(\mathbb{CP}^{n+1}, V \cup H)$ is the same as its link pair in $(\mathbb{C}^{n+1}, V_{\mathrm{aff}})$ with the induced stratification, or the link pair of $S$ in $(\mathbb{CP}^{n+1}, V)$. Let $x \in S - S \cap H$ be a point in an affine stratum of complex dimension $s$. The stalk cohomology calculation yields

$$\mathcal{H}^q(IC_{\bar{m}}^\bullet)_x \cong \begin{cases} IH_{-q-(2s+1)}^{\bar{m}}(S_x^{2n-2s+1}; \Gamma), & q \leq -n-s-2, \\ 0, & q > -n-s-2, \end{cases}$$

where $(S_x^{2n-2s+1}, K_x)$ is the link pair of the component containing $x$.

To obtain the desired vanishing, it suffices to prove that $IH_j^{\bar{m}}(S_x^{2n-2s+1}; \Gamma) \cong 0$ for $j \geq n-s+1$. We will show this in the following

**Lemma 3.2.** *If $S$ is an $s$-dimensional stratum of $V_{\mathrm{aff}}$ and $x$ is a point in $S$, then the intersection homology groups of its link pair $(S_x^{2n-2s+1}, K_x)$ in $(\mathbb{C}^{n+1}, V_{\mathrm{aff}})$ are characterized by the following properties:*

$$\begin{aligned} &IH_j^{\bar{m}}(S_x^{2n-2s+1}; \Gamma) \cong 0, && j \geq n-s+1 \\ &IH_j^{\bar{m}}(S_x^{2n-2s+1}; \Gamma) \cong H_j(S_x^{2n-2s+1} - K_x; \Gamma), && j \leq n-s. \end{aligned}$$

(here $\Gamma$ denotes the local coefficient system on the link complement, with stalk $\Gamma$ and action of an element $\alpha$ in the fundamental group of the complement given by multiplication by $t^{\mathrm{lk}(\alpha,K)}$; this is the same as the induced local system from $\mathcal{L}_H$.)

**Note.** The same property holds for link pairs of strata $S \in \mathcal{S}$ of a stratification of the pair $(\mathbb{CP}^{n+1}, V)$ since all of these are algebraic knots and have associated Milnor fibrations.

*Proof of the lemma.* We will prove the above claim by induction down on the dimension of singular strata of the pair $(\mathbb{C}^{n+1}, V_{\mathrm{aff}})$. To start the induction, note that the link pair of a component of the dense open subspace of $V_{\mathrm{aff}}$ (i.e., for $s = n$) is a circle $(S^1, \emptyset)$, that maps to $t$ under $\mathcal{L}_H$. Moreover, the (intersection) homology groups $IH_i^{\bar{m}}(S^1; \Gamma) \cong H_i(S^1; \Gamma)$ are zero, except for $i = 0$, hence the claim is trivially satisfied in this case.

Let $S$ be an $s$-dimensional stratum of $V_{\mathrm{aff}}$ and let $x$ be a point in $S$. Its link pair $(S_x^{2n-2s+1}, K_x)$ in $(\mathbb{C}^{n+1}, V_{\mathrm{aff}})$ is a singular algebraic knot, with a topological stratification induced by that of $(\mathbb{C}^{n+1}, V_{\mathrm{aff}})$. The link pairs of strata of $(S_x^{2n-2s+1}, K_x)$ are also link pairs of higher dimensional strata of $(\mathbb{C}^{n+1}, V_{\mathrm{aff}})$ (see for example [11]). Therefore, by the induction hypothesis, the claim holds for such link pairs.

Let $\mathcal{IC} = IC_{\bar{m}}^{\bullet}(S_x^{2n-2s+1}, \Gamma)$ be the middle-perversity intersection cohomology complex associated to the link pair of $S$ at $x$. In order to prove the claim, it suffices to show that its restriction to $K$ is quasi-isomorphic to the zero complex, i.e., $\mathcal{IC}_{|K} \cong 0$. Then the lemma will follow from the long exact sequence of compactly supported hypercohomology and from the fact that the fiber $F_x$ of the Milnor fibration associated with the algebraic knot $(S_x^{2n-2s+1}, K_x)$ has the homotopy type of an $(n-s)$-dimensional complex ([33], Theorem 5.1) and is homotopy equivalent to the infinite cyclic covering $\widetilde{S_x^{2n-2s+1} - K_x}$ of the knot complement, defined by the kernel of the total linking number homomorphism. More precisely, we obtain the isomorphisms of $\Gamma$-modules:

$$\begin{aligned} IH_j^{\bar{m}}(S_x^{2n-2s+1}; \Gamma) &\cong H_j(S_x^{2n-2s+1} - K_x; \Gamma) \\ &\cong H_j(\widetilde{S_x^{2n-2s+1} - K_x}; \mathbb{Q}) \cong H_j(F_x; \mathbb{Q}). \end{aligned}$$

Let $K' \supset K''$ be two consecutive terms in the filtration of $(S_x^{2n-2s+1}, K_x)$. Say $\dim_{\mathbb{R}}(K') = 2n - 2r - 1$, $r \geq s$. The stalk of $\mathcal{IC}$ at a point $y \in K' - K''$ is given by the following formula:

$$\mathcal{H}^q(\mathcal{IC})_y \cong \begin{cases} IH^{\bar{m}}_{-q-(2n-2r)}(S_y^{2r-2s+1}; \Gamma), & q \leq -2n + s + r - 1, \\ 0, & q > -2n + s + r - 1, \end{cases}$$

where $(S_y^{2r-2s+1}, K_y)$ is the link pair in $(S_x^{2n-2s+1}, K_x)$ of the component of $K' - K''$ containing $y$. Since $(S_y^{2r-2s+1}, K_y)$ is also the link pair of a higher dimensional stratum of $(\mathbb{C}^{n+1}, V_{\mathrm{aff}})$, the induction hypothesis yields $IH^{\bar{m}}_{-q-(2n-2r)}(S_y^{2r-2s+1}; \Gamma) \cong 0$ if $q \leq -2n + s + r - 1$. $\square$

**Remark 3.3.** The proof of Step 1 of the previous lemma provides a way of computing the modules $IH_i^{\bar{m}}(G; \Gamma)$, $i \leq k$, for $G \cong S^{2k+1} \cong S^1 * G_1$ the link of an $(n-k)$-dimensional stratum $S \cap H$, $S \in \mathcal{S}$:

$$IH_k^{\bar{m}}(G; \Gamma) \cong {}^{c}IH^{\bar{m}}_{k-1}(c^{\circ}G_1; \mathcal{L}_1) * IH_0^{\bar{m}}(S^1; \mathcal{L}_2) \cong IH^{\bar{m}}_{k-1}(G_1; \mathcal{L}_1) * IH_0^{\bar{m}}(S^1; \mathcal{L}_2)$$

and, for $i < k$,

$$IH_i^{\bar{m}}(G;\Gamma) \cong \{IH_i^{\bar{m}}(G_1;\mathcal{L}_1) \otimes IH_0^{\bar{m}}(S^1;\mathcal{L}_2)\} \oplus \{IH_{i-1}^{\bar{m}}(G_1;\mathcal{L}_1) * IH_0^{\bar{m}}(S^1;\mathcal{L}_2)\}.$$

The above formulas, as well as the claim of the first step of the previous lemma, can also be obtained from the formula for the intersection homology of a join ([17], Proposition 3), applied to $G \cong G_1 * S^1$.

If we denote by $I\gamma_i^{\bar{m}}(G) :=$ order $IH_i^{\bar{m}}(G;\Gamma)$ the intersection Alexander polynomial of the link pair $(G, F)$ (cf. [4], by noting that $F \subset G$ is of finite type), then we obtain

$$I\gamma_k^{\bar{m}}(G) = \gcd(I\gamma_{k-1}^{\bar{m}}(G_1), t^d - 1),$$

$$I\gamma_i^{\bar{m}}(G) = \gcd(I\gamma_i^{\bar{m}}(G_1), t^d - 1) \times \gcd(I\gamma_{i-1}^{\bar{m}}(G_1), t^d - 1), \quad i < k\,.$$

In particular, since $I\gamma_0^{\bar{m}}(G_1) \sim t-1$ ([11], Corollary 5.3), we have $I\gamma_0^{\bar{m}}(G) \sim t-1$, where $\sim$ stands for equality up to multiplication by a unit of $\Gamma$.

Note that the superduality isomorphism ([4], Corollary 3.4) yields the isomorphism $IH_j^{\bar{l}}(G;\Gamma) \cong IH_{2k-j}^{\bar{m}}(G;\Gamma)^{\mathrm{op}}$. Hence $IH_j^{\bar{l}}(G;\Gamma) \cong 0$ if $j < k$.

From the above considerations, the zeros of the polynomials $I\gamma_i^{\bar{m}}(G)$ and $I\gamma_i^{\bar{l}}(G)$ (in the non-trivial range) are all roots of unity of order $d$.

**Corollary 3.4.** *If $V$ is an $n$-dimensional reduced projective hypersurface, transversal to the hyperplane at infinity, then the intersection Alexander modules of $V$ are isomorphic to the classical Alexander modules of the hypersurface complement, i.e.,*

$$IH_*^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H) \cong H_*(\mathbb{CP}^{n+1} - V \cup H;\mathcal{L}_H) \cong H_*(\mathcal{U}^c;\mathbb{Q}).$$

*Proof.* The previous lemma and the hypercohomology spectral sequence yield

$$\mathbb{H}^i(V \cup H; IC_{\bar{m}}^{\bullet}) \cong 0.$$

Let $u$ and $v$ be the inclusions of $\mathbb{CP}^{n+1} - (V \cup H)$ and respectively $V \cup H$ into $\mathbb{CP}^{n+1}$. The distinguished triangle $u_!u^* \to \mathrm{id} \to v_*v^* \xrightarrow{[1]}$, upon applying the hypercohomology functor, yields the following long exact sequence:

$$\begin{aligned}\cdots \to \mathbb{H}_c^{-i-1}(V \cup H; IC_{\bar{m}}^{\bullet}) &\to \mathbb{H}_c^{-i}(\mathbb{CP}^{n+1} - (V \cup H); IC_{\bar{m}}^{\bullet}) \\ &\to \mathbb{H}_c^{-i}(\mathbb{CP}^{n+1}; IC_{\bar{m}}^{\bullet}) \to \mathbb{H}_c^{-i}(V \cup H; IC_{\bar{m}}^{\bullet}) \to \cdots\,.\end{aligned}$$

Therefore, we obtain the isomorphisms

$$\begin{aligned}IH_i^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H) &:= \mathbb{H}^{-i}(\mathbb{CP}^{n+1}; IC_{\bar{m}}^{\bullet}) \\ &\cong \mathbb{H}_c^{-i}(\mathbb{CP}^{n+1} - (V \cup H); IC_{\bar{m}}^{\bullet}) \\ &= {}^c IH_i^{\bar{m}}(\mathbb{CP}^{n+1} - (V \cup H);\mathcal{L}_H) \\ &\cong H_i(\mathbb{CP}^{n+1} - (V \cup H);\mathcal{L}_H) \\ &\cong H_i(\mathcal{U}^c;\mathbb{Q}).\end{aligned}$$

□

Our next goal is to show that, in our settings, the Alexander modules of the hypersurface complement, $H_i(\mathcal{U}^c;\mathbb{Q})$, are torsion $\Gamma$-modules if $i \leq n$. Based on the above corollary, it suffices to show this for the modules $IH_i^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H)$, $i \leq n$.

We will need the following result.

**Lemma 3.5.**
$$IH_i^{\bar{l}}(\mathbb{CP}^{n+1};\mathcal{L}_H) \cong 0 \quad \textit{for } i \leq n.$$

*Proof.* Let $u$ and $v$ be the inclusions of $\mathbb{CP}^{n+1} - V \cup H$ and respectively $V \cup H$ into $\mathbb{CP}^{n+1}$. Since $v^* IC^\bullet_{\bar{m}} \cong 0$, by superduality we obtain $0 \cong v^* \mathcal{D} IC^\bullet_{\bar{l}}[2n+2]^{\mathrm{op}} \cong \mathcal{D} v^! IC^\bullet_{\bar{l}}[2n+2]^{\mathrm{op}}$, so $v^! IC^\bullet_{\bar{l}} \cong 0$. Hence the distinguished triangle
$$v_* v^! IC^\bullet_{\bar{l}} \to IC^\bullet_{\bar{l}} \to u_* u^* IC^\bullet_{\bar{l}} \xrightarrow{[1]}$$
upon applying the hypercohomology functor, yields the isomorphism
$$IH_i^{\bar{l}}(\mathbb{CP}^{n+1};\mathcal{L}_H) \cong IH_i^{\bar{l}}(\mathbb{CP}^{n+1} - V \cup H;\mathcal{L}_H) \cong H_i^{BM}(\mathbb{CP}^{n+1} - V \cup H;\mathcal{L}_H),$$
where $H_*^{BM}$ denotes the Borel–Moore homology. By Artin's vanishing theorem ([38], Example 6.0.6), the latter module is 0 for $i < n+1$, since $\mathbb{CP}^{n+1} - V \cup H$ is a Stein space of dimension $n+1$. □

Now we can prove the main theorem of this section.

**Theorem 3.6.** *Let $V \subset \mathbb{CP}^{n+1}$ be a reduced, $n$-dimensional projective hypersurface, transversal to the hyperplane at infinity. Then for any $i \leq n$, the module $IH_i^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H) \cong H_i(\mathcal{U}^c;\mathbb{Q})$ is a finitely generated torsion $\Gamma$-module.*

*Proof.* Recall that the *peripheral complex* $\mathcal{R}^\bullet$, associated to the finite local type embedding $V \cup H \subset \mathbb{CP}^{n+1}$, is defined by the distinguished triangle ([4])
$$IC^\bullet_{\bar{m}} \to IC^\bullet_{\bar{l}} \to \mathcal{R}^\bullet \xrightarrow{[1]} .$$
Moreover, $\mathcal{R}^\bullet$ is a self-dual (i.e., $\mathcal{R}^\bullet \cong \mathcal{D}\mathcal{R}^{\bullet \mathrm{op}}[2n+3]$), torsion sheaf on $\mathbb{CP}^{n+1}$ (i.e., the stalks of its cohomology sheaves are torsion modules).

By applying the hypercohomology functor to the above distinguished triangle, and using the vanishing of Lemma 3.5, it follows that the natural maps
$$\mathbb{H}^{-i-1}(\mathbb{CP}^{n+1};\mathcal{R}^\bullet) \to IH_i^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H)$$
are isomorphisms for all $i \leq n-1$, and epimorphism for $i = n$.

Now, since $\mathcal{R}^\bullet$ is a torsion sheaf (having finite dimensional rational vector spaces as stalks), the spectral sequence for hypercohomology implies that the groups $\mathbb{H}^q(\mathbb{CP}^{n+1};\mathcal{R}^\bullet)$, $q \in \mathbb{Z}$, are also finite dimensional rational vector spaces, thus torsion $\Gamma$-modules. This finishes the proof of the theorem. □

**Note.** It follows from the proof of the previous theorem that for $i \leq n$, $H_i(\mathcal{U}^c; \mathbb{Q})$ is actually a finite dimensional rational vector space, thus its order coincides with the characteristic polynomial of the $\mathbb{Q}$-linear map induced by a generator of the group of covering transformations (see [34]).

**Definition 3.7.** For $i \leq n$, we denote by $\delta_i(t)$ the polynomial associated to the torsion module $H_i(\mathcal{U}^c; \mathbb{Q})$, and call it the $i$-th global Alexander polynomial of the hypersurface $V$. These polynomials will be well-defined up to multiplication by $ct^k$, $c \in \mathbb{Q}$.

As a consequence of Theorem 3.6, we may calculate the rank of the free $\Gamma$-module $H_{n+1}(\mathcal{U}^c; \mathbb{Q})$ in terms of the Euler characteristic of the complement.

**Corollary 3.8.** *Let $V \subset \mathbb{CP}^{n+1}$ be a reduced, $n$-dimensional projective hypersurface, in general position at infinity. Then the $\Gamma$-rank of $H_{n+1}(\mathcal{U}^c; \mathbb{Q})$ is expressed in terms of the Euler characteristic $\chi(\mathcal{U})$ of the complement by the formula*

$$(-1)^{n+1}\chi(\mathcal{U}) = \operatorname{rank}_\Gamma H_{n+1}(\mathcal{U}^c; \mathbb{Q}).$$

*Proof.* The equality follows from the Theorem 3.6, from the fact that for $q > n+1$ the Alexander modules $H_q(\mathcal{U}^c; \mathbb{Q})$ vanish, and from the formula 2.10(v) of [9]:

$$\chi(\mathcal{U}) = \sum_q (-1)^q \operatorname{rank}_\Gamma H_q(\mathcal{U}^c; \mathbb{Q}). \qquad \square$$

## 4. The main theorems

We will now state and prove the main theorems of this paper. These results are generalizations of the ones obtained by Libgober ([25], [26], [27]) in the case of hypersurfaces with only isolated singularities, and will lead to results on the monodromy of the Milnor fiber of a projective hypersurface arrangement, similar to those obtained by Libgober ([30]), Dimca ([7], [5]) (see §5).

The first theorem provides a characterization of the zeros of global Alexander polynomials. For hypersurfaces with only isolated singularities, it specializes to Corollary 4.8 of [25]. It also gives a first obstruction on the prime divisors of the global Alexander polynomials of hypersurfaces.

**Theorem 4.1.** *If $V$ is an $n$-dimensional reduced projective hypersurface of degree $d$, transversal to the hyperplane at infinity, then for $i \leq n$, any root of the global Alexander polynomial $\delta_i(t)$ is a root of unity of order $d$.*

*Proof.* Let $k$ and $l$ be the inclusions of $\mathbb{C}^{n+1}$ and respectively $H$ into $\mathbb{CP}^{n+1}$. For a fixed perversity $\bar{p}$, we will denote the intersection complexes $IC^\bullet_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ by $IC^\bullet_{\bar{p}}$. We will also drop the letter $R$ when using right derived functors. The distinguished triangle $l_*l^! \to \mathrm{id} \to k_*k^* \xrightarrow{[1]}$, upon applying the hypercohomology functor, yields the following exact sequence:

$$\cdots \to \mathbb{H}^{-i}_H(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \to IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}_H) \\ \to \mathbb{H}^{-i}(\mathbb{C}^{n+1}; k^*IC^\bullet_{\bar{m}}) \to \mathbb{H}^{-i+1}_H(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \to \cdots.$$

Note that the complex $k^*IC^\bullet_{\bar{m}}[-n-1]$ is perverse with respect to the middle perversity (since $k$ is the open inclusion and the functor $k^*$ is t-exact; [1]). Therefore, by Artin's vanishing theorem for perverse sheaves ([38], Corollary 6.0.4), we obtain

$$\mathbb{H}^{-i}(\mathbb{C}^{n+1}; k^*IC^\bullet_{\bar{m}}) \cong 0 \quad \text{for } i < n+1.$$

Hence

$$\mathbb{H}^{-i}_H(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \cong IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}_H) \quad \text{for } i < n,$$

and $IH^{\bar{m}}_n(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ is a quotient of $\mathbb{H}^{-n}_H(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}})$.

The superduality isomorphism $IC^\bullet_{\bar{m}} \cong \mathcal{D}IC^{\bullet\mathrm{op}}_{\bar{l}}[2n+2]$, and the fact that the stalks over $H$ of the complex $IC^\bullet_{\bar{l}}$ are torsion $\Gamma$-modules (recall that $l^*IC^\bullet_{\bar{l}} \cong l^*\mathcal{R}^\bullet$, and $\mathcal{R}^\bullet$ is a torsion sheaf by [4]), yield the isomorphisms

$$\begin{aligned}
\mathbb{H}^{-i}_H(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) &= \mathbb{H}^{-i}(H; l^!IC^\bullet_{\bar{m}}) \\
&\cong \mathbb{H}^{-i+2n+2}(H; \mathcal{D}l^*IC^{\bullet\mathrm{op}}_{\bar{l}}) \\
&\cong \mathrm{Hom}(\mathbb{H}^{i-2n-2}(H; l^*IC^{\bullet\mathrm{op}}_{\bar{l}}); \Gamma) \\
&\qquad \oplus \mathrm{Ext}(\mathbb{H}^{i-2n-1}(H; l^*IC^{\bullet\mathrm{op}}_{\bar{l}}); \Gamma) \\
&\cong \mathrm{Ext}(\mathbb{H}^{i-2n-1}(H; l^*IC^{\bullet\mathrm{op}}_{\bar{l}}); \Gamma) \\
&\cong \mathbb{H}^{i-2n-1}(H; l^*IC^{\bullet\mathrm{op}}_{\bar{l}}) \\
&\cong \mathbb{H}^{i-2n-1}(H; l^*\mathcal{R}^{\bullet\mathrm{op}}).
\end{aligned}$$

Then, in order to finish the proof of the theorem, it suffices to study the order of the module $\mathbb{H}^{i-2n-1}(H; l^*\mathcal{R}^{\bullet\mathrm{op}})$, for $i \leq n$, and to show that the zeros of its associated polynomial are roots of unity of order $d$. This follows by using the hypercohomology spectral sequence, since the stalks of $\mathcal{R}^{\bullet\mathrm{op}}$ at points of $H$ are torsion modules whose associated polynomials have the desired property: their zeros are roots of unity of order $d$ (see Remark 3.3 concerning the local intersection Alexander polynomials associated to link pairs of strata of $V \cap H$). □

**Note.** The above theorem is also a generalization of the following special case. If $V$ is the projective cone on a degree $d$ reduced hypersurface $Y = \{f = 0\} \subset \mathbb{CP}^n$, then there is a $\Gamma$-module isomorphism $H_i(\mathcal{U}^c; \mathbb{Q}) \cong H_i(F; \mathbb{Q})$, where $F = f^{-1}(1)$ is the fiber of the global Milnor fibration $\mathbb{C}^{n+1} - f^{-1}(0) \xrightarrow{f} \mathbb{C}^*$ associated to the homogeneous polynomial $f$, and the module structure on $H_i(F; \mathbb{Q})$ is induced by the monodromy action (see [6], p. 106–107). Therefore the zeros of the global Alexander polynomials of $V$ coincide with the eigenvalues of the monodromy operators acting on the homology of $F$. Since the monodromy homeomorphism has finite order $d$, all these eigenvalues are roots of unity of order $d$.

Next we show that the zeros of the global Alexander polynomials $\delta_i(t)$ $(i \leq n)$ are controlled by the local data, i.e., by the local Alexander polynomials of link pairs of singular strata in a stratification of the pair $(\mathbb{CP}^{n+1}, V)$. This is an extension to the case of non-isolated singularities of a result due to A. Libgober ([25], Theorem 4.3; [27], Theorem 4.1.a), which gives a similar fact for hypersurfaces with only isolated singularities.

**Theorem 4.2.** *Let $V$ be a reduced hypersurface in $\mathbb{CP}^{n+1}$, which is transversal to the hyperplane at infinity, $H$. Fix an arbitrary irreducible component of $V$, say $V_1$. Let $\mathcal{S}$ be a stratification of the pair $(\mathbb{CP}^{n+1}, V)$. Then for a fixed integer $1 \leq i \leq n$, the prime factors of the global Alexander polynomial $\delta_i(t)$ of $V$ are among the prime factors of local polynomials $\xi_l^s(t)$ associated to the local Alexander modules $H_l(S^{2n-2s+1} - K^{2n-2s-1}; \Gamma)$ of link pairs $(S^{2n-2s+1}, K^{2n-2s-1})$ of components of strata $S \in \mathcal{S}$ such that $S \subset V_1$, $n - i \leq s = \dim S \leq n$, and $l$ is in the range $2n - 2s - i \leq l \leq n - s$.*

**Note.** The 0-dimensional strata of $V$ may only contribute to $\delta_n(t)$, the 1-dimensional strata may only contribute to $\delta_n(t)$ and $\delta_{n-1}(t)$ and so on. This observation will play a key role in the proof of Proposition 5.1 of the next section.

*Proof.* We will use the Lefschetz hyperplane section theorem and induction down on $i$. The beginning of the induction is the characterization of the 'top' Alexander polynomial of $V$: *the prime divisors of $\delta_n(t)$ are among the prime factors of local polynomials $\xi_l^s(t)$ corresponding to strata $S \in \mathcal{S}$ with $S \subset V_1$, $0 \leq s = \dim S \leq n$, and $n - 2s \leq l \leq n - s$.* This follows from the following more general fact.

*Claim.* For any $1 \leq i \leq n$, the prime divisors of $\delta_i(t)$ are among the prime factors of the local polynomials $\xi_l^s(t)$ corresponding to strata $S \in \mathcal{S}$ such that $S \subset V_1$, $0 \leq s = \dim S \leq n$, and $i - 2s \leq l \leq n - s$.

*Proof of Claim.* Since $V_1$ is an irreducible component of $V$, it acquires the induced stratification from that of $V$. By the transversality assumption, the stratification $\mathcal{S}$ of the pair $(\mathbb{CP}^{n+1}, V)$ induces a stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$.

Let $j$ and $i$ be the inclusions of $\mathbb{CP}^{n+1} - V_1$ and respectively $V_1$ into $\mathbb{CP}^{n+1}$. For a fixed perversity $\bar{p}$ we will denote the intersection complexes $IC^\bullet_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ by $IC^\bullet_{\bar{p}}$. The distinguished triangle $i_*i^! \to \mathrm{id} \to j_*j^* \xrightarrow{[1]}$, upon applying the hypercohomology functor, yields the following long exact sequence:

$$\begin{aligned}\cdots \to \mathbb{H}^{-i}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \to IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}_H)&\\ \to \mathbb{H}^{-i}(\mathbb{CP}^{n+1} - V_1; IC^\bullet_{\bar{m}}) \to \mathbb{H}^{-i+1}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) &\to \cdots .\end{aligned}$$

Note that the complex $j^*IC^\bullet_{\bar{m}}[-n-1]$ on $\mathbb{CP}^{n+1} - V_1$ is perverse with respect to the middle perversity (since $j$ is the open inclusion and the functor $j^*$ is t-exact; [1]). Therefore, by Artin's vanishing theorem for perverse sheaves ([38], Corollary 6.0.4) and noting that $\mathbb{CP}^{n+1} - V_1$ is affine ([6], (1.6.7)), we obtain

$$\mathbb{H}^{-i}(\mathbb{CP}^{n+1} - V_1; j^*IC^\bullet_{\bar{m}}) \cong 0 \quad \text{for } i < n+1.$$

Therefore

$$\mathbb{H}^{-i}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \cong IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}_H) \quad \text{for } i < n,$$

and $IH^{\bar{m}}_n(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ is a quotient of $\mathbb{H}^{-n}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}})$.

Now using the superduality isomorphism $IC^\bullet_{\bar{m}} \cong \mathcal{D}IC^{\bullet\mathrm{op}}_{\bar{l}}[2n+2]$ and the fact that the stalks over $V_1$ of the complex $IC^{\bullet\mathrm{op}}_{\bar{l}}$ are torsion $\Gamma$-modules, and $IC^\bullet_{\bar{m}}|_{V_1} \cong 0$, we have the isomorphisms

$$\begin{aligned}\mathbb{H}^{-i}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) &= \mathbb{H}^{-i}(V_1; i^!IC^\bullet_{\bar{m}})\\ &\cong \mathbb{H}^{-i+2n+2}(V_1; \mathcal{D}i^*IC^{\bullet\mathrm{op}}_{\bar{l}})\\ &\cong \mathrm{Hom}(\mathbb{H}^{i-2n-2}(V_1; i^*IC^{\bullet\mathrm{op}}_{\bar{l}}); \Gamma)\\ &\qquad \oplus \mathrm{Ext}(\mathbb{H}^{i-2n-1}(V_1; i^*IC^{\bullet\mathrm{op}}_{\bar{l}}); \Gamma)\\ &\cong \mathrm{Ext}(\mathbb{H}^{i-2n-1}(V_1; i^*IC^{\bullet\mathrm{op}}_{\bar{l}}); \Gamma)\\ &\cong \mathbb{H}^{i-2n-1}(V_1; i^*IC^{\bullet\mathrm{op}}_{\bar{l}})\\ &\cong \mathbb{H}^{i-2n-1}(V_1; i^*\mathcal{R}^{\bullet\mathrm{op}}).\end{aligned}$$

Therefore it suffices to study the order of the module $\mathbb{H}^{i-2n-1}(V_1; i^*\mathcal{R}^{\bullet\mathrm{op}})$, for fixed $i \leq n$.

By the compactly supported hypercohomology long exact sequence and induction on the strata of $V_1$, the polynomial associated to $\mathbb{H}^{i-2n-1}(V_1; i^*\mathcal{R}^{\bullet\mathrm{op}})$ will divide the product of the polynomials associated with all the modules $\mathbb{H}^{i-2n-1}_c(\mathcal{V}; \mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})$, where $\mathcal{V}$ runs over the strata of $V_1$ in the stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$, i.e., $\mathcal{V}$ is of the form $S \cap H$ or $S - S \cap H$, for $S \in \mathcal{S}$ and $S \subset V_1$.

Next, we will need the following lemma.

**Lemma 4.3.** *Let $\mathcal{V}$ be a $j$-(complex) dimensional stratum of $V_1$ (or $V$) in the stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$. Then the prime factors of the polynomial associated to $\mathbb{H}_c^{i-2n-1}(\mathcal{V}; \mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})$ must divide one of the polynomials $\xi_l^j(t) = \mathrm{order}\{IH_l^{\bar{m}}(S^{2n-2j+1}; \Gamma)\}$, in the range $0 \leq l \leq n-j$ and $0 \leq i-l \leq 2j$, where $(S^{2n-2j+1}, K^{2n-2j-1})$ is the link pair of $\mathcal{V}$ in $(\mathbb{CP}^{n+1}, V \cup H)$.*

Once the lemma is proved, the *Claim* (and thus the beginning of the induction) follows from Remark 3.3 which describes the polynomials of link pairs of strata $S \cap H$ of $V \cap H$ in $(\mathbb{CP}^{n+1}, V \cup H)$ in terms of the polynomials of link pairs of strata $S \in \mathcal{S}$ of $V$ in $(\mathbb{CP}^{n+1}, V)$, and Lemma 3.2 which relates the local intersection Alexander polynomials of links of strata $S \in \mathcal{S}$ to the classical local Alexander polynomials.

In order to finish the proof of the theorem we use the Lefschetz hyperplane theorem and induction down on $i$. We denote the Alexander polynomials of $V$ by $\delta_i^V(t)$ and call $\delta_n^V(t)$ the 'top' Alexander polynomial of $V$.

Let $1 \leq i = n-k < n$ be fixed. Consider $L \cong \mathbb{CP}^{n-k+1}$ a generic codimension $k$ linear subspace of $\mathbb{CP}^{n+1}$, so that $L$ is transversal to $V \cup H$. Then $W = L \cap V$ is a $(n-k)$-dimensional, degree $d$, reduced hypersurface in $L$, which is transversal to the hyperplane at infinity $H \cap L$ of $L$. Moreover, by the transversality assumption, the pair $(L, W)$ has a Whitney stratification induced from that of the pair $(\mathbb{CP}^{n+1}, V)$, with strata of the form $\mathcal{V} = S \cap L$, for $S \in \mathcal{S}$. The local coefficient system $\mathcal{L}_H$ defined on $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$ restricts to a coefficient system on $\mathcal{U} \cap L$ defined by the same representation (here we already use the Lefschetz theorem).

By applying the Lefchetz hyperplane section theorem ([6], (1.6.5)) to $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$ and its section by $L$, we obtain the isomorphisms

$$\pi_i(\mathcal{U} \cap L) \xrightarrow{\cong} \pi_i(\mathcal{U}) \quad \text{for } i \leq n-k,$$

and a surjection for $i = n-k+1$. Therefore the homotopy type of $\mathcal{U}$ is obtained from that of $\mathcal{U} \cap L$ by adding cells of dimension $> n-k+1$. Hence the same is true for the infinite cyclic covers $\mathcal{U}^c$ and $(\mathcal{U} \cap L)^c$ of $\mathcal{U}$ and $\mathcal{U} \cap L$ respectively. Therefore,

$$H_i((\mathcal{U} \cap L)^c; \mathbb{Q}) \xrightarrow{\cong} H_i(\mathcal{U}^c; \mathbb{Q}) \quad \text{for } i \leq n-k.$$

Since the maps above are induced by embeddings, these maps are isomorphisms of $\Gamma$-modules. We conclude that $\delta_{n-k}^W(t) = \delta_{n-k}^V(t)$.

Next, note that $\delta_{n-k}^W(t)$ is the 'top' Alexander polynomial of $W$ as a hypersurface in $L \cong \mathbb{CP}^{n-k+1}$, therefore by the induction hypothesis, the prime factors of $\delta_{n-k}(t)$ are restricted to those of the local Alexander polynomials $\xi_l^r(t)$ associated to link pairs of strata $\mathcal{V} = S \cap L \subset W_1 = V_1 \cap L$, with $0 \leq r = \dim(\mathcal{V}) \leq n-k$ and $(n-k) - 2r \leq l \leq (n-k) - r$. Now, using the fact that the link pair of a stratum $\mathcal{V} = S \cap L$ in $(L, W)$ is the same as the link pair of $S$ in $(\mathbb{CP}^{n+1}, V)$, the conclusion follows by reindexing (replace $r$ by $s-k$, where $s = \dim(S)$). □

**Note.** The Lefschetz argument in the above proof may be replaced by a similar argument for intersection homology modules. More precisely, the Lefschetz hyperplane theorem for intersection homology ([13] or [38], Example 6.0.4(3)) yields the following isomorphisms of $\Gamma$-modules:

$$IH_i^{\bar{m}}(L, \mathcal{L}_H|_L) \xrightarrow{\cong} IH_i^{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H), \quad \text{for } i \leq n-k.$$

On the other hand, by Corollary 3.4, there are isomorphisms of $\Gamma$-modules:

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_i(\mathcal{U}^c; \mathbb{Q}) \quad \text{and} \quad IH_i^{\bar{m}}(L, \mathcal{L}_H|_L) \cong H_i((\mathcal{U} \cap L)^c; \mathbb{Q}).$$

*Proof of Lemma* 4.3. For simplicity, we let $r = i-2n-1$. The module $\mathbb{H}_c^r(\mathcal{V}; \mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})$ is the abutment of a spectral sequence with $E_2$ term given by

$$E_2^{p,q} = H_c^p(\mathcal{V}; \mathcal{H}^q(\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})).$$

Since $\mathcal{R}^{\bullet\mathrm{op}}$ is a constructible complex, $\mathcal{H}^q(\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})$ is a local coefficient system on $\mathcal{V}$. Therefore, by the orientability of $\mathcal{V}$ and the Poincaré duality isomorphism ([3], V.9.3), $E_2^{p,q}$ is isomorphic to the module $H_{2j-p}(\mathcal{V}; \mathcal{H}^q(\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}}))$. As in Lemma 9.2 of [11], we can show that the latter is a finitely generated module. More precisely, by deformation retracting $\mathcal{V}$ to a closed, hence finite, subcomplex of $V_1$ (or $\mathbb{CP}^{n+1}$), we can use simplicial homology with local coefficients to calculate the above $E_2$ terms.

We will keep the cohomological indexing in the study of the above spectral sequence (see, for example, [11]). By the above considerations, we may assume that $\mathcal{V}$ is a finite simplicial complex.

$E_2^{p,q}$ is the $p$-th homology of a cochain complex $C_c^*(\mathcal{V}; \mathcal{H}^q(\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}}))$ whose $p$-th cochain group is a subgroup of $C^p(\mathcal{V}; \mathcal{H}^q(\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}}))$, which in turn is the direct sum of modules of the form $\mathcal{H}^q(\mathcal{R}^{\bullet\mathrm{op}})_{x(\sigma)}$, where $x(\sigma)$ is the barycenter of a $p$-simplex $\sigma$ of $\mathcal{V} \subset V_1$. By the stalk calculation ([2], V.3.15), and by using $IC_{\bar{m}}^{\bullet}|_{V_1} \cong 0$ and the superduality isomorphism for link pairs ([4], Corollary 3.4), we obtain

$$\mathcal{H}^q(\mathcal{R}^{\bullet\mathrm{op}})_{x(\sigma)} \cong \mathcal{H}^q(IC_{\bar{l}}^{\bullet\mathrm{op}})_{x(\sigma)} \cong \begin{cases} 0, & q > -n-j-1, \\ IH_{2n+1+q}^{\bar{m}}(L_{x(\sigma)}; \Gamma), & q \leq -n-j-1 \end{cases}$$

(where $L_{x(\sigma)} \cong S^{2n-2j+1}$ is the link of $\mathcal{V}$ in $(\mathbb{CP}^{n+1}, V \cup H)$). Given that $E_2^{p,q}$ is a quotient of $C_c^p(\mathcal{V}; \mathcal{H}^q(\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}}))$, we see that $E_2^{p,q}$ is a torsion module, and a prime element $\gamma \in \Gamma$ divides the order of $E_2^{p,q}$ only if it divides the order of one of the torsion modules $IH_{2n+1+q}^{\bar{m}}(L_{x(\sigma)}; \Gamma)$. Denote by $\xi_{2n+1+q}^j(t)$ the order of the latter module, where $j$ stands for the dimension of the stratum.

Each $E_r^{p,q}$ is a quotient of a submodule of $E_{r-1}^{p,q}$, so by induction on $r$, each of them is a torsion $\Gamma$-module whose associated polynomial has the same property as that of $E_2$. Since the spectral sequence converges in finitely many steps, the same property is satisfied by $E_\infty$.

By spectral sequence theory,

$$E_\infty^{p,q} \cong F^p\mathbb{H}_c^{p+q}(\mathcal{V};\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})/F^{p+1}\mathbb{H}_c^{p+q}(\mathcal{V};\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}}),$$

where the modules $F^p\mathbb{H}_c^{p+q}(\mathcal{V};\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})$ form a descending bounded filtration of $\mathbb{H}_c^{p+q}(\mathcal{V};\mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})$.

Now set $A^* = \mathbb{H}_c^*(\mathcal{V};\mathcal{R}^{\bullet\mathrm{op}})$ as a graded module which is filtered by $F^pA^*$ and set $E_0^p(A^*) = F^pA^*/F^{p+1}A^*$. Then, for some $N$, we have

$$0 \subset F^NA^* \subset F^{N-1}A^* \subset \cdots \subset F^1A^* \subset F^0A^* \subset F^{-1}A^* = A^*.$$

This yields the following series of short exact sequences:

$$\begin{array}{l}
0 \longrightarrow F^NA^* \xrightarrow{\cong} E_0^N(A^*) \longrightarrow 0\\
0 \longrightarrow F^NA^* \longrightarrow F^{N-1}A^* \longrightarrow E_0^{N-1}(A^*) \longrightarrow 0\\
\qquad\qquad\qquad\vdots\\
0 \longrightarrow F^kA^* \longrightarrow F^{k-1}A^* \longrightarrow E_0^{k-1}(A^*) \longrightarrow 0\\
\qquad\qquad\qquad\vdots\\
0 \longrightarrow F^1A^* \longrightarrow F^0A^* \longrightarrow E_0^0(A^*) \longrightarrow 0\\
0 \longrightarrow F^0A^* \longrightarrow A^* \longrightarrow E_0^{-1}(A^*) \longrightarrow 0.
\end{array}$$

Let us see what happens at the $r$th grade of these graded modules. For clarity, we will indicate the grade with a superscript following the argument. For any $p$,

$$\begin{aligned}
E_0^p(A^*)^r &= (F^pA^*/F^{p+1}A^*)^r\\
&= F^pA^r/F^{p+1}A^r\\
&= F^pA^{p+r-p}/F^{p+1}A^{p+r-p}\\
&= E_\infty^{p,r-p}.
\end{aligned}$$

We know that each of the prime factors of the polynomial of this module must be a prime factor of some $\xi^j_{2n+1+(r-p)}(t)$. Further, by dimension considerations and stalk calculation, we know that $E_\infty^{p,r-p}$ can be non-trivial only if $0 \le p \le 2j$ and $-2n-1 \le r-p \le -n-j-1$. Hence, as $p$ varies, the only prime factors under consideration are those of $\xi^j_{2n+1+(r-p)}(t)$ in this range, i.e., they are the only possible prime factors of the $E_0^p(A^*)^r$, collectively in $p$ (but within the grade $r$).

By induction down the above list of short exact sequences, we conclude that $F^NA^r = E_0^N(A^*)^r$, and subsequently $F^{N-1}A^r, F^{N-2}A^r, \dots, F^0A^r$, and $A^r$, have the property of being torsion modules whose polynomials are products of polynomials whose prime factors are all factors of one of the $\xi^j_{2n+1+a}(t)$, where $a$ must be chosen in the range $0 \leq r-a \leq 2j$ and $-2n-1 \leq a \leq -n-j-1$. Since $\mathbb{H}^r_c(\mathcal{V}; \mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})$ is the submodule of $A^*$ corresponding to the $r$th grade, it too has this property. Using the fact that $r = i - 2n - 1$ and reindexing, we conclude that the prime factors of the polynomial of $\mathbb{H}^{i-2n-1}_c(\mathcal{V}; \mathcal{R}^{\bullet\mathrm{op}}|_{\mathcal{V}})$ must divide one of the polynomials $\xi^j_l(t) = \mathrm{order}\{IH^{\bar{m}}_l(S^{2n-2j+1}; \Gamma)\}$, in the range $0 \leq l \leq n-j$ and $0 \leq i-l \leq 2j$, where $S^{2n-2j+1}$ is the link of (a component of) $\mathcal{V}$. □

**Remark 4.4** (*Isolated singularities*). In the case of hypersurfaces with only isolated singularities, Theorem 4.2 can be strengthen as follows.

Assume that $V$ is an $n$-dimensional reduced projective hypersurface, transversal to the hyperplane at infinity, and having only isolated singularities. If $n \geq 2$ this assumption implies that $V$ is irreducible. If $n = 1$, we fix an irreducible component, say $V_1$. The only interesting global Alexander module is $IH^{\bar{m}}_n(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_n(\mathcal{U}^c; \mathbb{Q})$. As in the proof of the Theorem 4.2, the latter is a quotient of the torsion module $\mathbb{H}^{-n-1}(V_1; \mathcal{R}^{\bullet\mathrm{op}})$. Let $\Sigma_0 = \mathrm{Sing(V)} \cap V_1$ be the set of isolated singular points of $V$ which are contained in $V_1$. Note that $V_1$ has the stratification

$$V_1 \supset (V_1 \cap H) \cup \Sigma_0 \supset \Sigma_0$$

induced from that of $(\mathbb{CP}^{n+1}, V \cup H)$. The long exact sequence of the compactly supported hypercohomology yields

$$\to \mathbb{H}^{-n-1}_c(V_1 - \Sigma_0; \mathcal{R}^{\bullet\mathrm{op}}) \to \mathbb{H}^{-n-1}(V_1; \mathcal{R}^{\bullet\mathrm{op}}) \to \mathbb{H}^{-n-1}(\Sigma_0; \mathcal{R}^{\bullet\mathrm{op}}) \to,$$

and by the local calculation on stalks we obtain

$$\begin{aligned}
\mathbb{H}^{-n-1}(\Sigma_0; \mathcal{R}^{\bullet\mathrm{op}}) &\cong \oplus_{p\in\Sigma_0} \mathcal{H}^{-n-1}(\mathcal{R}^{\bullet})^{\mathrm{op}}_p \\
&\cong \oplus_{p\in\Sigma_0} IH^{\bar{l}}_n(S^{2n+1}_p; \Gamma)^{\mathrm{op}} \\
&\overset{(1)}{\cong} \oplus_{p\in\Sigma_0} IH^{\bar{m}}_n(S^{2n+1}_p; \Gamma) \\
&\cong \oplus_{p\in\Sigma_0} H_n(S^{2n+1}_p - S^{2n+1}_p \cap V; \Gamma),
\end{aligned}$$

where $(S^{2n+1}_p, S^{2n+1}_p \cap V)$ is the (smooth) link pair of the singular point $p \in \Sigma_0$, and $H_n(S^{2n+1}_p - S^{2n+1}_p \cap V; \Gamma)$ is the local Alexander module of the algebraic link. (1) follows from the superduality isomorphism for intersection Alexander polynomials of links ([4], Corollary 3.4; [11], Theorem 5.1).

By Remark 3.3, Lemma 4.3 and the long exact sequences of compactly supported hypercohomology, it can be shown that the modules $\mathbb{H}^{-n-1}_c(V_1 - \Sigma_0; \mathcal{R}^{\bullet\mathrm{op}})$ and $\mathbb{H}^{-n}_c(V_1 - \Sigma_0; \mathcal{R}^{\bullet\mathrm{op}})$ are annihilated by powers of $t-1$.

Thus we obtain the following divisibility theorem (compare [25], Theorem 4.3; [27], Theorem 4.1(1); [7], Corollary 6.4.16).

**Theorem 4.5.** *Let $V$ be a projective hypersurface in $\mathbb{CP}^{n+1}$ ($n \geq 1$), which is transversal to the hyperplane at infinity, $H$, and has only isolated singularities. Fix an irreducible component of $V$, say $V_1$, and let $\Sigma_0 = V_1 \cap \mathrm{Sing}(V)$. Then $H_n(\mathcal{U}^c; \mathbb{Q})$ is a torsion $\Gamma$-module, whose associated polynomial $\delta_n(t)$ divides the product $\prod_{p \in \Sigma_0} \Delta_p(t) \cdot (t-1)^r$ of the local Alexander polynomials of link pairs of the singular points $p$ of $V$ which are contained in $V_1$.*

An immediate consequence of the previous theorems is the triviality of the global polynomials $\delta_i(t)$, $1 \leq i \leq n$, if none of the roots of the local Alexander polynomials along some irreducible component of $V$ is a root of unity of order $d$.

**Example 4.6.** Suppose that $V$ is a degree $d$ reduced projective hypersurface which is also a rational homology manifold, has no codimension 1 singularities, and is transversal to the hyperplane at infinity. Assume that the local monodromies of link pairs of strata contained in some irreducible component $V_1$ of $V$ have orders which are relatively prime to $d$ (e.g., the transversal singularities along strata of $V_1$ are Brieskorn-type singularities, having all exponents relatively prime to $d$). Then, by Theorem 4.1, Theorem 4.2 and Proposition 2.1, it follows that $\delta_i(t) \sim 1$, for $1 \leq i \leq n$.

Further obstructions on the global Alexander modules/polynomials are provided by the relation with the 'modules/polynomials at infinity'. The following is an extension of Theorem 4.5 of [25] or, in the case $n = 1$, of Theorem 4.1(2) of [27].

**Theorem 4.7.** *Let $V$ be a degree $d$ reduced hypersurface in $\mathbb{CP}^{n+1}$, which is transversal to the hyperplane at infinity, $H$. Let $S_\infty$ be a sphere of sufficiently large radius in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$ (or equivalently, the boundary of a sufficiently small tubular neighborhood of $H$ in $\mathbb{CP}^{n+1}$). Then for all $i < n$,*

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong \mathbb{H}^{-i-1}(S_\infty; IC^\bullet_{\bar{m}}) \cong H_i(\mathcal{U}^c_\infty; \mathbb{Q})$$

*and $IH_n^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ is a quotient of $\mathbb{H}^{-n-1}(S_\infty; IC^\bullet_{\bar{m}}) \cong H_n(\mathcal{U}^c_\infty; \mathbb{Q})$, where $\mathcal{U}^c_\infty$ is the infinite cyclic cover of $S_\infty - (V \cap S_\infty)$ corresponding to the linking number with $V \cap S_\infty$ (cf. [25]).*

*Proof.* Choose coordinates $(z_0 : \cdots : z_{n+1})$ in the projective space such that $H = \{z_{n+1} = 0\}$ and $\mathbb{O} = (0 : \cdots : 0 : 1)$ is the origin in $\mathbb{CP}^{n+1} - H$. Define

$$\alpha : \mathbb{CP}^{n+1} \to \mathbb{R}_+, \quad \alpha := \frac{|z_{n+1}|^2}{\sum_{i=0}^{n+1} |z_i|^2}.$$

Note that $\alpha$ is well-defined, it is real analytic and proper,

$$0 \leq \alpha \leq 1, \quad \alpha^{-1}(0) = H \quad \text{and} \quad \alpha^{-1}(1) = \mathbb{O}.$$

Since $\alpha$ has only finitely many critical values, there is $\varepsilon$ sufficiently small such that the interval $(0, \varepsilon]$ contains no critical values. Set $U_\varepsilon = \alpha^{-1}([0, \varepsilon))$, a tubular neighborhood of $H$ in $\mathbb{CP}^{n+1}$ and note that $\mathbb{CP}^{n+1} - U_\varepsilon$ is a closed large ball of radius $R = \frac{1-\varepsilon}{\varepsilon} \xrightarrow{\varepsilon \to 0} \infty$ in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$.

Lemma 8.4.7(a) of [20] applied to $\alpha$ and $IC^{\bullet}_{\bar{m}}$, together with $IC^{\bullet}_{\bar{m}}|_H \cong 0$, yield

$$\mathbb{H}^*(U_\varepsilon; IC^{\bullet}_{\bar{m}}) \cong \mathbb{H}^*(H; IC^{\bullet}_{\bar{m}}) \cong 0,$$

and therefore, by the hypercohomology long exact sequence, we obtain the isomorphism

$$\mathbb{H}^*(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}}) \cong \mathbb{H}^*_{\mathbb{CP}^{n+1}-U_\varepsilon}(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}}).$$

Note that, for $i : \mathbb{CP}^{n+1} - U_\varepsilon \hookrightarrow \mathbb{CP}^{n+1}$ the inclusion,

$$\begin{aligned}\mathbb{H}^*_{\mathbb{CP}^{n+1}-U_\varepsilon}(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}}) &= \mathbb{H}^*(\mathbb{CP}^{n+1} - U_\varepsilon; i^! IC^{\bullet}_{\bar{m}}) \cong \mathbb{H}^*(\mathbb{CP}^{n+1}; i_* i^! IC^{\bullet}_{\bar{m}})\\ &\cong \mathbb{H}^*(\mathbb{CP}^{n+1}; i_! i^! IC^{\bullet}_{\bar{m}}) \overset{\text{def}}{=} \mathbb{H}^*(\mathbb{CP}^{n+1}, U_\varepsilon; IC^{\bullet}_{\bar{m}})\\ &\cong \mathbb{H}^*(\mathbb{C}^{n+1}, U_\varepsilon - H; IC^{\bullet}_{\bar{m}}),\end{aligned}$$

where the last isomorphism is the excision of $H$ (see for example [31], §1; [6], Remark 2.4.2(ii)).

If $k$ is the open inclusion of the affine space in $\mathbb{CP}^{n+1}$, then $k^* IC^{\bullet}_{\bar{m}}[-n-1]$ is perverse with respect to the middle perversity (since $k$ is the open inclusion and the functor $k^*$ is t-exact). Therefore, by Artin's vanishing theorem for perverse sheaves ([38], Corollary 6.0.4), we obtain

$$\mathbb{H}^{-i}(\mathbb{C}^{n+1}; k^* IC^{\bullet}_{\bar{m}}) \cong 0 \quad \text{for } i < n+1.$$

The above vanishing and the long exact sequence of the pair $(\mathbb{C}^{n+1}, U_\varepsilon - H)$ yield the isomorphisms

$$\mathbb{H}^{-i}_{\mathbb{CP}^{n+1}-U_\varepsilon}(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}}) \cong \mathbb{H}^{-i-1}(U_\varepsilon - H; IC^{\bullet}_{\bar{m}}) \quad \text{if } i < n,$$

and

$$\mathbb{H}^{-n-1}(U_\varepsilon - H; IC^{\bullet}_{\bar{m}}) \to \mathbb{H}^{-n}_{\mathbb{CP}^{n+1}-U_\varepsilon}(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}}) \quad \text{is an epimorphism.}$$

Note that $U_\varepsilon - H = \alpha^{-1}((0, \varepsilon))$ and by Lemma 8.4.7(c) of [20] we obtain the isomorphism

$$\mathbb{H}^*(U_\varepsilon - H; IC^{\bullet}_{\bar{m}}) \cong \mathbb{H}^*(S_\infty; IC^{\bullet}_{\bar{m}}),$$

where $S_\infty = \alpha^{-1}(\varepsilon')$, $0 < \varepsilon' < \varepsilon$.

Next, using the fact that $IC^\bullet_{\bar{m}}|_{V\cup H} \cong 0$, we obtain a sequence of isomorphisms as follows: for $i \le n$,

$$\begin{aligned}
\mathbb{H}^{-i-1}(S_\infty; IC^\bullet_{\bar{m}}) &= \mathbb{H}_c^{-i-1}(S_\infty - (V \cap S_\infty); IC^\bullet_{\bar{m}}) \\
&= \mathbb{H}_c^{-i-1}(S_\infty - (V \cap S_\infty); \mathcal{L}|_{S_\infty - V}[2n+2]) \\
&= H_c^{-i+2n+1}(S_\infty - (V \cap S_\infty); \mathcal{L}) \\
&\overset{(1)}{\cong} H_i(S_\infty - (V \cap S_\infty); \mathcal{L}) \\
&\cong H_i(\mathcal{U}^c_\infty; \mathbb{Q}),
\end{aligned}$$

where $\mathcal{L}$ is given on $S_\infty - (V \cap S_\infty)$ by the linking number with $V \cap S_\infty$, (1) is the Poincaré duality isomorphism ([3], Theorem V.9.3), and $\mathcal{U}^c_\infty$ is the infinite cyclic cover of $S_\infty - (V \cap S_\infty)$ corresponding to the linking number with $V \cap S_\infty$ (cf. [25]). □

**Remark 4.8.** Subsequently, A. Libgober has found a simpler proof of Theorem 4.7, using a purely topological argument based on the Lefschetz theorem. As a corollary to Theorem 4.7 it follows readily (cf. [29]) that the Alexander modules of the hypersurface complement are semi-simple, thus generalizing Libgober's result for the case of hypersurfaces with only isolated singularities (see [25], Corollary 4.8). The details will be given below.

**Proposition 4.9.** *Let $V \subset \mathbb{CP}^{n+1}$ be a degree $d$ reduced hypersurface which is transversal to the hyperplane at infinity, $H$. Then for each $i \le n$, the Alexander module $H_i(\mathcal{U}^c; \mathbb{C})$ is a semi-simple $\mathbb{C}[t, t^{-1}]$-module, which is annihilated by $t^d - 1$.*

*Proof.* By Theorem 4.7, it suffices to prove this fact for the modules 'at infinity' $H_i(\mathcal{U}^c_\infty; \mathbb{C})$, $i \le n$.

Note that since $V$ is transversal to $H$, the space $S_\infty - (V \cap S_\infty)$ is a circle fibration over $H - V \cap H$ which is homotopy equivalent to the complement in $\mathbb{C}^{n+1}$ to the affine cone over the projective hypersurface $V \cap H$. Let $\{h = 0\}$ be the polynomial defining $V \cap H$ in $H$. Then the infinite cyclic cover $\mathcal{U}^c_\infty$ of $S_\infty - (V \cap S_\infty)$ is homotopy equivalent to the Milnor fiber $\{h = 1\}$ of the (homogeneous) hypersurface singularity at the origin defined by $h$ and, in particular, $H_i(\mathcal{U}^c_\infty; \mathbb{C})$ $(i \le n)$ is a torsion finitely generated $\mathbb{C}[t, t^{-1}]$-module. Since the monodromy on the Milnor fiber $\{h = 1\}$ has finite order $d$ (given by multiplication by roots of unity), it also follows that the modules at infinity are semi-simple torsion modules, annihilated by $t^d - 1$ (see [23]). □

**Note.** Proposition 4.9 supplies alternative proofs to Theorem 3.6 and Theorem 4.1.

## 5. On the Milnor fiber of a projective arrangement of hypersurfaces

In this section, we apply the preceding results to the case of a hypersurface $V \subset \mathbb{CP}^{n+1}$, which is the projective cone over a reduced hypersurface $Y \subset \mathbb{CP}^n$. As an application to Theorem 4.2, we obtain restrictions on the eigenvalues of the monodromy operators associated to the Milnor fiber of the hypersurface arrangement defined by $Y$ in $\mathbb{CP}^n$.

Let $f: \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree $d > 1$, and let $Y = \{f = 0\}$ be the projective hypersurface in $\mathbb{CP}^n$ defined by $f$. Assume that the polynomial $f$ is square-free and let $f = f_1 \dots f_s$ be the decomposition of $f$ as a product of irreducible factors. Then $Y_i = \{f_i = 0\}$ are precisely the irreducible components of the hypersurface $Y$, and we refer to this situation by saying that we have a *hypersurface arrangement* $\mathcal{A} = (Y_i)_{i=1,s}$ in $\mathbb{CP}^n$.

The *Milnor fiber of the arrangement* $\mathcal{A}$ is defined as the fiber $F = f^{-1}(1)$ of the global Milnor fibration $f: \mathcal{U} \to \mathbb{C}^*$ of the (homogeneous) polynomial $f$; here $\mathcal{U} := \mathbb{C}^{n+1} - f^{-1}(0)$ is the complement of the central arrangement $A = f^{-1}(0)$ in $\mathbb{C}^{n+1}$, the cone on $\mathcal{A}$. $F$ has as characteristic homeomorphism $h: F \to F$ the mapping given by $h(x) = \tau \cdot x$ with $\tau = exp(2\pi i/d)$. This formula shows that $h^d = \mathrm{id}$ and hence the induced morphisms $h_q: H_q(F) \to H_q(F)$ at the homology level are all diagonalizable over $\mathbb{C}$, with eigenvalues among the $d$-th roots of unity. Denote by $P_q(t)$ the characteristic polynomial of the monodromy operator $h_q$.

Note that the Milnor fiber $F$ is homotopy equivalent to the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$, corresponding to the homomorphism $\mathbb{Z}^s = H_1(\mathcal{U}) \to \mathbb{Z}$ sending a meridian generator about a component of $A$ to the positive generator of $\mathbb{Z}$. With this identification, the monodromy homeomorphism $h$ corresponds precisely to a generator of the group of covering transformations (see [6], p. 106–107).

It is easy to see that $V \subset \mathbb{CP}^{n+1}$, the projective cone on $Y$, is in general position at infinity, where we identify the hyperplane at infinity, $H$, with the projective space on which $Y$ is defined as a hypersurface. Denote the irreducible components of $V$ by $V_i$, $i = 1, \dots, s$, each of which is the projective cone over the corresponding component of $Y$. Theorem 4.2 when applied to $F \simeq \mathcal{U}^c$ and to the hypersurface $V$, provides obstructions on the eigenvalues of the monodromy operators associated to the Milnor fiber $F$. More precisely, we obtain the following result concerning the prime divisors of the polynomials $P_q(t)$, for $q \leq n-1$ (compare [30], Theorem 3.1).

**Proposition 5.1.** *Let $Y = (Y_i)_{i=1,s}$ be a hypersurface arrangement in $\mathbb{CP}^n$, and fix an arbitrary component, say $Y_1$. Let $F$ be the Milnor fibre of the arrangement. Fix a Whitney stratification of the pair $(\mathbb{CP}^n, Y)$ and denote by $\mathcal{Y}$ the set of (open) singular strata. Then for $q \leq n-1$, a prime $\gamma \in \Gamma$ divides the characteristic polynomial $P_q(t)$ of the monodromy operator $h_q$ only if $\gamma$ divides one of the polynomials $\xi_l^s(t)$ associated to the local Alexander modules $H_l(S^{2n-2s-1} - K^{2n-2s-3}; \Gamma)$*

*corresponding to link pairs* $(S^{2n-2s-1}, K^{2n-2s-3})$ *of components of strata* $\mathcal{V} \in \mathcal{Y}$*, of complex dimension* $s$ *with* $\mathcal{V} \subset Y_1$*, such that* $n - q - 1 \leq s \leq n - 1$ *and* $2(n-1) - 2s - q \leq l \leq n - s - 1$.

*Proof.* There is an identification $P_q(t) \sim \delta_q(t)$, where $\delta_q(t)$ is the global Alexander polynomial of the hypersurface $V$, i.e., the order of the torsion module $H_q(\mathcal{U}^c; \mathbb{Q}) \cong IH_q^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H)$. We consider a topological stratification $\mathcal{S}$ on $V$ induced by that of $Y$, having the cone point as a zero-dimensional stratum. From Theorem 4.2 we recall that, for $q \leq n-1$, the local polynomials of the zero-dimensional strata of $V_1$ do not contribute to the prime factors of the global polynomial $\delta_q(t)$. Notice that link pairs of strata $S$ of $V_1$ in $(\mathbb{CP}^{n+1}, V)$ (with $\dim(S) \geq 1$) are the same as the link pairs of strata of $Y_1 = V_1 \cap H$ in $(H = \mathbb{CP}^n, V \cap H = Y)$. The desired conclusion follows from Theorem 4.2 by reindexing. □

**Note.** The polynomials $P_i(t)$, $i = 0, \dots, n$, are related by the formula (see [6], (4.1.21) or [7], (6.1.10))

$$\prod_{q=0}^{n} P_q(t)^{(-1)^{q+1}} = (1 - t^d)^{-\chi(F)/d},$$

where $\chi(F)$ is the Euler characteristic of the Milnor fiber. Therefore, it suffices to compute only the polynomials $P_0(t), \dots, P_{n-1}(t)$ and the Euler characteristic of $F$.

If $Y \subset \mathbb{CP}^n$ has only isolated singularities, the proof of the previous proposition can be strengthened to obtain the following result, similar to [6], (6.3.29) or [7], Corollary 6.4.16.

**Proposition 5.2.** *With the above notations, if* $Y$ *has only isolated singularities, then the polynomial* $P_{n-1}$ *divides (up to a power of* $t-1$*) the product of the local Alexander polynomials associated to the singular points of* $Y$ *which are contained in* $Y_1$.

A direct consequence of Proposition 5.1 is the next result.

**Corollary 5.3.** *If* $\lambda \neq 1$ *is a* $d$*-th root of unity such that* $\lambda$ *is not an eigenvalue of any of the local monodromies corresponding to link pairs of singular strata of* $Y_1$ *in a stratification of the pair* $(\mathbb{CP}^n, Y)$*, then* $\lambda$ *is not an eigenvalue of the monodromy operators acting on* $H_q(F)$ *for* $q \leq n-1$.

Using the fact that normal crossing divisor germs have trivial monodromy operators ([6], (5.2.21.ii); [7], (6.1.8.i)), we also obtain the following (compare [5], Corollary 16):

**Corollary 5.4.** *Let* $\mathcal{A} = (Y_i)_{i=1,s}$ *be a hypersurface arrangement in* $\mathbb{CP}^n$ *and fix one irreducible component, say* $Y_1$*. Assume that* $\bigcup_{i=1,s} Y_i$ *is a normal crossing divisor at any point* $x \in Y_1$*. Then the monodromy action on* $H_q(F; \mathbb{Q})$ *is trivial for* $q \leq n-1$.

## 6. Examples

We will now show, by explicit calculations on examples, how to combine Theorems 4.1 and 4.2 in order to obtain information on the Alexander modules of a hypersurface.

**Example 6.1** (*One-dimensional singular locus*). Let $V$ be the trifold in $\mathbb{CP}^4 = \{(x : y : z : t : v)\}$, defined by the polynomial $y^2z + x^3 + tx^2 + v^3 = 0$. The singular locus of $V$ is the projective line $\mathrm{Sing}(V) = \{(0 : 0 : z : t : 0); z, t \in \mathbb{C}\}$. We let $\{t = 0\}$ be the hyperplane $H$ at infinity. Then $V \cap H$ is the surface in $\mathbb{CP}^3 = \{(x : y : z : v)\}$ defined by the equation $y^2z + x^3 + v^3 = 0$, having the point $(0 : 0 : 1 : 0)$ as its singular set. Thus, $\mathrm{Sing}(V \cap H) = \mathrm{Sing}(V) \cap H$. Let $X$ be the affine part of $V$, i.e., defined by the polynomial $y^2z + x^3 + x^2 + v^3 = 0$. Then $\mathrm{Sing}(X) = \{(0, 0, z, 0)\} \cong \{(0 : 0 : z : 1 : 0)\} = \mathrm{Sing}(V) \cap X$ is the $z$-axis of $\mathbb{C}^4 = \{(x, y, z, v)\}$, and it is clear that the origin $(0, 0, 0, 0) = (0 : 0 : 0 : 1 : 0)$ looks different than any other point on the $z$-axis: the tangent cone at the point $(0, 0, \lambda, 0)$ is represented by two planes for $\lambda \neq 0$ and degenerates to a double plane for $\lambda = 0$. Therefore we give the pair $(\mathbb{CP}^4, V)$ the following Whitney stratification:

$$\mathbb{CP}^4 \supset V \supset \mathrm{Sing}(V) \supset (0 : 0 : 0 : 1 : 0)$$

and note that $V$ is transversal to the hyperplane at infinity.

In our example ($n = 3, k = 1$) we are interested in describing the prime factors of the global Alexander polynomials $\delta_2(t)$ and $\delta_3(t)$ (note that $\delta_1(t) \sim 1$, as $n - k \geq 2$; cf. [25]). In order to describe the local Alexander polynomials of link pairs of singular strata of $V$, we will use the results of [35] and [37].

The link pair of the top stratum of $V$ is $(S^1, \emptyset)$, and the only prime factor that may contribute to the global Alexander polynomials is $t - 1$, the order of $H_0(S^1, \Gamma)$.

Next, the link of the stratum $\mathrm{Sing}(V) - \{(0 : 0 : 0 : 1 : 0)\}$ is the algebraic knot in a 5-sphere $S^5 \subset \mathbb{C}^3$ given by the intersection of the affine variety $\{y^2 + x^3 + v^3 = 0\}$ in $\mathbb{C}^3 = \{(x, y, v)\}$ (where $t = 0$, $z = 1$) with a small sphere about the origin $(0, 0, 0)$. (To see this, choose the hyperplane $V(t) = \{(x : y : z : 0 : v)\}$ which is transversal to the singular set $V(x, y, v)$, and consider an affine neighborhood of their intersection $(0 : 0 : 1 : 0)$ in $V(t) \cong \mathbb{CP}^3$.) The polynomial $y^2 + x^3 + v^3$ is weighted homogeneous of Brieskorn type, hence ([33], [35]) the associated Milnor fibre is simply-connected, homotopy equivalent to $\{2 \text{ points}\} * \{3 \text{ points}\} * \{3 \text{ points}\}$, and the characteristic polynomial of the monodromy operator acting on $H_2(F; \mathbb{Q})$ is $(t + 1)^2(t^2 - t + 1)$.

Finally, the link pair of the zero-dimensional stratum, $\{(0 : 0 : 0 : 1 : 0)\}$, (the origin in the affine space $\{t = 1\}$), is the algebraic knot in a 7-sphere, obtained by intersecting the affine variety $y^2z + x^3 + x^2 + v^3 = 0$ in $\mathbb{C}^4 = \{(x, y, z, v)\}$ with a small sphere about the origin. Since we work in a neighborhood of the origin, by an analytic change of coordinates, this is the same as the link pair of the origin in the variety $y^2z+x^2+v^3=0$. Therefore the Milnor fiber of the associated Milnor fibration is the join of $\{x^2=1\}$, $\{y^2z=1\}$ and $\{v^3=1\}$, i.e., it is homotopy equivalent to $S^2 * \{3 \text{ points}\}$, that is, $S^3 \vee S^3$ ([33], [35]). Moreover, denoting by $\delta_f$ the characteristic polynomials of monodromy of the weighted homogeneous polynomial $f$, we obtain ([35], Theorem 6): $\delta_{x^2+v^3+y^2z}(t) = \delta_{x^2+v^3+yz}(t) = \delta_{x^2+v^3}(t) = t^2-t+1$.

Note that the above links, $K^3 \subset S^5$ and $K^5 \subset S^7$, are rational homology spheres since none of the characteristic polynomials of their associated Milnor fibers has the trivial eigenvalue 1 (see [37], Proposition 3.6). Therefore $V$, and hence $V \cap H$, are rational homology manifolds (see the discussion preceding Proposition 2.1). Then, by Proposition 2.1, $t-1$ cannot be a prime factor of the global Alexander polynomials of $V$. Also note that the local Alexander polynomials of links of the singular strata of $V$ have prime divisors none of which divides $t^3-1$, thus, by Theorem 4.1 and 4.2, they cannot appear among the prime divisors of $\delta_2(t)$ and $\delta_3(t)$.

Altogether, we conclude that $\delta_0(t) \sim t-1$, $\delta_1(t) \sim 1$, $\delta_2(t) \sim 1$ and $\delta_3(t) \sim 1$.

**Note.** The above example can be easily generalized to provide hypersurfaces of any dimension, with a one-dimensional singular locus and trivial global Alexander polynomials. This can be done by adding cubes of new variables to the polynomial $y^2z+x^3+tx^2+v^3$.

**Example 6.2** (*Manifold singularity*). Consider the hypersurface $V$ in $\mathbb{CP}^{n+1}$ defined by the zeros of the polynomial $f(z_0,\dots,z_n) = (z_0)^2+(z_1)^2+\dots+(z_{n-k})^2$. Assume that $n-k \geq 2$ is even. The singular set $\Sigma = V(z_0,\dots,z_{n-k}) \cong \mathbb{CP}^k$ is non-singular. Choose a generic hyperplane, $H$, for example $H=\{z_n=0\}$. The link of $\Sigma$ is the algebraic knot in a $(2n-2k+1)$-sphere $S_\varepsilon^{2n-2k+1}$ given by the intersection of the affine variety $(z_0)^2+(z_1)^2+\dots+(z_{n-k})^2=0$ in $\mathbb{C}^{n-k+1}$ with a small sphere about the origin. As $n-k$ is even, the link of the singularity (in the sense of [33]) is a rational homology sphere and the associate Alexander polynomial of the knot complement is $t-(-1)^{n-k+1}=t+1$. Hence the prime factors of the intersection Alexander polynomials of the hypersurface are either $t+1$ or $t-1$. However, since the links of singular strata are rational homology spheres, we conclude (by using Proposition 2.1 and [25], Corollary 4.9) that $\delta_{n-k}(t) \sim t+1$ and all the $\delta_j(t)$, $n-k<j\leq n$, are multiples of $t+1$. Also note that in this case, $\delta_j(t) \sim 1$ for $1 \leq j \leq n-k-1$.

Sometimes it is possible to calculate explicitly the global Alexander polynomials, as we will see in the next example.

**Example 6.3.** Let $V$ be the surface in $\mathbb{CP}^3$, defined by the following homogeneous polynomial of degree $d$: $f(x,y,z,t) = x^{d-1}z + xt^{d-1} + y^d + xyt^{d-2}$, $d \geq 3$. The singular locus of $V$ is a point: $\mathrm{Sing}(V) = \{(0:0:1:0)\}$. We fix a generic hyperplane, $H$. Then the (intersection) Alexander modules of $V$ are defined and the only 'non-trivial' Alexander polynomial of the hypersurface is $\delta_2(t)$. Note that, by Corollary 4.9 of [25], there is an isomorphism of $\Gamma$-modules: $H_2(\mathbb{CP}^3 - V \cup H; \Gamma) \cong H_2(M_f; \mathbb{Q})$, where $M_f$ is the Milnor fiber at the origin, as a non-isolated hypersurface singularity in $\mathbb{C}^4$, of the polynomial $f\colon \mathbb{C}^4 \to \mathbb{C}$, $f(x,y,z,t) = x^{d-1}z + xt^{d-1} + y^d + xyt^{d-2}$, and where the module structure on $H_2(M_f; \mathbb{Q})$ is given by the action on the monodromy operator. Moreover, by [6], Example 4.1.26, the characteristic polynomial of the latter is $t^{d-1} + t^{d-2} + \cdots + 1$. Therefore, $\delta_2(t) = t^{d-1} + t^{d-2} + \cdots + 1$.

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