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On Frobenius-destabilized rank-2 vector bundles over curves

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Abstract. Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. Let $\mathcal{M}_X$ be the moduli space of semistable rank-2 vector bundles over X with trivial determinant. The relative Frobenius map $F\colon X \to X_1$ induces by pull-back a rational map $V\colon \mathcal{M}_{X_1} \dashrightarrow \mathcal{M}_X$. In this paper we show the following results.

(1) For any line bundle L over X, the rank-p vector bundle F_*L is stable.

(2) The rational map V has base points, i.e., there exist stable bundles E over X_1 such that F^*E is not semistable.

(3) Let $\mathcal{B} \subset \mathcal{M}_{X_1}$ denote the scheme-theoretical base locus of V. If $g = 2$, $p > 2$ and X ordinary, then $\mathcal{B}$ is a 0-dimensional local complete intersection of length $\frac{2}{3}p(p^2-1)$ and the degree of V equals $\frac{1}{3}p(p^2+2)$.

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Introduction

Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. Denote by $F\colon X \to X_1$ the relative k-linear Frobenius map. Here $X_1 = X \times_{k,\sigma} k$, where $\sigma\colon \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is the Frobenius of k (see e.g. [R], Section 4.1). We denote by $\mathcal{M}_X$, respectively $\mathcal{M}_{X_1}$, the moduli space of semistable rank-2 vector bundles on X, respectively X_1, with trivial determinant. The Frobenius F induces by pull-back a rational map (the Verschiebung)

$$V\colon \mathcal{M}_{X_1} \dashrightarrow \mathcal{M}_X, \quad [E] \mapsto [F^*E].$$

Here $[E]$ denotes the S-equivalence class of the semistable bundle E. It is shown [MS] that V is generically étale, hence separable and dominant, if X or equivalently X_1 is an ordinary curve. Our first result is

Theorem 1. *Over any smooth projective curve X_1 of genus $g \geq 2$ there exist stable rank-2 vector bundles E with trivial determinant, such that F^*E is not semistable. In other words, V has base points.*

Note that this is a statement for an arbitrary curve of genus $g \geq 2$ over k, since associating X_1 to X induces an automorphism of the moduli space of curves of genus g over k. The existence of Frobenius-destabilized bundles was already proved in [LP2], Theorem A.4, by specializing the so-called Gunning bundle on a Mumford-Tate curve. The proof given in this paper is much simpler than the previous one. Given a line bundle L over X, the generalized Nagata–Segre theorem asserts the existence of rank-2 subbundles E of the rank-p bundle F_*L of a certain (maximal) degree. Quite surprisingly, these subbundles E of maximal degree turn out to be stable and Frobenius-destabilized.

In the case $g = 2$ the moduli space $\mathcal{M}_X$ is canonically isomorphic to the projective space $\mathbb{P}^3_k$ and the set of strictly semistable bundles can be identified with the Kummer surface $\mathrm{Kum}_X \subset \mathbb{P}^3_k$ associated to X. According to [LP2], Proposition A.2, the rational map

$$V : \mathbb{P}^3_k \dashrightarrow \mathbb{P}^3_k$$

is given by polynomials of degree p, which are explicitly known in the cases $p = 2$ [LP1] and $p = 3$ [LP2]. Let $\mathcal{B}$ be the scheme-theoretical base locus of V, i.e., the subscheme of $\mathbb{P}^3_k$ determined by the ideal generated by the 4 polynomials of degree p defining V. Clearly its underlying set equals (see [O1], Theorem A.6)

$$\operatorname{supp} \mathcal{B} = \{E \in \mathcal{M}_{X_1} \cong \mathbb{P}^3_k \mid F^*E \text{ is not semistable}\}$$

and $\operatorname{supp} \mathcal{B} \subset \mathbb{P}^3_k \setminus \mathrm{Kum}_{X_1}$. Since V has no base points on the ample divisor Kum_{X_1}, we deduce that $\dim \mathcal{B} = 0$. Then we show

Theorem 2. *Assume $p > 2$. Let X_1 be an ordinary curve of genus $g = 2$. Then the 0-dimensional scheme $\mathcal{B}$ is a local complete intersection of length*

$$\frac{2}{3}p(p^2 - 1).$$

Since $\mathcal{B}$ is a local complete intersection, the degree of V equals $\deg V = p^3 - l(\mathcal{B})$ where $l(\mathcal{B})$ denotes the length of $\mathcal{B}$ (see e.g. [O1], Proposition 2.2). Hence we obtain the

Corollary. *Under the assumption of Theorem* 2

$$\deg V = \frac{1}{3}p(p^2 + 2).$$

The underlying idea of the proof of Theorem 2 is rather simple: we observe that a vector bundle $E \in \operatorname{supp} \mathcal{B}$ corresponds via adjunction to a subbundle of the rank-p vector bundle $F_*(\theta^{-1})$ for some theta characteristic θ on X (Proposition 3.1). This is the motivation to introduce Grothendieck's Quot-scheme $\mathcal{Q}$ parametrizing rank-2 subbundles of degree 0 of the vector bundle $F_*(\theta^{-1})$. We prove that the two 0-dimensional schemes $\mathcal{B}$ and $\mathcal{Q}$ decompose as disjoint unions $\coprod \mathcal{B}_\theta$ and $\coprod \mathcal{Q}_\eta$ where θ and η vary over theta characteristics on X and p-torsion points of JX_1 respectively and that $\mathcal{B}_\theta$ and $\mathcal{Q}_0$ are isomorphic, if X is ordinary (Proposition 4.6). In particular since $\mathcal{Q}$ is a local complete intersection, $\mathcal{B}$ also is.

In order to compute the length of $\mathcal{B}$ we show that $\mathcal{Q}$ is isomorphic to a determinantal scheme $\mathcal{D}$ defined intrinsically by the 4-th Fitting ideal of some sheaf. The non-existence of a universal family over the moduli space of rank-2 vector bundles of degree 0 forces us to work over a different parameter space constructed via the Hecke correspondence and carry out the Chern class computations on this parameter space.

The underlying set of points of $\mathcal{B}$ has already been studied in the literature. In fact, using the notion of p-curvature, S. Mochizuki [Mo] describes points of $\mathcal{B}$ as "dormant atoms" and obtains, by degenerating the genus-2 curve X to a singular curve, the above mentioned formula for their number ([Mo], Corollary 3.7, p. 267). Moreover he shows that for a general curve X the scheme $\mathcal{B}$ is reduced. In this context we also mention the recent work of B. Osserman [O1], [O2], which explains the relationship of supp $\mathcal{B}$ with Mochizuki's theory.

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1. Stability of the direct image F_*L

Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p > 0$ and let $F: X \to X_1$ denote the relative Frobenius map. Let L be a line bundle over X with

$$\deg L = g - 1 + d,$$

for some integer d. Applying the Grothendieck–Riemann–Roch theorem to the morphism F, we obtain

Lemma 1.1. *The slope of the rank-p vector bundle F_*L equals*

$$\mu(F_*L) = g - 1 + \frac{d}{p}.$$

The following result will be used in Section 3.

Proposition 1.2. *If $g \geq 2$, then the vector bundle F_*L is stable for any line bundle L on X.*

Proof. Suppose that the contrary holds, i.e., F_*L is not stable. Consider its Harder–Narasimhan filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l = F_*L,$$

such that the quotients E_i/E_{i-1} are semistable with $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ for all $i \in \{1, \ldots, l-1\}$. If F_*L is not semistable, we denote $E := E_1$. If F_*L is semistable, we denote by E any proper semistable subbundle of the same slope. Then clearly

$$\mu(E) \geq \mu(F_*L). \tag{1}$$

In case $r = \operatorname{rk} E > \frac{p-1}{2}$, we observe that the quotient bundle

$$Q = \begin{cases} F_*L/E_{l-1} & \text{if } F_*L \text{ is not semistable,} \\ F_*L/E & \text{if } F_*L \text{ is semistable,} \end{cases}$$

is also semistable and that its dual Q^* is a subbundle of $(F_*L)^*$. Moreover, by relative duality $(F_*L)^* = F_*(L^{-1} \otimes \omega_X^{\otimes 1-p})$ and by assumption $\operatorname{rk} Q^* \leq p - r \leq \frac{p-1}{2}$. Hence, replacing if necessary E and L by Q^* and $L^{-1} \otimes \omega_X^{\otimes 1-p}$, we may assume that E is semistable and $r \leq \frac{p-1}{2}$.

Now, by [SB], Corollary 2, we have the inequality

$$\mu_{\max}(F^*E) - \mu_{\min}(F^*E) \leq (r-1)(2g-2), \tag{2}$$

where $\mu_{\max}(F^*E)$ (resp. $\mu_{\min}(F^*E)$) denotes the slope of the first (resp. last) graded piece of the Harder–Narasimhan filtration of F^*E. The inclusion $E \subset F_*L$ gives, by adjunction, a nonzero map $F^*E \to L$. Hence

$$\deg L \geq \mu_{\min}(F^*E) \geq \mu_{\max}(F^*E) - (r-1)(2g-2) \geq p\mu(E) - (r-1)(2g-2).$$

Combining this inequality with (1) and using Lemma 1.1, we obtain

$$g - 1 + \frac{d}{p} = \mu(F_*L) \leq \mu(E) \leq \frac{g-1+d}{p} + \frac{(r-1)(2g-2)}{p},$$

which simplifies to

$$(g-1) \leq (g-1)\left(\frac{2r-1}{p}\right).$$

This is a contradiction, since we have assumed $r \leq \frac{p-1}{2}$ and therefore $\frac{2r-1}{p} < 1$. $\square$

Remark 1.3. We observe that the vector bundles F_*L are destabilized by Frobenius, because of the nonzero canonical map $F^*F_*L \to L$ and clearly $\mu(F^*F_*L) > \deg L$. For further properties of the bundles F_*L, see [JRXY], Section 5.

Remark 1.4. In the context of Proposition 1.2 we mention the following open question: given a finite separable morphism between smooth curves $f\colon Y \to X$ and a line bundle $L \in \mathrm{Pic}(Y)$, is the direct image f_*L stable? For a discussion, see [B2].

2. Existence of Frobenius-destabilized bundles

Let the notation be as in the previous section. We recall the generalized Nagata–Segre theorem, proved by Hirschowitz, which says

Theorem 2.1. *For any vector bundle G of rank r and degree δ over any smooth curve X and for any integer n, $1 \leq n \leq r-1$, there exists a rank-n subbundle $E \subset G$, satisfying*

$$\mu(E) \geq \mu(G) - \left(\frac{r-n}{r}\right)(g-1) - \frac{\varepsilon}{rn}, \tag{3}$$

where ε is the unique integer with $0 \leq \varepsilon \leq r-1$ and $\varepsilon + n(r-n)(g-1) \equiv n\delta \bmod r$.

Remark 2.2. The previous theorem can be deduced (see [L], Remark 3.14) from the main theorem of [Hir] (for its proof, see http://math.unice.fr/~ah/math/Brill/).

Proof of Theorem 1. We apply Theorem 2.1 to the rank-p vector bundle F_*L on X_1 and $n = 2$, where L is a line bundle of degree $g-1+d$ on X, with $d \equiv -2g+2 \bmod p$: There exists a rank-2 vector bundle $E \subset F_*L$ such that

$$\mu(E) \geq \mu(F_*L) - \frac{p-2}{p}(g-1). \tag{4}$$

Note that our assumption on d was made to have $\varepsilon = 0$.

Now we will check that any E satisfying inequality (4) is stable with F^*E not semistable.

(i) E is stable: Let N be a line subbundle of E. The inclusion $N \subset F_*L$ gives, by adjunction, a nonzero map $F^*N \to L$, which implies (see also [JRXY], Proposition 3.2 (i))

$$\deg N \leq \mu(F_*L) - \frac{p-1}{p}(g-1).$$

Comparing with (4) we see that $\deg N < \mu(E)$.

(ii) F^*E is not semistable. In fact, we claim that L destabilizes F^*E. For the proof note that Lemma 1.1 implies

$$\mu(F_*L) - \frac{p-2}{p}(g-1) = \frac{2g-2+d}{p} > \frac{g-1+d}{p} = \frac{\deg L}{p} \tag{5}$$

since $g \geq 2$. Together with (4) this gives $\mu(E) > \frac{\deg L}{p}$ and hence

$$\mu(F^*E) > \deg L.$$

This implies the assertion, since by adjunction we obtain a nonzero map $F^*E \to L$.

Replacing E by a subsheaf of suitable degree, we may assume that inequality (4) is an equality. In that case, because of our assumption on d, $\mu(E)$ is an integer, hence $\deg E$ is even. In order to get trivial determinant, we may tensorize E with a suitable line bundle.

This shows the existence of a stable rank-2 vector bundle E with F^*E not semistable, which is equivalent to the existence of base points of V (see e.g. [O1], Theorem A.6). □

3. Frobenius-destabilized bundles in genus 2.

From now on we assume that X is an ordinary curve of genus $g = 2$ and the characteristic of k is $p > 2$. Recall that $\mathcal{M}_X$ denotes the moduli space of semistable rank-2 vector bundles with trivial determinant over X and $\mathcal{B}$ the scheme-theoretical base locus of the rational map

$$V : \mathcal{M}_{X_1} \cong \mathbb{P}^3_k \dashrightarrow \mathbb{P}^3_k \cong \mathcal{M}_X,$$

which is given by polynomials of degree p.

First of all we will show that the 0-dimensional scheme $\mathcal{B}$ is the disjoint union of subschemes $\mathcal{B}_\theta$ indexed by theta characteristics of X.

Proposition 3.1. (a) *Let E be a vector bundle on X_1 such that $E \in \operatorname{supp} \mathcal{B}$. Then we have*

(i) *There exists a unique theta characteristic θ on X such that* $\operatorname{Hom}(E, F_*(\theta^{-1})) \neq 0$.

(ii) *Any rank-2 vector bundle E of degree 0 satisfying* $\operatorname{Hom}(E, F_*(\theta^{-1})) \neq 0$ *is a subbundle of $F_*(\theta^{-1})$, i.e. the quotient $F_*(\theta^{-1})/E$ is torsion free.*

(b) *Let θ be a theta characteristic on X. Any rank-2 subbundle $E \subset F_*(\theta^{-1})$ of degree 0 has the following properties*

(i) *E is stable and F^*E is not semistable,*

(ii) $F^*(\det E) = \mathcal{O}_X$,

(iii) $\dim \mathrm{Hom}(E, F_*(\theta^{-1})) = 1$ *and* $\dim H^1(E^* \otimes F_*(\theta^{-1})) = 5$,

(iv) *E is a rank-2 subbundle of maximal degree.*

Proof. (a) By [LS], Corollary 2.6, we know that, for every $E \in \operatorname{supp} \mathcal{B}$ the bundle F^*E is the nonsplit extension of θ^{-1} by θ, for some theta characteristic θ on X (note that $\mathrm{Ext}^1(\theta^{-1}, \theta) \cong k$). By adjunction we get a nonzero homomorphism $\psi \colon E \to F_*(\theta^{-1})$, which shows (i). Uniqueness of θ will be proved below.

As for (ii), we have to show that ψ is of maximal rank. Suppose it is not, then there is a line bundle N on the curve X_1 such that ψ factorizes as $E \to N \to F_*(\theta^{-1})$. By stability of E we have $\deg N > 0$. On the other hand, by adjunction, we get a nonzero homomorphism $F^*N \to \theta^{-1}$ implying $p \cdot \deg N \leq -1$, a contradiction. Hence $\psi \colon E \to F_*(\theta^{-1})$ is injective. Moreover E is even a subbundle of $F_*(\theta^{-1})$, since otherwise there exists a subbundle $E' \subset F_*(\theta^{-1})$ with $\deg E' > 0$ and which fits into the exact sequence

$$0 \longrightarrow E \longrightarrow E' \xrightarrow{\pi} T \longrightarrow 0,$$

where T is a torsion sheaf supported on an effective divisor. Varying π, we obtain a family of bundles $\ker \pi \subset E'$ of dimension > 0 and $\det \ker \pi = \mathcal{O}_{X_1}$. This would imply (see proof of Theorem 1) $\dim \mathcal{B} > 0$, a contradiction.

Finally, since θ is the maximal destabilizing line subbundle of F^*E, it is unique.

(b) We observe that inequality (4) holds for the pair $E \subset F_*(\theta^{-1})$. Hence, by the proof of Theorem 1, E is stable and F^*E is not semistable.

Let $\varphi \colon F^*E \to \theta^{-1}$ denote the homomorphism adjoint to the inclusion $E \subset F_*(\theta^{-1})$. The homomorphism φ is surjective, since otherwise F^*E would contain a line subbundle of degree > 1, contradicting [LS], Satz 2.4. Hence we get an exact sequence

$$0 \to \ker \varphi \to F^*E \to \theta^{-1} \to 0. \tag{6}$$

On the other hand, let N denote a line bundle on X_1 such that $E \otimes N$ has trivial determinant, i.e. $N^{-2} = \det E$. Applying Corollary 2.6 in [LS] to the bundle $F^*(E \otimes N)$ we get an exact sequence

$$0 \to \tilde{\theta} \otimes F^*N^{-1} \to F^*E \to \tilde{\theta}^{-1} \otimes F^*N^{-1} \to 0,$$

for some theta characteristic $\tilde{\theta}$. By uniqueness of the destabilizing subbundle of maximal degree of F^*E, this exact sequence must coincide with (6) up to a nonzero constant. This implies that $F^*N \otimes \tilde{\theta} = \theta$, hence $(F^*N)^2 = \mathcal{O}_X$. So we obtain that $\mathcal{O}_X = \det(F^*E) = F^*(\det E)$ proving (ii).

By adjunction the equality $\dim \mathrm{Hom}(E, F_*(\theta^{-1})) = \dim \mathrm{Hom}(F^*E, \theta^{-1}) = 1$ holds. Moreover by Riemann–Roch we obtain $\dim H^1(E^* \otimes F_*(\theta^{-1})) = 5$. This proves (iii).

Finally, suppose that there exists a rank-2 subbundle $E' \subset F_*(\theta^{-1})$ with $\deg E' \geq 1$. Then we can consider the kernel $E = \ker \pi$ of a surjective morphism $\pi : E' \to T$ onto a torsion sheaf with length equal to $\deg E'$. By varying π and after tensoring $\ker \pi$ with a suitable line bundle of degree 0, we construct a family of dimension > 0 of stable rank-2 vector bundles with trivial determinant which are Frobenius-destabilized, contradicting $\dim \mathcal{B} = 0$. This proves (iv). □

It follows from Proposition 3.1 (a) that the scheme $\mathcal{B}$ decomposes as a disjoint union

$$\mathcal{B} = \coprod_{\theta} \mathcal{B}_\theta,$$

where θ varies over the set of all theta characteristics of X and

$$\operatorname{supp} \mathcal{B}_\theta = \{E \in \operatorname{supp} \mathcal{B} \mid E \subset F_*(\theta^{-1})\}.$$

Tensor product with a 2-torsion point $\alpha \in JX_1[2] \cong JX[2]$ induces an isomorphism of $\mathcal{B}_\theta$ with $\mathcal{B}_{\theta \otimes \alpha}$ for every theta characteristic θ. We denote by $l(\mathcal{B})$ and $l(\mathcal{B}_\theta)$ the length of the schemes $\mathcal{B}$ and $\mathcal{B}_\theta$. From the preceding we deduce the relations

$$l(\mathcal{B}) = 16 \cdot l(\mathcal{B}_\theta) \quad \text{for every theta characteristic } \theta. \tag{7}$$

4. Grothendieck's Quot-scheme

Let θ be a theta characteristic on X. We consider the functor $\underline{\mathcal{Q}}$ from the opposite category of k-schemes of finite type to the category of sets defined by

$$\begin{aligned} \underline{\mathcal{Q}}(S) = \{\sigma : \pi_{X_1}^*(F_*(\theta^{-1})) \to \mathcal{G} \to 0 \mid \mathcal{G} \text{ coherent over } X_1 \times S,\ \text{flat over } S, \\ \deg \mathcal{G}|_{X_1 \times \{s\}} = \operatorname{rk} \mathcal{G}|_{X_1 \times \{s\}} = p - 2 \text{ for all } s \in S\}/\cong \end{aligned}$$

where $\pi_{X_1} : X_1 \times S \to X_1$ denotes the natural projection and $\sigma \cong \sigma'$ for quotients σ and σ' if and only if there exists an isomorphism $\lambda : \mathcal{G} \to \mathcal{G}'$ such that $\sigma' = \lambda \circ \sigma$.

Grothendieck showed in [G] (see also [HL], Section 2.2) that the functor $\underline{\mathcal{Q}}$ is representable by a k-scheme, which we denote by $\mathcal{Q}$. A k-point of $\mathcal{Q}$ corresponds to a quotient $\sigma : F_*(\theta^{-1}) \to G$, or equivalently to a rank-2 subsheaf $E = \ker \sigma \subset F_*(\theta^{-1})$ of degree 0 on X_1. By Proposition 3.1 (a) (ii) any subsheaf E of degree 0 is a subbundle of $F_*(\theta^{-1})$, which implies that any sheaf $\mathcal{G} \in \underline{\mathcal{Q}}(S)$ is locally free (see also [MuSa] or [L], Lemma 3.8). Moreover we note that by Proposition 3.1 (b) (iv) the bundle E has maximal degree as a subbundle of $F_*(\theta^{-1})$.

Hence taking the kernel of σ induces a bijection of $\underline{\mathcal{Q}}(S)$ with the following set, which we also denote by $\underline{\mathcal{Q}}(S)$

$$\begin{aligned} \underline{\mathcal{Q}}(S) = \{\mathcal{E} \hookrightarrow \pi_{X_1}^*(F_*(\theta^{-1})) \mid \mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2, \\ \pi_{X_1}^*(F_*(\theta^{-1}))/\mathcal{E} \text{ locally free},\ \deg \mathcal{E}|_{X_1 \times \{s\}} = 0 \text{ for all } s \in S\}/\cong \end{aligned}$$

By Proposition 3.1 (b) the scheme $\mathcal{Q}$ decomposes as a disjoint union

$$\mathcal{Q} = \coprod_{\eta} \mathcal{Q}_{\eta},$$

where η varies over the p-torsion points $\eta \in JX_1[p]_{\mathrm{red}} = \ker(V\colon JX_1 \to JX)$. We also denote by V the Verschiebung of JX_1, i.e. $V(L) = F^*L$, for $L \in JX_1$. The set-theoretical support of $\mathcal{Q}_\eta$ equals

$$\operatorname{supp} \mathcal{Q}_\eta = \{E \in \operatorname{supp} \mathcal{Q} \mid \det E = \eta\}.$$

Because of the projection formula, the tensor product with a p-torsion point $\beta \in JX_1[p]_{\mathrm{red}}$ induces an isomorphism of $\mathcal{Q}_\eta$ with $\mathcal{Q}_{\eta\otimes\beta}$. This implies the relation

$$l(\mathcal{Q}) = p^2 \cdot l(\mathcal{Q}_0), \tag{8}$$

since X_1 is assumed to be ordinary. Moreover, by Proposition 3.1 we have the set-theoretical equality

$$\operatorname{supp} \mathcal{Q}_0 = \operatorname{supp} \mathcal{B}_\theta.$$

Proposition 4.1. (a) $\dim \mathcal{Q} = 0$.

(b) *The scheme $\mathcal{Q}$ is a local complete intersection at any k-point $e = (E \subset F_*(\theta^{-1})) \in \mathcal{Q}$.*

Proof. Assertion (a) follows from the preceding remarks and $\dim \mathcal{B} = 0$. By [HL], Proposition 2.2.8, assertion (b) follows from the equality $\dim_{[E]} \mathcal{Q} = 0 = \chi(\underline{\mathrm{Hom}}(E, G))$, where $E = \ker(\sigma\colon F_*(\theta^{-1}) \to G)$ and $\underline{\mathrm{Hom}}$ denotes the sheaf of homomorphisms. □

Let $\mathcal{N}_{X_1}$ denote the moduli space of semistable rank-2 vector bundles of degree 0 over X_1. We denote by $\mathcal{N}^s_{X_1}$ and $\mathcal{M}^s_{X_1}$ the open subschemes of $\mathcal{N}_{X_1}$ and $\mathcal{M}_{X_1}$ corresponding to stable vector bundles. Recall (see [La1], Theorem 4.1) that $\mathcal{N}^s_{X_1}$ and $\mathcal{M}^s_{X_1}$ universally corepresent the functors (see e.g. [HL], Definition 2.2.1) from the opposite category of k-schemes of finite type to the category of sets defined by

$$\begin{aligned}\underline{\mathcal{N}^s_{X_1}}(S) = \{&\mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2 \mid \mathcal{E}|_{X_1\times\{s\}} \text{ stable},\\ &\deg \mathcal{E}|_{X_1\times\{s\}} = 0 \text{ for all } s \in S\}/\sim,\end{aligned}$$

$$\begin{aligned}\underline{\mathcal{M}^s_{X_1}}(S) = \{&\mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2 \mid \mathcal{E}|_{X_1\times\{s\}} \text{ stable}\\ &\text{for all } s \in S, \det \mathcal{E} = \pi_S^* M \text{ for some line bundle } M \text{ on } S\}/\sim,\end{aligned}$$

where $\pi_S\colon X_1 \times S \to S$ denotes the natural projection and $\mathcal{E}' \sim \mathcal{E}$ if and only if there exists a line bundle L on S such that $\mathcal{E}' \cong \mathcal{E} \otimes \pi_S^* L$. We denote by $\langle\mathcal{E}\rangle$ the equivalence class of the vector bundle $\mathcal{E}$ for the relation $\sim$.

Consider the determinant morphism

$$\det\colon \mathcal{N}_{X_1} \to JX_1, \quad [E] \mapsto \det E,$$

and denote by $\det^{-1}(0)$ the scheme-theoretical fibre over the trivial line bundle on X_1. Since $\mathcal{N}^s_{X_1}$ universally corepresents the functor $\underline{\mathcal{N}}^s_{X_1}$, we have an isomorphism

$$\mathcal{M}^s_{X_1} \cong \mathcal{N}^s_{X_1} \cap \det^{-1}(0).$$

Remark 4.2. If $p > 0$, it is not known whether the canonical morphism $\mathcal{M}_{X_1} \to \det^{-1}(0)$ is an isomorphism (see e.g. [La2], Section 3).

In the sequel we need the following relative version of Proposition 3.1 (b) (ii). By a k-scheme we always mean a k-scheme of finite type.

Lemma 4.3. *Let S be a connected k-scheme and let $\mathcal{E}$ be a locally free sheaf of rank-2 over $X_1 \times S$ such that $\deg \mathcal{E}|_{X_1\times\{s\}} = 0$ for all points s of S. Suppose that $\mathrm{Hom}(\mathcal{E}, \pi^*_{X_1}(F_*(\theta^{-1})) \neq 0$. Then we have the exact sequence*

$$0 \longrightarrow \pi^*_X(\theta) \longrightarrow (F \times \mathrm{id}_S)^*\mathcal{E} \longrightarrow \pi^*_X(\theta^{-1}) \longrightarrow 0.$$

In particular

$$(F \times \mathrm{id}_S)^*(\det \mathcal{E}) = \mathcal{O}_{X_1\times S}.$$

Proof. First we note that by flat base change for $\pi_{X_1}\colon X_1 \times S \to X_1$, we have an isomorphism $\pi^*_{X_1}(F_*(\theta^{-1})) \cong (F \times \mathrm{id}_S)_*(\pi^*_X(\theta^{-1}))$. Hence the nonzero morphism $\mathcal{E} \to \pi^*_{X_1}(F_*(\theta^{-1}))$ gives via adjunction a nonzero morphism

$$\varphi\colon (F \times \mathrm{id}_S)^*\mathcal{E} \longrightarrow \pi^*_X(\theta^{-1}).$$

We know by the proof of Proposition 3.1 (b) that the fibre $\varphi_{(x,s)}$ over any closed point $(x, s) \in X \times S$ is a surjective k-linear map. Hence φ is surjective by Nakayama and we have an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow (F \times \mathrm{id}_S)^*\mathcal{E} \longrightarrow \pi^*_X(\theta^{-1}) \longrightarrow 0,$$

with $\mathcal{L}$ locally free sheaf of rank 1. By [K], Section 5, the rank-2 vector bundle $(F \times \mathrm{id}_S)^*\mathcal{E}$ is equipped with a canonical connection

$$\nabla\colon (F \times \mathrm{id}_S)^*\mathcal{E} \longrightarrow (F \times \mathrm{id}_S)^*\mathcal{E} \otimes \Omega^1_{X\times S/S}.$$

We note that $\Omega^1_{X\times S/S} = \pi^*_X(\omega_X)$, where ω_X denotes the canonical line bundle of X. The first fundamental form of the connection ∇ is an $\mathcal{O}_{X\times S}$-linear homomorphism

$$\psi_\nabla\colon \mathcal{L} \longrightarrow \pi^*_X(\theta^{-1}) \otimes \pi^*_X(\omega_X) = \pi^*_X(\theta).$$

The restriction of ψ_∇ to the curve $X \times \{s\} \subset X \times S$ for any closed point $s \in S$ is an isomorphism (see e.g. proof of [LS], Corollary 2.6). Hence the fibre of ψ_∇ is a k-linear isomorphism over any closed point $(x, s) \in X \times S$. We conclude that ψ_∇ is an isomorphism, by Nakayama's lemma and because $\mathcal{L}$ is a locally free sheaf of rank 1.

We obtain the second assertion of the lemma, since

$$(F \times \mathrm{id}_S)^*(\det \mathcal{E}) = \det(F \times \mathrm{id}_S)^*\mathcal{E} = \mathcal{L} \otimes \pi_X^*(\theta^{-1}) = \mathcal{O}_{X_1 \times S}. \qquad \square$$

Proposition 4.4. *We assume X ordinary.*

(a) *The forgetful morphism*

$$i : \mathcal{Q} \hookrightarrow \mathcal{N}^s_{X_1}, \quad e = (E \subset F_*(\theta^{-1})) \mapsto E$$

is a closed embedding.

(b) *The restriction i_0 of i to the subscheme $\mathcal{Q}_0 \subset \mathcal{Q}$ factors through $\mathcal{M}^s_{X_1}$, i.e. there is a closed embedding*

$$i_0 : \mathcal{Q}_0 \hookrightarrow \mathcal{M}^s_{X_1}.$$

Proof. (a) Let $e = (E \subset F_*(\theta^{-1}))$ be a k-point of $\mathcal{Q}$. To show that i is a closed embedding at $e \in \mathcal{Q}$, it is enough to show that the differential $(di)_e : T_e\mathcal{Q} \to T_{[E]}\mathcal{N}_{X_1}$ is injective – note that $\mathcal{Q}$ is proper. Since the bundle E is stable, the Zariski tangent spaces identify with $\mathrm{Hom}(E, G)$ and $\mathrm{Ext}^1(E, E)$ respectively (see e.g. [HL], Proposition 2.2.7 and Corollary 4.5.2). Moreover, if we apply the functor $\mathrm{Hom}(E, \cdot)$ to the exact sequence associated with $e \in \mathcal{Q}$

$$0 \longrightarrow E \longrightarrow F_*(\theta^{-1}) \longrightarrow G \longrightarrow 0,$$

the coboundary map δ of the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(E, E) &\longrightarrow \mathrm{Hom}(E, F_*(\theta^{-1})) \\ &\longrightarrow \mathrm{Hom}(E, G) \xrightarrow{\delta} \mathrm{Ext}^1(E, E) \longrightarrow \cdots \end{aligned}$$

identifies with the differential $(di)_e$. By Proposition 3.1 (b) we obtain that the map $\mathrm{Hom}(E, E) \to \mathrm{Hom}(E, F_*(\theta^{-1}))$ is an isomorphism. Thus $(di)_e$ is injective.

(b) We consider the composite map

$$\alpha : \mathcal{Q} \xrightarrow{i} \mathcal{N}^s_{X_1} \xrightarrow{\det} JX_1 \xrightarrow{V} JX,$$

where the last map is the isogeny given by the Verschiebung on JX_1, i.e. $V(L) = F^*L$ for $L \in JX_1$. The morphism α is induced by the natural transformation of functors $\underline{\alpha} : \underline{\mathcal{Q}} \Rightarrow \underline{JX}$, defined by

$$\underline{\mathcal{Q}}(S) \longrightarrow \underline{JX}(S), \quad (\mathcal{E} \hookrightarrow \pi_X^*(F_*(\theta^{-1}))) \mapsto (F \times \mathrm{id}_S)^*(\det \mathcal{E}).$$

Using Lemma 4.3 this immediately implies that α factors through the inclusion of the reduced point $\{\mathcal{O}_X\} \hookrightarrow JX$. Hence the image of $\mathcal{Q}$ under the composite morphism $\det \circ i$ is contained in the kernel of the isogeny V, which is the reduced scheme $JX_1[p]_{\mathrm{red}}$, since we have assumed X ordinary. Taking connected components we see that the image of $\mathcal{Q}_0$ under $\det \circ i$ is the reduced point $\{\mathcal{O}_{X_1}\} \hookrightarrow JX_1$, which implies that $i_0(\mathcal{Q}_0)$ is contained in $\mathcal{N}^s_{X_1} \cap \det^{-1}(0) \cong \mathcal{M}^s_{X_1}$. □

In order to compare the two schemes $\mathcal{B}_\theta$ and $\mathcal{Q}_0$ we need the following lemma.

Lemma 4.5. (1) *The closed subscheme $\mathcal{B} \subset \mathcal{M}^s_{X_1}$ corepresents the functor $\underline{\mathcal{B}}$ which associates to a k-scheme S the set*

$$\begin{aligned}\underline{\mathcal{B}}(S) = \{&\mathcal{E} \textit{ locally free sheaf over } X_1 \times S \textit{ of rank } 2 \mid \mathcal{E}|_{X_1 \times \{s\}} \textit{ stable for all } s \in S,\\ &0 \to \mathcal{L} \to (F \times \mathrm{id}_S)^*\mathcal{E} \to \mathcal{M} \to 0 \textit{ for some locally free sheaves } \mathcal{L}, \mathcal{M}\\ &\textit{over } X \times S \textit{ of rank } 1, \deg \mathcal{L}|_{X\times\{s\}} = -\deg \mathcal{M}|_{X \times \{s\}} = 1 \textit{ for all } s \in S,\\ &\det \mathcal{E} = \pi_S^* M \textit{ for some line bundle } M \textit{ on } S\}/\sim .\end{aligned}$$

(2) *The closed subscheme $\mathcal{B}_\theta \subset \mathcal{M}^s_{X_1}$ corepresents the subfunctor $\underline{\mathcal{B}}_\theta$ of $\underline{\mathcal{B}}$ defined by $\langle \mathcal{E} \rangle \in \underline{\mathcal{B}}_\theta(S)$ if and only if the set-theoretical image of the classifying morphism of $\mathcal{L}$*

$$\Phi_{\mathcal{L}} : S \longrightarrow \mathrm{Pic}^1(X), \quad s \longmapsto \mathcal{L}|_{X\times\{s\}},$$

is the point $\theta \in \mathrm{Pic}^1(X)$.

Proof. We denote by $\mathfrak{M}_X$ the algebraic stack parametrizing rank-2 vector bundles with trivial determinant over X. Let $\mathfrak{M}^{ss}_X$ and $\mathfrak{M}^s_X$ denote the open substacks of $\mathfrak{M}_X$ parametrizing semistable and stable bundles. We similarly denote the corresponding stacks of bundles over X_1. The Shatz stratification [Sh] of $\mathfrak{M}_X$ induced by the degree of the first piece of the Harder–Narasimhan filtration reduces in the case of rank-2 vector bundles to a filtration of the stack $\mathfrak{M}_X$

$$\mathfrak{M}^{ss}_X \subset \mathfrak{M}^{\leq 1}_X \subset \mathfrak{M}^{\leq 2}_X \subset \cdots \subset \mathfrak{M}^{\leq n}_X \subset \cdots \subset \mathfrak{M}_X$$

by open substacks $\mathfrak{M}^{\leq n}_X$. It follows from the semicontinuity of the Harder–Narasimhan filtration ([Sh], Section 5) that, for every integer n, there is a closed reduced substack $\mathfrak{M}^n_X$ of $\mathfrak{M}^{\leq n}_X$ parametrizing vector bundles having a maximal destabilizing line subbundle of degree n. Note that $\mathfrak{M}^n_X$ is the complement of $\mathfrak{M}^{\leq n-1}_X$ in $\mathfrak{M}^{\leq n}_X$. It can be shown (see e.g. [He], Folgerung 2.1.10) that the stacks $\mathfrak{M}^n_X$ and $\mathfrak{M}_X$ are smooth. Let $\mathfrak{V} : \mathfrak{M}_{X_1} \to \mathfrak{M}_X$ denote the morphism of stacks induced by pull-back under the Frobenius map $F : X \to X_1$. It follows from [LS], Corollary 2.6, that the restriction of $\mathfrak{V}$ to the open substack $\mathfrak{M}^{ss}_{X_1}$ determines a morphism of stacks

$$\mathfrak{V}^{ss} : \mathfrak{M}^{ss}_{X_1} \longrightarrow \mathfrak{M}^{\leq 1}_X .$$

We will use the following facts about the stack $\mathfrak{M}_X$.

- The pull-back of $\mathcal{O}_{\mathbb{P}^3}(1)$ by the natural map $\mathfrak{M}_X^{ss} \to \mathcal{M}_X \cong \mathbb{P}^3$ extends to a line bundle, which we denote by $\mathcal{O}(1)$, over the moduli stack $\mathfrak{M}_X^{\leq 1}$ and $\mathrm{Pic}(\mathfrak{M}_X^{\leq 1}) = \mathbb{Z} \cdot \mathcal{O}(1)$. Moreover, for any positive integer l, there is a natural isomorphism $H^0(\mathfrak{M}_X^{\leq 1}, \mathcal{O}(l)) \cong H^0(\mathcal{M}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(l))$ (see [BL], Propositions 8.3 and 8.4).
- The closed substack $\mathfrak{M}_X^1$ is the base locus of the linear system $|\mathcal{O}(1)|$ over the stack $\mathfrak{M}_X^{\leq 1}$ (see Proposition A).

In order to prove part (1) it will be enough to show that the functor $\underline{\mathcal{B}}$ defined in the lemma coincides with the fibre product functor $\mathcal{B} \times_{\mathcal{M}_{X_1}^s} \underline{\mathcal{M}}_{X_1}^s$ – we recall that $\mathcal{M}_{X_1}^s$ universally corepresents the functor $\underline{\mathcal{M}}_{X_1}^s$.

We now compute the fibre product functor $\mathcal{B} \times_{\mathcal{M}_{X_1}^s} \underline{\mathcal{M}}_{X_1}^s$. Let S be a k-scheme and consider a vector bundle $\mathcal{E} \in \mathfrak{M}_{X_1}^s(S)$. Since the subscheme $\mathcal{B}$ is defined as the base locus of the linear system $V^*|\mathcal{O}_{\mathbb{P}^3}(1)|$, we obtain that $\langle \mathcal{E} \rangle \in \left[\mathcal{B} \times_{\mathcal{M}_{X_1}^s} \underline{\mathcal{M}}_{X_1}^s\right](S)$ if and only if $\mathcal{E}$ lies in the base locus of $\mathfrak{V}^{ss*}|\mathcal{O}(1)|$ – here we use the isomorphism $|\mathcal{O}_{\mathbb{P}^3}(1)| \cong |\mathcal{O}(1)|$, or equivalently $\mathfrak{V}^{ss}(\mathcal{E}) := (F \times \mathrm{id}_S)^* \mathcal{E} \in \mathfrak{M}_X^{\leq 1}(S)$ lies in the base locus of $|\mathcal{O}(1)|$, which is the closed substack $\mathfrak{M}_X^1$.

We now consider the universal exact sequence defined by the Harder–Narasimhan filtration over $\mathfrak{M}_X^1$:

$$0 \to \mathcal{L} \to (F \times \mathrm{id}_S)^* \mathcal{E} \to \mathcal{M} \to 0,$$

with $\mathcal{L}, \mathcal{M}$ locally free sheaves over $X \times S$ such that $\deg \mathcal{L}|_{X \times \{s\}} = -\deg \mathcal{M}|_{X \times \{s\}} = 1$ for any $s \in S$. This shows that the two sets $\left[\mathcal{B} \times_{\mathcal{M}_{X_1}^s} \underline{\mathcal{M}}_{X_1}^s\right](S)$ and $\underline{\mathcal{B}}(S)$ coincide. This proves (1).

As for (2), we add the condition that the family $\mathcal{E}$ is Frobenius-destabilized by the theta-characteristic θ. □

Remark 4.6. Note that in Lemma 4.5 we do not need to assume X ordinary.

Proposition 4.7. *We assume X ordinary. There is a scheme-theoretical equality*

$$\mathcal{B}_\theta = \mathcal{Q}_0$$

as closed subschemes of $\mathcal{M}_{X_1}$.

Proof. Since $\mathcal{B}_\theta$ and $\mathcal{Q}_0$ corepresent the two functors $\underline{\mathcal{B}}_\theta$ and $\underline{\mathcal{Q}}_0$ it will be enough to show that there is a canonical bijection between the set $\underline{\mathcal{B}}_\theta(S)$ and $\underline{\mathcal{Q}}_0(S)$ for any k-scheme S. We recall that

$$\begin{aligned}\underline{\mathcal{Q}}_0(S) = \{\mathcal{E} \hookrightarrow \pi_{X_1}^*(F_*(\theta^{-1})) \mid\ & \mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2,\\ & \pi_X^*(F_*(\theta^{-1}))/\mathcal{E} \text{ locally free},\ \det \mathcal{E} \cong \mathcal{O}_{X_1 \times S}\}/\cong\end{aligned}$$

Note that the property $\det \mathcal{E} \cong \mathcal{O}_{X_1 \times S}$ is implied as follows: by Proposition 4.4 (b) we have $\det \mathcal{E} \cong \pi_S^* L$ for some line bundle L over S and by Lemma 4.3 we conclude that $L = \mathcal{O}_S$.

First we show that the natural map $\underline{\mathcal{Q}}_0(S) \longrightarrow \underline{\mathcal{M}}^s_{X_1}(S)$ is injective. Suppose that there exist $\mathcal{E}, \mathcal{E}' \in \underline{\mathcal{Q}}_0(S)$ such that $\langle \mathcal{E} \rangle = \langle \mathcal{E}' \rangle$, i.e. $\mathcal{E}' \cong \mathcal{E} \otimes \pi_S^*(L)$ for some line bundle L on S. Then by Lemma 4.3 we have two inclusions

$$i \colon \pi_X^*(\theta) \longrightarrow (F \times \mathrm{id}_S)^* \mathcal{E}, \quad i' \colon \pi_X^*(\theta) \otimes \pi_S^*(L^{-1}) \longrightarrow (F \times \mathrm{id}_S)^* \mathcal{E}.$$

Composing with the projection $\sigma \colon (F \times \mathrm{id}_S)^* \mathcal{E} \to \pi_X^*(\theta^{-1})$ we see that the composite map $\sigma \circ i'$ is identically zero. Hence the two subbundles $\pi_X^*(\theta)$ and $\pi_X^*(\theta) \otimes \pi_S^*(L^{-1})$ coincide, which implies $\pi_S^*(L) = \mathcal{O}_{X_1 \times S}$.

Therefore the two sets $\underline{\mathcal{Q}}_0(S)$ and $\underline{\mathcal{B}}_\theta(S)$ are naturally subsets of $\underline{\mathcal{M}}^s_{X_1}(S)$.

We now show that $\underline{\mathcal{Q}}_0(S) \subset \underline{\mathcal{B}}_\theta(S)$. Consider $\mathcal{E} \in \underline{\mathcal{Q}}_0(S)$. By Proposition 3.1 (b) the bundle $\mathcal{E}|_{X_1 \times \{s\}}$ is stable for all $s \in S$. By Lemma 4.3 we can take $\mathcal{L} = \pi_X^*(\theta)$ and $\mathcal{M} = \pi_X^*(\theta^{-1})$, so that $\langle \mathcal{E} \rangle \in \underline{\mathcal{B}}_\theta(S)$.

Hence it remains to show that $\underline{\mathcal{B}}_\theta(S) \subset \underline{\mathcal{Q}}_0(S)$. Consider a sheaf $\mathcal{E}$ with $\langle \mathcal{E} \rangle \in \underline{\mathcal{B}}_\theta(S)$ – see Lemma 4.5 (2). As in the proof of Lemma 4.3 we consider the canonical connection ∇ on $(F \times \mathrm{id}_S)^* \mathcal{E}$. Its first fundamental form is an $\mathcal{O}_{X \times S}$-linear homomorphism

$$\psi_\nabla \colon \mathcal{L} \longrightarrow \mathcal{M} \otimes \pi_X^*(\omega_X),$$

which is surjective on closed points $(x, s) \in X \times S$. Hence we can conclude that ψ_∇ is an isomorphism. Moreover taking the determinant, we obtain

$$\mathcal{L} \otimes \mathcal{M} = \det(F \times \mathrm{id}_S)^* \mathcal{E} = \pi_S^* M,$$

for some line bundle M on S. Combining both isomorphisms we deduce that

$$\mathcal{L} \otimes \mathcal{L} = \pi_X^*(\omega_X) \otimes \pi_S^* M.$$

Hence its classifying morphism $\Phi_{\mathcal{L} \otimes \mathcal{L}} \colon S \longrightarrow \mathrm{Pic}^2(X)$ factorizes through the inclusion of the reduced point $\{\omega_X\} \hookrightarrow \mathrm{Pic}^2(X)$. Moreover the composite map of $\Phi_{\mathcal{L}}$ with the duplication map [2]

$$\Phi_{\mathcal{L} \otimes \mathcal{L}} \colon S \xrightarrow{\Phi_{\mathcal{L}}} \mathrm{Pic}^1(X) \xrightarrow{[2]} \mathrm{Pic}^2(X)$$

coincides with $\Phi_{\mathcal{L} \otimes \mathcal{L}}$. We deduce that $\Phi_{\mathcal{L}}$ factorizes through the inclusion of the reduced point $\{\theta\} \hookrightarrow \mathrm{Pic}^1(X)$. Note that the fibre $[2]^{-1}(\omega_X)$ is reduced, since $p > 2$. Since $\mathrm{Pic}^1(X)$ is a fine moduli space, there exists a line bundle N over S such that

$$\mathcal{L} = \pi_X^*(\theta) \otimes \pi_S^*(N).$$

We introduce the vector bundle $\mathcal{E}_0 = \mathcal{E} \otimes \pi_S^*(N^{-1})$. Then $\langle \mathcal{E}_0 \rangle = \langle \mathcal{E} \rangle$ and we have an exact sequence

$$0 \longrightarrow \pi_X^*(\theta) \longrightarrow (F \times \mathrm{id}_S)^* \mathcal{E}_0 \xrightarrow{\sigma} \pi_X^*(\theta^{-1}) \longrightarrow 0,$$

since $\pi_S^* M = \pi_S^* N^2$. By adjunction the morphism σ gives a nonzero morphism

$$j \colon \mathcal{E}_0 \longrightarrow (F \times \mathrm{id}_S)_*(\pi_X^*(\theta^{-1})) \cong \pi_{X_1}^*(F_*(\theta^{-1})).$$

We now show that j is injective. Suppose it is not. Then there exists a subsheaf $\tilde{\mathcal{E}}_0 \subset \pi_{X_1}^*(F_*(\theta^{-1}))$ and a surjective map $\tau : \mathcal{E}_0 \to \tilde{\mathcal{E}}_0$. Let $\mathcal{K}$ denote the kernel of τ. Again by adjunction we obtain a map $\alpha \colon (F \times \mathrm{id}_S)^* \tilde{\mathcal{E}}_0 \to \pi_X^*(\theta^{-1})$ such that the composite map

$$(F \times \mathrm{id}_S)^* \mathcal{E}_0 \xrightarrow{\tau^*} (F \times \mathrm{id}_S)^* \tilde{\mathcal{E}}_0 \xrightarrow{\alpha} \pi_X^*(\theta^{-1})$$

coincides with σ. Here τ^* denotes the map $(F \times \mathrm{id}_S)^* \tau$. Since σ is surjective, α is also surjective. We denote by $\mathcal{M}$ the kernel of α. The induced map $\bar{\tau} : \pi_X^*(\theta) = \ker \sigma \to \mathcal{M}$ is surjective, because τ^* is surjective. Moreover the first fundamental form of the canonical connection $\tilde{\nabla}$ on $(F \times \mathrm{id}_S)^* \tilde{\mathcal{E}}_0$ induces an $\mathcal{O}_{X \times S}$-linear homomorphism $\psi_{\tilde{\nabla}} : \mathcal{M} \to \pi_X^*(\theta)$ and the composite map

$$\psi_\nabla \colon \pi_X^*(\theta) \xrightarrow{\bar{\tau}} \mathcal{M} \xrightarrow{\psi_{\tilde{\nabla}}} \pi_X^*(\theta)$$

coincides with the first fundamental form of ∇ of $(F \times \mathrm{id}_S)^* \mathcal{E}_0$, which is an isomorphism. Therefore $\bar{\tau}$ is an isomorphism too. So τ^* is an isomorphism and $(F \times \mathrm{id}_S)^* \mathcal{K} = 0$. We deduce that $\mathcal{K} = 0$.

In order to show that $\mathcal{E}_0 \in \underline{\mathcal{Q}}_0(S)$, it remains to verify that the quotient sheaf $\pi_{X_1}(F_*(\theta^{-1}))/\mathcal{E}_0$ is flat over S. We recall that flatness implies locally freeness because of maximality of degree. But flatness follows from [HL], Lemma 2.1.4, since the restriction of j to $X_1 \times \{s\}$ is injective for any closed $s \in S$ by Proposition 3.1 (a). □

Since $\mathcal{Q}_0$ represents the functor $\underline{\mathcal{Q}}_0$, we obtain the following

Corollary 4.8. *The scheme $\mathcal{B}_\theta$ represents the functor $\underline{\mathcal{B}}_\theta$ defined in Lemma* 4.5.

Combining Proposition 4.7 with relations (7) and (8), we obtain

Corollary 4.9. *We have*

$$l(\mathcal{B}) = \frac{16}{p^2} \cdot l(\mathcal{Q}).$$

5. Determinantal subschemes

In this section we introduce a determinantal subscheme $\mathcal{D} \subset \mathcal{N}_{X_1}$, whose length will be computed in the next section. We also show that $\mathcal{D}$ is isomorphic to Grothendieck's Quot-scheme $\mathcal{Q}$. We first define a determinantal subscheme $\tilde{\mathcal{D}}$ of a variety $JX_1 \times Z$ covering $\mathcal{N}_{X_1}$ and then we show that $\tilde{\mathcal{D}}$ is a $\mathbb{P}^1$-fibration over an étale cover of $\mathcal{D} \subset \mathcal{N}_{X_1}$.

Since there does not exist a universal bundle over $X_1 \times \mathcal{M}_{X_1}$, following an idea of Mukai [Mu], we consider the moduli space $\mathcal{M}_{X_1}(x)$ of stable rank-2 vector bundles on X_1 with determinant $\mathcal{O}_{X_1}(x)$ for a fixed point $x \in X_1$. According to [N1] the variety $\mathcal{M}_{X_1}(x)$ is a smooth intersection of two quadrics in $\mathbb{P}^5$. Let $\mathcal{U}$ denote a universal bundle on $X_1 \times \mathcal{M}_{X_1}(x)$ and denote

$$\mathcal{U}_x := \mathcal{U}|_{\{x\}\times \mathcal{M}_{X_1}(x)}$$

considered as a rank-2 vector bundle on $\mathcal{M}_{X_1}(x)$. Then the projectivized bundle

$$Z := \mathbb{P}(\mathcal{U}_x)$$

is a $\mathbb{P}^1$-bundle over $\mathcal{M}_{X_1}(x)$. The variety Z parametrizes pairs (F_z, l_z) consisting of a stable vector bundle $F_z \in \mathcal{M}_{X_1}(x)$ and a non-trivial linear form $l_z\colon F_z(x) \to k_x$ on the fibre of F_z over x defined up to a non-zero constant. Thus to any $z \in Z$ one can associate an exact sequence

$$0 \to E_z \to F_z \to k_x \to 0$$

uniquely determined up to a multiplicative constant. Clearly E_z is semistable, since F_z is stable, and $\det E_z = \mathcal{O}_{X_1}$. Hence we get a diagram (the so-called Hecke correspondence)

$$\begin{tikzcd} Z \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathcal{M}_{X_1} \cong \mathbb{P}^3 \\ \mathcal{M}_{X_1}(x) & \end{tikzcd}$$

with $\varphi(z) = [E_z]$ and $\pi(z) = F_z$. We note that there is an isomorphism $\varphi^{-1}(E) \cong \mathbb{P}^1$ (see e.g. [Mu], (3.7)) and that $\pi(\varphi^{-1}(E)) \subset \mathcal{M}_{X_1}(x) \subset \mathbb{P}^5$ is a conic for any stable $E \in \mathcal{M}^s_{X_1}$ (see e.g. [NR2]). On $X_1 \times Z$ there exists a "universal" bundle, which we denote by $\mathcal{V}$ (see [Mu], (3.8)). It has the property

$$\mathcal{V}|_{X_1\times\{z\}} \cong E_z, \quad \text{for all } z \in Z.$$

Let $\mathcal{L}$ denote a Poincaré bundle on $X_1 \times JX_1$. By abuse of notation we also denote by $\mathcal{V}$ and $\mathcal{L}$ their pull-backs to $X_1 \times JX_1 \times Z$. We denote by π_{X_1} and q the canonical projections

$$X_1 \xleftarrow{\pi_{X_1}} X_1 \times JX_1 \times Z \xrightarrow{q} JX_1 \times Z.$$

We consider the map m given by tensor product

$$m\colon JX_1 \times \mathcal{M}_{X_1} \longrightarrow \mathcal{N}_{X_1}, \quad (L, E) \longmapsto L \otimes E.$$

Note that the restriction of m to the stable locus $m^s\colon JX_1 \times \mathcal{M}^s_{X_1} \longrightarrow \mathcal{N}^s_{X_1}$ is an étale map of degree 16. We denote by ψ the composite map

$$\psi\colon JX_1 \times Z \xrightarrow{\mathrm{id}_{JX_1}\times\varphi} JX_1 \times \mathcal{M}_{X_1} \xrightarrow{m} \mathcal{N}_{X_1}, \quad \psi(L, z) = L \otimes E_z.$$

Let $D \in |\omega_{X_1}|$ be a smooth canonical divisor on X_1. We introduce the following sheaves over $JX_1 \times Z$

$$\mathcal{F}_1 = q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi^*_{X_1}(F_*(\theta^{-1}) \otimes \omega_{X_1}))$$

and

$$\mathcal{F}_0 = \oplus_{y\in D}\big(\mathcal{L}^* \otimes \mathcal{V}^*|_{\{y\}\times JX_1\times Z}\big) \otimes k^{\oplus p}.$$

The next proposition is an even degree analogue of [LN], Theorem 3.1.

Proposition 5.1. (a) *The sheaves $\mathcal{F}_0$ and $\mathcal{F}_1$ are locally free of rank $4p$ and $4p-4$ respectively and there is an exact sequence*

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{\gamma} \mathcal{F}_0 \longrightarrow R^1q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi^*_{X_1}(F_*(\theta^{-1}))) \longrightarrow 0.$$

Let $\tilde{\mathcal{D}} \subset JX_1 \times Z$ denote the subscheme defined by the 4-th Fitting ideal of the sheaf $R^1q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi^*_{X_1}(F_*(\theta^{-1})))$. We have set-theoretically

$$\operatorname{supp}\tilde{\mathcal{D}} = \{(L, z) \in JX_1 \times Z \mid \dim \operatorname{Hom}(L \otimes E_z, F_*(\theta^{-1})) = 1\},$$

and $\dim \tilde{\mathcal{D}} = 1$.

(b) *Let δ denote the l-adic ($l \neq p$) cohomology class of $\tilde{\mathcal{D}}$ in $JX_1 \times Z$. Then*

$$\delta = c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{10}(JX_1 \times Z, \mathbb{Z}_l).$$

Proof. We consider the canonical exact sequence over $X_1 \times JX_1 \times Z$ associated to the effective divisor $\pi^*_{X_1}D$

$$\begin{aligned} 0 \to \mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi^*_{X_1}F_*(\theta^{-1}) &\xrightarrow{\otimes D} \mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi^*_{X_1}(F_*(\theta^{-1}) \otimes \omega_{X_1}) \\ &\to \mathcal{L}^* \otimes \mathcal{V}^*|_{\pi^*_{X_1}D} \otimes k^{\oplus p} \to 0. \end{aligned}$$

By Proposition 1.2 the rank-p vector bundle $F_*(\theta^{-1})$ is stable and since

$$1 - \frac{2}{p} = \mu(F_*(\theta^{-1})) > \mu(L \otimes E) = 0 \quad \text{for all } (L, E) \in JX_1 \times \mathcal{M}_{X_1},$$

we obtain

$$\dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1}) \otimes \omega_{X_1}) = \dim \operatorname{Hom}(F_*(\theta^{-1}), L \otimes E) = 0.$$

This implies

$$R^1 q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}) \otimes \omega_{X_1})) = 0.$$

By the base change theorems the sheaf $\mathcal{F}_1$ is locally free. Taking direct images by q (note that $q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* F_*(\theta^{-1})) = 0$ because it is a torsion sheaf), we obtain the exact sequence

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{\gamma} \mathcal{F}_0 \longrightarrow R^1 q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))) \longrightarrow 0.$$

with $\mathcal{F}_1$ and $\mathcal{F}_0$ as in the statement of the proposition. Note that by Riemann–Roch we have

$$\operatorname{rk} \mathcal{F}_1 = 4p - 4 \quad \text{and} \quad \operatorname{rk} \mathcal{F}_0 = 4p.$$

It follows from the proof of Proposition 3.1 (a) that any nonzero homomorphism $L \otimes E \longrightarrow F_*(\theta^{-1})$ is injective. Moreover by Proposition 3.1 (b) (iii) for any subbundle $L \otimes E \subset F_*(\theta^{-1})$ we have $\dim \operatorname{Hom}(L \otimes E, F_*(\theta^{-1})) = 1$, or equivalently $\dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1})) = 5$. Using the base change theorems we obtain the following series of equivalences

$$\begin{aligned}
(L, z) \in \operatorname{supp} \tilde{\mathcal{D}} &\iff \operatorname{rk} \gamma_{(L,z)} < 4p - 4 = \operatorname{rk} \mathcal{F}_1 \\
&\iff \dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1})) \geq 5 \\
&\iff \dim \operatorname{Hom}(L \otimes E, F_*(\theta^{-1})) \geq 1 \\
&\iff \dim \operatorname{Hom}(L \otimes E, F_*(\theta^{-1})) = 1.
\end{aligned}$$

Finally we clearly have the equality $\operatorname{supp} \psi(\tilde{\mathcal{D}}) = \operatorname{supp} \mathcal{Q}$. Since $\dim \mathcal{Q} = 0$ and since $\varphi^{-1}(E) \cong \mathbb{P}^1$ for E stable, we deduce that $\dim \tilde{\mathcal{D}} = 1$. This proves part (a).

Part (b) follows from Porteous' formula, which says that the fundamental class $\delta \in H^{10}(JX_1 \times Z, \mathbb{Z}_l)$ of the determinantal subscheme $\tilde{\mathcal{D}}$ is given (with the notation of [ACGH], p. 86) by

$$\begin{aligned}
\delta &= \Delta_{4p-(4p-5), 4p-4-(4p-5)}(c_t(\mathcal{F}_0 - \mathcal{F}_1)) \\
&= \Delta_{5,1}(c_t(\mathcal{F}_0 - \mathcal{F}_1)) \\
&= c_5(\mathcal{F}_0 - \mathcal{F}_1).
\end{aligned}$$

□

Let M be a sheaf over a k-scheme S. We denote by

$$\mathrm{Fitt}_n[M] \subset \mathcal{O}_S$$

the n-th Fitting ideal sheaf of M.

We now define the 0-dimensional subscheme $\mathcal{D} \subset \mathcal{N}^s_{X_1}$, which is supported on $\operatorname{supp} \mathcal{Q}$, by defining a scheme structure $\mathcal{D}_E$ for every $E \in \operatorname{supp} \mathcal{Q}$. Note that

$$\mathcal{D} = \coprod_{E \in \operatorname{supp} \mathcal{Q}} \mathcal{D}_E.$$

Consider a bundle $E \in \mathcal{N}^s_{X_1}$ with $E \in \operatorname{supp} \mathcal{Q}$, i.e.

$$\dim \mathrm{Hom}(E, F_*(\theta^{-1})) \geq 1 \iff \dim H^1(E^* \otimes F_*(\theta^{-1})) \geq 5.$$

The GIT-construction of the moduli space $\mathcal{N}^s_{X_1}$ realizes $\mathcal{N}^s_{X_1}$ as a quotient of an open subset $\mathcal{U}$ of a Quot-scheme by the group $\mathbb{P}\mathrm{GL}(N)$ for some N. It can be shown (see e.g. [La2], Section 3) that $\mathcal{U}$ is a principal $\mathbb{P}\mathrm{GL}(N)$-bundle for the étale topology over $\mathcal{N}^s_{X_1}$. Hence there exists an étale neighbourhood $\tau : \overline{U} \to U$ of E over which the $\mathbb{P}\mathrm{GL}(N)$-bundle is trivial, i.e., admits a section. The universal bundle over the Quot-scheme restricts to a bundle $\mathcal{E}$ over $X_1 \times \overline{U}$. Choose a point $\overline{E} \in \overline{U}$ over E. We denote by $\mathcal{D}_{\overline{E}}$ the connected component supported at $\overline{E}$ of the scheme defined by the Fitting ideal sheaf

$$\mathrm{Fitt}_4[R^1\pi_{\overline{U}*}(\mathcal{E}^* \otimes \pi^*_{X_1} F_*(\theta^{-1}))].$$

Lemma 5.2. *Let $\tau : \overline{U} \to U$ be an étale map and $y \in \overline{U}$, $x \in U$ such that $\tau(y) = x$. Let $\overline{\Lambda} \subset \overline{U}$ be a 0-dimensional scheme supported at y. Then the restriction of τ to $\overline{\Lambda}$ induces an isomorphism of $\overline{\Lambda}$ with its scheme-theoretical image in $\Delta = \tau(\overline{\Lambda}) \subset U$, i.e.*

$$\tau|_{\overline{\Lambda}} : \overline{\Lambda} \xrightarrow{\sim} \Delta \subset U.$$

Proof. We denote by $A = \mathcal{O}_{\overline{U},y}$, $B = \mathcal{O}_{U,x}$ the local rings at the points y, x and by $\mathfrak{m}_A, \mathfrak{m}_B$ their maximal ideals. Let $I \subset \mathfrak{m}_A$ denote the ideal defining the scheme $\overline{\Lambda}$. Since $\dim \overline{\Lambda} = 0$ there exists an integer n such that $\mathfrak{m}_A^n \subset I$. The natural map $B \hookrightarrow A \twoheadrightarrow A/I$ factorizes as follows

$$\beta : B \twoheadrightarrow B/\mathfrak{m}_B^n \xrightarrow{\alpha} A/\mathfrak{m}_A^n \twoheadrightarrow A/I.$$

Note that α is an isomorphism, since τ is étale (see e.g. [Mum], Corollary 1 of Theorem III.5.3). This shows that β is surjective, hence $\tau|_{\overline{\Lambda}}$ is an isomorphism. $\square$

Proposition–Definition 5.3. *For $E \in \operatorname{supp}\mathcal{Q}$ we define $\mathcal{D}_E$ as the scheme-theoretical image $\tau(\mathcal{D}_{\overline{E}}) \subset \mathcal{N}^s_{X_1}$ under the étale map τ. Then the scheme $\mathcal{D}_E$ does not depend on the étale neighbourhood $\tau : \overline{U} \to U$ of E and the point $\overline{E}$.*

Proof. Consider for $i = 1, 2$ étale neighbourhoods $\tau_i : \overline{U_i} \to U$ such that universal bundles $\mathcal{E}_i$ exist over $X_1 \times \overline{U_i}$, and points $\overline{E_i} \in \overline{U_i}$ lying over $E \in U$. Because of Lemma 5.2 it will be enough to show that the schemes $\mathcal{D}_{\overline{E_1}}$ and $\mathcal{D}_{\overline{E_2}}$ are isomorphic.

Consider the fibre product $\overline{U} = \overline{U_1} \times_U \overline{U_2}$ and the point $\overline{E} = (\overline{E_1}, \overline{E_2}) \in \overline{U}$. The two projections $\pi_i : \overline{U} \to \overline{U_i}$ for $i = 1, 2$ are étale. Moreover $(\mathrm{id}_{X_1} \times \pi_i)^* \mathcal{E}_i \sim \mathcal{E}$, where $\mathcal{E}$ denotes the universal bundle over $X_1 \times \overline{U}$. Since the formation of the Fitting ideal and taking the higher direct image $R^1 \pi_{\overline{U}*}$ commutes with the flat base changes π_1 and π_2 (see [E], Corollary 20.5), we obtain for $i = 1, 2$

$$\pi_i^{-1}\big[\mathrm{Fitt}_4(R^1\pi_{\overline{U}_1*}(\mathcal{E}_i^* \otimes \pi_{X_1}^* F_*(\theta^{-1}))\big] \cdot \mathcal{O}_{\overline{U}} = \mathrm{Fitt}_4(R^1\pi_{\overline{U}*}(\mathcal{E}^* \otimes \pi_{X_1}^* F_*(\theta^{-1}))).$$

This shows that the connected components supported at $\overline{E}$ of the fibres $\pi_i^{-1}(\mathcal{D}_{\overline{E_i}})$ equal $\mathcal{D}_{\overline{E}}$. Applying Lemma 5.2 to π_i and $\mathcal{D}_{\overline{E}}$ we obtain isomorphisms $\pi_i : \mathcal{D}_{\overline{E}} \to \mathcal{D}_{\overline{E_i}}$ and we are done. □

Lemma 5.4. (a) *Let S be a k-scheme and $\mathcal{E}$ a sheaf over $X_1 \times S$ with $\langle \mathcal{E} \rangle \in \mathcal{N}^s_{X_1}(S)$. We suppose that the set-theoretical image of the classifying morphism of $\mathcal{E}$*

$$\Phi_{\mathcal{E}} : S \longrightarrow \mathcal{N}^s_{X_1}, \quad s \longmapsto \mathcal{E}|_{X_1 \times \{s\}}$$

is a point. Then there exists an Artinian ring A, a morphism $\varphi : S \longrightarrow \Delta := \mathrm{Spec}(A)$ and a locally free sheaf $\mathcal{E}_0$ over $X_1 \times \Delta$ such that

(1) $\mathcal{E} \sim (\mathrm{id}_{X_1} \times \varphi)^* \mathcal{E}_0$

(2) *the natural map $\mathcal{O}_\Delta \longrightarrow \varphi_* \mathcal{O}_S$ is injective.*

(b) *There exists a universal family $\mathcal{E}_0$ over $X_1 \times \mathcal{D}$.*

Proof. (a) Since the set-theoretical support of $\operatorname{im} \Phi_{\mathcal{E}}$ is a point $x \in \mathcal{N}^s_{X_1}$, there exists an Artinian ring A such that $\Phi_{\mathcal{E}}$ factorizes through the inclusion $\Delta := \mathrm{Spec}(A) \hookrightarrow \mathcal{N}^s_{X_1}$. As explained above there exists an étale neighbourhood $\tau : \overline{U} \to U$ of x such that there is a universal bundle $\mathcal{E}^{\mathrm{univ}}$ over $X_1 \times \overline{U}$. Choose $y \in \overline{U}$ such that $\tau(y) = x$ and denote by $\overline{\Lambda} \subset \overline{U}$ the connected component supported at y of the fibre $\tau^{-1}(\Delta)$. By Lemma 5.2 there is an isomorphism $\tau : \overline{\Lambda} \xrightarrow{\sim} \Delta$. Denote by $\mathcal{E}_0$ the restriction of $\mathcal{E}^{\mathrm{univ}}$ to $X_1 \times \overline{\Lambda} \cong X_1 \times \Delta$. This shows property (1). As for (2), we consider the ideal $I \subset A$ defined by $\tilde{I} = \ker(\mathcal{O}_{\mathrm{Spec}(A)} \to \varphi_* \mathcal{O}_S)$, where $\tilde{I}$ denotes the associated $\mathcal{O}_{\mathrm{Spec}(A)}$-module. If $I \neq 0$, we replace A by A/I and we are done.

(b) We take $\Delta = \mathcal{D}_E$ and $\overline{\Lambda} = \mathcal{D}_{\overline{E}}$ and proceed as in (a). □

Proposition 5.5. *The subscheme $\mathcal{D} \subset \mathcal{N}^s_{X_1}$ represents the functor $\underline{\mathcal{D}}$ which associates to any k-scheme S the set*

$$\underline{\mathcal{D}}(S) = \{\mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2 \mid \deg \mathcal{E}|_{X_1\times\{s\}} = 0$$
$$\text{for all } s \in S, \operatorname{Fitt}_4[R^1\pi_{S*}(\mathcal{E}^* \otimes \pi^*_{X_1}(F_*(\theta^{-1})))] = 0\}/\sim .$$

Proof. Consider a sheaf $\mathcal{E}$ over $X_1 \times S$ with $\langle\mathcal{E}\rangle \in \underline{\mathcal{N}}^s_{X_1}(S)$. Then $\langle\mathcal{E}\rangle$ is an element of $\big[\mathcal{D} \times_{\mathcal{N}^s_{X_1}} \underline{\mathcal{N}}^s_{X_1}\big](S)$ if and only if the classifying map $\Phi_{\mathcal{E}} : S \to \mathcal{N}^s_{X_1}$ factorizes as $S \xrightarrow{\varphi} \mathcal{D} \subset \mathcal{N}^s_{X_1}$. By Lemma 5.4 (b) there exists a universal family $\mathcal{E}_0$ over $X_1 \times \mathcal{D}$ and we have $\mathcal{E} \sim (\mathrm{id}_{X_1} \times \varphi)^*\mathcal{E}_0$. Since $\mathcal{D}$ is defined (over an étale cover) by a Fitting ideal and since the formation of the Fitting ideal commutes with any base change, we deduce that $\big[\mathcal{D} \times_{\mathcal{N}^s_{X_1}} \underline{\mathcal{N}}^s_{X_1}\big](S) = \underline{\mathcal{D}}(S)$. Since $\mathcal{N}^s_{X_1}$ universally corepresents the functor $\underline{\mathcal{N}}^s_{X_1}$, this shows that $\mathcal{D}$ corepresents the functor $\underline{\mathcal{D}}$. The existence of a universal family $\mathcal{E}_0$ over $X \times \mathcal{D}$ implies that $\mathcal{D}$ represents the functor $\underline{\mathcal{D}}$. □

Proposition 5.6. *There is a scheme-theoretical equality*

$$\tilde{\mathcal{D}} = \psi^{-1}\mathcal{D}.$$

Proof. In order to show that the subschemes $\tilde{\mathcal{D}}$ and $\psi^{-1}\mathcal{D}$ of $JX_1 \times Z$ coincide, it is enough to show that the two subsets $\operatorname{Mor}(S, \tilde{\mathcal{D}})$ and $\operatorname{Mor}(S, \psi^{-1}\mathcal{D})$ of $\operatorname{Mor}(S, JX_1\times Z)$ coincide for any k-scheme S. Consider $\Phi \in \operatorname{Mor}(S, JX_1 \times Z)$ and denote $\mathcal{E}_\Phi := (\mathrm{id}_{X_1} \times \Phi)^*(\mathcal{L}\otimes\mathcal{V})$. By definition of $\tilde{\mathcal{D}}$ we have $\Phi \in \operatorname{Mor}(S, \tilde{\mathcal{D}})$ if and only if $\operatorname{Fitt}_4[R^1\pi_{S*}(\mathcal{E}^*_\Phi \otimes \pi^*_{X_1}(F_*(\theta^{-1})))] = 0$. On the other hand $\Phi \in \operatorname{Mor}(S, \psi^{-1}(\mathcal{D}))$ if and only if $\psi \circ \Phi \in \operatorname{Mor}(S, \mathcal{D})$. The latter set equals $\underline{\mathcal{D}}(S)$ by Proposition 5.5. Since $(\psi \circ \Phi)^*\mathcal{E}_0 \sim \mathcal{E}_\Phi$, we are done. □

Proposition 5.7. *There is a scheme-theoretical equality*

$$\mathcal{D} = \mathcal{Q}.$$

Proof. We note that $\underline{\mathcal{D}}(S)$ and $\underline{\mathcal{Q}}(S)$ are subsets of $\underline{\mathcal{N}}^s_{X_1}(S)$ (the injectivity of the map $\underline{\mathcal{Q}}(S) \to \underline{\mathcal{N}}^s_{X_1}(S)$ is proved similarly as in the proof of Proposition 4.7). Since $\mathcal{D}$ and $\mathcal{Q}$ corepresent the two functors $\underline{\mathcal{D}}$ and $\underline{\mathcal{Q}}$, it will be enough to show that the set $\underline{\mathcal{D}}(S)$ coincides with $\underline{\mathcal{Q}}(S)$ for any k-scheme S.

We first show that $\underline{\mathcal{D}}(S) \subset \underline{\mathcal{Q}}(S)$. Consider a sheaf $\mathcal{E}$ with $\langle\mathcal{E}\rangle \in \underline{\mathcal{D}}(S)$. For simplicity we denote the sheaf $\mathcal{E}^* \otimes \pi^*_{X_1}(F_*(\theta^{-1}))$ by $\mathcal{H}$. By [Ha], Theorem 12.11, there is an isomorphism

$$R^1\pi_{S*}\mathcal{H} \otimes k(s) \cong H^1(X_1 \times \{s\}, \mathcal{H}|_{X_1\times\{s\}}) \quad \text{for all } s \in S.$$

From our assumption $\operatorname{Fitt}_4[R^1\pi_{S*}\mathcal{H}] = 0$, we obtain $\dim H^1(X_1 \times \{s\}, \mathcal{H}|_{X_1\times\{s\}}) \geq 5$, or equivalently $\dim H^0(X_1 \times \{s\}, \mathcal{H}|_{X_1\times\{s\}}) \geq 1$, i.e., the vector bundle $\mathcal{E}|_{X_1\times\{s\}}$

is a subsheaf, hence by Proposition 3.1 (a) (ii) a subbundle, of $F_*(\theta^{-1})$. This implies that the set-theoretical image of the classifying map $\Phi_{\mathcal{E}}$ is contained in supp$\mathcal{Q}$. Taking connected components of S, we can assume that the image of $\Phi_{\mathcal{E}}$ is a point. Therefore we can apply Lemma 5.4: there exists a locally free sheaf $\mathcal{E}_0$ over $X_1 \times \Delta$ such that $\mathcal{E} \sim (\mathrm{id}_{X_1} \times \varphi)^* \mathcal{E}_0$. For simplicity we write $\mathcal{H}_0 = \mathcal{E}_0^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))$. In particular $\mathcal{H} = (\mathrm{id}_{X_1} \times \varphi)^* \mathcal{H}_0$. Since the projection $\pi_\Delta \colon X_1 \times \Delta \to \Delta$ is of relative dimension 1, taking the higher direct image $R^1\pi_{\Delta *}$ commutes with the (not necessarily flat) base change $\varphi \colon S \to \Delta$ ([Ha], Proposition 12.5), i.e., there is an isomorphism

$$\varphi^* R^1 \pi_{\Delta *} \mathcal{H}_0 \cong R^1 \pi_{S*} \mathcal{H}.$$

Since the formation of Fitting ideals also commutes with any base change (see [E], Corollary 20.5), we obtain

$$\mathrm{Fitt}_4[R^1\pi_{S*}\mathcal{H}] = \mathrm{Fitt}_4[R^1\pi_{\Delta *}\mathcal{H}_0] \cdot \mathcal{O}_S.$$

Since $\mathrm{Fitt}_4[R^1\pi_{S*}\mathcal{H}]$ is equal to 0 and $\mathcal{O}_\Delta \to \varphi_* \mathcal{O}_S$ is injective, we deduce that $\mathrm{Fitt}_4[R^1\pi_{\Delta *}\mathcal{H}_0] = 0$. Since by Proposition 3.1 (b) (iii) $\dim R^1\pi_{\Delta *}\mathcal{H}_0 \otimes k(s_0) = 5$ for the closed point $s_0 \in \Delta$, we have $\mathrm{Fitt}_5[R^1\pi_{\Delta *}\mathcal{H}_0] = \mathcal{O}_\Delta$. We deduce by [E], Proposition 20.8, that the sheaf $R^1\pi_{\Delta *}\mathcal{H}_0$ is a free A-module of rank 5. By [Ha], Theorem 12.11 (b), we deduce that there is an isomorphism

$$\pi_{\Delta *}\mathcal{H}_0 \otimes k(s_0) \cong H^0(X_1 \times s_0, \mathcal{H}|_{X_1 \times \{s_0\}})$$

Again by Proposition 3.1 (b) (iii) we obtain $\dim \pi_{\Delta *}\mathcal{H}_0 \otimes k(s_0) = 1$. In particular the $\mathcal{O}_\Delta$-module $\pi_{\Delta *}\mathcal{H}_0$ is not zero and therefore there exists a nonzero global section $i \in H^0(\Delta, \pi_{\Delta *}\mathcal{H}_0) = H^0(X_1 \times \Delta, \mathcal{E}_0^* \otimes \pi_{X_1}^* F_*(\theta^{-1}))$. We pull-back i under the map $\mathrm{id}_{X_1} \times \varphi$ and we obtain a nonzero section

$$j = (\mathrm{id}_{X_1} \times \varphi)^* i \in H^0(X_1 \times S, \mathcal{E}^* \otimes \pi_{X_1}^* F_*(\theta^{-1})).$$

Now we apply Lemma 4.3 and we continue as in the proof of Proposition 4.7. This shows that $\langle \mathcal{E} \rangle \in \underline{\mathcal{Q}}(S)$.

We now show that $\underline{\mathcal{Q}}(S) \subset \underline{\mathcal{D}}(S)$. Consider a sheaf $\mathcal{E} \in \underline{\mathcal{Q}}(S)$. The nonzero global section $j \in H^0(X_1 \times S, \mathcal{H}) = H^0(S, \pi_{S*}\mathcal{H})$ determines by evaluation at a point $s \in S$ an element $\alpha \in \pi_{S*}\mathcal{H} \otimes k(s)$. The image of α under the natural map

$$\varphi^0(s) \colon \pi_{S*}\mathcal{H} \otimes k(s) \longrightarrow H^0(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}})$$

coincides with $j|_{X_1 \times \{s\}}$ which is nonzero. Also, as $\dim H^0(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}}) = 1$, we obtain that $\varphi^0(s)$ is surjective. Hence by [Ha], Theorem 12.11, the sheaf $R^1\pi_{S*}\mathcal{H}$ is locally free of rank 5. Again by [E], Proposition 20.8, this is equivalent to $\mathrm{Fitt}_4[R^1\pi_{S*}\mathcal{H}] = 0$ and $\mathrm{Fitt}_5[R^1\pi_{S*}\mathcal{H}] = \mathcal{O}_S$ and we are done. □

6. Chern class computations

In this section we will compute the length of the determinantal subscheme $\mathcal{D} \subset \mathcal{N}_{X_1}$ by evaluating the Chern class $c_5(\mathcal{F}_0 - \mathcal{F}_1)$ – see Proposition 5.1 (b).

Let l be a prime number different from p. We have to recall some properties of the cohomology ring $H^*(X_1 \times JX_1 \times Z, \mathbb{Z}_l)$ (see also [LN]). In the sequel we identify all classes of $H^*(X_1, \mathbb{Z}_l)$, $H^*(JX_1, \mathbb{Z}_l)$ etc. with their preimages in $H^*(X_1 \times JX_1 \times Z, \mathbb{Z}_l)$ under the natural pull-back maps.

Let $\Theta \in H^2(JX_1, \mathbb{Z}_l)$ denote the class of the theta divisor in JX_1. Let f denote a positive generator of $H^2(X_1, \mathbb{Z}_l)$. The cup product $H^1(X_1, \mathbb{Z}_l) \times H^1(X_1, \mathbb{Z}_l) \to H^2(X_1, \mathbb{Z}_l) \simeq \mathbb{Z}_l$ gives a symplectic structure on $H^1(X_1, \mathbb{Z}_l)$. Choose a symplectic basis e_1, e_2, e_3, e_4 of $H^1(X_1, \mathbb{Z}_l)$ such that $e_1e_3 = e_2e_4 = -f$ and all other products $e_ie_j = 0$. We can then normalize the Poincaré bundle $\mathcal{L}$ on $X_1 \times JX_1$ so that

$$c(\mathcal{L}) = 1 + \xi_1 \tag{9}$$

where $\xi_1 \in H^1(X_1, \mathbb{Z}_l) \otimes H^1(JX_1, \mathbb{Z}_l) \subset H^2(X_1 \times JX_1, \mathbb{Z}_l)$ can be written as

$$\xi_1 = \sum_{i=1}^{4} e_i \otimes \varphi_i$$

with $\varphi_i \in H^1(JX_1, \mathbb{Z}_l)$. Moreover, we have by the same reasoning, applying [ACGH], p. 335 and p. 21,

$$\xi_1^2 = -2\Theta f \quad \text{and} \quad \Theta^2[JX_1] = 2. \tag{10}$$

Since the variety $\mathcal{M}_{X_1}(x)$ is a smooth intersection of 2 quadrics in $\mathbb{P}^5$, one can work out that the l-adic cohomology groups $H^i(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$ for $i = 0, \dots, 6$ are (see e.g. [Re], p. 0.19)

$$\mathbb{Z}_l, \quad 0, \quad \mathbb{Z}_l, \quad \mathbb{Z}_l^4, \quad \mathbb{Z}_l, \quad 0, \quad \mathbb{Z}_l.$$

In particular $H^2(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$ is free of rank 1 and, if α denotes a positive generator of it, then

$$\alpha^3[\mathcal{M}_{X_1}(x)] = 4. \tag{11}$$

According to [N2] p. 338 and applying reduction mod p and a comparison theorem, the Chern classes of the universal bundle $\mathcal{U}$ are of the form

$$c_1(\mathcal{U}) = \alpha + f \quad \text{and} \quad c_2(\mathcal{U}) = \chi + \xi_2 + \alpha f \tag{12}$$

with $\chi \in H^4(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$ and $\xi_2 \in H^1(X_1, \mathbb{Z}_l) \otimes H^3(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$. As in [N2] and [KN] we write

$$\beta = \alpha^2 - 4\chi \quad \text{and} \quad \xi_2^2 = \gamma f \quad \text{with } \gamma \in H^6(\mathcal{M}_{X_1}(x), \mathbb{Z}_l). \tag{13}$$

Then the relations of [KN] give

$$\alpha^2 + \beta = 0 \quad \text{and} \quad \alpha^3 + 5\alpha\beta + 4\gamma = 0.$$

Hence $\beta = -\alpha^2,\ \gamma = \alpha^3$. Together with (12) and (13) this gives

$$c_2(\mathcal{U}) = \frac{\alpha^2}{2} + \xi_2 + \alpha f \quad \text{and} \quad \xi_2^2 = \alpha^3 f \tag{14}$$

Define $\Lambda \in H^1(JX_1, \mathbb{Z}_l) \otimes H^3(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$ by

$$\xi_1 \xi_2 = \Lambda f. \tag{15}$$

Then we have for dimensional reasons and noting that $H^5(\mathcal{M}_{X_1}(x), \mathbb{Z}_l) = 0$, that the following classes are all zero:

$$f^2, \quad \xi_1^3, \quad \alpha^4, \quad \xi_1 f, \quad \xi_2 f, \quad \alpha\xi_2, \quad \alpha\Lambda, \quad \Theta^2\Lambda, \quad \Theta^3. \tag{16}$$

Finally, Z is the $\mathbb{P}^1$-bundle associated to the vector bundle $\mathcal{U}_x$ on $\mathcal{M}_{X_1}(x)$. Let $H \in H^2(Z, \mathbb{Z}_l)$ denote the first Chern class of the tautological line bundle on Z. We have, using the definition of the Chern classes $c_i(\mathcal{U})$ and (11),

$$H^2 = \alpha H - \frac{\alpha^2}{2}, \quad H^4 = 0, \quad \alpha^3 H[Z] = 4 \tag{17}$$

and we get for the "universal" bundle $\mathcal{V}$,

$$c_1(\mathcal{V}) = \alpha \quad \text{and} \quad c_2(\mathcal{V}) = \frac{\alpha^2}{2} + \xi_2 + Hf. \tag{18}$$

Lemma 6.1. (a) *The cohomology class $\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{12}(JX_1 \times Z, \mathbb{Z}_l)$ is a multiple of the class $\alpha^3 H \Theta^2$.*

(b) *The pull-back under the map $\varphi\colon Z \longrightarrow \mathcal{M}_{X_1} \cong \mathbb{P}^3$ of the class of a point is the class $H^3 = \frac{\alpha^2}{2}H - \frac{\alpha^3}{2}$.*

Proof. For part (a) it is enough to note that all other relevant cohomology classes vanish, since $\alpha^4 = 0$ and $\alpha\Lambda = 0$.

As for part (b), it suffices to show that $c_1(\varphi^* \mathcal{O}_{\mathbb{P}^3}(1)) = H$. The line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$ is the inverse of the determinant line bundle [KM] over the moduli space $\mathcal{M}_{X_1}$. Since the formation of the determinant line bundle commutes with any base change (see [KM]), the pull-back $\varphi^*\mathcal{O}_{\mathbb{P}^3}(1)$ is the inverse of the determinant line bundle associated to the family $\mathcal{V} \otimes \pi_{X_1}^* N$ for any line bundle N of degree 1 over X_1. Hence the first Chern class of $\varphi^*\mathcal{O}_{\mathbb{P}^3}(1)$ can be computed by the Grothendieck–Riemann–Roch theorem applied to the sheaf $\mathcal{V} \otimes \pi_{X_1}^* N$ over $X_1 \times Z$ and the morphism

$\pi_Z \colon X_1 \times Z \to Z$. We have

$$\begin{aligned} ch(\mathcal{V} \otimes \pi_{X_1}^* N) \cdot \pi_{X_1}^* td(X_1) &= (2 + \alpha + (-\xi_2 - Hf) + \text{h.o.t.})\,(1+f)(1-f) \\ &= 2 + \alpha + (-\xi_2 - Hf) + \text{h.o.t.}, \end{aligned}$$

and therefore G-R-R implies that $c_1(\varphi^* \mathcal{O}_{\mathbb{P}^3}(1)) = H$ – note that $\pi_{Z*}(\xi_2) = 0$. □

Proposition 6.2. *We have*

$$l(\mathcal{D}) = \frac{1}{24} p^3 (p^2 - 1).$$

Proof. Let λ denote the length of the subscheme $m^{-1}(\mathcal{D}) \subset JX_1 \times \mathcal{M}_{X_1}$ Since the map m^s is étale of degree 16, we obviously have the relation $\lambda = 16 \cdot l(\mathcal{D})$. According to Lemma 6.1 (b) we have in $H^{10}(JX_1 \times Z, \mathbb{Z}_l)$

$$[(\mathrm{id} \times \varphi)^{-1}(pt)] = H^3 \cdot \frac{\Theta^2}{2} = \frac{1}{4}\alpha^2 H \Theta^2 - \frac{1}{4}\alpha^3 \Theta^2,$$

where pt denotes the class of a point in $JX_1 \times \mathcal{M}_{X_1}$. Using Proposition 5.6 we obtain that the class $\delta = c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{10}(JX_1 \times Z, \mathbb{Z}_l)$ equals $\lambda \cdot (\frac{1}{4}\alpha^2 H\Theta^2 - \frac{1}{4}\alpha^3\Theta^2)$. Intersecting with α we obtain with Lemma 6.1 (a) and (16)

$$\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1) = \frac{\lambda}{4}\alpha^3 H \Theta^2. \tag{19}$$

So we have to compute the class $\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1)$. By (9) and (10),

$$ch(\mathcal{L}) = 1 + \xi_1 - \Theta f$$

whereas by (14), (16) and (18),

$$ch(\mathcal{V}) = 2 + \alpha + (-\xi_2 - Hf) + \frac{1}{12}(-\alpha^3 - 6\alpha H f) + \frac{1}{12}(\alpha^3 f - \alpha^2 H f).$$

Moreover

$$ch(\pi_{X_1}^*(F_*(\theta^{-1}) \otimes \omega_{X_1})) \cdot \pi_{X_1}^* td(X_1) = p + (2p-2) f.$$

So using (14), (15) and (16),

$$\begin{aligned} &ch(\mathcal{V}^* \otimes \mathcal{L}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}) \otimes \omega_{X_1})) \cdot \pi_{X_1}^* td(X_1) \\ &\quad = 2p + [(4p-4)f - p\alpha - 2p\xi_1] \\ &\qquad + \Big[p\alpha\xi_1 - 2p\Theta f - (2p-2)\alpha f - p\xi_2 - pHf\Big] \\ &\qquad + \Big[\frac{p}{12}\alpha^3 + \frac{p}{2}\alpha H f + p\Lambda f + p\alpha\Theta f\Big] \\ &\qquad + \Big[\frac{3p-2}{12}\alpha^3 f - \frac{p}{12}\alpha^3\xi_1 - \frac{p}{12}\alpha^2 H f\Big] + \Big[-\frac{p}{12}\alpha^3\Theta f\Big]. \end{aligned}$$

Hence by Grothendieck–Riemann–Roch for the morphism q we get

$$ch(\mathcal{F}_1) = 4p - 4 + [-(2p-2)\alpha - 2p\Theta - pH] + \left[\frac{p}{2}\alpha H + p\Lambda + p\alpha\Theta\right] + \left[\frac{3p-2}{12}\alpha^3 - \frac{p}{12}\alpha^2 H\right] + \left[-\frac{p}{12}\alpha^3\Theta\right].$$

From (10) and (18) we easily obtain

$$ch(\mathcal{F}_0) = 4p - 2p\alpha + \frac{p}{6}\alpha^3.$$

So

$$ch(\mathcal{F}_0 - \mathcal{F}_1) = 4 + [2p\Theta - 2\alpha + pH] + \left[-\frac{p}{2}\alpha H - p\Lambda - p\alpha\Theta\right] + \left[-\frac{p+1}{12}\alpha^3 + \frac{p}{12}\alpha^2 H\right] + \left[\frac{p}{12}\alpha^3\Theta\right].$$

Defining $p_n := n! \cdot ch_n(\mathcal{F}_0 - \mathcal{F}_1)$ we have according to Newton's recursive formula ([F] p. 56),

$$c_5(\mathcal{F}_0 - \mathcal{F}_1) = \frac{1}{5}\left(p_5 - \frac{5}{6}p_2p_3 - \frac{5}{4}p_1p_4 + \frac{5}{6}p_1^2p_3 + \frac{5}{8}p_1p_2^2 - \frac{5}{12}p_1^3p_2 + \frac{1}{24}p_1^5\right)$$

with

$$p_1 = 2p\Theta - 2\alpha + pH, \quad p_2 = -p(\alpha H + 2\Lambda + 2\alpha\Theta),$$

$$p_3 = \frac{1}{2}(-(p+1)\alpha^3 + p\alpha^2 H), \quad p_4 = 2p\alpha^3\Theta, \quad p_5 = 0.$$

Now an immediate computation using (16) and (17) gives

$$\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1) = \frac{p^3(p^2-1)}{6}\alpha^3 H\Theta^2.$$

We conclude from (19) that $\lambda = \frac{2}{3}p^3(p^2-1)$ and we are done. □

Remark 6.3. If $k = \mathbb{C}$, the number of maximal subbundles of a general vector bundle has recently been computed by Y. Holla by using Gromov–Witten invariants [Ho]. His formula ([Ho], Corollary 4.6) coincides with ours.

7. Proof of Theorem 2

The proof of Theorem 2 is now straightforward. It suffices to combine Corollary 4.9, Proposition 5.7 and Proposition 6.2 to obtain the length $l(\mathcal{B})$.

The fact that $\mathcal{B}$ is a local complete intersection follows from the isomorphism $\mathcal{B}_\theta = \mathcal{Q}_0$ (Proposition 4.6) and Proposition 4.1. □

8. Questions and remarks

(1) Is the rank-p vector bundle F_*L very stable, i.e. F_*L has no nilpotent ω_{X_1}-valued endomorphisms, for a general line bundle?

(2) Is $F_*(\theta^{-1})$ very stable for a general curve X? Note that very-stability of $F_*(\theta^{-1})$ implies reducedness of $\mathcal{B}$ (see e.g. [LN], Lemma 3.3).

(3) If $g = 2$, we have shown that for a general stable $E \in \mathcal{M}_X$ the fibre $V^{-1}(E)$ consists of $\frac{1}{3}p(p^2+2)$ stable vector bundles $E_1 \in \mathcal{M}_{X_1}$, i.e. bundles E_1 such that $F^*E_1 \cong E$ or equivalently (via adjunction) $E_1 \subset F_*E$. The Quot-scheme parametrizing rank-2 subbundles of degree 0 of the rank-$2p$ vector bundle F_*E has expected dimension 0, contains the fibre $V^{-1}(E)$, but it also has a 1-dimensional component arising from Frobenius-destabilized bundles as follows: for any $M \in \mathrm{Pic}^1(X)$ with $\mathrm{Hom}(M^{-1}, E) \neq 0$ consider a stable degree 0 rank-2 bundle E_1 such that F^*E_1 has a nonzero map to M^{-1}.

(4) If $p = 3$ the base locus $\mathcal{B}$ consists of 16 reduced points, which correspond to the 16 nodes of the Kummer surface associated to JX (see [LP2], Corollary 6.6). For general p, does the configuration of points determined by $\mathcal{B}$ have some geometric significance?

Appendix on base loci and substack of non-semistable vector bundles.

For lack of a suitable reference, we include a detailed proof of the following fact, which was used in Lemma 4.5. We use the notation of Lemma 4.5.

Proposition A. *Let X be a smooth curve of genus 2. The closed substack $\mathfrak{M}_X^1$ equals the base locus $\mathrm{Bs}|\mathcal{O}(1)|$ of the linear system $|\mathcal{O}(1)|$ over the moduli stack $\mathfrak{M}_X^{\leq 1}$.*

Proof. Let E be a rank-2 vector bundle with trivial determinant over X. It follows from [R], Proposition 1.6.2, that E is semistable if and only if there exists a line bundle M of degree 1 such that $h^0(X, E \otimes M) = h^1(X, E \otimes M) = 0$. Consider the determinant divisor θ_M associated to M. Then $\theta_M \in |\mathcal{O}(1)|$ and for an S-valued point $\mathcal{E}$ of $\mathfrak{M}_X^{\leq 1}$

$$\mathrm{supp}(\theta_M) = \{s \in S \mid h^0(X, \mathcal{E}_s \otimes M) > 0\}.$$

We know (see e.g. [B1], Proposition 2.5) that the linear system $|\mathcal{O}(1)|$ is linearly generated by the divisors θ_M when M varies in $\mathrm{Pic}^1(X)$. The previous equivalence implies that the open complements of the closed substacks $\mathrm{Bs}|\mathcal{O}(1)|$ and $\mathfrak{M}_X^1$ coincide. To conclude the proposition it remains to show that the base locus $\mathrm{Bs}|\mathcal{O}(1)|$ is a reduced substack of $\mathfrak{M}_X^{\leq 1}$.

The normal bundle N of the closed substack $\mathfrak{M}_X^1$ in $\mathfrak{M}_X^{\leq 1}$ can be described as follows(e.g. [He], Behauptung 2.1.12, p. 44 or [VL], exposé 4, Théorème 4, p. 90): let $\mathcal{E}$ denote the universal bundle over $X \times \mathfrak{M}_X$ restricted to $X \times \mathfrak{M}_X^1$. There is a canonical inclusion

$$\mathrm{End}_0(\mathcal{E})^{\mathrm{filt}} \subset \mathrm{End}_0(\mathcal{E}),$$

where $\mathrm{End}_0(\mathcal{E})^{\mathrm{filt}}$ denotes the sheaf of tracefree endomorphisms preserving the Harder–Narasimhan filtration. We denote by $\mathrm{End}_0'(\mathcal{E})$ the quotient. Then the normal bundle N equals $R^1 p_* \mathrm{End}_0'(\mathcal{E})$, where p denotes projection onto $\mathfrak{M}_X^1$. In the rank-2 case the universal Harder–Narasimhan filtration over $X \times \mathfrak{M}_X^1$ is of the form

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1} \longrightarrow 0,$$

where $\mathcal{L}$ is a degree 1 line bundle. In that case we have $\mathrm{End}_0'(\mathcal{E}) = \mathrm{Hom}(\mathcal{L}, \mathcal{L}^{-1})$ and therefore $N = R^1 p_* \mathcal{L}^{-2}$.

Consider an S-point $\mathcal{E} \in \mathfrak{M}_X^{\leq 1}(S)$ and $x \in S$ such that the vector bundle $\mathcal{E}_x = E \in \mathfrak{M}_X^1(k)$, i.e., E is destabilized by L of degree 1. Consider a line bundle M of degree 1 and its associated determinant divisor θ_M. Then the divisor θ_M contains the closed substack $\mathfrak{M}_X^1$. The Kodaira–Spencer map at the point $x \in S$ associated to $\mathcal{E}$ is a k-linear map

$$\kappa \colon T_x S \longrightarrow H^1(X, \mathrm{End}_0(E)).$$

Note that we consider bundles with trivial determinant, hence κ takes values in $H^1(X, \mathrm{End}_0(E))$. By [Las], Sections II and III, the linear form on $T_x S$ defining the tangent space $T_x\theta_M$ to the determinant divisor θ_M is the map $\Phi \circ \kappa$, where Φ is given by cup product

$$\Phi \colon H^1(X, \mathrm{End}_0(E)) \longrightarrow \mathrm{Hom}(H^0(X, E \otimes M), H^1(X, E \otimes M)), \quad e \mapsto \cup e.$$

Using Serre duality we identify $H^1(X, \mathrm{End}_0(E))^*$ with $H^0(X, \mathrm{End}_0(E) \otimes \omega)$ and $H^1(X, E \otimes M)$ with $H^0(X, E \otimes \omega M^{-1})^*$. The dual of Φ equals the symmetrized multiplication map of global sections (note that $\mathrm{End}_0(E) = \mathrm{Sym}^2 E$ and $E = E^*$)

$$\mu \colon H^0(X, E \otimes M) \otimes H^0(X, E \otimes \omega M^{-1}) \longrightarrow H^0(X, \mathrm{End}_0(E) \otimes \omega).$$

Note that both spaces on the left have dimension equal to 1 for general M and that $H^0(X, E \otimes M) = H^0(X, L \otimes M)$ and $H^0(X, E \otimes \omega M^{-1}) = H^0(X, L \otimes \omega M^{-1})$ for general M. This implies that $\dim \mathrm{im}(\mu) = 1$ and

$$\mathrm{im}(\mu) \subset H^0(X, L^2\omega) \subset H^0(X, \mathrm{End}_0(E) \otimes \omega).$$

We denote by h a generator of $\mathrm{im}(\mu)$. We obtain that for general M the conormal vector defined by $T_x\theta_M$ is given (up to a scalar) by

$$h \in H^0(X, L^2\omega) = H^1(X, L^{-2})^* = N_x^*.$$

The corresponding rational map

$$\mathrm{Pic}^1(X) \longrightarrow \mathbb{P}H^0(X, L^2\omega) = \mathbb{P}^2, \quad M \mapsto h,$$

is easily seen to be dominant. In particular its image is non degenerate. This shows that the point E is a reduced point of $\mathrm{Bs}|\mathcal{O}(1)|$, because the linear span of the family of conormal vectors defined by $T_x\theta_M$ when M varies in an open set of $\mathrm{Pic}^1(X)$ equals the full space N_x^*. □

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