The rational homotopy Lie algebra of function spaces

Comment. Math. Helv. 83 (2008), 723–739

Commentarii Mathematici Helvetici

The rational homotopy Lie algebra of function spaces

Urtzi Buijs and Aniceto Murillo*

Abstract. In this paper we fully describe the rational homotopy Lie algebra of any component of a given (free or pointed) function space. Also, we characterize higher order Whitehead products on these spaces. From this, we deduce the existence of H-structures on a given component of a pointed mapping space $\mathcal{F}_*(X, Y; f)$ between rational spaces, assuming the cone length of X is smaller than the order of any non trivial generalized Whitehead product in $\pi_*(Y)$.

Mathematics Subject Classification (2000). 55P62, 54C35.

Keywords. Function space, homotopy Lie algebra, Sullivan model, rational homotopy theory.

1. Introduction

Starting with the work of Thom [15] and followed by that of Haefliger [9], the rational homotopy type of function spaces has been extensively studied. However, there is no explicit and complete description of the homotopy Lie algebra structure of such spaces, and only special cases are known.

Denote by $\mathcal{F}(X, Y)$ (resp. $\mathcal{F}_*(X, Y)$) the space of free (resp. based) maps from X to Y. From now on, X and Y are assumed to be nilpotent complexes with X finite and Y of finite type over $\mathbb{Q}$. In this way the components of both $\mathcal{F}(X, Y)$ and $\mathcal{F}_*(X, Y)$ are nilpotent CW-complexes of finite type over $\mathbb{Q}$ and can be rationalized in the classical sense.

If $\dim X < \operatorname{conn} Y$ (so that $\mathcal{F}(X, Y)$ is connected) M. Vigué [16] showed that the homotopy Lie algebra $\pi_*\mathcal{F}(X, Y)_{\mathbb{Q}}$ (resp. $\pi_*\mathcal{F}_*(X, Y)_{\mathbb{Q}}$) is isomorphic as Lie algebra to $H^*(X; \mathbb{Q}) \otimes \pi_*(Y_{\mathbb{Q}})$ (resp. $H^+(X; \mathbb{Q}) \otimes \pi_*(Y_{\mathbb{Q}})$). Later on, Y. Félix [6] used essential properties of this homotopy Lie algebra to show, among other deep results, that the Lusternik–Schnirelmann category of the mentioned components is often infinite. It is also important to remark that in [5], F. Da Silveira describes a Lie model for any component of the space of sections of a given fibration which, in particular, yields a Lie model for function spaces.

*Partially supported by the Ministerio de Ciencia y Tecnología grant MTM2004-60016 and by the Junta de Andalucía grant FQM-213 and FQM-02863.

Following the Brown–Szczarba approach [3] to the Haefliger model of function spaces [9], we first obtain a natural description of its rational homotopy groups in terms of derivations. Then we give a full and explicit description of the homotopy Lie algebra structure of $\mathcal{F}(X,Y)_{\mathbb{Q}}$ and $\mathcal{F}_*(X,Y)_{\mathbb{Q}}$. Let us be more precise:

Let $(\Lambda V, d)$ be a Sullivan model, not necessarily minimal, of Y, i.e., a cofibrant replacement of a commutative differential graded algebra (CDGA henceforth) homotopy equivalent to $C^*(Y;\mathbb{Q})$, and let B be a finite dimensional CDGA of the homotopy type of $C^*(X;\mathbb{Q})$. Then there is a model of $\mathcal{F}(X;Y)$ of the form $(\Lambda(V\otimes B_*),\tilde{d})$ (see the next section for proper definitions and details). By a model of a non connected space (or a map between non connected spaces), we mean a $\mathbb{Z}$-graded CDGA (or a CDGA morphism), whose simplicial realization has the homotopy type of the singular simplicial approximation of the chosen space or map.

Moreover, given a map $f\colon X\to Y$, there is a standard procedure [13] to produce a Sullivan model $(\Lambda S_\phi,\bar{d})$ (in fact, the Haefliger model) of the nilpotent space $\mathcal{F}(X,Y;f)$, the path component of $\mathcal{F}(X,Y)$ containing f.

Our first result is that the space of the indecomposables of this model $\big(S_\phi, Q(\bar{d})\big)$ is isomorphic as differential vector space to $(\mathrm{Der}(\Lambda V, B;\phi),\delta)$, the ϕ-derivations from ΛV to B, where $\phi\colon \Lambda V\to B$ is a model of f. From this, via the classical characterization of rational homotopy groups in terms of the indecomposables of a cofibrant model [2, Theorem 12.7], we immediately obtain:

Theorem 1. (i) *For $n\geq 2$:*

$$\pi_n(\mathcal{F}(X,Y;f)_{\mathbb{Q}})\cong H_n(\mathrm{Der}(\Lambda V,B;\phi),\delta),$$
$$\pi_n(\mathcal{F}_*(X,Y;f)_{\mathbb{Q}})\cong H_n(\mathrm{Der}(\Lambda V,B_+;\phi),\delta).$$

(ii) *For $n=1$:*

$$\Gamma\big(\pi_1(\mathcal{F}(X,Y;f)_{\mathbb{Q}}\big)\cong H_1(\mathrm{Der}(\Lambda V,B;\phi),\delta);$$
$$\Gamma\big(\pi_1(\mathcal{F}_*(X,Y;f)_{\mathbb{Q}}\big)\cong H_1(\mathrm{Der}(\Lambda V,B_+;\phi),\delta).$$

Remark 2. (i) For a given nilpotent space Z, $\Gamma(\pi_1 Z_{\mathbb{Q}})$ denotes the rational vector space

$$\bigoplus_{i=1}^{m}\Gamma_i/\Gamma_{i+1}\otimes\mathbb{Q},$$

where

$$\pi_1 Z=\Gamma_1\supset\Gamma_2\supset\cdots\supset\Gamma_m=\{1\}$$

is the lower central series of $\pi_1 Z$. In particular, $\dim\Gamma(\pi_1 Z_{\mathbb{Q}})=\mathrm{rk}\,\pi_1 Z$.

(ii) The extended version of this theorem in Corollary 7 includes as particular cases the main results in [11, Theorem 2.1] and [12, Theorem 1].

Then we proceed to fully and explicitly describe the Lie bracket on $\pi_*\mathcal{F}(X, Y; f)_{\mathbb{Q}}$ and $\pi_*\mathcal{F}_*(X, Y; f)_{\mathbb{Q}}$ in terms of derivations:

Theorem 3. *The differential linear map of degree* 1

$$[\ ,\ ]\colon \operatorname{Der}_*(\Lambda V, B; \phi) \otimes \operatorname{Der}_*(\Lambda V, B; \phi) \longrightarrow \operatorname{Der}_*(\Lambda V, B; \phi),$$

defined by

$$[\varphi, \psi](v) = (-1)^{|\varphi|+|\psi|-1} \sum \Big(\sum_{i \neq j} \varepsilon\, \phi(v_1 \dots \hat{v}_i \dots \hat{v}_j \dots v_k) \varphi(v_i) \psi(v_j) \Big),$$

in which $dv = \sum v_1 \dots v_k$ *and* ε *is the sign defined in Remark* 11 *below, induces the Whitehead product in homology. Moreover, the restriction to*

$$[\ ,\ ]\colon \operatorname{Der}_*(\Lambda V, B_+; \phi) \otimes \operatorname{Der}_*(\Lambda V, B_+; \phi) \longrightarrow \operatorname{Der}_*(\Lambda V, B_+; \phi),$$

also induces the Lie bracket in $\pi_*\mathcal{F}_*(X, Y; f)_{\mathbb{Q}}$.

A similar result gives also an explicit description of higher order Whitehead products (see Theorem 15).

As an immediate application we generalize the result of Vigué stated above: If we denote by $*\colon X \to Y$ the constant map, $\pi_n(\mathcal{F}(X, Y; *)_{\mathbb{Q}})$ (resp. $\pi_n(\mathcal{F}_*(X, Y; *)_{\mathbb{Q}})$) is isomorphic as Lie algebras to $H^*(X; \mathbb{Q}) \otimes \pi_*(Y_{\mathbb{Q}})$ (resp. $H^+(X; \mathbb{Q}) \otimes \pi_*(Y_{\mathbb{Q}})$).

Finally, from Theorem 3 we may generalize [8] and [10, Theorem 1.2]. For a given space Y, denote by $\operatorname{dl} Y$ the least n (or ∞) for which there is a non trivial Whitehead product of order n in $\pi_*(Y_{\mathbb{Q}})$ (see Section 3 for more about this invariant).

Theorem 4. *If* $\operatorname{cl} X_{\mathbb{Q}} < \operatorname{dl} Y_{\mathbb{Q}}$, *then* $\mathcal{F}_*(X, Y; f)_{\mathbb{Q}}$ *is an* H*-space for all* f. *Equivalently, its rational cohomology algebra is free.*

Here, $\operatorname{cl} X$ denotes the cone length of the space X which is a well-known numerical invariant [4]. It coincides with the strong LS-category, and therefore, it is bounded by $\operatorname{cat} X$ and $\operatorname{cat} X + 1$.

We thank Professor Barry Jessup for very helpful advise.

2. Basics of rational homotopy theory of function spaces

We shall be using known results on rational homotopy theory for which [7] is a very good and standard reference. We now recall some specific facts on the rational homotopy type of a function space $\mathcal{F}(X, Y)$ starting by its Brown–Szczarba model. Consider $A = (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(Y)$ a Sullivan model, not necessarily minimal, of

Y and $B \xrightarrow{\simeq} A_{PL}(X)$ a quasi-isomorphism with B a connected finite dimensional CDGA. Let $B_* = \hom(B, \mathbb{Q})$ be the differential graded coalgebra dual of B, and consider the $\mathbb{Z}$-graded CDGA $\Lambda(A \otimes B_*)$ with the natural differential induced by the one on A and by the dual of the differential of B. Now, consider the differential ideal $I \subset \Lambda(A \otimes B_*)$ generated by $1 \otimes 1^* - 1$ and by the elements of the form

$$a_1 a_2 \otimes \beta - \sum_j (-1)^{|a_2||\beta_j{}'|}(a_1 \otimes \beta_j{}')(a_2 \otimes \beta_j{}''),$$

$a_1, a_2 \in A,\ \beta \in B_*$, and $\Delta\beta = \sum_j \beta_j{}' \otimes \beta_j{}''$. Then, the composition

$$\rho\colon \Lambda(V \otimes B_*) \subset \Lambda(A \otimes B_*) \longrightarrow \Lambda(A \otimes B_*)/I$$

is an isomorphism of graded algebras [3, Theorem 1.2], and therefore, considering on $\Lambda(V \otimes B_*)$ the differential $\tilde{d} = \rho^{-1} d\rho$, ρ is also an isomorphism of CDGA's. Then, $(\Lambda(V \otimes B_*), \tilde{d})$ is a model of $\mathcal{F}(X, Y_{\mathbb{Q}})$ [3, Theorem 1.3]. In other words, $S_*\mathcal{F}(X, Y_{\mathbb{Q}})$ and the simplicial realization of $(\Lambda(V \otimes B_*), \tilde{d})$ are homotopy equivalent.

In order to explicitly determine $\tilde{d}$ on $v \otimes \beta \in V \otimes B_*$, calculate $(dv) \otimes \beta + (-1)^{|v|} v \otimes d\beta$ and then use the relations which generate the ideal I to express $(dv) \otimes \beta$ as an element of $\Lambda(V \otimes B_*)$.

We now explain how to obtain Sullivan models (in fact the Haefliger models) of the different components of $\mathcal{F}(X, Y)$ [3], [9], [13]. For this we need some algebraic tools: let $(\Lambda W, d)$ be a CDGA in which W is $\mathbb{Z}$-graded, and let $u\colon \Lambda W \to \mathbb{Q}$ be an augmentation. Given $\Phi = \alpha \cdot \Psi, \alpha \in (\Lambda^+ W^0)$ and $\Psi \in \Lambda(W^{\neq 0})$, we denote by Φ/u the element $u(\alpha)\Psi$. Define a linear map $\partial\colon W^0 \to W^1$ as follows: given $w \in W^0$, write $dw = \Phi_0 + \Phi_1 + \Phi_2$, with $\Phi_0 \in (\Lambda^+ W^{<0}) \cdot (\Lambda W)$, $\Phi_1 \in (\Lambda^+ W^0) \cdot W^1$, $\Phi_2 \in W^1$, and define $\partial(w) = \Phi_1/u + \Phi_2$.

Call $\overline{W}^1$ a complement of the image of this map, $W^1 = \partial W^0 \oplus \overline{W}^1$, and define the CDGA $(\Lambda \overline{W}^1 \oplus W^{\geq 2}, \bar{d})$ as follows:

Given $w \in \Lambda(\overline{W}^1 \oplus W^{\geq 2})$ write $dw = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3$, in which $\Phi_0 \in \Lambda^+ W^{<0} \cdot \Lambda W$, $\Phi_1 \in \Lambda^+(\partial W^0) \cdot \Lambda W^{\geq 0}$, $\Phi_2 \in (\Lambda^+ W^0) \cdot \Lambda(\overline{W}^1 \oplus W^{\geq 2})$ and $\Phi_3 \in \Lambda(\overline{W}^1 \oplus W^{\geq 2})$. Define $\bar{d}w = \Phi_2/u + \Phi_3$.

Note that if we have in W a basis $\{w_i\}$ for which $dw_i \in \Lambda W_{<i}$, then the image of this basis in $(\Lambda \overline{W}^1 \oplus W^{\geq 2}, \bar{d})$ makes it a Sullivan model. However, even when d is decomposable in ΛW, $\bar{d}$ might not be, i.e., $(\Lambda \overline{W}^1 \oplus W^{\geq 2}, \bar{d})$ is not necessarily minimal. This depends on u. In fact, as we just remarked, for each $w \in W$, Φ_2/u could contain a linear part.

Next, consider $(\Lambda(V \otimes B_*), \tilde{d})$ the model of the function space $\mathcal{F}(X, Y)$ and let $\phi\colon (\Lambda V, d) \to B$ be a model of a given map $f\colon X \to Y$. The morphism ϕ clearly induces a natural augmentation which shall be denoted also by $\phi\colon (\Lambda(V \otimes B_*), \tilde{d}) \to \mathbb{Q}$. Applying the process above to this particular case yields a CDGA $(\Lambda S_\phi, \bar{d}) =$

$(\Lambda\overline{V\otimes B}^1_* \oplus (V\otimes B_*)^{\geq 2}, \bar d)$ which turns out to be a Sullivan model of $\mathcal{F}(X,Y;f)$. Moreover, the CDGA morphism

$$\omega_0\colon (\Lambda V,d) \longrightarrow (\Lambda S_\phi,\bar d) = (\Lambda\overline{V\otimes B}^1_* \oplus (V\otimes B_*)^{\geq 2},\bar d),$$

$\omega_0(v) = v\otimes 1^*$ if $v\in V^{\geq 2}$, or its projection over $\overline{V\otimes B}^1_*$ if $v\in V^1$, is a Sullivan model of the evaluation at the base point $\omega_0\colon \mathcal{F}(X,Y;f)\to Y$ [13, Corollary 22].

While $\omega_0(v)$ could vanish if $|v|=1$, when $(\Lambda V,d)$ is 1-connected,

$$\omega_0\colon (\Lambda V,d) \longrightarrow (\Lambda S_\phi,\bar d),$$

is a KS-extension or a relative Sullivan algebra. The fibre, which is of the form

$$(\Lambda(S_\phi/V),\bar d) \cong (\Lambda\overline{V\otimes B}^1_+ \oplus (V\otimes B_+)^{\geq 2},\bar d),$$

is a Sullivan model of the fibre of $\omega_0\colon \mathcal{F}(X,Y;f)\to Y$, i.e., of $\mathcal{F}_*(X,Y;f)$.

Finally, we set some notation: for any pair V, B of $\mathbb{Z}$-graded vector spaces, denote by $\mathcal{L}(V,B) = \{\mathcal{L}_n(V,B)\}_{n\geq 0}$ the graded vector space of its homomorphisms. In particular, the dual of a given object (except for B_*) shall be denoted by $\mathcal{L}(-,\mathbb{Q})$. There is a natural isomorphism

$$\Theta\colon \mathcal{L}(V,B) \xrightarrow{\cong} \mathcal{L}(V\otimes B_*,\mathbb{Q}),\qquad \Theta(\theta)(v\otimes\beta) = (-1)^{|\beta|(|v|+|\theta|)}\beta(\theta(v)).$$

Given a CDGA morphism $\phi\colon A\to B$, call $(\mathrm{Der}(A,B;\phi),\delta)$ the differential graded vector space where $\mathrm{Der}_n(A,B;\phi)$ are the ϕ-derivations of degree n, i.e., linear maps $\theta\colon A^*\to B^{*-n}$ for which $\theta(ab) = \theta(a)\phi(b) + (-1)^{n|b|}\phi(a)\theta(b)$. The differential is defined as usual $\delta\theta = d\circ\theta + (-1)^{n+1}\theta\circ d$. Note that when $A = \Lambda V$, $\mathrm{Der}(\Lambda V,B;\phi)\cong\mathcal{L}(V,B)$ as graded vector spaces via the identification $\theta\mapsto\theta|_V$. We shall denote also by

$$\Theta\colon \mathrm{Der}(\Lambda V,B;\phi) \xrightarrow{\cong} \mathcal{L}(V\otimes B_*,\mathbb{Q})$$

the isomorphism above under this identification.

3. Rational homotopy groups of function spaces

In this section we prove Theorem 1 and extract some consequences. Hereafter, and to avoid excessive notation, given any nilpotent space Z, whenever we write $\pi_1 Z$ we shall mean $\Gamma(\pi_1 Z)$ (see (i) of Remark 2).

With this in mind, consider the Sullivan model $(\Lambda S_\phi,\bar d)$ of $\mathcal{F}(X,Y;f)$ and recall that [2, Theorem 12.7] $\pi_n\mathcal{F}(X,Y;f)_\mathbb{Q}$ is naturally isomorphic to the dual of $H^n\big(S_\phi, Q(\bar d)\big)$, $n\geq 1$, with $S_\phi\cong Q(\Lambda S_\phi) = \Lambda S_\phi/(\Lambda^+S_\phi\cdot\Lambda^+S_\phi)$ being the space of indecomposables. In other words, the rational homotopy of the f-component of the function space is encoded in the dual of the homology of the following complex:

$$0 \longrightarrow \overline{V \otimes B_*}^1 \xrightarrow{Q(\bar d)} (V \otimes B_*)^2 \xrightarrow{Q(\bar d)} (V \otimes B_*)^3 \xrightarrow{Q(\bar d)} \cdots .$$

However, as $(V \otimes B_*)^1 = \partial(V \otimes B_*)^0 \oplus \overline{V \otimes B_*}^1$, this is exactly the homology of this slightly different complex:

$$(V \otimes B_*)^0 \xrightarrow{\partial} (V \otimes B_*)^1 \xrightarrow{0 \oplus Q(\bar d)} (V \otimes B_*)^2 \xrightarrow{Q(\bar d)} (V \otimes B_*)^3 \xrightarrow{Q(\bar d)} \cdots .$$

Our main result in this section is that the dual of the complex above is isomorphic to $(\mathrm{Der}(\Lambda V, B; \phi), \delta)$ via the map Θ defined in Section 2. We prove:

Theorem 5. *The following diagram commutes:*

$$\begin{array}{ccccccc}
\mathcal{L}_0(V \otimes B_*, \mathbb{Q}) & \xleftarrow{\partial^*} & \mathcal{L}_1(V \otimes B_*, \mathbb{Q}) & \xleftarrow{(0 \oplus Q(\bar d))^*} & \mathcal{L}_2(V \otimes B_*, \mathbb{Q}) & \xleftarrow{Q(\bar d)^*} & \cdots \\
\cong \uparrow \Theta & & \cong \uparrow \Theta & & \cong \uparrow \Theta & & \\
\mathrm{Der}_0(\Lambda V, B; \phi) & \xleftarrow{\delta} & \mathrm{Der}_1(\Lambda V, B; \phi) & \xleftarrow{\delta} & \mathrm{Der}_2(\Lambda V, B; \phi) & \xleftarrow{\delta} & \cdots .
\end{array}$$

Proof. Here, for simplicity in the notation, we write $Q(\bar d)^*$ instead of $\mathcal{L}(Q(\bar d), \mathbb{Q})$. For the same purpose we shall omit signs and write just $\pm$. However, a careful use of Koszul convention leads to proper sign adjustments.

We first show that, for $n \geq 2$, the square

$$\begin{array}{ccc}
\mathcal{L}_n(V \otimes B_*, \mathbb{Q}) & \xleftarrow{Q(\bar d)^*} & \mathcal{L}_{n+1}(V \otimes B_*, \mathbb{Q}) \\
\cong \uparrow \Theta & & \cong \uparrow \Theta \\
\mathrm{Der}_n(\Lambda V, B; \phi) & \xleftarrow{\delta} & \mathrm{Der}_{n+1}(\Lambda V, B; \phi)
\end{array}$$

commutes. On one hand, given $\theta \in \mathrm{Der}_{n+1}(\Lambda V, B; \phi)$ and $v \otimes \beta \in (V \otimes B_*)^n$,

$$(\Theta\delta\theta)(v \otimes \beta) = \pm\beta(\delta\theta(v)) = \pm\beta(d(\theta(v))) \pm \beta(\theta(dv)). \qquad (*)$$

On the other hand,

$$\begin{aligned}
(Q(\bar d)^*\Theta\theta)(v \otimes \beta) &= \pm\Theta\theta(Q(\bar d)(v \otimes \beta)) = \pm\Theta\theta(\{dv \otimes \beta\} \pm v \otimes d\beta) \\
&= \pm\Theta\theta(\{dv \otimes \beta\}) \pm (d\beta)(\theta(v)) = \pm\Theta\theta(\{dv \otimes \beta\}) \pm \beta(d(\theta(v))).
\end{aligned} \qquad (**)$$

Here $\{dv \otimes \beta\}$ denotes the indecomposable part of the image of $[dv \otimes \beta]$ through the morphism

$$A \otimes B_*/I \xrightarrow[\cong]{\rho^{-1}} \Lambda(V \otimes B_*) \to \Lambda(\overline{V \otimes B_*}^1 \oplus (V \otimes B_*)^{\geq 2}) = \Lambda S_\phi.$$

To effectively compute $\{dv \otimes \beta\}$ use first the relations which generates I to write $[dv \otimes \beta]$ as an element of $\Lambda(V \otimes B_*)$. Then, cancel all elements of negative degree and their derivatives, and replace any element of degree zero by the corresponding scalar via ϕ. Finally, keep the linear part.

At the sight of $(*)$ and $(**)$, it will be enough to prove:

Lemma 6. *Given* $\Phi \in \Lambda V$, $\beta \in B_*$ *and* $\theta \in \operatorname{Der}_*(\Lambda V, B; \phi)$, $(\Theta\theta)(\{\Phi \otimes \beta\}) = (-1)^{|\beta|(|\theta|+|\Phi|)}\beta(\theta(\Phi))$.

Proof. Denote by $F_B \colon B \otimes B_* \to \mathbb{Q}$ and $F_{B\otimes B} \colon (B \otimes B) \otimes (B_* \otimes B_*) \to \mathbb{Q}$ the maps defined respectively by $F_B(b\otimes\beta) = (-1)^{|b|}\beta(b)$ and $F_{B\otimes B}(b\otimes b'\otimes\beta\otimes\beta') = (-1)^{|b||b'|+|b|+|b'|}\beta(b)\beta'(b')$. Then, if μ is multiplication in B, it is easy to see that the following diagram commutes:

$$\begin{array}{ccc} (B\otimes B)\otimes B_* & \xrightarrow{\mu\otimes 1_{B_*}} & B\otimes B_* \\ \downarrow{\scriptstyle 1_{B\otimes B}\otimes\Delta} & & \downarrow{\scriptstyle F_B} \\ (B\otimes B)\otimes(B_*\otimes B_*) & \xrightarrow{F_{B\otimes B}} & \mathbb{Q}. \end{array}$$

To prove the lemma, assume that $\Phi = \Lambda^k V$ and argue by induction on k. For $k = 1$, $\Phi = v \in V$ and $\{v \otimes \beta\} = v \otimes \beta$ for which the lemma holds by definition of Θ. Assume now $\Phi = \Psi \cdot v$ with $\Psi \in \Lambda^{k-1}V$. Again, to avoid excessive notation, we shall omit signs:

$$\begin{aligned} \beta\big(\theta(\Psi\cdot v)\big) &= \beta\big(\theta(\Psi)\phi(v) \pm \phi(\Psi)\theta(v)\big) \\ &= \pm F_B(\theta(\Psi)\phi(v)\otimes\beta) \pm F_B(\phi(\Psi)\theta(v)\otimes\beta) \\ &= \pm F_{B\otimes B}(\theta(\Psi)\otimes\phi(v)\otimes\Delta\beta) \pm F_{B\otimes B}(\phi(\Psi)\otimes\theta(v)\otimes\Delta\beta) \\ &= (a) + (b). \end{aligned}$$

On the other hand,

$$(\Theta\theta)\{\Psi\cdot v\otimes\beta\} = (\Theta\theta)\Big\{\sum_j \pm(\Psi\otimes\beta_j')(v\otimes\beta_j'')\Big\}$$

with $\Delta\beta = \sum_j \beta_j' \otimes \beta_j''$. By definition of $\{\,,\}$, we may keep only those summands for which one of the factors is of degree zero. Hence, the above equality becomes

$$\begin{aligned} &(\Theta\theta)\Big\{\sum_{|\Psi|+|\beta_j'|=0} \pm(\Psi\otimes\beta_j')(v\otimes\beta_j'')\Big\} + (\Theta\theta)\Big\{\sum_{|v|+|\beta_j''|=0} \pm(\Psi\otimes\beta_j')(v\otimes\beta_j'')\Big\} \\ &= (e) + (f). \end{aligned}$$

Using definition and induction hypothesis we get

$$\begin{aligned}(f) &= \sum_{|v|+|\beta_j''|=0} \pm(\Theta\theta)\{\Psi\otimes\beta_j'\}\phi(v\otimes\beta_j'')\\ &= \sum_{|v|+|\beta_j''|=0} \pm\beta_j'\big(\theta(\Psi)\big)\beta_j''\big(\phi(v)\big)\\ &= \sum_{|v|+|\beta_j''|=0} \pm F_{B\otimes B}(\theta(\Psi)\otimes\phi(v)\otimes\beta_j'\otimes\beta_j'')\\ &= \pm F_{B\otimes B}(\theta(\Psi)\otimes\phi(v)\otimes\triangle\beta) = (a).\end{aligned}$$

Using repeatedly a similar argument one checks that $(b) = (e)$ and the proof is complete. □

Finally, we see that

$$\begin{array}{ccccc}\mathcal{L}_0(V\otimes B_*,\mathbb{Q}) & \xleftarrow{\partial^*} & \mathcal{L}_1(V\otimes B_*,\mathbb{Q}) & \xleftarrow{(0\oplus Q(\bar d))^*} & \mathcal{L}_2(V\otimes B_*,\mathbb{Q})\\ \cong\uparrow\Theta & & \cong\uparrow\Theta & & \cong\uparrow\Theta\\ \mathrm{Der}_0(\Lambda V,B;\phi) & \xleftarrow{\delta} & \mathrm{Der}_1(\Lambda V,B;\phi) & \xleftarrow{\delta} & \mathrm{Der}_2(\Lambda V,B;\phi)\end{array}$$

commutes. For it note that given $v\otimes\beta\in(V\otimes B_*)^0$, $\partial(v\otimes\beta)=\{dv\otimes\beta\}+(-1)^{|v|}v\otimes d\beta$. Hence, using Lemma 6, and following exactly the above argument:

$$(\partial^*\circ\Theta)(\theta)(v\otimes\beta) = (-1)^{|\theta|+1}\Theta\theta(\{dv\otimes\beta\}+(-1)^{|v|}v\otimes d\beta) = (\Theta\circ\delta)(\theta)(v\otimes\beta),$$

which gives the commutativity of the left square. For the right square, write $w\in(V\otimes B_*)^1$ as a sum $x+v\otimes\beta$, $x\in\partial(V\otimes B_*)^0$, $v\otimes\beta\in\overline{V\otimes B_*}^1$. Then,

$$\begin{aligned}\big((0\oplus Q(\bar d))^*\circ\Theta\big)(\theta)(w) &= (-1)^{|\theta|+1}(\Theta\theta)\big(Q(\bar d)(v\otimes\beta)\big)\\ &= (-1)^{|\theta|+1}(\Theta\theta)(\{dv\otimes\beta\}+(-1)^{|v|}v\otimes d\beta)\\ &= (\Theta\circ\delta)(\theta)(w).\end{aligned}$$

and the proof of Theorem 5 is completed. □

Proof of Theorem 1. The free case of (i) and (ii) is immediate from Theorem 5. For the based case consider the Sullivan model $(\Lambda(S_\phi/V),\bar d)$ of $\mathcal{F}_*(X,Y;f)$ recalled in the past section, and observe that $\Gamma\big(\pi_1(\mathcal{F}_*(X,Y;f)_{\mathbb{Q}}\big)$ and $\pi_n\big(\mathcal{F}_*(X,Y;f)_{\mathbb{Q}}\big)$, $n\geq 2$, are then isomorphic to the dual of the homology of the following complex:

$$0\longrightarrow \overline{V\otimes B_+}^1 \xrightarrow{Q(\bar d)} (V\otimes B_+)^2 \xrightarrow{Q(\bar d)} (V\otimes B_+)^3 \xrightarrow{Q(\bar d)} \cdots.$$

To finish, restrict Theorem 5 to the dual of this complex. □

We now check that the above isomorphism is natural and respects the evaluation map at the base point. Fix a map $f\colon X \to Y$ between nilpotent complexes of finite type over $\mathbb{Q}$ and let Z be a finite nilpotent complex. Let $A = (\Lambda W, d) \xrightarrow[\simeq]{\varphi} A_{PL}(X)$ and $(\Lambda V, d) \xrightarrow[\simeq]{\varphi} A_{PL}(Y)$ be Sullivan models (again not necessarily minimal!) of X and Y respectively, let $C \xrightarrow[\simeq]{\nu} A_{PL}(Z)$ be a quasi-isomorphism with C connected finite dimensional, and let $\zeta\colon (\Lambda V, d) \to (\Lambda W, d)$ be a Sullivan model for f. Define

$$\xi\colon (\Lambda(V \otimes C_*), \tilde{d}) \longrightarrow (\Lambda(W \otimes C_*), \tilde{d}), \quad \xi(v \otimes c) = \rho^{-1}[\zeta(v) \otimes c],$$

$(\Lambda(V \otimes C_*), \tilde{d})$ and $(\Lambda(W \otimes C_*), \tilde{d})$ being the models of $\mathcal{F}(Z, Y)$ and $\mathcal{F}(Z, X)$ respectively, and $\rho\colon (\Lambda(W \otimes C_*), \tilde{d}) \xrightarrow{\cong} (\Lambda(A \otimes C_*), d)/I$ the CDGA isomorphism described in Section 2. In other words, to compute effectively $\xi(v \otimes c)$ use the relations which define I to express $\zeta(v) \otimes c$ as an element of $\Lambda(V \otimes C_*)$. For instance, if $\zeta(v) = w_1 w_2$ and $\Delta c = \sum_i c_i' \otimes c_j''$, $\xi(v \otimes c) = \sum_i (-1)^{|w_2||c_i'|}(w_1 \otimes c_i')(w_2 \otimes c_i'')$.

Finally, let $\phi\colon (\Lambda W, d) \to C$ and $\phi \circ \zeta\colon (\Lambda V, d) \to C$ be models of $g\colon Z \to X$ and $f \circ g\colon Z \to Y$ respectively. Then [13, Theorem 24], the diagram

$$\begin{array}{ccc}
(\Lambda S_\phi / W, \bar{d}) & \xleftarrow{\ \bar{\xi}\ } & (\Lambda S_{\phi\circ\zeta} / V, \bar{d}) \\
\uparrow & & \uparrow \\
(\Lambda S_\phi, \bar{d}) & \xleftarrow{\ \bar{\xi}\ } & (\Lambda S_{\phi\circ\zeta}, \bar{d}) \\
\uparrow{\scriptstyle \omega_0} & & \uparrow{\scriptstyle \omega_0} \\
(\Lambda W, d) & \xleftarrow{\ \zeta\ } & (\Lambda V, d)
\end{array}$$

is a Sullivan model of

$$\begin{array}{ccc}
\mathcal{F}_*(Z, X; g) & \xrightarrow{\ (f)_*\ } & \mathcal{F}_*(Z, Y; f \circ g) \\
\downarrow & & \downarrow \\
\mathcal{F}(Z, X; g) & \xrightarrow{\ (f)_*\ } & \mathcal{F}(Z, Y; f \circ g) \\
\downarrow{\scriptstyle \omega_0} & & \downarrow{\scriptstyle \omega_0} \\
X & \xrightarrow{\ f\ } & Y.
\end{array}$$

Hence in view of Theorem 1, the following corollary, which includes in particular the main results in [11, Theorem 2.1] and [12, Theorem 1], is an easy exercise:

Corollary 7. (1) *For* $n \geq 1$, $\pi_n(f_*)_{\mathbb{Q}} \colon \pi_n\mathcal{F}(Z, X; g)_{\mathbb{Q}} \to \pi_n\mathcal{F}(Z, Y; f \circ g)_{\mathbb{Q}}$ *is naturally equivalent to*

$$H(\zeta_*) \colon H_n\mathrm{Der}_*(\Lambda W, C; \phi) \longrightarrow H_n\mathrm{Der}_*(\Lambda V, C; \phi \circ \zeta).$$

(2) *Moreover,*

$$\begin{array}{ccc}
\pi_n\mathcal{F}_*(Z, X; g)_{\mathbb{Q}} & \xrightarrow{\pi_n(f_*)_{\mathbb{Q}}} & \pi_n\mathcal{F}_*(Z, Y; f \circ g)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\pi_n\mathcal{F}(Z, X; g)_{\mathbb{Q}} & \xrightarrow{\pi_n(f_*)_{\mathbb{Q}}} & \pi_n\mathcal{F}(Z, Y; f \circ g)_{\mathbb{Q}} \\
{\scriptstyle \pi_n(\omega_0)_{\mathbb{Q}}}\downarrow & & \downarrow{\scriptstyle \pi_n(\omega_0)_{\mathbb{Q}}} \\
\pi_n(X)_{\mathbb{Q}} & \xrightarrow{\pi_n(f)_{\mathbb{Q}}} & \pi_n(Y)_{\mathbb{Q}}
\end{array}$$

is equivalent to

$$\begin{array}{ccc}
H_n(\mathrm{Der}_*(\Lambda W, C_+; \phi)) & \xrightarrow{H(\zeta_*)} & H_n(\mathrm{Der}_*(\Lambda V, C_+; \phi \circ \zeta)) \\
\downarrow & & \downarrow \\
H_n(\mathrm{Der}_*(\Lambda W, C; \phi)) & \xrightarrow{H(\zeta_*)} & H_n(\mathrm{Der}_*(\Lambda V, C; \phi \circ \zeta)) \\
{\scriptstyle H(\varepsilon_*)}\downarrow & & \downarrow{\scriptstyle H(\varepsilon_*)} \\
H_n(\mathrm{Der}_*(\Lambda W, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\zeta)} & H_n(\mathrm{Der}_*(\Lambda V, \mathbb{Q}; \varepsilon)).
\end{array}$$

□

Remark 8. Note that $(\mathrm{Der}_*(\Lambda V, \mathbb{Q}; \varepsilon), \delta) \cong \big((\mathcal{L}(V, \mathbb{Q}), Q(d)^*\big)$ and therefore $H_n(\mathrm{Der}_*(\Lambda V, \mathbb{Q}; \varepsilon))$ is isomorphic to the dual of $H^*(V, Q(d))$.

4. The Lie algebra structure

This section is devoted to the proof of Theorem 3 and its consequences. For that, the following remark is essential:

Remark 9. Let $(\Lambda V, d)$ be a Sullivan model of a nilpotent space X. Recall that d can be written as the sum $d = \sum_{i \geq 1} d_i$, with $d_i(V) \subset \Lambda^i V$. The linear part $d_1 = Q(d)$ induces a differential on ΛV. The differential d' induced by d on $H^*(\Lambda V, d_1) = \Lambda H^*(V, d_1)$ has no linear term and $(\Lambda H^*(V, d_1), d')$ is the minimal model of X. The quadratic part, d_2' is then a differential which can be identified as the Lie bracket on $\pi_*(X_{\mathbb{Q}})$ [14, II.6.(16)]. More precisely, given the natural isomorphism $\pi_*(X_{\mathbb{Q}}) \cong \mathcal{L}_*\big(H^*(\Lambda V, d_1), \mathbb{Q}\big)$ and the multilinear map

$$\langle\, ;\ ,\ \rangle \colon \wedge^2 H^*(\Lambda V, d_1) \times \pi_*(X_{\mathbb{Q}}) \times \pi_*(X_{\mathbb{Q}}) \longrightarrow \mathbb{Q},$$

$$\langle \alpha \wedge \beta; \gamma_0, \gamma_1 \rangle = \gamma_1(\alpha)\gamma_0(\beta) + (-1)^{|\beta||\gamma_0|}\gamma_0(\alpha)\gamma_1(\beta),$$

it turns out that

$$\big[\gamma_0, \gamma_1\big](\alpha) = (-1)^{p+q-1}\langle d_2'\alpha; \gamma_0, \gamma_1\rangle,$$

in which $\alpha \in H^*(\Lambda V, d_1)$, $\gamma_0 \in \pi_p(X_{\mathbb{Q}})$, $\gamma_1 \in \pi_q(X_{\mathbb{Q}})$.

In the same way, given the multilinear map

$$\langle\, ;\, , \dots ,\, \rangle \colon \wedge^j V \times V^* \times \dots \times V^* \longrightarrow \mathbb{Q},$$

$$\langle v_1 \dots v_j; \gamma_0, \dots, \gamma_j\rangle = \sum_{i_1,\dots,i_j} \delta_{i_1\dots i_j}\gamma_1(v_{i_1})\dots\gamma_j(v_{i_j}),$$

where $\delta_{i_1\dots i_j}$ is the expected sign induced by the Koszul convention, the higher order Whitehead products on $\pi_*(X_{\mathbb{Q}})$ can be identified with the j-th part of d, via

$$[\gamma_1, \dots, \gamma_j](v) = (-1)^{p_1+\dots+p_j-1}\langle d_j v; \gamma_1, \dots, \gamma_j\rangle,$$

each γ_i being of degree p_i [1, Theorem 5.4] or [14, V.7(3)].

Consider now the component $\mathcal{F}(X, Y; f)$ of a given function space and let $(\Lambda S_\phi, \bar{d})$ be its Sullivan model defined in Section 2. We shall need a "quadratic" analogue of Lemma 6. Given $\Phi \in \Lambda V$ and $\beta \in B_*$, denote by $\{\Phi \otimes \beta\}_2$ the quadratic part of the image of $[\Phi \otimes \beta]$ through the morphism

$$A \otimes B_*/I \xrightarrow[\cong]{\rho^{-1}} \Lambda(V \otimes B_*) \to \Lambda\overline{V \otimes B_*}^1 \oplus (V \otimes B_*)^{\geq 2}.$$

To effectively compute $\{\Phi \otimes \beta\}_2$ use first the relations which generates I to write $[\Phi \otimes \beta]$ as an element of $\Lambda(V \otimes B_*)$. Then, cancel all elements of negative degree and their derivatives, and replace any element of degree zero by the corresponding scalar via ϕ. Finally, keep the quadratic part.

Lemma 10. *Let $\Phi = v_1 \dots v_k \in \Lambda^k V$, $\beta \in B_*$ and $\varphi, \psi \in \mathrm{Der}_*(\Lambda V, B; \phi)$ of strictly positive degrees. Then,*

$$\big\langle \{\Phi \otimes \beta\}_2; \Theta\varphi, \Theta\psi \big\rangle = (-1)^{|\beta|(|\varphi|+|\psi|+|\Phi|)} \sum_{i \neq j} \varepsilon\, \beta\big(\phi(v_1 \dots \hat{v}_i \dots \hat{v}_j \dots v_k)\varphi(v_i)\psi(v_j)\big),$$

where ε is the sign produced by the Koszul convention and, for completeness, it is explicitly given in Remark 11 *below.*

Proof. As in Lemma 6, to be clear in presenting our argument, we shall write $\pm$ instead of proper signs, and leave to the reader the straightforward task that the equality above holds with the given signs.

We proceed by induction on k. Let $\Phi = v_1 v_2$, assume $\Delta\beta = \sum_r \beta'_r \otimes \beta''_r$ and denote by Γ the sum of all terms of $\sum_r (-1)^{|\beta'_r||v_2|}(v_1 \otimes \beta'_r)(v_2 \otimes \beta''_r)$ in which at least one of the two factors is of degree 0. Then

$$\langle \{v_1 v_2 \otimes \beta\}_2; \Theta\varphi, \Theta\psi\rangle = \Big\langle \sum_r \pm(v_1 \otimes \beta'_r)(v_2 \otimes \beta''_r) - \Gamma; \Theta\varphi, \Theta\psi\Big\rangle.$$

However, as φ, ψ are of positive degree, $\langle \Gamma; \Theta\varphi, \Theta\psi\rangle = 0$ and the formula above becomes

$$\begin{aligned}
&\sum_r \pm\langle (v_1 \otimes \beta'_r)(v_2 \otimes \beta''_r); \Theta\varphi, \Theta\psi\rangle \\
&\quad = \sum_r \pm\Theta\varphi(v_1 \otimes \beta'_r)\Theta\psi(v_2 \otimes \beta''_r) \pm \Theta\psi(v_1 \otimes \beta'_r)\Theta\varphi(v_2 \otimes \beta''_r) \\
&\quad = \sum_r \pm\beta'_r(\varphi(v_1))\beta''_r(\psi(v_2)) \pm \beta'_r(\psi(v_1))\beta''_r(\varphi(v_2)) \\
&\quad = \pm F_{B\otimes B}\big(\varphi(v_1) \otimes \psi(v_2) \otimes \Delta\beta\big) \pm F_{B\otimes B}\big(\psi(v_1) \otimes \varphi(v_2) \otimes \Delta\beta\big) \\
&\quad = \pm\beta\big(\varphi(v_1)\psi(v_2)\big) \pm \beta\big(\psi(v_1)\varphi(v_2)\big)
\end{aligned}$$

which is the expected expression for $k = 2$.

Assume the lemma holds for $k-1$ and let $\Phi = v_1 \dots v_k$. On the one hand,

$$\begin{aligned}
&\sum_{i\neq j} \pm\beta\big(\phi(v_1 \dots \hat{v}_i \dots \hat{v}_j \dots v_k)\varphi(v_i)\psi(v_j)\big) \\
&\quad = \Big[\sum_{j\neq k} \pm\beta\big(\phi(v_1 \dots \hat{v}_j \dots v_{k-1})\varphi(v_k)\psi(v_j)\big) \\
&\qquad + \sum_{i\neq k} \pm\beta\big(\phi(v_1 \dots \hat{v}_i \dots v_{k-1})\varphi(v_i)\psi(v_k)\big)\Big] \\
&\qquad + \sum_{\substack{i\neq j\\ i,j\neq k}} \pm\beta\big(\phi(v_1 \dots \hat{v}_i \dots \hat{v}_j \dots v_k)\varphi(v_i)\psi(v_j)\big) = (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

On the other hand,

$$\langle \{v_1 \dots v_k \otimes \beta\}_2; \Theta\varphi, \Theta\psi\rangle = \sum_r \pm\langle \{(v_1 \dots v_{k-1} \otimes \beta'_r)(v_k \otimes \beta''_r)\}_2; \Theta\varphi, \Theta\psi\rangle.$$

In this formula, whenever $v_k \otimes \beta''_r$ is of degree 0, we can replace it by the scalar $\phi(v_k \otimes \beta''_r)$ resulting in

$$\begin{aligned}
&\sum_{|v_k\otimes\beta''_r|=0} \pm\phi(v_k \otimes \beta''_r)\langle \{v_1 \dots v_{k-1} \otimes \beta'_r)\}_2; \Theta\varphi, \Theta\psi\rangle \\
&\qquad + \sum_{|v_k\otimes\beta''_r|>0} \pm\langle \{v_1 \cdots v_{k-1} \otimes \beta'_r\}(v_k \otimes \beta''_r); \Theta\varphi, \Theta\psi\rangle = (\mathrm{II}') + (\mathrm{I}').
\end{aligned}$$

Applying induction we get

$$\begin{aligned}
(\mathrm{II}') &= \sum_{i\neq j,r} \pm\beta'_r\big(\phi(v_1\dots\hat{v}_i\dots\hat{v}_j\dots v_{k-1})\varphi(v_i)\psi(v_j)\big)\beta''_r(\phi(v_k))\\
&= \sum_{i\neq j,r} \pm F_{B\otimes B}\big(\phi(v_1\dots\hat{v}_i\dots\hat{v}_j\dots v_{k-1})\varphi(v_i)\psi(v_j)\otimes\phi(v_k)\otimes\beta'_r\otimes\beta''_r\big)\\
&= \sum_{i\neq j} \pm F_{B\otimes B}\big(\phi(v_1\dots\hat{v}_i\dots\hat{v}_j\dots v_{k-1})\varphi(v_i)\psi(v_j)\otimes\phi(v_k)\otimes\Delta\beta\big)\\
&= \sum_{i\neq j} \pm F_{B}\big(\phi(v_1\dots\hat{v}_i\dots\hat{v}_j\dots v_{k-1})\varphi(v_i)\psi(v_j)\phi(v_k))\otimes\beta\big)\\
&= \sum_{\substack{i\neq j\\ i,j\neq k}} \pm\beta\big(\phi(v_1\dots\hat{v}_i\dots\hat{v}_j\dots v_{k-1})\phi(v_k)\varphi(v_i)\psi(v_j)\big) = (\mathrm{II}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathrm{I}') = \sum_r \pm\Theta\varphi(\{v_1\dots v_{k-1}\otimes\beta'_r\})\Theta\psi(v_k\otimes\beta''_r)\\
+\sum_r \pm\Theta\psi(\{v_1\dots v_{k-1}\otimes\beta'_r\})\Theta\varphi(v_k\otimes\beta''_r).
\end{aligned}$$

Applying Lemma 6 to this formula gives the following:

$$\begin{aligned}
&= \sum_r \pm\beta'_r\big(\varphi(v_1\dots v_{k-1})\big)\beta''_r\big(\psi(v_k)\big) + \sum_r \pm\beta'_r\big(\psi(v_1\dots v_{k-1})\big)\beta''_r\big(\varphi(v_k)\big)\\
&= \sum_r \pm F_{B\otimes B}\big(\varphi(v_1\dots v_{k-1})\otimes\psi(v_k)\otimes\beta'_r\otimes\beta''_r\big)\\
&\qquad \pm F_{B\otimes B}\big(\psi(v_1\dots v_{k-1})\otimes\varphi(v_k)\otimes\beta'_r\otimes\beta''_r\big)\\
&= \pm F_{B\otimes B}\big(\varphi(v_1\dots v_{k-1})\otimes\psi(v_k)\otimes\Delta\beta\big)\\
&\qquad \pm F_{B\otimes B}\big(\psi(v_1\dots v_{k-1})\otimes\varphi(v_k)\otimes\Delta\beta\big)\\
&= \pm\beta\big(\varphi(v_1\cdots v_{k-1})\psi(v_k)\big)\pm\beta\big(\psi(v_1\cdots v_{k-1})\varphi(v_k)\big).
\end{aligned}$$

Finally, as φ and ψ are ϕ-derivations, this last equation results in

$$\begin{aligned}
\sum_{i\neq k} \pm\beta\big(\phi(v_1\dots\hat{v}_i\dots v_{k-1})\varphi(v_i)\psi(v_k)\big)\\
+\sum_{j\neq k} \pm\beta\big(\phi(v_1\dots\hat{v}_j\dots v_{k-1})\psi(v_j)\varphi(v_k)\big) = (\mathrm{I})
\end{aligned}$$

and the proof is complete. □

Remark 11. *The sign* ε *of Lemma* 10 *clearly depends on* $i, j, \varphi, \psi, v_1, \dots, v_k$. *To make it explicit consider the following notation: for each set of indices* $\Gamma \subset \{1, \dots, k\}$ *write*

$$\rho_\Gamma = \sum_{n \notin \Gamma} |v_n|.$$

For instance, $\rho_{\leq i,j}$ *denotes* $|v_{i+1}| + \dots + \widehat{|v_j|} + \dots + |v_k|$ *if* $i < j$ *or* $|v_{i+1}| + \dots + |v_k|$ *if* $i \geq j$. *Then,*

$$\varepsilon = (-1)^{\rho_{i,j}|\varphi| + \rho_j|\psi| + \rho_{\leq i,j}|v_i| + \rho_{i,\leq j}|v_j|}.$$

Proof of Theorem 3. Let $\varphi, \psi \in \mathrm{Der}(\Lambda V, B; \phi)$ be homogeneous derivations of positive degrees p and q respectively. In view of Theorem 1 and Remark 9, it is enough to show that, for any $v \otimes \beta \in S_\phi$,

$$\Theta[\varphi, \psi](v \otimes \beta) = (-1)^{p+q-1} \langle \bar{d}_2(v \otimes \beta); \Theta\varphi, \Theta\psi \rangle,$$

where, as always, $\bar{d}_2$ denotes the quadratic part of the differential in $(\Lambda S_\phi, \bar{d})$. But this holds by noting that φ and ψ are of positive degree, and applying Lemma 10. Indeed:

$$\begin{aligned}
&(-1)^{p+q-1} \langle \bar{d}_2(v \otimes \beta); \Theta\varphi, \Theta\psi \rangle \\
&\quad = (-1)^{p+q-1} \langle \{dv \otimes \beta\}_2; \Theta\varphi, \Theta\psi \rangle \\
&\quad = (-1)^{p+q-1} \sum \langle \{v_1 \dots v_k \otimes \beta\}_2; \Theta\varphi, \Theta\psi \rangle \\
&\quad = (-1)^{p+q-1} \sum (-1)^{|\beta|(p+q+|v|+1)} \sum_{i \neq j} \varepsilon\, \beta\big(\phi(v_1 \dots \hat{v}_i \dots \hat{v}_j \dots v_k)\varphi(v_i)\psi(v_j)\big) \\
&\quad = (-1)^{|\beta|(p+q+|v|+1)} \beta\big([\varphi, \psi](v)\big) = \Theta[\varphi, \psi](v \otimes \beta).
\end{aligned}$$

Exactly the same argument can be used to conclude that the suitable restriction of the bracket induces the Lie structure on $\pi_n(\mathcal{F}_*(X, Y; f)_{\mathbb{Q}})$. □

Remark 12. (i) To show that the restriction to

$$[\ ,\ ]\colon \mathrm{Der}_*(\Lambda V, B_+; \phi) \otimes \mathrm{Der}_*(\Lambda V, B_+; \phi) \longrightarrow \mathrm{Der}_*(\Lambda V, B_+; \phi),$$

also induces the Lie bracket in $\pi_n(\mathcal{F}_*(X, Y; *)_{\mathbb{Q}})$, when choosing the path component of the constant map, one may also proceed as follows: as the fibration

$$\mathcal{F}_*(X, Y; *) \longrightarrow \mathcal{F}(X, Y; *) \xrightarrow{\omega_0} Y$$

has a section, the exact sequence on rational homotopy induces an extension of Lie algebras

$$0 \longrightarrow \pi_*\mathcal{F}_*(X, Y; *)_{\mathbb{Q}} \longrightarrow \pi_*\mathcal{F}(X, Y; *)_{\mathbb{Q}} \longrightarrow \pi_* Y_{\mathbb{Q}} \longrightarrow 0.$$

Hence, the Lie bracket on $\pi_*\mathcal{F}_*(X, Y; *)_{\mathbb{Q}} = H_*\big(\mathrm{Der}(\Lambda V, B_+; \phi)\big)$ is the restriction of the one in $\pi_*\mathcal{F}(X, Y; *)_{\mathbb{Q}} = H_*\big(\mathrm{Der}(\Lambda V, B; \phi)\big)$.

(ii) At the sight of the proof above, which heavily relies on Remark 9, the fact that

$$[\ ,\ ]\colon \mathrm{Der}_*(\Lambda V, B; \phi) \otimes \mathrm{Der}_*(\Lambda V, B; \phi) \longrightarrow \mathrm{Der}_*(\Lambda V, B; \phi)$$

commutes with differential automatically holds. This is far from trivial if one uses only differential homological algebra tools.

As a first and immediate application of Theorem 3 we describe the Lie algebra structure on $\pi_*\mathcal{F}(X, Y; *)_{\mathbb{Q}}$ and $\pi_*\mathcal{F}_*(X, Y; *)_{\mathbb{Q}}$ when considering the constant map $*\colon X \to Y$, recovering in particular Vigué's result [16] stated in the introduction.

Theorem 13. *$\pi_n(\mathcal{F}(X, Y; *)_{\mathbb{Q}})$ (resp. $\pi_n(\mathcal{F}_*(X, Y; *)_{\mathbb{Q}})$) is isomorphic as Lie algebra to $H^*(X; \mathbb{Q}) \otimes \pi_*(Y_{\mathbb{Q}})$ (resp. $H^+(X; \mathbb{Q}) \otimes \pi_*(Y_{\mathbb{Q}})$).*

Proof. In this case, $\phi\colon (\Lambda V, d) \to B$ annihilates V. In view of Theorem 3,

$$[\varphi, \psi](v) = (-1)^{|\varphi|+|\psi|-1} \sum_i (-1)^{|\psi||v_i'|} \varphi(v_i')\psi(v_i'') + (-1)^{|\varphi|(|v_i''|+|\psi|)} \varphi(v_i'')\psi(v_i'),$$

with $d_2 v = \sum_i v_i' v_i''$. Via the isomorphism Θ of Theorem 5, this is taken to the Lie bracket induced by $\bar{d}_2$ on $H^*(V \otimes B_*, \bar{d}_1)$. However, this is precisely the $V \otimes H^*(B)$ with the usual Lie bracket. □

We may extend Lemma 6 to calculate in $H_*\big(\mathrm{Der}(\Lambda V, B; \phi)\big)$ Whitehead products of higher order.

Definition 14. Given $\varphi_1, \dots, \varphi_j \in \mathrm{Der}_*(\Lambda V, B; \phi)$, of strictly positive degrees $p_1, \dots, p_j$, define $[\varphi_1, \dots, \varphi_j] \in \mathrm{Der}(\Lambda V, B; \phi)$ by

$$\begin{aligned}[\varphi_1, \dots, \varphi_j](v) \\ = (-1)^{p_1+\cdots+p_j-1} \sum \Big(\sum_{i_1,\dots,i_j} \varepsilon\, \phi(v_1 \dots \hat{v}_{i_1} \dots \hat{v}_{i_j} \dots v_k)\varphi_1(v_{i_1}) \dots \varphi_j(v_{i_j})\Big),\end{aligned}$$

$dv = \sum v_1 \dots v_k$ and ε being the suitable sign given by the Koszul convention which, with the notation of Remark 11, can be explicitly described as $(-1)^\alpha$ where

$$\begin{aligned}\alpha = {}& \rho_{i_1\dots i_j}|\varphi_1| + \rho_{i_2\dots i_j}|\varphi_2| + \cdots + \rho_{i_j}|\varphi_j| \\ &+ \rho_{\le i_1, i_2,\dots,i_j}|v_{i_1}| + \rho_{\le i_2, i_3,\dots,i_j}|v_{i_2}| + \cdots + \rho_{\le i_j}|v_{i_j}|.\end{aligned}$$

Then, the exact analogue of the proof of Lemma 6 shows that given $\Phi = v_1 \dots v_k \in \Lambda^k V$ and $\beta \in B_*$,

$$\langle \{\Phi \otimes \beta\}_j; \Theta\varphi_1, \dots, \Theta\varphi_j \rangle$$
$$= (-1)^{|\beta|(p_1+\dots+p_j+|\Phi|)} \sum_{i_1,\dots,i_j} \varepsilon\, \beta\big(\phi(v_1 \dots \hat{v}_{i_1} \dots \hat{v}_{i_j} \dots v_k)\varphi_1(v_{i_1})\varphi_2(v_{i_2}) \dots \varphi_j(v_{i_j})\big).$$

Again, $\{\Phi \otimes \beta\}_j$ is defined as the j-th part of the image of $[\Phi \otimes \beta]$ through the morphism

$$A \otimes B_*/I \xrightarrow[\cong]{\rho^{-1}} \Lambda(V \otimes B_*) \to \Lambda(\overline{V \otimes B_*})^1 \oplus (V \otimes B_*)^{\geq 2}.$$

Thus, as in the proof of Theorem 3, we get the following which, in view of Remark 9, describes j-order Whitehead products on $\pi_*\mathcal{F}(X, Y; f)_{\mathbb{Q}}$ and $\pi_*\mathcal{F}_*(X, Y; f)_{\mathbb{Q}}$.

Theorem 15. $\Theta[\varphi_1, \dots, \varphi_j](v \otimes \beta) = (-1)^{p_1+\dots+p_j-1}\langle \bar{d}_j(v \otimes \beta); \Theta\varphi_1, \dots, \Theta\varphi_j \rangle.$ □

From this, we immediately deduce Theorem 4. For a given a space X, recall that $\mathrm{dl}\, X$ (dl stands for differential length) is the least n, or ∞, for which there is a non trivial Whitehead product of order n on $\pi_*(X_{\mathbb{Q}})$. This coincides with the least n for which d_n, the n-th part of the differential of the minimal model of X is non trivial. Another geometric description of this invariant is given in [8] in terms of the Ganea spaces of X.

Proof of Theorem 4. Assume $\mathrm{cl}\, X_{\mathbb{Q}} = m$. Then, by a deep result of Cornea [4], X has a finite dimensional model B for which any product of length greater than m of elements of B^+ vanishes. Hence, for $j > m$ and for all $v \otimes \beta$, given $\varphi_1, \dots, \varphi_j \in \mathrm{Der}(\Lambda V, B_+; \phi)$, $[\varphi_1, \dots, \varphi_j](v \otimes \beta) \in B^{>m} = 0$. However, as $\mathrm{dl}\, Y_{\mathbb{Q}} > m$, in view of Theorem 15, this implies that $\bar{d}_j$ vanishes for all $j \geq 2$. This means that the differential on the minimal model vanishes and the theorem follows. □

References

[1] P. Andrews and M. Arkowitz, Sullivan's minimal models and higher order Whitehead products. *Canad. J. Math.* **30** (8) (1978), 961–982. Zbl 0441.55012 MR 0506254

[2] A. K. Bousfield and V. K. A. M. Gugenheim, On PL De Rahm theory and rational homotopy type. *Mem. Amer. Math. Soc.* **179** (1976). Zbl 0338.55008 MR 0425956

[3] E. H. Brown and R. H. Szczarba, On the rational homotopy type of function spaces. *Trans. Amer. Math. Soc.* **349** (1997), 4931–4951. Zbl 0927.55012 MR 1407482

[4] O. Cornea, Cone length and LS-category. *Topology* **33** (1994), 95–111. Zbl 0811.55004 MR 1259517

[5] F. E. A. Da Silveira, Rational homotopy theory of fibrations. *Pacific J. Math.* **113**(1) (1984), 1–34. Zbl 0562.55012 MR 0745592

[6] Y. Félix, Rational category of the space of sections of a nilpotent bundle. *Comment. Math. Helv.* **65** (1990), 615–622. Zbl 0715.55009 MR 1078101

[7] Y. Félix, S. Halperin and J. C. Thomas, *Rational Homotopy Theory*. Grad. Texts in Math. 205, Springer-Verlag, New York 2000. Zbl 0961.55002 MR 1802847

[8] Y. Félix, and D. Tanré, H-space structure on pointed mapping spaces. *Algebr. Geom. Topol.* **5** (2005), 713–724. Zbl 1082.55006 MR 2153111

[9] A. Haefliger, Rational homotopy of the space of sections of a nilpotent bundle. *Trans. Amer. Math. Soc.* **273** (1982), 609–620. Zbl 0508.55019 MR 0667163

[10] Y. Kotani, Note on the rational cohomology of the function space of based maps. *Homology, Homotopy Appl.* **6** (2004), 341–350. Zbl 1065.55005 MR 2084591

[11] G. Lupton and S. Smith, Rationalized evaluation subgroups of a map I: Sullivan models, derivations and G-sequences. *J. Pure Appl. Algebra* **209** (2007), 159–171. Zbl 1112.55012 MR 2292124

[12] G. Lupton and S. Smith, Rank of the fundamental groups of any component of a function space. *Proc. Amer. Math. Soc.* **135** (2007), 2649–2659. Zbl 1116.55006 MR 2302588

[13] A. Murillo and U. Buijs, Basic constructions in rational homotopy theory of function spaces. *Ann. Inst. Fourier* **56** (3) (2006), 815–838. Zbl 1122.55008 MR 2244231

[14] D. Tanré, *Homotopie Rationelle: Modèles de Chen, Quillen, Sullivan*. Lecture Notes in Math. 1025, Springer-Verlag, Berlin 1983. Zbl 0539.55001 MR 0764769

[15] R. Thom, L'homologie des espaces fonctionnels. *Colloque de topologie algébrique* (Louvain 1957), 29–39. Zbl 0077.36301 MR 0089408

[16] M. Vigué-Poirrier, Sur l'homotopie rationnelle des espaces fonctionnels. *Manuscripta Math.* **56** (1986), 177–191. Zbl 0597.55008 MR 0850369

Received October 4, 2006

Urtzi Buijs, Departamento de Algebra, Geometría y Topología, Universidad de Málaga, Ap. 59, 29080 Málaga, Spain

E-mail: urtzi@agt.cie.uma.es

Aniceto Murillo, Departamento de Algebra, Geometría y Topología, Universidad de Málaga, Ap. 59, 29080 Málaga, Spain

E-mail: aniceto@agt.cie.uma.es