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Comparison of dynamical degrees for semi-conjugate meromorphic maps

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Abstract. Let $f: X \to X$ be a dominant meromorphic map on a projective manifold X which preserves a meromorphic fibration $\pi: X \to Y$ of X over a projective manifold Y. We establish formulas relating the dynamical degrees of f, the dynamical degrees of f relative to the fibration and the dynamical degrees of the map $g: Y \to Y$ induced by f. Applications are given.

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1. Introduction

Let (X, ω_X) be a compact Kähler manifold of dimension k and let $f: X \to X$ be a meromorphic map. We assume that f is *dominant*, i.e. the image of f contains an open subset of X. Let $\pi: X \to Y$ be a dominant meromorphic map from X onto a compact Kähler manifold (Y, ω_Y) of dimension $l \leq k$. The fibers of π define a fibration on X which might be singular. If f preserves this fibration, i.e. f sends generic fibers of π to fibers of π, it induces a dominant meromorphic map $g: Y \to Y$ such that $\pi \circ f = g \circ \pi$. In that case, we say that f is *semi-conjugate* to g. For simplicity, we assume that ω_Y is normalized so that ω_Y^l is a probability measure.

A natural question is how the dynamical system defined by f is similar to the one defined by g when f is semi-conjugate to g as above. One of the first steps towards understanding this question should be to find out the relations between some invariants associated to f and g. In this paper, we will compare their dynamical degrees.

Let $f^n := f \circ \cdots \circ f$, n times, denote the iterate of order n of f. The dynamical degree $d_p(f)$ of order p is the quantity which measures the growth of the norms of $(f^n)^*$ acting on the Hodge cohomology group $H^{p,p}(X, \mathbb{R})$ when n tends to infinity. By Poincaré duality, it also measures the growth of the norms of $(f^n)_*$ acting on $H^{k-p,k-p}(X, \mathbb{R})$. If X is a projective manifold, $d_p(f)$ represents the volume growth of $f^n(V)$ for p-dimensional (closed complex) submanifolds V of X.

It was shown by Sibony and the first author in [6], [7] that dynamical degrees are bi-meromorphic invariants, that is, if f and g are bi-meromorphically conjugate, they have the same dynamical degrees. Dynamical degrees capture important dynamical information, in particular, in the computation of the topological entropy or in the construction of Green currents and of measures of maximal entropy. We refer the reader to the above references and to [8], [10], [15], [20] for more results on this matter.

When f preserves a fibration $\pi: X \to Y$ as above, the dynamical degree $d_p(f|\pi)$ of order p of f relative to π measures the growth of $(f^n)^*$ acting on the subspace $H_\pi^{l+p,l+p}(X,\mathbb{R})$ of classes in $H^{l+p,l+p}(X,\mathbb{R})$ which can be supported by a generic fiber of π. It also measures the growth of $(f^n)_*$ acting on $H_\pi^{k-p,k-p}(X,\mathbb{R})$ and represents the volume growth of $f^n(V)$ for p-dimensional submanifolds V of a generic fiber of π when X is projective. Precise definitions and properties will be given in Section 3. Here is our main result.

Theorem 1.1. *Let X and Y be projective manifolds of dimension k and l respectively with $k \geq l$. Let $f: X \to X$, $g: Y \to Y$ and $\pi: X \to Y$ be dominant meromorphic maps such that $\pi \circ f = g \circ \pi$. Then the dynamical degrees $d_p(f)$ of f are related to the dynamical degrees $d_p(g)$ of g and the relative dynamical degrees $d_p(f|\pi)$ by the formulas*

$$d_p(f) = \max_{\max\{0,p-k+l\} \leq j \leq \min\{p,l\}} d_j(g) d_{p-j}(f|\pi)$$

for $0 \leq p \leq k$.

Note that the condition $\max\{0, p-k+l\} \leq j \leq \min\{p,l\}$ is equivalent to $0 \leq j \leq l$ and $0 \leq p-j \leq k-l$. It guarantees that $d_j(g)$ and $d_{p-j}(f|\pi)$ are meaningful[1]. We deduce from the above result that $\max d_p(f) \geq \max d_p(g)$. This gives an affirmative answer to the problem 9.3 in Hasselblatt–Propp [12]. When X and Y have the same dimension, generic fibers of π are finite and have the same cardinality. Moreover, f defines bijections between generic fibers of π. We deduce from the proof of Theorem 1.1 the following corollary which generalizes a result in [6], [7]. It was proved by Nakayama–Zhang for holomorphic maps in [14].

Corollary 1.2. *Let X and Y be compact Kähler manifolds of same dimension k. Let $f: X \to X$, $g: Y \to Y$ and $\pi: X \to Y$ be dominant meromorphic maps such that $\pi \circ f = g \circ \pi$. Then the dynamical degrees of f are equal to the dynamical degrees of g.*

[1] We will find later analogous conditions, essentially for the same reason but also to avoid expressions which always vanish, e.g. $\omega_Y^{l+1} = 0$.

Recall that by a theorem of Khovanskii [13], Teissier [17] and Gromov [9], the dynamical degrees of f are log-concave, i.e. $p \mapsto \log d_p(f)$ is concave. Therefore, there are integers $p \leq p'$ such that

$$1 = d_0(f) < \cdots < d_p(f) = \cdots = d_{p'}(f) > \cdots > d_k(f).$$

An instructive example with $p \neq p'$ is a map $f(x_1, x_2) = (h(x_1), x_2)$ on a product $X_1 \times X_2$ of projective manifolds. A natural problem is to find dynamically interesting examples of maps on projective manifolds. Therefore, it would be interesting to see construction of maps with distinct consecutive dynamical degrees, i.e. with $p = p'$. Somehow, this condition insures that there is no trivial direction in the associated dynamical systems. We have the following useful results.

Corollary 1.3. *Let f, π, g be as in Theorem 1.1. If the consecutive dynamical degrees of f are distinct, then the same property holds for g and for the consecutive dynamical degrees of f relative to π.*

The following result is obtained using the Iitaka fibrations of X.

Corollary 1.4. *Let X be a projective manifold admitting a dominant meromorphic map with distinct consecutive dynamical degrees. Then the Kodaira dimension of X is either equal to 0 or $-\infty$.*

Note that the same result was proved for compact Kähler surfaces by Cantat in [3] and Guedj in [11], and for holomorphic maps on compact Kähler manifolds by Nakayama and Zhang in [14], [21]. We also refer to Amerik–Campana [1] and Nakayama-Zhang [14], [22] for other invariant fibrations for which Theorem 1.1 may be applied in order to compute dynamical degrees.

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2. Positive closed currents

The proof of our main result uses a delicate calculus on positive closed currents on compact Kähler manifolds[2]. In this section, we prove some useful results which can be applied to currents of integration on varieties and may have independent interest. The reader will find in Demailly [4] and Voisin [19] the basic facts on currents and on Kähler geometry.

[2] In this paper, we only consider the strong positivity, see e.g. [8], A.2, for the terminology.

Let (X, ω_X) be a compact Kähler manifold of dimension k. Let $\mathcal{K}^p(X)$ denote the cone of classes of strictly positive closed (p,p)-forms in $H^{p,p}(X,\mathbb{R})$. This is an open cone which is salient, i.e. $\overline{\mathcal{K}^p(X)} \cap -\overline{\mathcal{K}^p(X)} = \{0\}$. If c, c' are two classes in $H^{p,p}(X,\mathbb{R})$, we write $c \leq c'$ and $c' \geq c$ when $c' - c$ is in $\mathcal{K}^p(X) \cup \{0\}$.

If T is a real closed (p,p)-current, denote by $\{T\}$ its class in $H^{p,p}(X,\mathbb{R})$. If moreover T is positive, the *mass* of T is defined by $\|T\| := \langle T, \omega_X^{k-p}\rangle$. We often use the properties that $\|T\|$ depends only on the class of T and $\{T\} \leq A\|T\|\{\omega_X^p\}$ for some constant $A > 0$ independent of T. The following semi-regularization of currents was proved by Sibony and the first author in [6], [7].

Proposition 2.1. *Let T be a positive closed (p,p)-current on a compact Kähler manifold (X,ω_X). Then there is a sequence of smooth positive closed (p,p)-forms T_n on X which converges weakly to a positive closed (p,p)-current T' such that $T' \geq T$, i.e. $T' - T \geq 0$, $\|T_n\| \leq A\|T\|$ and $\{T_n\} \leq A\|T\|\{\omega_X^p\}$, where $A > 0$ is a constant independent of T. Moreover, if T is smooth on an open set U, then for every compact set $K \subset U$, we have $T_n \geq T$ on K when n is large enough.*

We need the following lemma.

Lemma 2.2. *Let T and S be positive closed currents on X of bidegree (p,p) and (q,q) respectively with $p+q \leq k$. Assume that T is smooth on a dense Zariski open set U of X. Then $T_{|U} \wedge S_{|U}$ has a finite mass. More precisely, there is a constant $A > 0$ independent of T, S and U such that*

$$\|T_{|U} \wedge S_{|U}\| := \langle T_{|U} \wedge S_{|U}, \omega_X^{k-p-q}\rangle \leq A\|T\|\|S\|.$$

Proof. Let T_n and K be as in Proposition 2.1. Since $\|T_n \wedge S\|$ can be computed cohomologically, we have

$$\|T_{|K} \wedge S_{|K}\| \leq \liminf_{n\to\infty} \|T_n \wedge S\| \leq A\|T\|\|\omega_X^p \wedge S\| = A\|T\|\|S\|.$$

This property holds for every compact subset K of U. Therefore,

$$\|T_{|U} \wedge S_{|U}\| \leq A\|T\|\|S\|.$$

The lemma follows. □

Consider currents T and S as in Lemma 2.2. So, $T_{|U} \wedge S_{|U}$ has a finite mass. Therefore, by Skoda's theorem [16], its trivial extension defines a positive closed current on X. We denote by $T \overset{\circ}{\wedge} S$ this current obtained for the maximal Zariski open set U on which T is smooth (in that case $T_{|U}$ is the regular part of T). Observe that when S has no mass on proper analytic subsets of X, the current obtained in this way does not change if we replace U with another dense Zariski open set. We often use this property in the sequence. By Lemma 2.2, we always have

$$\|T \overset{\circ}{\wedge} S\| \leq A\|T\|\|S\|.$$

We will be interested in positive closed currents T on $Y \times \mathbb{P}^m$, where (Y, ω_Y) is a compact Kähler manifold of dimension l and $\mathbb{P}^m$ is the projective space of dimension m endowed with the standard Fubini–Study form ω_{FS}. We assume that ω_{FS} is normalized so that ω_{FS}^m is a probability measure. In practice, we will take $m := k - l = \dim X - \dim Y$. In order to simplify the notation, the pull-back of ω_Y and ω_{FS} to $Y \times \mathbb{P}^m$ under the canonical projections are also denoted by ω_Y and ω_{FS}. Consider on $Y \times \mathbb{P}^m$ the Kähler form $\omega := \omega_Y + \omega_{\mathrm{FS}}$. The pull-back of a class c in $H^*(Y, \mathbb{C})$ or $H^*(\mathbb{P}^m, \mathbb{C})$ to $H^*(Y \times \mathbb{P}^m, \mathbb{C})$ under the canonical projections is also denoted by c.

If T is a positive closed (p, p)-current on $Y \times \mathbb{P}^m$, define for $\max\{0, p - m\} \leq j \leq \min\{l, p\}$ (or equivalently, for $0 \leq j \leq l$ and $0 \leq p - j \leq m$)

$$\alpha_j(T) := \big\langle T, \omega_Y^{l-j} \wedge \omega_{\mathrm{FS}}^{m-p+j} \big\rangle. \tag{1}$$

Observe that $\alpha_j(T)$ depends only on the class $\{T\}$ of T. Denote by $\cup$ the cup-product on Hodge cohomology groups.

Proposition 2.3. *Let T be a positive closed (p, p)-current on $Y \times \mathbb{P}^m$ as above. Then*

$$\{T\} \leq A \sum_{\max\{0,p-m\} \leq j \leq \min\{l,p\}} \alpha_j(T)\{\omega_Y^j\} \cup \{\omega_{\mathrm{FS}}^{p-j}\},$$

where $A > 0$ is a constant independent of T.

Proof. By the Künneth formula (see e.g. [19], p. 266) we have

$$H^*(Y \times \mathbb{P}^m, \mathbb{C}) = H^*(Y, \mathbb{C}) \otimes H^*(\mathbb{P}^m, \mathbb{C}).$$

Therefore, there are classes $c_j \in H^{j,j}(Y, \mathbb{R})$ such that

$$\{T\} = \sum_{\max\{0,p-m\} \leq j \leq \min\{l,p\}} c_j \cup \{\omega_{\mathrm{FS}}^{p-j}\}.$$

Let S be a smooth positive closed $(l - j, l - j)$-form on Y and S' its canonical pull-back to $Y \times \mathbb{P}^m$. Recall that c_j denotes also the pull-back of c_j to $Y \times \mathbb{P}^m$. Since ω_{FS}^m is a probability measure on $\mathbb{P}^m$, a simple computation on bidegree gives

$$c_j \cup \{S\} = c_j \cup \{S'\} \cup \{\omega_{\mathrm{FS}}^m\} = \langle T, S' \wedge \omega_{\mathrm{FS}}^{m-p+j} \rangle \geq 0.$$

So, c_j belongs to the convex closed cone $\mathcal{K}$ of classes c in $H^{j,j}(Y,\mathbb{R})$ with $c \cup c' \geq 0$ for $c' \in \mathcal{K}^{l-j}(Y)$. Since $\mathcal{K}^{l-j}(Y)$ is open and since $\cup$ is non-degenerate, $\mathcal{K}$ is salient, i.e. $\mathcal{K} \cap -\mathcal{K} = \{0\}$. The fact that $\{\omega_Y^{l-j}\}$ is in the interior of $\mathcal{K}^{l-j}(Y)$ implies that $c_j \cup \{\omega_Y^{l-j}\} = 0$ only when $c_j = 0$. Moreover, we have

$$\|c_j\| \leq A' c_j \cup \{\omega_Y^{l-j}\} = A'\langle T, \omega_Y^{l-j} \wedge \omega_{\mathrm{FS}}^{m-p+j}\rangle = A'\alpha_j(T)$$

for a fixed norm $\| \ \|$ on $H^{j,j}(Y,\mathbb{R})$ and for some constant $A' > 0$. It follows that

$$c_j \leq A\alpha_j(T)\{\omega_Y^j\}$$

for some constant $A > 0$. The result follows. □

Proposition 2.4. *Let T be a positive closed (p,p)-current on $Y \times \mathbb{P}^m$ as above. Assume that Y is a projective manifold. Then there is a sequence of smooth positive closed (p,p)-forms T_n on $Y \times \mathbb{P}^m$ which converges weakly to a current $T' \geq T$ such that $\alpha_j(T_n) \leq A\alpha_j(T)$ for all j, where $A > 0$ is a constant independent of T. Moreover, if T is smooth on an open set U, then for every compact subset K of U and every $\epsilon > 0$, we have $T_n \geq T - \epsilon\omega^p$ on K when n is large enough.*

Proof. We first consider the case where $Y = \mathbb{P}^l$ and ω_Y is the Fubini–Study form normalized so that ω_Y^l is a probability measure. The Künneth formula applied to this particular case says that T is cohomologous to

$$\sum_{\max\{0,p-m\} \leq j \leq \min\{l,p\}} \alpha_j(T)\{\omega_Y^j\} \cup \{\omega_{\mathrm{FS}}^{p-j}\}.$$

Since $Y \times \mathbb{P}^m$ is homogeneous, we can regularize T using the automorphisms of $Y \times \mathbb{P}^m$ which are close to the identity.

More precisely, let ν_n be a sequence of smooth probability measures on the group of automorphisms $\mathrm{Aut}(Y \times \mathbb{P}^m)$ of $Y \times \mathbb{P}^m$ whose supports converge to the identity $\mathrm{id} \in \mathrm{Aut}(Y \times \mathbb{P}^m)$. Define

$$T_n := \int_{\tau \in \mathrm{Aut}(Y \times \mathbb{P}^m)} \tau_*(T)\, d\nu_n(\tau).$$

Then, the T_n are smooth positive closed (p,p)-forms and converge weakly to T. We also have $\{T_n\} = \{T\}$ and hence $\alpha_j(T_n) = \alpha_j(T)$. This gives the first assertion for $Y = \mathbb{P}^l$.

For the second assertion, we can prove a stronger property. Let Φ be a smooth positive (p,p)-form on U such that $\Phi \leq T$. We do not assume that T is smooth nor that Φ is closed on U. Then

$$\Phi_n := \int_{\tau \in \mathrm{Aut}(Y \times \mathbb{P}^m)} \tau_*(\Phi)\, d\nu_n(\tau)$$

converges uniformly to Φ on K. Since $\Phi_n \leq T_n$, we have $T_n \geq \Phi - \epsilon\omega^p$ on K for n large enough. With our hypothesis, T is smooth on U and we can replace Φ with T.

Assume now that Y is a general projective manifold. We may find a finite family of open holomorphic maps Ψ_i, $1 \leq i \leq s$, from Y onto $\mathbb{P}^l$ such that for every point $y \in Y$ at least one map Ψ_i is of maximal rank at y. To do this it suffices to embed Y into a projective space and take a family of central projections. Let $\Pi_i : Y \times \mathbb{P}^m \to \mathbb{P}^l \times \mathbb{P}^m$ be defined by

$$\Pi_i(y, z) := (\Psi_i(y), z), \quad (y, z) \in Y \times \mathbb{P}^m.$$

We apply the first case to the currents $T^{(i)} := (\Pi_i)_*(T)$.

We construct as above smooth positive closed (p, p)-forms $T_n^{(i)}$ on $\mathbb{P}^l \times \mathbb{P}^m$ converging to $T^{(i)}$ such that $\{T_n^{(i)}\} = \{T^{(i)}\}$. Define $T_n := \sum_i \Pi_i^*(T_n^{(i)})$. Since the cohomology classes of $T_n^{(i)}$ are bounded, the classes of T_n are also bounded. Therefore, the masses of T_n are bounded. Up to extracting a subsequence, we can assume that $\Pi_i^*(T_n^{(i)})$ converges and hence the T_n converge to a positive closed current T'. If (y, z) is a point in $Y \times \mathbb{P}^m$ and Ψ_i has maximal rank at y, then Π_i defines a local bi-holomorphic map on a neighbourhood of (y, z). In this neighbourhood, we have

$$T \leq \Pi_i^*(\Pi_i)_*(T) = \Pi_i^*(T^{(i)}) \leq \lim_{n\to\infty} \Pi_i^*(T_n^{(i)}) \leq T'.$$

The choice of Ψ_i implies that $T \leq T'$ on $Y \times \mathbb{P}^m$. The second assertion of the proposition is a local property. So, it is also easy to check.

It remains to prove the estimate on $\alpha_j(T_n)$. Let $\tilde{\omega}_{\mathrm{FS}}$ denote the Fubini–Study form of $\mathbb{P}^l$ normalized so that $\tilde{\omega}_{\mathrm{FS}}^l$ is a probability measure. Since $\tilde{\omega}_{\mathrm{FS}}$ is strictly positive, there is a constant $A_1 > 0$ such that $(\Psi_i)_*\{\omega_Y^{l-j}\} \leq A_1\{\tilde{\omega}_{\mathrm{FS}}^{l-j}\}$. We also have $(\Psi_i)^*(\tilde{\omega}_{\mathrm{FS}}^{l-j}) \leq A_2\omega_Y^{l-j}$ for some constant $A_2 > 0$. For simplicity, we will also denote by ω_Y, ω_{FS} and $\tilde{\omega}_{\mathrm{FS}}$ the pull-backs of these forms to $Y \times \mathbb{P}^m$ or to $\mathbb{P}^l \times \mathbb{P}^m$. In particular, $(\Pi_i)_*(\omega_Y^{l-j} \wedge \omega_{\mathrm{FS}}^{m-p+j})$ and $(\Psi_i)_*(\omega_Y^{l-j}) \wedge \omega_{\mathrm{FS}}^{m-p+j}$ represent the same form on $\mathbb{P}^l \times \mathbb{P}^m$. Since the $T_n^{(i)}$ are smooth and since the following integrals can be computed cohomologically, we have

$$\begin{aligned}
\big\langle \Pi_i^*(T_n^{(i)}), \omega_Y^{l-j} \wedge \omega_{\mathrm{FS}}^{m-p+j}\big\rangle &= \big\langle T_n^{(i)}, (\Psi_i)_*(\omega_Y^{l-j}) \wedge \omega_{\mathrm{FS}}^{m-p+j}\big\rangle \\
&\leq A_1\big\langle T_n^{(i)}, \tilde{\omega}_{\mathrm{FS}}^{l-j} \wedge \omega_{\mathrm{FS}}^{m-p+j}\big\rangle \\
&= A_1\big\langle T^{(i)}, \tilde{\omega}_{\mathrm{FS}}^{l-j} \wedge \omega_{\mathrm{FS}}^{m-p+j}\big\rangle \\
&= A_1\big\langle T, \Psi_i^*(\tilde{\omega}_{\mathrm{FS}}^{l-j}) \wedge \omega_{\mathrm{FS}}^{m-p+j}\big\rangle \\
&\leq A_1A_2\big\langle T, \omega_Y^{l-j} \wedge \omega_{\mathrm{FS}}^{m-p+j}\big\rangle.
\end{aligned}$$

It follows that $\alpha_j\big(\Pi_i^*(T_n^{(i)})\big) \leq A_1A_2\alpha_j(T)$ and hence $\alpha_j(T_n) \leq A\alpha_j(T)$ for some constant $A > 0$. $\square$

3. Dynamical degrees

Let $\pi\colon (X,\omega_X) \to (Y,\omega_Y)$ be a dominant meromorphic map between compact Kähler manifolds of dimension k and l respectively. The map π is holomorphic outside the indeterminacy set I_π which is an analytic subset of X of codimension at least 2. The closure Γ of its graph over $X \setminus I_\pi$ is an irreducible analytic subset of dimension k of $X\times Y$. If, τ_X and τ_Y denote the projections from $X\times Y$ onto its factors, then τ_X defines a bi-holomorphic map between $\Gamma \setminus \tau_X^{-1}(I_\pi)$ and $X\setminus I_\pi$. The fibers of $\tau_{X|\Gamma}$ over I_π have positive dimension. One can identify π with $\tau_Y\circ(\tau_{X|\Gamma})^{-1}$. For $A\subset X$ and $B\subset Y$, define $\pi(A) := \tau_Y(\tau_{X|\Gamma})^{-1}(A)$ and $\pi^{-1}(B) := \tau_X(\tau_{Y|\Gamma})^{-1}(B)$.

The map π induces linear operators on currents. If Φ is a smooth (p,q)-form on Y, then $\pi^*(\Phi)$ is the (p,q)-current defined by

$$\pi^*(\Phi) := (\tau_X)_*(\tau_Y^*(\Phi)\wedge[\Gamma]),$$

where $[\Gamma]$ is the current of integration on Γ. It is not difficult to see that $\pi^*(\Phi)$ is an L^1 form smooth outside I_π. If Ψ is a smooth (p,q)-form on X with $p,q\geq k-l$, then $\pi_*(\Psi)$ is the $(p-k+l, q-k+l)$-current defined by

$$\pi_*(\Psi) := (\tau_Y)_*(\tau_X^*(\Psi)\wedge[\Gamma]).$$

If Φ and Ψ are closed or positive, so are $\pi^*(\Phi)$ and $\pi_*(\Psi)$. Therefore, π^* and π_* induce linear operators on the Hodge cohomology groups of X and Y.

In general, the above operators do not extend continuously to positive closed currents. We will use instead the strict transforms of currents $\pi^\bullet$ and $\pi_\bullet$ which coincide with π^* and π_* on smooth positive closed forms. In this paper, we only need these operators in the case where X and Y have the same dimension k.

Let U be the maximal Zariski open set in $X\setminus I_\pi$ such that $\pi\colon U\to\pi(U)$ is locally invertible. The complement of U in X is called the *critical set* of π. If T is a positive closed (p,p)-current on Y, $(\pi_{|U})^*(T)$ is well-defined and is a positive closed (p,p)-current on U. Proposition 2.1 allows us to show that this current has finite mass. By Skoda's Theorem [16], its trivial extension to X is a positive closed (p,p)-current that we denote by $\pi^\bullet(T)$.

Let V be the maximal Zariski open set in $Y\setminus\pi(I_\pi)$ such that $\pi\colon \pi^{-1}(V)\to V$ is a non-ramified covering. The complement of V in Y is called the *set of critical values* of π. If S is a positive closed (p,p)-current on X, then $\pi_\bullet(S)$ is the trivial extension of $(\pi_{|\pi^{-1}(V)})_*(S)$ to Y. This is also a positive closed (p,p)-current. We will use the properties that $\|\pi^\bullet(T)\|\leq A\|T\|$ and $\|\pi_\bullet(S)\|\leq A\|S\|$ for some constant $A>0$ independent of T, S, see [6], [7] for details.

Consider now a dominant meromorphic self-map $f\colon X\to X$. The iterate of order n of f is defined by $f^n := f\circ\cdots\circ f$ (n times) on a dense Zariski open set and extends to a dominant meromorphic map on X. Define for $0\leq p\leq k$

$$\lambda_p(f^n) := \|(f^n)^*(\omega_X^p)\| = \langle (f^n)^*(\omega_X^p), \omega_X^{k-p}\rangle.$$

It is not difficult to see that

$$\lambda_p(f^n) = \|(f^n)_*(\omega_X^{k-p})\| = \langle (f^n)_*(\omega_X^{k-p}), \omega_X^p\rangle.$$

It was shown in [6], [7] that $[\lambda_p(f^n)]^{1/n}$ converges to a constant $d_p(f)$ which is the *dynamical degree of order* p of f. Note that the main difficulty here is that in general we do not have $(f^{n+s})^* = (f^n)^* \circ (f^s)^*$ on cohomology classes.

Let $\|\ \|_{H^{p,p}}$ denote the norm of an operator acting on $H^{p,p}(X,\mathbb{R})$ with respect to a fixed norm on that space. Since the mass of a positive closed current depends only on its cohomology class, we deduce from the above discussion that

$$A^{-1}\lambda_p(f^n) \le \|(f^n)^*\|_{H^{p,p}} \le A\lambda_p(f^n),$$

for some constant $A > 0$. It follows that

$$d_p(f) = \lim_{n\to\infty} \|(f^n)^*\|_{H^{p,p}}^{1/n}.$$

Note that we also have $d_p(f^n) = d_p(f)^n$ for $n \ge 1$. The last dynamical degree $d_k(f)$ is also called the *topological degree* of f. It is equal to the number of points in a generic fiber of f and we have $\lambda_k(f^n)\|\omega_X^k\|^{-1} = d_k(f^n) = d_k(f)^n$.

Proposition 3.1. *Let T be a positive closed (p,p)-current and S a positive closed $(k-p,k-p)$-current on X. Then*

$$\|(f^n)^\bullet(T)\| \le A\|T\|\lambda_p(f^n) \quad \textit{and} \quad \|(f^n)_\bullet(S)\| \le A\|S\|\lambda_p(f^n)$$

for some constant $A > 0$ independent of T, S and n. In particular, we have

$$\limsup_{n\to\infty} \|(f^n)^\bullet(T)\|^{1/n} \le d_p(f) \quad \textit{and} \quad \limsup_{n\to\infty} \|(f^n)_\bullet(S)\|^{1/n} \le d_p(f).$$

Proof. We show the first inequality. The second one is proved in the same way. Let T_i be smooth positive closed forms as in Proposition 2.1. It follows from the definition of $(f^n)^\bullet$ that any limit value of $(f^n)^*(T_i)$ is larger than or equal to $(f^n)^\bullet(T)$. So, it is enough to bound the mass of $(f^n)^*(T_i)$. Since this mass can be computed cohomologically and since $\{T_i\} \le A\|T\|\{\omega_X^p\}$, we obtain that $\|(f^n)^*(T_i)\| \le A\|T\|\lambda_p(f^n)$ for some constant $A > 0$. This completes the proof. □

The above proposition can be applied to currents of integration on submanifolds V of dimension $k-p$ or p of X and gives a upper bound for the volume growth of the preimage or image of V by f^n.

It was shown in [6], [7] that dynamical degrees are bi-meromorphic invariants, i.e. conjugate maps have the same dynamical degrees. This property allows us to define dynamical degrees for maps on singular manifolds having a kählerian desingularization. We will use the same argument in order to define dynamical degrees relative to an invariant meromorphic fibration.

Let f, g, π be as in Theorem 1.1. So, π defines a fibration and f preserves this fibration. Let us assume first that π is a *holomorphic* map. By the Bertini–Sard theorem, the set Z of critical values of π is a proper analytic subset of Y. Therefore, $\pi\colon X \setminus \pi^{-1}(Z) \to Y \setminus Z$ defines a regular holomorphic fibration. Its fibers form a continuous family of smooth submanifolds of dimension $k-l$ of X.

Let P_f and P_g denote the union of the critical set and the set of critical values of f and g respectively. They contain the indeterminacy sets of f and of g. A fiber $L_y := \pi^{-1}(y)$ with $y \in Y \setminus Z$ is called *generic* if for every $n \geq 0$

(a) $g^n(y)$ and $g^{-n}(y)$ do not intersect P_g;

(b) For every point b in $g^n(y) \cup g^{-n}(y)$, no component of L_b is contained in P_f.

Denote by Σ the set of y such that L_y is generic. Observe that $Y \setminus \Sigma$ is contained in a finite or countable union of proper analytic subsets of Y. So, Σ is connected. We also have $g(\Sigma) = g^{-1}(\Sigma) = \Sigma$. We will use the following lemma for $\nu = \omega_Y^l$ and for $\nu = [d_l(g)]^{-n}(g^n)^*(\omega_Y^l)$.

Lemma 3.2. *Let L_y be a generic fiber as above. Let ν be a probability measure on Y which has no mass on proper analytic subsets of Y. Then, for $0 \leq p \leq k-l$ and for $n \geq 0$, the 6 positive closed currents*

$$d_l(g)^{-n}(f^n)^\bullet(\omega_X^p \wedge [L_y]), \quad (f^n)^*(\omega_X^p) \overset{\circ}{\wedge} [L_y], \quad (f^n)^*(\omega_X^p) \overset{\circ}{\wedge} \pi^*(\nu)$$

and

$$(f^n)_\bullet(\omega_X^{k-l-p} \wedge [L_y]), \quad d_l(g)^{-n}(f^n)_*(\omega_X^{k-l-p}) \overset{\circ}{\wedge} [L_y],$$
$$d_l(g)^{-l}(f^n)_*(\omega_X^{k-l-p}) \overset{\circ}{\wedge} \pi^*(\nu)$$

have the same mass. In particular, their mass does not depend on $y \in \Sigma$.

Proof. For $y \in \Sigma$, define

$$\varphi(y) := d_l(g)^{-n}\big\|(f^n)^\bullet(\omega_X^p \wedge [L_y])\big\| \quad \text{and} \quad \psi(y) := \big\|(f^n)_\bullet(\omega_X^{k-l-p} \wedge [L_y])\big\|.$$

It is not difficult to see that these functions are continuous on Σ. We have

$$\begin{aligned}\varphi(y) &= d_l(g)^{-n}\big\langle (f^n)^\bullet(\omega_X^p \wedge [L_y]), \omega_X^{k-l-p}\big\rangle \\ &= d_l(g)^{-n}\big\langle \omega_X^p, [L_y] \overset{\circ}{\wedge} (f^n)_*(\omega_X^{k-l-p})\big\rangle.\end{aligned}$$

It follows that

$$\varphi = d_l(g)^{-n}\pi_*\big(\omega_X^p \wedge (f^n)_*(\omega_X^{k-l-p})\big)$$

in the sense of currents on Y. Therefore, φ defines a closed 0-current on Y and it should be constant on Σ.

We also deduce from the above computation that

$$\varphi(y) = d_l(g)^{-n} \big\| (f^n)_*(\omega_X^{k-l-p}) \overset{\circ}{\wedge} [L_y] \big\|.$$

Since ν has no mass on $Y \setminus \Sigma$, we obtain

$$\varphi = \int \varphi(y) d\nu = d_l(g)^{-n} \big\| (f^n)_*(\omega_X^{k-l-p}) \overset{\circ}{\wedge} \pi^*(\nu) \big\|.$$

In the same way, we prove that ψ is constant on Σ and

$$\psi = \big\| (f^n)^*(\omega_X^p) \overset{\circ}{\wedge} [L_y] \big\| = \big\| (f^n)^*(\omega_X^p) \overset{\circ}{\wedge} \pi^*(\nu) \big\|.$$

It remains to check that $\varphi = \psi$. Using that ψ is constant and $\# g^{-n}(y) = d_l(g)^n$, we have

$$\varphi = d_l(g)^{-n} \big\| (f^n)^\bullet(\omega_X^p \wedge [L_y]) \big\| = d_l(g)^{-n} \sum_{b \in g^{-n}(y)} \big\| (f^n)^*(\omega_X^p) \overset{\circ}{\wedge} [L_b] \big\| = \psi.$$

This completes the proof. □

Define $\lambda_p(f^n|\pi)$ the mass of the currents in Lemma 3.2. We have in particular

$$\lambda_p(f^n|\pi) = \big\| (f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^l) \big\|.$$

Recall that π is holomorphic and then $\pi^*(\omega_Y^l)$ is smooth.

Proposition 3.3. *The sequence $\lambda_p(f^n|\pi)^{1/n}$ converges to a constant $d_p(f|\pi)$. Let T be a positive closed $(p+l, p+l)$-current and S a positive closed $(k-p, k-p)$-current on X which are supported on a generic fiber L_y. Then*

$$\|(f^n)^\bullet(T)\| \leq A_y \|T\| d_l(g)^n \lambda_p(f^n|\pi) \quad \textit{and} \quad \|(f^n)_\bullet(S)\| \leq A_y \|S\| \lambda_p(f^n|\pi)$$

for some constant $A_y > 0$ independent of T and S. In particular, we have

$$\limsup_{n\to\infty} \|(f^n)^\bullet(T)\|^{1/n} \leq d_l(g) d_p(f|\pi)$$

and

$$\limsup_{n\to\infty} \|(f^n)_\bullet(S)\|^{1/n} \leq d_p(f|\pi).$$

Proof. Fix a generic fiber L_y with $y \in \Sigma$. We will show that

$$\lambda_p(f^{n+m}|\pi) \leq A_y \lambda_p(f^n|\pi)\lambda_p(f^m|\pi)$$

for some constant $A_y > 0$ and for all $n, m \geq 0$. This will imply the first assertion because the sequence $A_y \lambda_p(f^n|\pi)$ is sub-multiplicative.

Since L_y is a compact Kähler manifold, we can apply Proposition 2.1 to L_y. Let b be a point in Σ such that $g^m(b) = y$. Define $R := (f^m)_\bullet(\omega_X^{k-l-p} \wedge [L_b])$. This is a positive closed $(k-p, k-p)$-current on X which is also a $(k-l-p, k-l-p)$-current on L_y. By Lemma 3.2, we have $\|R\| = \lambda_p(f^m|\pi)$. Therefore, there are smooth positive closed $(k-l-p, k-l-p)$-forms Θ_i on L_y which converge to a current $\Theta \geq R$. Moreover, we have $\{\Theta_i\} \leq A_y \lambda_p(f^m|\pi)\{\omega_{X|L_y}^{k-l-p}\}$ for some constant $A_y > 0$, where the inequality is considered in $H^*(L_y, \mathbb{R})$.

Let h denote the restriction of f^n to L_y. It defines a meromorphic map from L_y to $L_{g^n(y)}$. Since the mass of a positive closed current can be computed cohomologically, we obtain

$$\begin{aligned}
\lambda_p(f^{n+m}|\pi) = \|(f^n)_\bullet(R)\| &\leq \liminf_{i\to\infty} \|h_*(\Theta_i)\| \leq A_y \lambda_p(f^m|\pi)\|h_*(\omega_{X|L_y}^{k-l-p})\| \\
&= A_y \lambda_p(f^m|\pi)\|(f^n)_\bullet(\omega_X^{k-l-p} \wedge [L_y])\| \\
&= A_y \lambda_p(f^m|\pi)\lambda_p(f^n|\pi).
\end{aligned}$$

This implies the first assertion in the proposition. The rest is proved in the same way using the semi-regularization result for T and S on L_y, see also Proposition 3.1. □

We call $d_p(f|\pi)$ the *dynamical degree of order* p *of* f *relative* to π. The convergence in Proposition 3.3 implies that $d_p(f^n|\pi) = d_p(f|\pi)^n$.

Remark 3.4. Our choice of Σ simplifies the calculus on currents but several properties above still hold for some $y \notin \Sigma$. For example, if y is a fixed point of g which is not a critical value of π and if no component of L_y is contained in the critical set of f, then $d_p(f|\pi) = d_p(f_{|L_y})$. The proof is left to the reader.

The next result shows that the relative dynamical degrees are bi-meromorphic invariants. Consider a bi-meromorphic map $\tau\colon (\widetilde{X}, \omega_{\widetilde{X}}) \to (X, \omega_X)$ between compact Kähler manifolds. Define $\tilde{\pi} := \pi \circ \tau$ and $\tilde{f} := \tau^{-1} \circ f \circ \tau$. Then, $\tilde{f}$ is a dominant meromorphic map conjugate to f and $\tilde{\pi} \circ \tilde{f} = g \circ \tilde{\pi}$.

Proposition 3.5. *Assume that* $\tilde{\pi}$ *is holomorphic. Then*

$$d_p(f|\pi) = d_p(\tilde{f}|\tilde{\pi})$$

for $0 \leq p \leq k-l$.

Proof. Since τ is bi-meromorphic, $\tau_\bullet \tilde{\pi}^*(\omega_Y^l) = \pi^*(\omega_Y^l)$ and $\tilde{f}^n = \tau^{-1} \circ f^n \circ \tau$, we have

$$\begin{aligned}\lambda_p(\tilde{f}^n|\tilde{\pi}) &= \langle (\tilde{f}^n)^*(\omega_{\tilde{X}}^p) \wedge \tilde{\pi}^*(\omega_Y^l), \omega_{\tilde{X}}^{k-l-p} \rangle \\ &= \langle \tau^\bullet (f^n)^\bullet \tau_*(\omega_{\tilde{X}}^p) \wedge \omega_{\tilde{X}}^{k-l-p}, \tilde{\pi}^*(\omega_Y^l) \rangle \\ &= \langle (f^n)^\bullet \tau_*(\omega_{\tilde{X}}^p) \overset{\circ}{\wedge} \tau_*(\omega_{\tilde{X}}^{k-l-p}), \pi^*(\omega_Y^l) \rangle.\end{aligned}$$

Using the semi-regularization result for $\tau_*(\omega_{\tilde{X}}^{k-l-p})$, we deduce that

$$\lambda_p(\tilde{f}^n|\tilde{\pi}) \leq A \langle (f^n)^\bullet \tau_*(\omega_{\tilde{X}}^p) \wedge \omega_X^{k-l-p}, \pi^*(\omega_Y^l) \rangle$$

for some constant $A > 0$. Then, using a semi-regularization of $\tau_*(\omega_{\tilde{X}}^p)$, we obtain

$$\lambda_p(\tilde{f}^n|\tilde{\pi}) \leq A' \langle (f^n)^*(\omega_X^p) \wedge \omega_X^{k-l-p}, \pi^*(\omega_Y^l) \rangle = A' \lambda_p(f^n|\pi)$$

for some constant $A' > 0$. It follows that $d_p(\tilde{f}|\tilde{\pi}) \leq d_p(f|\pi)$. The converse inequality is proved in the same way. □

The last proposition allows us to define relative dynamical degrees in the general case. Assume now that f preserves a *meromorphic* fibration $\pi: X \to Y$, i.e. $\pi \circ f = g \circ \pi$ as in Theorem 1.1. Let Γ denote the closure of the graph of π in $X \times Y$. Then Γ is an irreducible analytic set of dimension k which is bi-meromorphic to X. Let $\sigma: \tilde{X} \to \Gamma$ be a desingularization of Γ which can be constructed using a blow-up along the singularities. By Blanchard's theorem [2], $\tilde{X}$ is a compact Kähler manifold. Then, $\tau := \tau_X \circ \sigma$ is a bi-meromorphic map from $\tilde{X}$ to X. Define also $\tilde{\pi} := \tau_Y \circ \sigma$ and $\tilde{f} := \tau^{-1} \circ f \circ \tau$. The map $\tilde{\pi}$ is holomorphic and $\tilde{\pi} \circ \tilde{f} = g \circ \tilde{\pi}$. Define the *dynamical degree of order* p of f *relative* to π by

$$d_p(f|\pi) := d_p(\tilde{f}|\tilde{\pi}).$$

Proposition 3.5 implies that the definition does not depend on the choice of σ. The following result is a consequence of a theorem by Khovanskii, Teissier and Gromov.

Proposition 3.6. *The function $p \mapsto \log d_p(f|\pi)$ is concave for $0 \leq p \leq k - l$. In particular, $d_p(f|\pi) \geq 1$ for $0 \leq p \leq k - l$.*

Proof. We can assume that π is holomorphic. We have to show that

$$d_{p-1}(f|\pi) d_{p+1}(f|\pi) \leq d_p(f|\pi)^2.$$

For this purpose, it is enough to check that

$$\lambda_{p-1}(f^n|\pi) \lambda_{p+1}(f^n|\pi) \leq \lambda_p(f^n|\pi)^2.$$

Observe that for non-critical values y of π, the fibers L_y are not necessarily connected but they contain the same number s of components. The family of these components is connected since X is connected. It defines a covering of degree s over the set of non-critical values of π. Let Σ' denote the parameter space for the components L'_y of L_y with $y \in \Sigma$. We may think of Σ' as a covering of degree s over Σ. We can then prove as in Lemma 3.2 that the function $L'_y \mapsto \|(f^n)^*(\omega_X^p) \overset{\circ}{\wedge} [L'_y]\|$ is constant on Σ'. Therefore, it is equal to $s^{-1}\|(f^n)^*(\omega_X^p) \overset{\circ}{\wedge} [L_y]\|$ and then to $s^{-1}\lambda_p(f^n|\pi)$.

Let h be the restriction of f^n to $L_1 := L'_y$ and define $L_2 := h(L_1)$. Let Γ denote the graph of h in $L_1 \times L_2$ and $\tau\colon \widehat{\Gamma} \to \Gamma$ a desingularization of Γ using some blow-up along the singularities. By Blanchard's theorem [2], $\widetilde{\Gamma}$ is a compact Kähler manifold. Denote by $\tau_1\colon \widehat{\Gamma} \to L_1$ and $\tau_2\colon \widehat{\Gamma} \to L_2$ the canonical projections. We have $h = \tau_2 \circ \tau_1^{-1}$. Define $\omega_1 := \tau_1^*(\omega_X)$ and $\omega_2 := \tau_2^*(\omega_X)$. We deduce from the above discussion that

$$s^{-1}\lambda_p(f^n|\pi) = \|(h^n)^*(\omega_X^p)\| = \int_{\widehat{\Gamma}} \omega_1^{k-l-p} \wedge \omega_2^p.$$

If γ_p denotes the last integral, Gromov proved in [9] that $p \mapsto \log \gamma_p$ is concave, i.e. $\gamma_{p-1}\gamma_{p+1} \leq \gamma_p^2$, when ω_1 and ω_2 are Kähler forms. By continuity, this still holds in our case where these forms are only smooth positive and closed. Hence, $p \mapsto \log d_p(f|\pi)$ is concave.

In order to deduce the second assertion of the proposition, it is enough to show that $d_0(f|\pi) = 1$ and $d_{k-l}(f|\pi) \geq 1$. For y generic, we have

$$\lambda_0(f^n|\pi) = d_l(g)^{-n}\|(f^n)^\bullet[L_y]\| = d_l(g)^{-n} \sum_{b \in g^{-n}(y)} \|[L_b]\|.$$

Hence, $\lambda_0(f^n|\pi)$ is independent of n since $\#g^{-n}(y) = d_l(g)^n$ and the mass of $[L_b]$, with $b \in \Sigma$, is independent of b. It follows that $d_0(f|\pi) = 1$.

We also have for y generic and $b \in g^{-n}(y)$

$$\lambda_{k-l}(f^n|\pi) = \|(f^n)_\bullet[L_b]\| \geq \|[L_y]\|.$$

So, the sequence $\lambda_{k-l}(f^n|\pi)$ is bounded from below by a positive constant. Therefore, $d_{k-l}(f|\pi) \geq 1$. This completes the proof of the lemma. Note that we can show that $d_{k-l}(f|\pi)$ is the number of points in a generic fiber of the restriction of f to L_y. □

Consider now some examples, see also [1], [14], [21], [22].

Example 3.7. Let $X = Y \times Z$ be the product of two compact Kähler manifolds and $\pi\colon X \to Y$ the canonical projection. Consider $f(y,z) := (g(y), h(z))$ where

$g: Y \to Y$ and $h: Z \to Z$ are dominant meromorphic maps. So, f is semi-conjugate to g. The relative dynamical degree $d_p(f|\pi)$ is equal to $d_p(h)$. We easily deduce from the definition of dynamical degrees that

$$d_p(f) = \max_{\max\{0,p-k+l\} \leq j \leq \min\{p,l\}} d_j(g)d_{p-j}(h).$$

There are more interesting examples of maps on the product $Y \times Z$. Let F be a compact Kähler manifold. Assume that F is also the parameter space of a meromorphic family of meromorphic self-maps of Z. Let $\tau: Y \to F$ be a meromorphic map. Then $f(y,z) := (g(y), \tau(y)(z))$ is a meromorphic self-map of $Y \times Z$ which preserves the fibration π. The example is also interesting when $\tau(y)$ is holomorphic for generic y or when a Zariski open set G of F is a Lie group and τ is a morphism from G to the group of bi-meromorphic maps of Z.

Example 3.8. Let $g: Y \to Y$ be a dominant meromorphic map on a compact Kähler manifold Y. It induces a meromorphic self-map f on the projectivization $X := \mathbb{P}T_Y$ of the holomorphic tangent bundle of Y. The map f preserves the fibration associated to the canonical projection from X onto Y and is semi-conjugate to g. This example and some applications were considered in [5].

4. Proofs of the results

We first prove Theorem 1.1. Since the dynamical degrees are bi-meromorphic invariants, Proposition 3.5 allows us to assume that π is a holomorphic map. Since X is projective, we can construct a dominant meromorphic map $v: X \to \mathbb{P}^{k-l}$. Indeed, it is enough to embed X in a projective space and choose a generic central projection on $\mathbb{P}^{k-l}$. Replacing X with a desingularization of the graph of v allows to assume that v is holomorphic. Consider the holomorphic map $\Pi: X \to Y \times \mathbb{P}^{k-l}$ defined by

$$\Pi(x) := (\pi(x), v(x)).$$

Since the chosen central projection is generic, the intersection of a generic fiber of π and a generic fiber of v is finite. Therefore, Π is dominant.

Our proof is based on a delicate calculus on currents. If $X = Y \times \mathbb{P}^{k-l}$ and π is the canonical projection onto Y, the proof is simpler and the properties obtained in Section 2 can be directly applied. A rough idea is to reduce the general case to the particular case using the map Π. In other words, we use the fact that f is, in some sense, "semi-conjugate" to the multi-valued map $\Pi \circ f \circ \Pi^{-1}$ which is defined on $Y \times \mathbb{P}^{k-l}$.

Let ω_{FS} denote the Fubini–Study form on $\mathbb{P}^{k-l}$. For simplicity, the canonical pull-back of ω_Y and ω_{FS} to $Y \times \mathbb{P}^{k-l}$ are still denoted by ω_Y and ω_{FS}. In particular,

$\Pi^*(\omega_Y)$ and $\pi^*(\omega_Y)$ represent the same form on X. We consider on $Y \times \mathbb{P}^{k-l}$ the Kähler form $\omega := \omega_Y + \omega_{\mathrm{FS}}$. Our calculus will involve the quantities $a_{q,p}(n)$ defined for $n \geq 0$, $0 \leq q \leq k-l$ and $q \leq p \leq l+q$ by

$$\begin{aligned} a_{q,p}(n) &:= \|\Pi_*(f^n)^*\Pi^*(\omega^p) \wedge \omega_Y^{l-p+q}\| \\ &= \langle \Pi_*(f^n)^*\Pi^*(\omega^p), \omega_Y^{l-p+q} \wedge \omega^{k-l-q} \rangle \\ &= \langle (f^n)^*\Pi^*(\omega^p), \Pi^*(\omega_Y^{l-p+q} \wedge \omega^{k-l-q}) \rangle \\ &= \langle (f^n)^*\Pi^*(\omega^p) \wedge \pi^*(\omega_Y^{l-p+q}), \Pi^*(\omega^{k-l-q}) \rangle. \end{aligned}$$

Observe that

$$a_{q,p}(n) \geq \alpha_{p-q}\big(\Pi_*(f^n)^*\Pi^*(\omega^p)\big),$$

where $\alpha_{p-q}(\cdot)$ is defined in (1).

Lemma 4.1. *There is a constant $A > 0$ independent of p, n such that*

$$A^{-1}\lambda_p(f^n|\pi) \leq a_{p,p}(n) \leq A\lambda_p(f^n|\pi)$$

In particular, $[a_{p,p}(n)]^{1/n}$ converges to $d_p(f|\pi)$.

Proof. Since the pull-back of a smooth form under Π is smooth, we have

$$\begin{aligned} a_{p,p}(n) &= \langle (f^n)^*\Pi^*(\omega^p) \wedge \pi^*(\omega_Y^l), \Pi^*(\omega^{k-l-p}) \rangle \\ &\leq A\langle (f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^l), \omega_X^{k-l-p} \rangle = A\lambda_p(f^n|\pi) \end{aligned}$$

for some constant $A > 0$. This gives the second inequality in the lemma.

Define $T := \Pi_*(\omega_X^p)$. Since $\Pi^\bullet(T) \geq \omega_X^p$, we have

$$\lambda_p(f|\pi) = \|(f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^l)\| \leq \|(f^n)^\bullet\Pi^\bullet(T) \wedge \pi^*(\omega_Y^l)\|.$$

We apply Proposition 2.1 to the current T on $Y \times \mathbb{P}^{k-l}$ which is an L^1 form smooth on a Zariski open set. Let T_i be as in that proposition with $\{T_i\} \leq A\{\omega^p\}$ for some constant $A > 0$. If $S := \Pi_*(\omega_X^{k-l-p})$, we have $\Pi^\bullet(S) \geq \omega_X^{k-l-p}$ and hence

$$\begin{aligned} \lambda_p(f|\pi) &\leq \liminf_{i\to\infty} \|(f^n)^*\Pi^*(T_i) \wedge \pi^*(\omega_Y^l)\| \\ &\leq A\|(f^n)^*\Pi^*(\omega^p) \wedge \pi^*(\omega_Y^l)\| \\ &= A\langle (f^n)^*\Pi^*(\omega^p) \wedge \pi^*(\omega_Y^l), \omega_X^{k-l-p} \rangle \\ &\leq A\big\|(f^n)^*\Pi^*(\omega^p) \wedge \pi^*(\omega_Y^l) \mathbin{\mathring{\wedge}} \Pi^\bullet(S)\big\|. \end{aligned}$$

Now, we apply again Proposition 2.1, in particular its last assertion, to the current S which is an L^1 form smooth on a Zariski open set. If S_i are smooth forms satisfying that proposition, the latter expression is bounded from above by

$$\begin{aligned}\liminf_{i\to\infty} &\big\langle (f^n)^*\Pi^*(\omega^p)\wedge\pi^*(\omega_Y^l),\Pi^*(S_i)\big\rangle \\ &\lesssim \big\langle (f^n)^*\Pi^*(\omega^p)\wedge\pi^*(\omega_Y^l),\Pi^*(\omega^{k-l-p})\big\rangle.\end{aligned}$$

The last integral is equal to $a_{p,p}(n)$. The first inequality in the lemma follows. □

Define for $0\le p\le k$

$$b_p(n) := \sum_{\max\{0,p-l\}\le q\le \min\{p,k-l\}} a_{q,p}(n).$$

We have the following lemma.

Lemma 4.2. *The sequence $b_p(n)^{1/n}$ converges to $d_p(f)$.*

Proof. Since $\Pi^*(\omega^p)$, $\pi^*(\omega_Y^{l-p+q})$ and $\Pi^*(\omega^{k-l-q})$ are smooth on X, we have

$$\begin{aligned}a_{q,p}(n) &= \big\langle (f^n)^*\Pi^*(\omega^p)\wedge\pi^*(\omega_Y^{l-p+q}),\Pi^*(\omega^{k-l-q})\big\rangle \\ &\le A\|(f^n)^*(\omega_X^p)\| = A\lambda_p(f^n)\end{aligned}$$

for some constant $A>0$. We deduce that $\limsup b_p(n)^{1/n}\le d_p(f)$.

It remains to check that $\liminf b_p(n)^{1/n}\ge d_p(f)$. For this purpose, we only need to show that $\lambda_p(f^n)\le Ab_p(n)$ for some constant $A>0$. Define $T := \Pi_*(f^n)^*\Pi^*(\omega^p)$. We prove that $\lambda_p(f^n)\lesssim \|T\|\lesssim b_p(n)$ which will imply the result.

Define $S := \Pi_*(\omega_X^p)$. We have $\Pi^\bullet(S)\ge\omega_X^p$. Therefore,

$$\lambda_p(f^n) = \big\langle (f^n)^*(\omega_X^p),\omega_X^{k-p}\big\rangle \le \big\langle (f^n)^\bullet\Pi^\bullet(S),\omega_X^{k-p}\big\rangle.$$

Using a semi-regularization of S, we deduce that

$$\lambda_p(f^n)\lesssim \big\langle (f^n)^*\Pi^*(\omega^p),\omega_X^{k-p}\big\rangle.$$

Define $R := \Pi_*(\omega_X^{k-p})$. We also have $\Pi^\bullet(R)\ge\omega_X^{k-p}$. We obtain as above using a semi-regularization of R that

$$\begin{aligned}\lambda_p(f^n) &\lesssim \big\|(f^n)^*\Pi^*(\omega^p)\mathbin{\mathring{\wedge}}\Pi^\bullet(R)\big\| \lesssim \big\|(f^n)^*\Pi^*(\omega^p)\wedge\Pi^*(\omega^{k-p})\big\| \\ &= \big\langle \Pi_*(f^n)^*\Pi^*(\omega^p),\omega^{k-p}\big\rangle = \|T\|.\end{aligned}$$

Now, since $\omega_Y^{l+1} = 0$ and $\omega_{\text{FS}}^{k-l+1} = 0$, we have

$$\|T\| = \langle T, (\omega_Y + \omega_{\text{FS}})^{k-p}\rangle \lesssim \sum_{\max\{0,p-l\}\le q\le\min\{p,k-l\}} \langle T, \omega_Y^{l-p+q}\wedge\omega_{\text{FS}}^{k-l-q}\rangle$$
$$\le \sum_{\max\{0,p-l\}\le q\le\min\{p,k-l\}} a_{q,p}(n) = b_p(n).$$

This completes the proof of the lemma. □

For every $n \ge 0$ and $0 \le p \le l$ define

$$c_p(n) := \lambda_p(g^n) = \|(g^n)^*(\omega_Y^p)\| = \langle (g^n)^*(\omega_Y^p), \omega_Y^{l-p}\rangle.$$

We have the following lemma.

Lemma 4.3. *There is a constant $A > 0$ such that*

$$\langle \Pi_*(f^n)^*\Pi^*(\omega_Y^{p-q}\wedge\omega^q), \omega_Y^{l-p+p_0}\wedge\omega^{k-l-p_0}\rangle \le A a_{p_0,q}(n) c_{p-q}(n)$$

for $0\le p_0\le k-l$, $p_0\le p\le l+p_0$, $p_0\le q\le p$ and $n\ge 0$. Moreover, the above integral vanishes when $q < p_0$.

Proof. Observe that by definition of Π_*

$$\Pi_*(f^n)^*\Pi^*(\omega_Y^{p-q}\wedge\omega^q) = \Pi_*\big[(f^n)^*\Pi^*(\omega_Y^{p-q}) \overset{\circ}{\wedge} (f^n)^*\Pi^*(\omega^q)\big]$$
$$\le \Pi_*(f^n)^*\Pi^*(\omega_Y^{p-q}) \overset{\circ}{\wedge} \Pi_*(f^n)^*\Pi^*(\omega^q).$$

Hence, the left hand side of the inequality in the lemma is smaller than or equal to

$$\langle \Pi_*(f^n)^*\Pi^*(\omega_Y^{p-q}) \overset{\circ}{\wedge} \Pi_*(f^n)^*\Pi^*(\omega^q), \omega_Y^{l-p+p_0}\wedge\omega^{k-l-p_0}\rangle.$$

Define $T := \Pi_*(f^n)^*\Pi^*(\omega_Y^{p-q})\wedge\omega_Y^{l-p+p_0}$ and $S := \Pi_*(f^n)^*\Pi^*(\omega^q)\wedge\omega^{k-l-p_0}$. Note that T and S are of bidegree $(l-q+p_0, l-q+p_0)$ and $(k-l+q-p_0, k-l+q-p_0)$ respectively. The quantity considered above is equal to the mass of the measure $T \overset{\circ}{\wedge} S$.

We first show that $\alpha_j(T) = 0$ when $j < l-q+p_0$ and $\alpha_{l-q+p_0}(T) \le Ac_{p-q}(n)$ for some constant $A > 0$. Since $\pi\circ f^n = g^n\circ\pi$, we have

$$T = \Pi_*(f^n)^*\pi^*(\omega_Y^{p-q})\wedge\omega_Y^{l-p+p_0} = \Pi_*\pi^\bullet(g^n)^*(\omega_Y^{p-q})\wedge\omega_Y^{l-p+p_0}.$$

Hence,

$$\alpha_j(T) = \langle \Pi_*\pi^\bullet(g^n)^*(\omega_Y^{p-q})\wedge\omega_Y^{l-p+p_0}, \omega_Y^{l-j}\wedge\omega_{\text{FS}}^{k-2l+q-p_0+j}\rangle$$
$$= \langle \pi^\bullet(g^n)^*(\omega_Y^{p-q})\wedge\pi^*(\omega_Y^{l-p+p_0}), \pi^*(\omega_Y^{l-j})\wedge\Pi^*(\omega_{\text{FS}}^{k-2l+q-p_0+j})\rangle$$
$$= \langle \pi^\bullet\big[(g^n)^*(\omega_Y^{p-q})\wedge\omega_Y^{2l-p+p_0-j}\big], \Pi^*(\omega_{\text{FS}}^{k-2l+q-p_0+j})\rangle.$$

When $j < l-q+p_0$, the form in the brackets has bidegree $\geq (l+1, l+1)$ and should vanish because $\dim Y = l$. Therefore, $\alpha_j(T) = 0$ in that case. When $j = l-q+p_0$, this form defines a positive measure of mass $\lambda_{p-q}(g^n)$. Its cohomology class is equal to $\lambda_{p-q}(g^n)\{\omega_Y^l\}$. Therefore, using a semi-regularization as above, we obtain

$$\alpha_{l-q+p_0}(T) \lesssim \lambda_{p-q}(g^n)\langle \pi^*(\omega_Y^l), \Pi^*(\omega_{\mathrm{FS}}^{k-l})\rangle \leq Ac_{p-q}(n)$$

for some constant $A > 0$.

We deduce from Proposition 2.3 that $\{T\} \lesssim c_{p-q}(n)\{\omega_Y^{l-q+p_0}\}$. Using the semi-regularization in Proposition 2.4 for T, we obtain

$$\{T \overset{\circ}{\wedge} S\} \lesssim c_{p-q}(n)\|\omega_Y^{l-q+p_0} \wedge S\| = c_{p-q}(n)a_{p_0,q}(n).$$

This completes the proof of the first assertion in the lemma. For the second one, it is enough to observe that when $q < p_0$, we have $\alpha_j(T) = 0$ for every j and hence $T = 0$. □

The following lemma is crucial in our proof.

Lemma 4.4. *There exists a constant $A > 0$ such that for all $0 \leq p_0 \leq k - l$, $p_0 \leq p \leq l + p_0$ and all $n, r \geq 1$*

$$a_{p_0,p}(nr) \leq A^r \sum \prod_{s=1}^{r} a_{p_{s-1},p_s}(n)c_{p-p_s}(n),$$

where the sum is taken over $(p_1, \dots, p_r)$ with $p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_r \leq p$ and $p_{r-1} \leq k - l$.

Proof. We proceed by induction on r. Clearly, the lemma is true for $r = 1$. Suppose the lemma true for r, we need to prove it for $r + 1$. In what follows, the constants A_i depend only on the geometry of X and Y.

Define $T^{(r)} := \Pi_*(f^{nr})^*\Pi^*(\omega^p)$. This is a positive closed L^1 form, smooth on a dense Zariski open set. Observe that $\Pi^\bullet\Pi_* \geq \mathrm{id}$ on positive closed currents having no mass on proper analytic subsets of X. Therefore,

$$T^{(r+1)} \leq \Pi_*(f^n)^\bullet\Pi^\bullet\Pi_*(f^{nr})^\bullet\Pi^*(\omega^p) = \Pi_*(f^n)^\bullet\Pi^\bullet(T^{(r)}).$$

On the other hand, by Proposition 2.4, we can find a sequence of smooth positive closed (p, p)-forms $T_i^{(r)}$ converging weakly to a positive closed current $\widetilde{T}^{(r)} \geq T^{(r)}$ such that

$$\alpha_{p-q}(T_i^{(r)}) \leq A_1\alpha_{p-q}(T^{(r)}) \leq A_1 a_{q,p}(nr)$$

for $\max\{0, p-l\} \le q \le \min\{p, k-l\}$ and $A_1 > 0$ a constant. By Proposition 2.3, there is a constant $A_2 > 0$ such that

$$\{T_i^{(r)}\} \le A_2 \sum_{\max\{0,p-l\}\le q\le\min\{p,k-l\}} a_{q,p}(nr)\{\omega_Y^{p-q}\} \cup \{\omega_{\mathrm{FS}}^q\}.$$

We deduce from the above discussion and Lemma 4.3 that

$$\begin{aligned} a_{p_0,p}(n(r+1)) &= \langle T^{(r+1)}, \omega_Y^{l-p+p_0} \wedge \omega^{k-l-p_0}\rangle \\ &\le \liminf_{i\to\infty} \langle \Pi_*(f^n)^*\Pi^*(T_i^{(r)}), \omega_Y^{l-p+p_0} \wedge \omega^{k-l-p_0}\rangle \\ &\le A_2 \sum_{\substack{\max\{0,p-l\}\le q \\ \le \min\{p,k-l\}}} a_{q,p}(nr)\langle \Pi_*(f^n)^*\Pi^*(\omega_Y^{p-q} \wedge \omega_{\mathrm{FS}}^q), \omega_Y^{l-p+p_0} \wedge \omega^{k-l-p_0}\rangle \\ &\le A_3 \sum_{p_0 \le q \le \min\{p,k-l\}} a_{q,p}(nr)a_{p_0,q}(n)c_{p-q}(n) \end{aligned}$$

for some constant $A_3 > 0$. Consequently, the induction hypothesis implies the result. □

Theorem 1.1 is a consequence of the next two propositions.

Proposition 4.5. *We have*

$$d_p(f) \ge \max_{\max\{0,p-k+l\}\le j\le\min\{p,l\}} d_j(g)d_{p-j}(f|\pi)$$

for $0 \le p \le k$.

Proof. Since $\Pi^*(\omega_Y^j \wedge \omega^{p-j})$ is a smooth form, we have for some constant $A > 0$

$$\|(f^n)^*\Pi^*(\omega_Y^j \wedge \omega^{p-j})\| \le A\lambda_p(f^n).$$

So, by definition of dynamical degrees and Lemma 4.1, it is enough to bound $\|(f^n)^*\Pi^*(\omega_Y^j \wedge \omega^{p-j})\|$ from below by a constant times $\lambda_j(g^n)a_{p-j,p-j}(n)$.

Fix a constant $A > 0$ large enough. Using the identity $\pi \circ f^n = g^n \circ \pi$ and that $\Pi^*(\omega_Y^{l-j} \wedge \omega^{k-l-p+j})$ is smooth, we obtain

$$\begin{aligned} &A\|(f^n)^*\Pi^*(\omega_Y^j \wedge \omega^{p-j})\| \\ &\quad\ge \langle (f^n)^*\Pi^*(\omega_Y^j \wedge \omega^{p-j}), \Pi^*(\omega_Y^{l-j} \wedge \omega^{k-l-p+j})\rangle \\ &\quad= \langle (f^n)^*\pi^*(\omega_Y^j) \mathring{\wedge} (f^n)^*\Pi^*(\omega^{p-j}), \pi^*(\omega_Y^{l-j}) \wedge \Pi^*(\omega^{k-l-p+j})\rangle \\ &\quad= \|(f^n)^*\pi^*(\omega_Y^j) \wedge \pi^*(\omega_Y^{l-j}) \mathring{\wedge} (f^n)^*\Pi^*(\omega^{p-j}) \wedge \Pi^*(\omega^{k-l-p+j})\| \\ &\quad= \|\pi^*[(g^n)^*(\omega_Y^j) \wedge \omega_Y^{l-j}] \mathring{\wedge} (f^n)^*\Pi^*(\omega^{p-j}) \wedge \Pi^*(\omega^{k-l-p+j})\|. \end{aligned}$$

Observe that $(g^n)^*(\omega_Y^j) \wedge \omega_Y^{l-j}$ is a positive measure of mass $\lambda_j(g^n)$. As in Lemma 3.2, we show that the last expression in the previous identities is equal to $\lambda_j(g^n)$ times the mass of the restriction of $(f^n)^*\Pi^*(\omega^{p-j}) \wedge \Pi^*(\omega^{k-l-p+j})$ to a generic fiber L_y of π. Therefore, it is also equal to

$$\lambda_j(g^n)\big\langle \pi^*(\omega_Y^l), (f^n)^*\Pi^*(\omega^{p-j}) \wedge \Pi^*(\omega^{k-l-p+j})\big\rangle = \lambda_j(g^n) a_{p-j,p-j}(n).$$

This completes the proof. □

Proposition 4.6. *We have*

$$d_p(f) \le \max_{\max\{0,p-k+l\} \le j \le \min\{p,l\}} d_j(g) d_{p-j}(f|\pi)$$

for $0 \le p \le k$.

Proof. For every $0 \le p \le k$ and $n \ge 0$ let

$$\mu_p(n) := \max_{\max\{0,p-k+l\} \le j \le \min\{p,l\}} c_j(n) a_{p-j,p-j}(n).$$

Observe that for $r > p$, in Lemma 4.4, there are at most p indices s such that $p_{s-1} < p_s$. Moreover, the sum in that lemma contains at most $(k+1)^r$ terms and the sum in the definition of $b_p(n)$ contains at most $p+1$ terms. We infer the following estimate

$$b_p(rn) \le \Big[(p+1)(k+1)^r A^r b_0(n) \cdots b_p(n) \prod_{j=0}^{l} c_j(n)\Big] \mu_p(n)^r.$$

We deduce that

$$[b_p(rn)]^{1/rn} \le (p+1)^{1/nr}(k+1)^{1/n} A^{1/n} \big[b_0(n)^{1/n} \cdots b_p(n)^{1/n}\big]^{1/r} \cdot \Big[\prod_{j=0}^{l} c_j(n)^{1/n}\Big]^{1/r} \mu_p(n)^{1/n}.$$

Letting n tend to infinity, we obtain using Lemma 4.2 that

$$d_p(f) \le \big[d_0(f) \ldots d_p(f)\big]^{1/r} \Big[\prod_{j=0}^{l} d_j(g)\Big]^{1/r} \liminf_{n\to\infty} \mu_p(n)^{1/n}.$$

Now, letting $r \to \infty$, the first two factors in the right hand side tend to 1. Therefore, using Lemma 4.1, we obtain

$$d_p(f) \le \liminf_{n\to\infty} \mu_p(n)^{1/n} = \max_{\max\{0,p-k+l\} \le j \le \min\{p,l\}} d_j(g) d_{p-j}(f|\pi).$$

This completes the proof. □

Proof of Corollary 1.2. When X and Y are projective, the corollary is a direct consequence of Theorem 1.1. We only used the projectivity in Proposition 2.4 applied to $m := k - l$ and for the existence of $v : X \to \mathbb{P}^{k-l}$. This is superfluous when X and Y have the same dimension, i.e. $k = l$. □

Proof of Corollary 1.3. Let j, p be such that $d_j(g) = \max_q d_q(g)$ and $d_{p-j}(f|\pi) = \max_q d_q(f|\pi)$. We have $0 \leq j \leq l$ and $0 \leq p - j \leq k - l$. By Theorem 1.1, $d_p(f)$ is the maximal dynamical degree of f and $d_p(f) = d_j(g)d_{p-j}(f|\pi)$. We have $d_{p-1}(f) < d_p(f) < d_{p+1}(f)$. Theorem 1.1 implies that

$$d_{j-1}(g) < d_j(g) < d_{j+1}(g) \quad \text{and} \quad d_{p-j-1}(f|\pi) < d_{p-j}(f|\pi) < d_{p-j+1}(f|\pi).$$

The log-concavity of $d_q(g)$ and $d_q(f|\pi)$ implies the result. Note that when $j = 0, l$ or $p - j = 0, k - l$, in the above inequalities, one has to remove the expressions which are not meaningful. □

In the rest of the paper, we prove Corollary 1.4. Let K_X denote the canonical lines bundle of X. Let $H^0(X, K_X^n)$ denote the space of holomorphic sections of K_X^n and $H^0(X, K_X^n)^*$ its dual space. Assume that $H^0(X, K_X^n)$ has positive dimension. If x is a generic point in X, the family H_x of sections which vanish at x is a hyperplane of $H^0(X, K_X^n)$ passing through 0. So, the correspondence $x \mapsto H_x$ defines a meromorphic map

$$\pi_n : X \to \mathbb{P}H^0(X, K_X^n)^*$$

from X to the projectivization of $H^0(X, K_X^n)^*$ which is called an *Iitaka fibration* of X. Let Y_n denote the image of X by π_n. The *Kodaira dimension* of X is $\kappa_X := \max_{n \geq 1} \dim Y_n$. When $H^0(X, K_X^n) = 0$ for every $n \geq 1$, the Kodaira dimension of X is $-\infty$. We have the following result.

Theorem 4.7 ([14], [18]). *Let $f : X \to X$ be a dominant meromorphic map. Assume that $\kappa_X \geq 1$. Then f preserves the Iitaka fibration $\pi_n : X \to Y_n$. Moreover, the map $g : Y_n \to Y_n$ induced by f is periodic, i.e. $g^N = \mathrm{id}$ for some integer $N \geq 1$.*

Proof of Corollary 1.4. Assume in order to get a contradiction that $\kappa_X \geq 1$. Let $n \geq 1$ be such that $l := \dim Y_n \geq 1$. Replacing f with an iterate, we can assume that $g = \mathrm{id}$. A priori, Y_n may be singular, but we can use a blow-up and assume that Y_n is smooth. We have $d_j(g) = 1$ for $0 \leq j \leq l$. This contradicts Corollary 1.3. Note that in order to prove that $d_j(g) = 1$, instead of Theorem 4.7, it is enough to use the weaker result that g is induced by a linear endomorphism of $\mathbb{P}H^0(X, K_X^n)^*$. □

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