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Gehring–Hayman Theorem for conformal deformations

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Abstract. We study conformal deformations of a uniform space that satisfies the Ahlfors Q-regularity condition on balls of Whitney type. We verify the Gehring–Hayman Theorem by using a Whitney covering of the space.

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1. Introduction

Given $x, y \in B^2(0,1)$, the hyperbolic geodesic $[x, y]$ is essentially the shortest curve joining x to y in $B^2(0,1)$. More precisely

$$\ell([x, y]) \leq \frac{\pi}{2}\ell(\gamma)$$

whenever γ is a path that joins x to y in $B^2(0,1)$. This simple fact is an instance of a theorem of Gehring and Hayman in [GH]: If $f : B^2(0,1) \to \Omega \subset \mathbb{C}$ is a conformal mapping and γ is a path joining points x and y, then

$$\int_{[x,y]} |f'(z)|\, ds \leq C \int_{\gamma} |f'(z)|\, ds, \tag{1.1}$$

where $C \geq 1$ is an absolute constant. The density $\rho(z) = |f'(z)|$ satisfies a Harnack inequality

$$\frac{\rho(z)}{A} \leq \rho(w) \leq A\rho(z)$$

whenever $z \in B^2(0,1)$ and $w \in B(z, (1-|z|)/2)$. It also satisfies the area growth estimate

$$\int_{B\rho(z,r)} \rho^2\, dA \leq \pi r^2,$$

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where $B_\rho(z,r)$ refers to the ball with centre z and radius r in the path metric

$$d_\rho(x,y) = \inf \int_\gamma \rho\, ds,$$

where the infimum is taken over all curves γ joining points x and y.

In [BKR] the Gehring–Hayman inequality (1.1) was extended to $B^n(0,1)$, $n \geq 2$, for conformal deformations of the Euclidean metric. By a conformal deformation (a conformal density) ρ we mean a continuous function $\rho\colon B^n(0,1) \to (0,\infty)$ that satisfies a Harnack inequality with a constant $A \geq 1$,

$$\frac{\rho(z)}{A} \leq \rho(w) \leq A\rho(z) \quad \text{for all } w \in B(z,(1-|z|)/2) \text{ and all } z \in B^n(0,1),$$

and a volume growth condition with a constant $B > 0$,

$$\int_{B_\rho(z,r)} \rho^n \, dm_n \leq Br^n \quad \text{for all } z \in B^n(0,1) \text{ and all } r > 0,$$

with respect to n-dimensional Lebesgue measure m_n.

Subsequently, Herron showed in [H1] that $B^n(0,1)$ can be replaced by any uniform space (Ω,d) of bounded geometry. In this setting conformal densities are defined by conditions analogous to those given above – see Section 2 for details. Here uniformity is a substitute for the "roundness" of $B^n(0,1)$. The assumption of bounded geometry includes two conditions. First, it requires that Ω carries a Borel regular measure μ that satisfies the *(Ahlfors) Q-regularity condition on balls of Whitney type* for some $Q > 1$. That is, there is a constant $C_1 \geq 1$ such that if $r \leq d(z,\partial\Omega)/2$, then

$$C_1^{-1} r^Q \leq \mu(B(z,r)) \leq C_1 r^Q.$$

Secondly, it requires that balls $B(z,d(z,\partial\Omega)/2)$ allow for nice lower bounds for the Q-modulus (see e.g. [HK], [BHK]). In fact, the Q-regularity condition on balls of Whitney type is not explicitly stated in [H1] but it follows from the other assumptions. The precise definition of a uniform space is given in Section 2 below. This concept, introduced in [BHK], generalizes the notion of a uniform domain introduced by Jones [Jo] and Martio and Sarvas [MaSa], see also [GO]. The volume growth condition for ρ then refers to integrals of ρ^Q with respect to the measure μ. For predecessors of the results in [H1], see [HN], [HR]. For connections to Gromov hyperbolicity, see [Gr], [BHK] and [BB].

In this paper we show that, surprisingly, lower bounds on the Q-modulus are not needed to prove the Gehring–Hayman inequality.

Theorem 1.1 (Gehring–Hayman Theorem). *Let $Q > 1$ and let (Ω,d,μ) be a non-complete uniform space equipped with a measure that is Q-regular on balls* *of Whitney type. If $\rho\colon \Omega \to (0,\infty)$ is a conformal density on Ω, then there is a constant $C \geq 1$ that depends only on the data associated with Ω and ρ such that*

$$\ell_\rho([x,y]) \leq C\ell_\rho(\gamma),$$

whenever $[x,y]$ is a quasihyperbolic geodesic and γ is a curve joining x to y in Ω.

The definition of a quasihyperbolic geodesic is given in Section 2 and the proof of the theorem is in Section 4. Especially Subcase D of the proof is the novelty, that allows us to avoid the use of lower bounds for the Q-modulus. The previous arguments [BKR], [H1], [HN] and [HR] rely on modulus estimates.

The Gehring–Hayman Theorem was a central tool in [BHR], [BKR], [H1] and [H2]. We expect that Theorem 1.1 will allow one to remove the use of modulus bounds in [BHR], [BKR], [H1] and [H2] and thus extend large parts of those papers to a much more general setting. A very simple example of a space that satisfies the assumptions of Theorem 1.1 but does not support lower bounds for the Q-modulus is

$$\Omega = \{(x,y) \in \mathbb{R}^2 : |y| \leq |x|,\ -1 < x < 1\}$$

equipped with the path metric and Lebesgue measure.

2. Preliminaries

Let (Ω, d) be a metric space. A *curve* means a continuous map $\gamma\colon [a,b] \to \Omega$ from an interval $[a,b] \subset \mathbb{R}$ to Ω. We also denote the image set $\gamma([a,b])$ of γ by γ. The *length* $\ell_d(\gamma)$ of γ with respect to the metric d is defined as

$$\ell_d(\gamma) = \sup \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_m = b$ of the interval $[a,b]$. If $\ell_d(\gamma) < \infty$, then γ is said to be a *rectifiable curve*. When the parameter interval is open or half-open, we set

$$\ell_d(\gamma) = \sup \ell_d(\gamma|_{[c,d]}),$$

where the supremum is taken over all compact subintervals $[c,d]$. For a rectifiable curve γ we define the *arc length* $s\colon [a,b] \to [0,\infty)$ along γ by

$$s(t) = \ell_d(\gamma|_{[a,t]}).$$

Next, let us assume that $\rho\colon \Omega \to [0,\infty]$ is a Borel function. For each rectifiable curve $\gamma\colon [a,b] \to \Omega$ we define the *ρ-length* $\ell_\rho(\gamma)$ of γ by

$$\ell_\rho(\gamma) = \int_\gamma \rho\, ds = \int_a^b \rho(\gamma(t))\, ds(t).$$

If Ω is *rectifiably connected* – that is, every pair of points in Ω can be joined by a rectifiable curve – then ρ determines a distance function

$$d_\rho(x, y) = \inf \ell_\rho(\gamma),$$

where the infimum is taken over all rectifiable curves γ joining $x, y \in \Omega$. In general, the distance function d_ρ need not be a metric. However, it is a metric – called a *ρ-metric* – if ρ is positive and continuous. If $\rho \equiv 1$, then $\ell_\rho(\gamma) = \ell_d(\gamma)$ is the length of the curve γ with respect to the metric d. Furthermore, if $\ell_d(\gamma) = d(x, y)$ for some curve γ joining points $x, y \in \Omega$, then γ is said to be a *geodesic*. If every pair of points in Ω can be joined by a geodesic, then (Ω, d) is called a *geodesic space*.

Let (Ω, d) be a locally compact, rectifiably connected and non-complete metric space and denote by $\overline{\Omega}$ its metric completion. Then the *boundary* $\partial\Omega := \overline{\Omega} \setminus \Omega$ is nonempty. We write

$$d(z) = \operatorname{dist}_d(z, \partial\Omega) = \inf\{d(z, x) : x \in \partial\Omega\}$$

for $z \in \Omega$. If we choose

$$\rho(z) = \frac{1}{d(z)},$$

we obtain the *quasihyperbolic metric k* in Ω. In this special case we denote the metric d_ρ by k and the quasihyperbolic length of the curve γ by $\ell_k(\gamma)$. That $\ell_k(\gamma) = \ell_\rho(\gamma)$ is shown in [BHK], Appendix. Moreover, $[x, y]$ refers to a quasihyperbolic geodesic joining points x and y in Ω.

Given a real number $D \geq 1$, a curve $\gamma\colon [a, b] \to (\Omega, d)$ is called a *D-uniform curve* if it is *quasiconvex:*

$$\ell_d(\gamma) \leq Dd(\gamma(a), \gamma(b)), \tag{2.1}$$

and

$$\min\{\ell_d(\gamma|_{[a,t]}), \ell_d(\gamma|_{[t,b]})\} \leq Dd(\gamma(t)) \tag{2.2}$$

for every $t \in [a, b]$. A metric space (Ω, d) is called a *D-uniform space* if every pair of points in it can be joined by a D-uniform curve.

If (Ω, d) is a uniform space, then by Proposition 2.8 and Theorem 2.10 of [BHK] the quasihyperbolic space (Ω, k) is complete, proper (closed balls are compact), and geodesic. Furthermore, each quasihyperbolic geodesic $[x, y]$ is a D'-uniform curve for every $x, y \in \Omega$, where $D' = D'(D) \geq 1$. Quasihyperbolic geodesics are also *locally D'-uniform curves* – that is, every subcurve $[u, v] \subset [x, y]$ is a D'-uniform curve – because $[u, v]$ is a quasihyperbolic geodesic as well. We also have an estimate for a quasihyperbolic distance of every pair of points x and y in the D-uniform space (Ω, d) (see [BHK], Lemma 2.13):

$$k(x, y) \leq 4D^2 \log\left(1 + \frac{d(x, y)}{\min\{d(x), d(y)\}}\right). \tag{2.3}$$

Let us consider a continuous function $\rho\colon \Omega \to (0,\infty)$, called a *density*. The metric d_ρ is then well defined. We use the subscript ρ for metric notations which refer to d_ρ, and similarly for k and d. For example, $B_\rho(a,r)$, $B_k(a,r)$ and $B_d(a,r)$ are open balls with centre a and radius r in metrics d_ρ, k and d. Furthermore, we abbreviate the "Whitney ball" $B_d(z,\frac{1}{2}d(z))$ to B_z.

Let μ be a Borel regular measure on (Ω,d) with dense support. We call ρ a *conformal density* provided it satisfies both a *Harnack type inequality*, HI(A), for some constant $A \geq 1$:

$$\frac{1}{A} \leq \frac{\rho(x)}{\rho(y)} \leq A \quad \text{for all } x,y \in B_z \text{ and all } z \in \Omega, \tag{HI(A)}$$

and a *volume growth condition*, VG(B), for some constant $B > 0$:

$$\mu_\rho(B_\rho(z,r)) \leq Br^Q \quad \text{for all } z \in \Omega \text{ and } r > 0. \tag{VG(B)}$$

Here μ_ρ is the Borel measure on Ω defined by

$$\mu_\rho(E) = \int_E \rho^Q\, d\mu \quad \text{for a Borel set } E \subset \Omega,$$

and Q is a positive real number. Generally Q will be the Hausdorff dimension of our space (Ω,d).

We defined in the introduction the concept of Q-regularity on balls of Whitney type. The immediate consequence is that the measure μ is also *doubling on balls of Whitney type:* there exists a constant $C_2 \geq 1$ such that

$$\mu(B_d(z,2r)) \leq C_2\mu(B_d(z,r)) \tag{2.4}$$

for every $z \in \Omega$ and every $0 < r \leq \frac{1}{4}d(z)$.

3. Whitney covering

In this section we assume that (Ω,d,μ) is a locally compact, rectifiably connected, and non-complete metric measure space such that the measure μ is doubling on balls of Whitney type. Let $r(z) = d(z)/50$. From the family of balls $\{B_d(z,r(z))\}_{z\in\Omega}$ we select a maximal (countable) subfamily $\{B_d(z_i,r(z_i)/5)\}_{i\in I}$ of pairwise disjoint balls. Let $\mathcal{B} = \{B_i\}_{i\in I}$, where $B_i = B_d(z_i,r_i)$ and $r_i = r(z_i)$. We call the family $\mathcal{B}$ the *Whitney covering* of Ω. Let us list a few facts concerning the Whitney covering. The last property is a consequence of the doubling on balls of Whitney type property of the measure μ. For more properties of the Whitney covering, see e.g. Theorem III.1.3 of [CW], Lemma 2.9 of [MaSe], Lemma 7 of [HKT], and [BS], Theorem 5.3 and Lemma 5.5.

Lemma 3.1. *There is* $N \in \mathbb{N}$ *such that*

(i) *the balls* $B_d(z_i, r_i/5)$ *are pairwise disjoint,*

(ii) $\Omega = \bigcup_{i \in I} B_d(z_i, r_i)$,

(iii) $B_d(z_i, 5r_i) \subset \Omega$,

(iv) $\sum_{i=1}^{\infty} \chi_{B_d(z_i, 5r_i)}(x) \leq N$ *for all* $x \in \Omega$.

The family $\mathcal{B}$ has the same kind of properties as the usual Whitney decomposition $\mathcal{W}$ of a domain $\Omega \subset \mathbb{R}^n$ and next we prove a couple of them. In addition to the assumptions above, we assume that for each pair of points in $B \in \mathcal{B}$ for every $B \in \mathcal{B}$ can be joined by a D-uniform curve in Ω.

Lemma 3.2. *Let* $x, y \in (\Omega, d, \mu)$ *and* $d(x, y) \geq d(x)/2$. *There is a constant* $C = C(C_2, D) > 0$ *such that*

$$C^{-1} N(x, y) \leq k(x, y) \leq C N(x, y),$$

where $N(x, y)$ *is the number of balls* $B \in \mathcal{B}$ *intersecting a quasihyperbolic geodesic* $[x, y]$.

Proof. Let $x, y \in \Omega$ be points so that $d(x, y) \geq d(x)/2$. Since $24 \operatorname{diam}_d(B) \leq d(z)$ for every $B \in \mathcal{B}$ and for every $z \in B$, then the basic estimate (2.3) implies

$$\operatorname{diam}_k(B) \leq 4D^2 \log\left(1 + \frac{\operatorname{diam}_d(B)}{24 \operatorname{diam}_d(B)}\right) = 4D^2 \log \frac{25}{24}.$$

Thus

$$N(x, y) \geq \frac{k(x, y)}{4D^2 \log \frac{25}{24}}.$$

Lemma 3.1 (iv) says that there are only N balls $B \in \mathcal{B}$ that contain x. Fix one of them and denote it by B_1. A *neighbour* of the ball B_1 is a ball $B \in \mathcal{B}$ which intersects the ball $5B_1 = B_d(z_1, 5r_1) = B_d(z_1, d(z_1)/10)$. Because the measure μ is doubling in every ball $B_d(z, r)$ with radius $0 < r \leq d(z)/4$, the ball B_1 has a uniformly bounded number of neighbours. Let this number be $N' \in \mathbb{N}$ and let $y_1 \in [x, y]$ be the first point such that y_1 does not belong to any neighbour of B_1. This choice is possible because $d(x, y) \geq d(x)/2$. The geodesic $[x, y_1]$ intersects at most N' balls $B \in \mathcal{B}$ and

$$\begin{aligned} k(x, y_1) &= \int_{[x, y_1]} \frac{1}{d(z)}\, ds \geq \int_{5B_1 \cap [x, y_1]} \frac{10}{11 d(z_1)}\, ds \\ &\geq \frac{10}{11 d(z_1)} \left(\frac{d(z_1)}{10} - \frac{d(z_1)}{50}\right) = \frac{4}{55}. \end{aligned} \tag{3.1}$$

Let $B_2 \in \mathcal{B}$ be a ball such that $y_1 \in B_2$ and $B_2 \cap B \neq \emptyset$ for some neighbour $B \in \mathcal{B}$ of B_1. Again there are only N' balls $B \in \mathcal{B}$ which are neighbours of B_2. Let $y_2 \in [x, y]$ be the first point so that y_2 does not belong to any neighbour of B_2. Then the geodesic $[y_1, y_2]$ intersects at most N' balls $B \in \mathcal{B}$ and $k(y_1, y_2) \geq \frac{4}{55}$, by the same way than in inequality (3.1). We continue this process until we end up with a ball B_m whose neighbours contain $[y_{m-1}, y]$. This process really ends and $m < \infty$, because $[x, y]$ is compact. We may start doing this process from every ball B that contains x. Thus we obtain the upper bound to the number of balls that intersects the quasihyperbolic geodesic $[x, y]$:

$$N(x, y) \leq \frac{55}{4} N N' k(x, y). \qquad \square$$

Fix a ball B_0 from the Whitney covering $\mathcal{B}$ and let z_0 be its centre point. For each $B_i \in \mathcal{B}$ we fix a geodesic $[z_0, z_i]$. Furthermore, for each $B_i \in \mathcal{B}$ we set $P(B_i) = \{B \in \mathcal{B} : B \cap [z_0, z_i] \neq \emptyset\}$ and define the *shadow* $S(B)$ of a ball $B \in \mathcal{B}$ by

$$S(B) = \bigcup_{\substack{B_i \in \mathcal{B} \\ B \in P(B_i)}} B_i.$$

For $n \in \mathbb{N}$ we set

$$\mathcal{B}_n = \{B_i \in \mathcal{B} : n \leq k(z_0, z_i) < n + 1\}.$$

The next two lemmas are metric space analogues of [KL], Lemma 2.1 and Lemma 2.2.

Lemma 3.3. *Let γ be a quasihyperbolic geodesic in Ω starting at the point z_0. Then there is a constant $C = C(C_2, D) > 0$ such that, for each $n \in \mathbb{N}$,*

$$\#\{B \in \mathcal{B}_n : B \cap \gamma \neq \emptyset\} \leq C.$$

Proof. Put

$$a_n := \#\{B \in \mathcal{B}_n : B \cap \gamma \neq \emptyset\} < \infty.$$

Let $B_1, \ldots, B_{a_n} \in \mathcal{B}_n$ be the balls intersecting γ, ordered so that if $k < l$, then there exists $x_k \in B_k \cap \gamma$ such that for every $z \in B_l \cap \gamma$, we have $k(z_0, x_k) \leq k(z_0, z)$. We may assume that $d(x_1, x_{a_n}) \geq d(x_1)/2$, otherwise $x_{a_n} \in B_{x_1}$ and we get the result by doubling on balls of Whitney type. Thus by Lemma 3.2, $k(x_1, x_{a_n}) \geq \frac{a_n}{C}$. Since $k(z_i, x_i) \leq \frac{1}{49} < 1$ for all $i = 1, \ldots, a_n$, we may compute

$$\begin{aligned} \frac{a_n}{C} &\leq k(x_1, x_{a_n}) = k(z_0, x_{a_n}) - k(z_0, x_1) \\ &\leq k(z_0, z_{a_n}) + k(z_{a_n}, x_{a_n}) - (k(z_0, z_1) - k(x_1, z_1)) \\ &\leq (n + 1) + 1 - n + 1 = 3. \end{aligned}$$

Hence $a_n \leq 3C$. $\square$

Lemma 3.4. *There is a constant* $C = C(C_2, D) > 0$ *such that, for each* $n \in \mathbb{N}$,

$$\sum_{B \in \mathcal{B}_n} \chi_{S(B)}(x) \le C$$

whenever $x \in \Omega$.

Proof. Let $x \in \Omega$. The number of balls $B \in \mathcal{B}$ containing x is bounded, so we may assume that there is a unique ball, denote it by B_1, in $\mathcal{B}$ such that $x \in B_1$. Let $[z_0, z_1]$ be the fixed geodesic joining z_0 to z_1. Then $x \in S(B)$ for $B \in \mathcal{B}_n$ if and only if $[z_0, z_1] \cap B \neq \emptyset$. By Lemma 3.3, the number of balls $B \in \mathcal{B}_n$ is bounded by a constant that is independent of n. □

4. Gehring–Hayman Theorem

We begin with *Frostman's Lemma.* First we recall the definitions of the Hausdorff measure and the weighted Hausdorff measure.

Let (X, d) be a compact metric space. Let $0 \le s < \infty$ and $0 < \delta \le \infty$. We set

$$\lambda^s_\delta(X) = \inf\left\{\textstyle\sum_{i=1}^\infty c_i \operatorname{diam}_d(E_i)^s : \chi_X \le \sum_i c_i \chi_{E_i},\ c_i > 0,\ \operatorname{diam}_d(E_i) \le \delta\right\}.$$

The *weighted Hausdorff s-measure* of X is

$$\lambda^s(X) = \lim_{\delta \to 0} \lambda^s_\delta(X).$$

In the special case, where $c_i = 1$ for every $i = 1, 2, \dots$, we set $\mathcal{H}^s_\delta(X) = \lambda^s_\delta(X)$, and we obtain the *Hausdorff s-measure*

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(X).$$

The *Hausdorff s-content* of X is

$$\mathcal{H}^s_\infty(X) = \inf\left\{\textstyle\sum_{i=1}^\infty \operatorname{diam}_d(E_i)^s : X \subset \bigcup_{i=1}^\infty E_i\right\}.$$

By Lemma 8.16 of [Ma] we know that $\mathcal{H}^s(X) \le 30^s \lambda^s(X)$, but in fact from the proof of that lemma one obtains that

$$\mathcal{H}^s_{30\delta}(X) \le 30^s \lambda^s_\delta(X) \quad \text{for every } 0 < \delta \le \infty.$$

In particular

$$\mathcal{H}^s_\infty(X) \le 30^s \lambda^s_\infty(X).$$

The following formulation of Frostman's Lemma (cf. [Ma], Theorem 8.17, and [BO], Theorem 2) is suitable for our purposes.

Theorem 4.1 (Frostman's Lemma). *For any $s \geq 0$ there is a Radon measure ω on X such that*

$$\omega(X) = \lambda^s_\infty(X)$$

and

$$\omega(E) \leq \operatorname{diam}_d(E)^s \quad \text{for all } E \subset X.$$

In particular, when $s = 1$ and X is connected, we obtain

$$\omega(X) \geq \frac{1}{30}\mathcal{H}^1_\infty(X) \geq \frac{\operatorname{diam}_d(X)}{60}.$$

In this paper we apply the version of Frostman's Lemma, where X is connected and $s = 1$.

For the rest of the paper we assume that (Ω, d, μ) is a locally compact, non-complete and D-uniform metric measure space such that the measure μ is Q-regular on balls of Whitney type for some $Q > 1$. Let ρ be a conformal density such that the number Q in the definition VG(B) coincides with the previous $Q > 1$.

Proof of Theorem 1.1. Let x and y be points in $\overline{\Omega}$ and let $[x, y]$ be a quasihyperbolic geodesic in Ω joining points x and y. Because quasihyperbolic geodesics are D'-uniform curves, $[x, y]$ is rectifiable in the metric d.

Let γ be another rectifiable curve in Ω joining points x and y. Let $a \in [x, y]$ be the point such that $\ell_d([x, a]) = \ell_d([a, y])$, and write $p = d(x, a)$. Moreover, for each $j = 0, 1, 2, \ldots$, write $A_j = (\overline{B}_d(x, 2^{-j}p) \setminus B_d(x, 2^{-(j+1)}p)) \cap \Omega$. Let $[x_{j+1}, x_j] \subset [x, a] \subset [x, y]$ be a subcurve, where x_{j+1} is the last point of $[x, y]$ in $\overline{B}(x, 2^{-(j+1)}p)$ and x_j is the last point of $[x, y]$ in $\overline{B}(x, 2^{-j}p)$, and set $\gamma_j = \gamma \cap A_j$. We may clearly assume that γ_j is connected. By summing and symmetry it suffices to prove that

$$\ell_\rho([x_{j+1}, x_j]) \leq C\ell_\rho(\gamma_j) \tag{4.1}$$

for every $j = 0, 1, 2, \ldots$.

Let $j = 0, 1, 2, \ldots$. From the definition of the curve γ_j it follows that

$$\ell_d(\gamma_j) \geq 2^{-(j+1)}p. \tag{4.2}$$

From the definition of the quasihyperbolic geodesic $[x_{j+1}, x_j]$ and from the local D'-uniformity of the curve $[x, y]$, we have that

$$\ell_d([x_{j+1}, x_j]) \leq D'd(x_{j+1}, x_j) \leq D'2^{-j+1}p, \tag{4.3}$$

$$2^{-(j+1)}p \leq \ell_d([x, z]) \leq D'd(z) \quad \text{for every } z \in [x_{j+1}, x_j], \tag{4.4}$$

and

$$k(x_{j+1}, x_j) = \int_{[x_{j+1}, x_j]} \frac{1}{d(z)}\, ds \le \frac{D'}{p} 2^{j+1} \ell_d([x_{j+1}, x_j]) \le 4D'^2. \tag{4.5}$$

The proof consists of two parts: the "easy part", Case A, and the "hard part", Case B. Furthermore, Case B is divided into two parts, Subcase C and Subcase D. Here Subcase D is the hardest part and the novelty of our proof.

Case A. We first prove that inequality (4.1) holds when the curves $[x_{j+1}, x_j]$ and γ_j are "close" to each other in the quasihyperbolic metric k. Let

$$M > \max\left\{4D^2 \frac{\log(4D'^2)}{\log 2} + 1, 4D^2 \frac{\log(B(2 + A^2/6)^Q/c_1)}{\log 2}\right\},$$

where $c_1 > 0$ is a sufficiently small constant depending on A, C_1, D and Q, and let us assume that $\operatorname{dist}_k([x_{j+1}, x_j], \gamma_j) \le M$. Let $y_j \in [x_{j+1}, x_j]$ and $\tilde{y}_j \in \gamma_j$ be points such that $k(y_j, \tilde{y}_j) \le M$. Let us show that we may estimate the ρ-length of the quasihyperbolic geodesic $[x_{j+1}, x_j]$ from above by $2^{-j} p\rho(y_j)$ in the following way

$$\ell_\rho([x_{j+1}, x_j]) \le A^b D' \rho(y_j) 2^{-j+1} p, \tag{4.6}$$

where $b = 4c_2 D'^2$ and $c_2 = c_2(C_1, D) > 0$ is the constant from Lemma 3.2.

If there exists $z \in [x_{j+1}, x_j]$ such that $[x_{j+1}, x_j] \subset B_z = B_d(z, d(z)/2)$, we obtain from HI($A$) and (4.3)

$$\ell_\rho([x_{j+1}, x_j]) \le A\rho(y_j)\ell_d([x_{j+1}, x_j]) \le AD'\rho(y_j)2^{-j+1} p.$$

Otherwise we may assume that $d(x_{j+1}, x_j) \ge d(x_{j+1})/2$. From Lemma 3.2 and inequality (4.5), it follows that

$$N(x_{j+1}, x_j) \le 4c_2 D'^2 =: b,$$

where the constant $c_2 = c_2(C_1, D) > 0$ is the constant from Lemma 3.2. Then by HI(A), every $z \in [x_{j+1}, x_j]$ satisfies

$$\rho(z) \le A^b \rho(y_j).$$

This with (4.3) gives us inequality (4.6)

$$\begin{aligned}\ell_\rho([x_{j+1}, x_j]) &\le A^b \rho(y_j)\ell_d([x_{j+1}, x_j]) \\ &\le A^b D' \rho(y_j) 2^{-j+1} p.\end{aligned}$$

Next we estimate the ρ-length of the curve γ_j from below by $2^{-j} p\rho(y_j)$. If $[x_{j+1}, x_j] \cap B_{\tilde{y}_j} \ne \emptyset$, we easily get from HI($A$) an estimate for $\ell_\rho(\gamma_j)$:

$$\ell_\rho(\gamma_j) \ge \frac{1}{A^{b+1}} \rho(y_j) \ell_d(\gamma_j \cap B_{\tilde{y}_j}). \tag{4.7}$$

Furthermore, for every $z \in [x_{j+1}, x_j] \cap B_{\tilde{y}_j}$, using inequalities (4.2) and (4.4) it holds that

$$\ell_d(\gamma_j \cap B_{\tilde{y}_j}) \geq \begin{cases} 2^{-(j+1)} p & \text{if } \gamma_j \subset B_{\tilde{y}_j}, \\ \frac{1}{2} d(\tilde{y}_j) \geq \frac{1}{2}\big(\frac{3}{2} d(z)\big) \geq \frac{3}{4D'} 2^{-(j+1)} p & \text{if } \gamma_j \not\subset B_{\tilde{y}_j}. \end{cases} \tag{4.8}$$

In this case, combining (4.6), (4.7) and (4.8) we obtain the desired result (4.1)

$$\ell_\rho([x_{j+1}, x_j]) \leq \frac{16}{3} A^{2b+1} D'^2 \ell_\rho(\gamma_j).$$

Therefore we may assume that $[x_{j+1}, x_j] \cap B_{\tilde{y}_j} = \emptyset$. This implies that $d(y_j, \tilde{y}_j) \geq d(\tilde{y}_j)/2$. By Lemma 3.2 there are at most $h := Mc_2$ balls in the Whitney covering $\mathcal{B}$ that intersect $[y_j, \tilde{y}_j]$ and hence, by HI(A),

$$\rho(y_j) \leq A^h \rho(\tilde{y}_j). \tag{4.9}$$

On the other hand, by HI(A) and (4.2),

$$\ell_\rho(\gamma_j) \geq \frac{1}{A} \rho(\tilde{y}_j) \ell_d(\gamma_j \cap B_{\tilde{y}_j}) \geq \begin{cases} \frac{1}{A} \rho(\tilde{y}_j) 2^{-(j+1)} p & \text{if } \gamma_j \subset B_{\tilde{y}_j}, \\ \frac{1}{2A} \rho(\tilde{y}_j) d(\tilde{y}_j) & \text{if } \gamma_j \not\subset B_{\tilde{y}_j}. \end{cases} \tag{4.10}$$

If $\gamma_j \subset B_{\tilde{y}_j}$, again we obtain the desired inequality (4.1) by combining inequalities (4.6), (4.9) and (4.10). If $\gamma_j \not\subset B_{\tilde{y}_j}$, then (4.10) with (4.9) gives

$$\rho(y_j) \leq A^{h+1} \frac{2}{d(\tilde{y}_j)} \ell_\rho(\gamma_j). \tag{4.11}$$

By elementary inequalities in [GP], Lemma 2.1, and [BHK], Inequality (2.4), we obtain

$$\log\left(1 + \frac{d(y_j, \tilde{y}_j)}{\min\{d(y_j), d(\tilde{y}_j)\}}\right) \leq k(y_j, \tilde{y}_j) \leq M$$

and further,

$$\frac{1}{d(\tilde{y}_j)} \leq \frac{e^M - 1}{d(y_j, \tilde{y}_j)}. \tag{4.12}$$

Moreover, the assumption $d(y_j, \tilde{y}_j) \geq d(\tilde{y}_j)/2$ gives us

$$d(y_j) \leq d(y_j, \tilde{y}_j) + d(\tilde{y}_j) \leq 3d(y_j, \tilde{y}_j).$$

This, along with inequalities (4.11), (4.12) and (4.4), yields an estimate for the ρ-length of γ_j:

$$\begin{aligned} \rho(y_j) &\leq 2A^{h+1} \frac{e^M - 1}{d(y_j, \tilde{y}_j)} \ell_\rho(\gamma_j) \leq 6A^{h+1} \frac{e^M - 1}{d(y_j)} \ell_\rho(\gamma_j) \\ &\leq 6A^{h+1}(e^M - 1) \frac{D'}{p} 2^{j+1} \ell_\rho(\gamma_j). \end{aligned} \tag{4.13}$$

Now combining (4.6) and (4.13) we obtain

$$\ell_\rho([x_{j+1}, x_j]) \le 24(e^M - 1)A^{b+h+1}D'^2\ell_\rho(\gamma_j).$$

Thus (4.1) is proven when the curves $[x_{j+1}, x_j]$ and γ_j are "close" to each other in the quasihyperbolic metric.

Case B. By Case A we may assume that $\mathrm{dist}_k([x_{j+1}, x_j], \gamma_j) > M$. Let $w_j \in [x_{j+1}, x_j]$ satisfy $d(x, w_j) = 3 \cdot 2^{-(j+2)}p$. Let $r := \ell_\rho(\gamma_j)$ and let $w \in \gamma_j$. Let us consider the ρ-ball $B_\rho(w, 2r)$.

Subcase C. If $\mathrm{dist}_k(w_j, B_\rho(w, 2r)) < M$, there exists $u \in B_\rho(w, 2r)$ such that $k(w_j, u) \le M$ and hence $\rho(w_j) \le A^h\rho(u)$ (cf. inequality (4.9)). We may assume that $\gamma_j \cap B_u = \emptyset$. Otherwise $\mathrm{dist}_k([x_{j+1}, x_j], \gamma_j) \le M + 1$ and replacing M with $M + 1$ we obtain the result by the case A. As we have assumed $\gamma_j \cap B_u = \emptyset$,

$$\begin{aligned}
2\ell_\rho(\gamma_j) = 2r &> \mathrm{dist}_\rho(u, \gamma_j) \\
&\overset{\mathrm{HI}(A)}{\ge} \frac{1}{2A}\rho(u)d(u) \\
&\overset{(4.9)}{\ge} \frac{1}{2A^{h+1}}\rho(w_j)d(u) \\
&\overset{(*)}{\ge} \frac{1}{2A^{h+1}e^M}\rho(w_j)d(w_j) \\
&\overset{(4.4)}{\ge} \frac{2^{-(j+1)}p}{2A^{h+1}D'e^M}\rho(w_j) \\
&\overset{(4.6)}{\ge} \frac{1}{8A^{b+h+1}D'^2e^M}\ell_\rho([x_{j+1}, x_j]).
\end{aligned}$$

The inequality $(*)$ above follows from the elementary estimate ([GP], Lemma 2.1, [BHK], Inequality (2.3))

$$\left|\log\frac{d(w_j)}{d(u)}\right| \le k(w_j, u) \le M.$$

Again we find a constant $C \ge 1$ such that $\ell_\rho([x_{j+1}, x_j]) \le C\ell_\rho(\gamma_j)$. So (4.1) is satisfied.

Subcase D. By Subcase C we may assume that the ρ-ball $B_\rho(w, 2r)$ is "far away" from the quasihyperbolic geodesic $[x_{j+1}, x_j]$. More precisely, we may assume that $\mathrm{dist}_k(w_j, B_\rho(w, 2r)) \ge M$. Our plan is to prove that the volume growth condition VG(B) does not hold for such a ρ-ball. This is done by considering subcurves of ρ-length r of quasihyperbolic geodesics $[z, w_j]$ with $z \in \gamma_j$ and "averaging over γ_j" with respect to a suitable Frostman measure.

Let for every $z \in \gamma_j$, $[z, w_j]$ be a quasihyperbolic geodesic which joins z and w_j. Cover $[z, w_j]$ with balls $\{B_1, \dots, B_{n(z)}\} \subset \mathcal{B}$ ordered so that if $m < n$, then there exists $z_m \in B_m \cap [z, w_j]$ such that for every $\tilde{z} \in B_n \cap [z, w_j]$, we have $k(z, z_m) \le k(z, \tilde{z})$. Recall that $n(z) < \infty$.

Let $[z, w_z] \subset [z, w_j]$, where w_z is the first point which does not belong to $B_\rho(w, 2r)$. Thus $\ell_\rho([z, w_z]) \ge r$. Let $\{B_1, \dots, B_{n_r(z)}\} \subset \{B_1, \dots, B_{n(z)}\}$ be those balls which cover $[z, w_z]$. So by HI(A) and by the local D'-uniformity (quasiconvexity) of quasihyperbolic geodesics we obtain

$$\begin{aligned} r \le \ell_\rho([z, w_z]) &\le \sum_{i=1}^{n_r(z)} A\rho(z_i)\ell_d([z, w_z] \cap B_i) \\ &\le AD' \sum_{i=1}^{n_r(z)} \rho(z_i)\,\mathrm{diam}_d(B_i). \end{aligned} \tag{4.14}$$

We next provide a tool that will be used to estimate the μ_ρ-measure of the ρ-ball $B_\rho(w, 2r)$. We claim that if $B \in \mathcal{B}$ intersects $B_\rho(w, 2r)$, then $B \subset B_\rho(w, (2+\frac{A^2}{6})r)$. To show this, it suffices to prove that if $B \in \mathcal{B}$ intersects $B_\rho(w, 2r)$ then

$$\mathrm{diam}_\rho(B) \le \frac{A^2}{6} r. \tag{4.15}$$

Consider such a ball $B \in \mathcal{B}$. It follows from HI(A) that

$$\mathrm{diam}_\rho(B) \le A\rho(z_B)\,\mathrm{diam}_d(B) = \frac{A}{25}\rho(z_B)d(z_B)$$

for each $B \in \mathcal{B}$, where z_B is the centre of B. Hence it actually suffices to prove that

$$\rho(z_B)d(z_B) \le \frac{25}{6}Ar. \tag{4.16}$$

Let $y \in B \cap B_\rho(w, 2r)$. If $w \notin B_{z_B}$, then there exists a curve γ, which joins points w and y and

$$\begin{aligned} 2r &\ge \int_\gamma \rho(z)\,ds \ge \frac{1}{A}\rho(z_B)\ell_d(\gamma \cap B_{z_B}) \\ &\ge \Big(\frac{1}{2} - \frac{1}{50}\Big)\frac{1}{A}\rho(z_B)d(z_B) = \frac{12}{25A}\rho(z_B)d(z_B), \end{aligned}$$

and the inequality (4.16) is proven.

Let us assume that $w \in B_{z_B}$. The elementary estimate (2.3) implies

$$M \le k(w_j, w) \le 4D^2 \log\left(1 + \frac{d(w_j, w)}{\min\{d(w_j), d(w)\}}\right).$$

Along with the assumption that $M > 4D^2\frac{\log(4D'^2)}{\log 2} + 1$, we see that

$$\min\{d(w_j), d(w)\} \le \frac{d(w_j, w)}{e^{M/4D^2} - 1} \le 2^{-j+1-(M-1)/4D^2} p. \tag{4.17}$$

The assumption $M > 4D^2\frac{\log(4D'^2)}{\log 2} + 1$ and (4.4) give us

$$\begin{aligned} d(w_j) &\geq \frac{p}{D'}2^{-(j+1)} = 2^{-j+1-(M-1)/4D^2} p \frac{2^{(M-1)/4D^2}}{2^2 D'} \\ &\geq 2^{-j+1-(M-1)/4D^2} p. \end{aligned} \tag{4.18}$$

Thus it follows from inequality (4.17) that

$$d(w) \leq 2^{-j+1-(M-1)/4D^2} p \leq 2^{-(j+1)} p.$$

Hence, from the definition of the curve γ_j and inequality (4.2) we know that γ_j cannot be a subset of B_w. Then by HI(A)

$$r = \int_{\gamma_j} \rho(z)\, ds \geq \frac{1}{2A}\rho(z_B)d(w) \geq \frac{1}{4A}\rho(z_B)d(z_B),$$

and (4.16) is proven.

Now we know that if $B \in \mathcal{B}$ intersects $B_\rho(w, 2r)$, then $B \subset B_\rho(w, (2 + \frac{1}{6}A^2)r)$. Then by HI($A$), Lemma 3.1 (iv) and Q-regularity on balls of Whitney type, we have

$$\begin{aligned} \mu_\rho(B_\rho(w, (2 + \tfrac{1}{6}A^2)r)) &= \int_{B\rho(w,(2+\frac{1}{6}A^2)r)} \rho^Q \, d\mu \\ &\geq \sum_{\substack{B\in\mathcal{B} \\ B\cap B_\rho(w,2r)\neq\emptyset}} \frac{1}{NA^Q}\rho(z_B)^Q \mu(B) \\ &\geq \sum_{\substack{B\in\mathcal{B} \\ B\cap B_\rho(w,2r)\neq\emptyset}} c_3 \rho(z_B)^Q \left(\frac{\operatorname{diam}_d(B)}{2}\right)^Q, \end{aligned} \tag{4.19}$$

where $c_3 = \dfrac{1}{NC_1A^Q}$.

Let us choose the basepoint z_0 to be w_j. According to Frostman's Lemma (Theorem 4.1) there is a Radon measure ω supported on γ_j such that $\omega(\gamma_j) \geq \frac{\operatorname{diam}_d(\gamma_j)}{60}$ and $\omega(E) \leq \operatorname{diam}_d(E)$ for every $E \subset \gamma_j$. Then with (4.14) we obtain (a version of Fubini's theorem)

$$\begin{aligned} \omega(\gamma_j)r &\leq AD' \int_{\gamma_j} \sum_{i=1}^{n_r(z)} \rho(z_i) \operatorname{diam}_d(B_i)\, d\omega(z) \\ &\leq AD' \sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n \\ B\cap[z,w_z]\neq\emptyset \\ z\in\gamma_j}} \rho(z_B) \operatorname{diam}_d(B)\omega(S(B)\cap\gamma_j). \end{aligned} \tag{4.20}$$

By Hölder's inequality we obtain that

$$\begin{aligned}
&\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n\\ B\cap[z,w_z]\neq\emptyset\\ z\in\gamma_j}} \rho(z_B)\operatorname{diam}_d(B)\omega(S(B)\cap\gamma_j)\\
&\leq \Big(\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n\\ B\cap[z,w_z]\neq\emptyset\\ z\in\gamma_j}} \rho(z_B)^Q\operatorname{diam}_d(B)^Q\Big)^{\frac{1}{Q}}\\
&\quad \Big(\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n\\ B\cap[z,w_z]\neq\emptyset\\ z\in\gamma_j}} \omega(S(B)\cap\gamma_j)^{\frac{Q}{Q-1}}\Big)^{\frac{Q-1}{Q}}.
\end{aligned}$$

Combining this with (4.20), (4.19) and the assumption $\operatorname{dist}_k(w_j, B_\rho(w,2r)) \geq M$ we obtain the estimate

$$\begin{aligned}
\omega(\gamma_j)r &\leq AD'\Big(\frac{2^Q}{c_3}\mu_\rho\big(B_\rho\big(w,\big(2+\tfrac{1}{6}A^2\big)r\big)\big)\Big)^{\frac{1}{Q}}\\
&\quad \Big(\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n\\ B\cap[z,w_z]\neq\emptyset\\ z\in\gamma_j}} \omega(S(B)\cap\gamma_j)^{\frac{Q}{Q-1}}\Big)^{\frac{Q-1}{Q}}\\
&= c_4\big(\mu_\rho\big(B_\rho\big(w,\big(2+\tfrac{1}{6}A^2\big)r\big)\big)\big)^{\frac{1}{Q}}\Big(\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n\\ B\cap[z,w_z]\neq\emptyset\\ z\in\gamma_j}} \omega(S(B)\cap\gamma_j)^{\frac{Q}{Q-1}}\Big)^{\frac{Q-1}{Q}},
\end{aligned}\tag{4.21}$$

where $c_4 = 2AD'c_3^{-\frac{1}{Q}} = 2(NC_1)^{\frac{1}{Q}}A^2D'$.

In order to estimate the measure of the shadow of the ball $B \in \mathcal{B}_n$, let us make a couple of preliminary estimates. For every $v \in B \cap [z, w_j]$, where $B \in \mathcal{B}$ and $z \in \gamma_j$, we have by uniformity (quasiconvexity) and inequality (4.3) that

$$d(w_j, v) \leq \ell_d([w_j, v]) \leq \ell_d([w_j, z]) \leq D'd(w_j, z) \leq 2^{-j+1}pD'.$$

In the same way as in inequalities (4.17) and (4.18), we obtain from inequality (4.4) and the assumption $n \geq M-1 \geq 4D^2\frac{\log(4D'^2)}{\log 2}$ that for every $v \in B\cap[z, w_j]$, where $B \in \mathcal{B}_n$ and $z \in \gamma_j$, it holds that

$$d(v) \leq 2^{-j+1-n/4D^2}pD'.$$

Furthermore, for every centre point $z_B \in B \in \mathcal{B}_n$, such that $B \cap [z, w_j] \neq \emptyset$ for some $z \in \gamma_j$, it holds that

$$d(z_B) \leq \frac{50}{49} d(v) \leq 2^{-j+1-n/4D^2} p \frac{50D'}{49}. \tag{4.22}$$

Also from the uniformity of the space (Ω, d) and inequality (4.22) it follows that there exist a constant $c_5 = c_5(C_1, D) \geq 1$ such that for every $B \in \mathcal{B}_n$, so that $B \cap [z, w_j] \neq \emptyset$ for some $z \in \gamma_j$, it holds

$$\operatorname{diam}_d(S(B)) \leq c_5 \operatorname{diam}_d(B) \leq 2^{-j+2-n/4D^2} p c_5 \frac{50D'}{49}. \tag{4.23}$$

Now for every $n \geq M - 1$ it holds by Lemma 3.4, Frostman's Lemma and inequality (4.23) that

$$\begin{aligned}
&\sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{Q}{Q-1}} \\
&\quad \leq \max_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{1}{Q-1}} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j) \\
&\quad \leq c_6 \omega(\gamma_j) \max_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{1}{Q-1}} \\
&\quad \leq c_6 \omega(\gamma_j) \max_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \operatorname{diam}_d(S(B) \cap \gamma_j)^{\frac{1}{Q-1}} \\
&\quad \leq c_6 \Big(\frac{200 D' c_5}{49} \Big)^{\frac{1}{Q-1}} \omega(\gamma_j) (2^{-j-n/4D^2} p)^{\frac{1}{Q-1}},
\end{aligned}$$

where $c_6 = c_6(C_1, D)$ is from Lemma 3.4. Furthermore, using this we may compute that

$$\begin{aligned}
&\sum_{n=M-1}^{\infty} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{Q}{Q-1}} \\
&\quad \leq c_6 \Big(\frac{200 D' c_5}{49} \Big)^{\frac{1}{Q-1}} \omega(\gamma_j) \sum_{n=M-1}^{\infty} (2^{-j-n/4D^2} p)^{\frac{1}{Q-1}} \\
&\quad \leq c_7 \omega(\gamma_j) p^{\frac{1}{Q-1}} 2^{\frac{-j}{Q-1}} 2^{\frac{-M}{4D^2(Q-1)}},
\end{aligned}$$

where $c_7 = c_6\big(\frac{200D'c_5}{49}\big)^{\frac{1}{Q-1}} \dfrac{2^{\frac{2}{4D^2(Q-1)}}}{2^{\frac{1}{4D^2(Q-1)}} - 1}$. Thus with (4.21) we have

$$\omega(\gamma_j)^Q r^Q \le c_4^Q c_7^{Q-1} \mu_\rho(B_\rho(w,(2+\tfrac{1}{6}A^2)r))\omega(\gamma_j)^{Q-1}2^{-j-\frac{M}{4D^2}}p.$$

Furthermore $\omega(\gamma_j) \ge \frac{\mathrm{diam}_d(\gamma_j)}{60}$, and this gives us

$$\begin{aligned}\mu_\rho(B_\rho(w,(2+\tfrac{1}{6}A^2)r)) &\ge \omega(\gamma_j)\frac{1}{c_4^Q c_7^{Q-1}} \frac{2^{j+\frac{M}{4D^2}}}{p} r^Q \\ &\ge \frac{2^{-j-1}p}{60}\frac{1}{c_4^Q c_7^{Q-1}}\frac{2^{j+\frac{M}{4D^2}}}{p} r^Q \\ &= 2^{\frac{M}{4D^2}} c_1 r^Q,\end{aligned}$$

where $c_1 = \dfrac{49 \cdot 2^{\frac{-2}{4D^2}-1}\big(2^{\frac{1}{4D^2(Q-1)}}-1\big)^{Q-1}}{12000 c_5 N C_1 (2A^2)^Q D'^{Q+1} c_6^{Q-1}}$.

This is a contradiction because when M is sufficiently big, the volume growth condition VG(B) will not hold. Consequently, if $k([x_{j+1},x_j],\gamma_j) > M$ then our ρ-ball is in the quasihyperbolic metric k so big that $\mathrm{dist}_k(w_j, B_\rho(w,2r)) \le M$. Thus the conclusion is that $\ell_\rho([x_{j+1},x_j]) \le C\ell_\rho(\gamma_j)$, where $C = C(A,B,C_1,D,Q)$. □

There is nothing special about the constant $\frac{1}{2}$ in condition HI(A) and the constants $\frac{1}{50}$ and 5 in Whitney covering. The only restriction in the Whitney covering is that if $\lambda_1 B_d(z_1, d(z_1)/\lambda_2) \cap \lambda_1 B_d(z_2, d(z_2)/\lambda_2) \ne \emptyset$, then $\lambda_1 B_d(z_1, d(z_1)/\lambda_2)$ must be included in some ball $B_d(z_2, d(z_2)/\lambda_3)$ on which the measure μ is doubling. Otherwise one can choose the constants as desired.

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References

[BB] Z. M. Balogh and S. M. Buckley, Geometric characterization of Gromov hyperbolicity. *Invent. Math.* **153** (2003), 261–301. Zbl 1059.30038 MR 1992014

[BO] J. Björn and J. Onninen, Orlicz capacities and Hausdorff measures on metric spaces. *Math. Z.* **251** (2005), 131–146. Zbl 1084.31004 MR 2176468

[BS] J. Björn and N. Shanmugalingam, Poincaré inequalities, uniform domains and extension properties for Newton–Sobolev functions in metric spaces. *J. Math. Anal. Appl.* **332** (2007), 190–208. Zbl 1132.46021 MR 2319654

[BHK] M. Bonk, J. Heinonen and P. Koskela, Uniformizing Gromov hyperbolic spaces. *Astérisque* **270** (2001), 1–99. Zbl 0970.30010 MR 1829896

[BHR] M. Bonk, J. Heinonen and S. Rohde, Doubling conformal densities. *J. Reine Angew. Math.* **541** (2001), 117–141. Zbl 0987.30009 MR 1876287

[BKR] M. Bonk, P. Koskela and S. Rohde, Conformal metrics on the unit ball in Euclidean space. *Proc. London Math. Soc.* (3) **77** (1998), 635–664. Zbl 0916.30017 MR 1643421

[CW] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*. Lecture Notes in Math. 242, Springer–Verlag, Berlin 1971. Zbl 224.43006 MR 0499948

[GH] F. W. Gehring and W. K. Hayman, An inequality in the theory of conformal mapping. *J. Math. Pures Appl.* (9) **41** (1962), 353–361. Zbl 0105.28002 MR 0148884

[GO] F. W. Gehring and B. G. Osgood, Uniform domains and the quasi-hyperbolic metric. *J. Anal. Math.* **36** (1979), 50–74. Zbl 0449.30012 MR 0581801

[GP] F. W. Gehring and B. P. Palka, Quasiconformally homogeneous domains. *J. Anal. Math.* **30** (1976), 172–199. Zbl 0349.30019 MR 0437753

[Gr] M. Gromov, Hyperbolic Groups. In *Essays in group theory* (S. Gersten, ed.), MSRI Publication, Springer-Verlag 1987, 75–265. Zbl 0634.20015 MR 0919829

[HKT] P. Hajłasz, P. Koskela and H. Tuominen, Measure density and extendability of Sobolev functions. *Rev. Mat. Iberoamericana* **24** (2008), no. 2, 645–669. Zbl 1226.46029 MR 2459208

[HK] J. Heinonen and P. Koskela, Quasiconformal maps in metric space with controlled geometry. *Acta Math.* **181** (1998), 1–61. Zbl 0915.30018 MR 1654771

[HN] J. Heinonen and R. Näkki, Quasiconformal distortion on arcs. *J. Anal. Math.* **63** (1994), 19–53. Zbl 0804.30016 MR 1269214

[HR] J. Heinonen and S. Rohde, The Gehring–Hayman inequality for quasihyperbolic geodesics. *Math. Proc. Cambridge Phil. Soc.* **114** (1993), 393–404. Zbl 0791.30015 MR 1235987

[H1] D. A. Herron, Conformal deformations of uniform Loewner spaces. *Math. Proc. Cambridge Philos. Soc.* **136** (2004), 325–360. Zbl 1046.30027 MR 2040578

[H2] D. A. Herron, Quasiconformal deformations and volume growth. *Proc. London Math. Soc.* (3) **92** (2006), 161–199. Zbl 1088.30012 MR 2192388

[Jo] P. W. Jones, Extension Theorems for BMO. *Indiana Univ. Math. J.* **29** (1980), no. 1, 41–66. Zbl 0432.42017 MR 0554817

[KL] P. Koskela and J. Lehrbäck, Quasihyperbolic growth conditions and compact embeddings of Sobolev sapces. *Michigan Math. J.* **55** (2007), 183–193. Zbl 1135.46015 MR 2320179

[MaSe] R. A. Macías and C. Segovia, A decomposition into atoms of distributions on spaces of homogeneous type. *Adv. in Math.* **33** (1979), 271–309. Zbl 0431.46019 MR 0546296

[MaSa] O. Martio and J. Sarvas, Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **4** (1979), 383–401. Zbl 0406.30013 MR 0565886

[Ma] P. Mattila, *Geometry of sets and measures in euclidean spaces: fractals and rectifiability.* Cambridge University Press, Cambridge 1995. Zbl 0819.28004 MR 1333890

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