Vertices of closed curves in Riemannian surfaces

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Vertices of closed curves in Riemannian surfaces

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Abstract. We uncover some connections between the topology of a complete Riemannian surface M and the minimum number of *vertices*, i.e., critical points of geodesic curvature, of closed curves in M. In particular we show that the space forms with finite fundamental group are the only surfaces in which every simple closed curve has more than two vertices. Further we characterize the simply connected space forms as the only surfaces in which every closed curve bounding a compact immersed surface has more than two vertices.

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Keywords. Four vertex theorem, Riemannian surface, surface of constant curvature, space form, hyperbolic surface, geodesic curvature.

1. Introduction

In this paper we study the relation between the topology of a complete Riemannian surface M and the minimum number of *vertices*, i.e., critical points of geodesic curvature, of closed curves in M. Our prime motivation here is the classical theorem of Kneser [25], [29], which states that any ($\mathcal{C}^2$) simple closed curve in Euclidean plane $\mathbb{R}^2$ has at least four vertices. It is known that this result also holds in the sphere $\mathbb{S}^2$ and hyperbolic plane $\mathbb{H}^2$, since the stereographic projection and the inclusion map of the Poincaré disk preserve vertices [26], [9]. Further, it follows from another classical result due to Möbius [28], [39] that any simple closed curve in the projective plane $\mathbb{R}\mathbf{P}^2$ has at least three vertices. We show that, up to a rescaling, these are the only surfaces where every simple closed curve has more than two vertices:

Theorem 1.1. *The only complete Riemannian surfaces where every simple closed curve has more than two vertices are the space forms with finite fundamental group.*

Thus the simply connected space forms are the only complete Riemannian surfaces where Kneser's four vertex theorem holds, and real projective planes are the only

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other surfaces where every simple closed curve has at least three vertices. Another classification result we obtain in this work uses the extension of Kneser's theorem to self-intersecting curves by Pinkall [34] who showed that every closed curve in $\mathbb{R}^2$ (and hence $\mathbb{H}^2$ and $\mathbb{S}^2$) which bounds a compact immersed region must have at least four vertices. Our next theorem shows that this property also characterizes the simply connected space forms:

Theorem 1.2. *The only complete Riemannian surfaces where every closed curve which bounds a compact immersed surface has more than two vertices are the simply connected space forms.*

Finally one may ask what are all surfaces which satisfy the condition of the last theorem if the word "immersed" is replaced by "embedded". Obviously, surfaces of the above theorems would then fall into that category, but there are more examples:

Theorem 1.3. *The only complete Riemannian surfaces where every closed curve which bounds a compact* embedded surface *has more than two vertices are orientable space forms of genus zero, flat tori, and rescalings of* $\mathbb{R}\mathbf{P}^2$.

We recall that an orientable surface of genus zero is one that is homeomorphic to $\mathbb{S}^2$ minus a totally disconnected subset [37]. The above theorems are proved in the next three sections below. The proofs of Theorems 1.1 and 1.3 employ some basic curve shortening and certain perturbation results for closed geodesics or horocycles in hyperbolic surfaces. An important lemma utilized here is a result of Jackson [24] who showed that Kneser's four vertex property holds only in surfaces with constant curvature. The proof of Theorem 1.2 also uses this lemma, together with an explicit example of a rather remarkable cylindrical curve which has only two vertices but bounds a compact immersed surface (Figure 5). A general result for perturbing closed geodesics to curves with only two vertices is discussed in the appendix.

Four vertex theorems have spawned a vast and diverse literature since the first version of the theorem was proved in 1909 by Mukhopadhyaya [30], who showed that convex planar curves have four vertices. This is the only version mentioned in nearly all differential geometry textbooks, with the exceptions of [18], [29] where the more general theorem of Kneser is discussed, see also [23], [31], [20]. Pinkall's paper [34] and related work [48], [7] offer other more general proofs as well, based on a number of ideas. For more recent proofs see [1], [45] which use curvature flow and Sturm theory. Another recent development is concerned with a converse of Kneser's theorem [10], [15]. There are also interesting generalizations to space curves [41], [49], [46], connections with contact geometry [2], [50], polygonal analogues [33], and a version for surfaces with boundary [14]. See [36] for a physical application, and [13], [32] for some more references and historical remarks.

2. Proof of Theorem 1.1

We begin by recording the result of Jackson mentioned above:

Lemma 2.1 ([24]). *Let M be a Riemannian surface with curvature K and let p be a point of M. Suppose that p is not a stationary point of K, i.e., $dK_p \neq 0$. Then there exists an $\epsilon > 0$ such that for all $0 < r \leq \epsilon$, the metric circle of radius r centered at p has only two vertices.* □

Thus we may confine our attention to complete Riemannian surfaces M of constant curvature K, or 2-dimensional *space forms*, as far as proving Theorem 1.1 is concerned. Furthermore, we may assume that M is not simply connected due to the following fact: let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere and $\mathbb{H}^2 \subset \mathbb{R}^2$ be the Poincaré half-plane with its standard metric of constant curvature -1; then we have:

Lemma 2.2 ([26]). *The stereographic projection $\pi : \mathbb{S}^2 - \{(0,0,1)\} \to \mathbb{R}^2$ and the inclusion map $i : \mathbb{H}^2 \to \mathbb{R}^2$ preserve the sign of the derivative of the geodesic curvature of curves.* □

Thus Kneser's theorem holds in all simply connected 2-dimensional space forms. The property of the stereographic projection mentioned above was already known to Kneser [25], and the property of the inclusion map for the Poincaré disk was proved by Maeda [26]; but the Poincaré disk and half-plane are equivalent up to a Möbius transformation, which establishes the corresponding fact for the Poincaré half-plane, since Möbius transformations preserve vertices [34], [20]. Essentially, these observations are consequences of the fact that the stereographic projection and the inclusion map of the upper half-plane send circles to circles, and the vertices occur when the curve has third order contact with its osculating circle.

It remains to consider non-simply connected 2-dimensional space forms M. We may assume, after a rescaling, that the curvature K of M is 0, 1, or -1. Then by the Hopf–Killing theorem [44], [38], $M = X/G$ where $X = \mathbb{R}^2$, $\mathbb{S}^2$, or $\mathbb{H}^2$ and G is a discrete subgroup of isometries of X which acts freely and properly discontinuously on X. In particular, the projection $\pi : X \to X/G = M$ is a Riemannian covering map. We also recall that G is isometric to the fundamental group $\pi_1(M)$. Indeed, for any point $o \in M$, $\bar{o} \in X$ with $\pi(\bar{o}) = o$, and closed curve $\gamma : [0, L] \to M$ with $\gamma(0) = o$, if $\bar{\gamma} : [0, L] \to X$ is the lifting of γ with $\bar{\gamma}(0) = \bar{o}$, then there exists a unique element $g \in G$ such $g(\bar{\gamma}(0)) = \bar{\gamma}(L)$. This correspondence, which will be utilized a number of times below, establishes the isomorphism between G and $\pi_1(M)$.

2.1. The elliptic case. If $K = 1$, then M is the real projective plane $\mathbb{R}\mathbf{P}^2 = \mathbb{S}^2/\{\pm 1\}$. By a theorem of Möbius [28], [39], see Note 2.8 below for other references,

any simple closed noncontractible curve Γ in $\mathbb{R}\mathbf{P}^2$ has at least three *inflection points*, i.e., points where the geodesic curvature vanishes. But there must be at least one vertex between every pair of inflections. Thus Γ must have at least three vertices. On the other hand, if Γ is contractible, then it lifts to a pair of simple closed curves in $\mathbb{S}^2$, each of which must have four vertices by Kneser's theorem. Since the covering map is one-to-one on each of these curves, it then follows that Γ must have at least four vertices. So we conclude then that all simple closed curves in $\mathbb{R}\mathbf{P}^2$ must have at least three vertices. (Note that in a nonorientable surface, the geodesic curvature in only well-defined locally and up to a sign; however, this is enough to allow one to talk about vertices and inflection points.)

In the remaining cases we construct explicit examples of simple closed curves with only two vertices, by perturbing geodesics or horocycles of M.

2.2. The parabolic case. If $K = 0$, then $M = \mathbb{R}^2/G$ where there are exactly four types of possibilities for G corresponding to the cases of cylinder, twisted cylinder, torus, or Klein bottle [44]. In each of these cases M contains a simple closed geodesic Γ, as may be easily seen by looking at the fundamental region of these surfaces which is either a rectangle or an infinite strip in $\mathbb{R}^2$. Let $\gamma\colon \mathbb{R}/L \to M$ be a parametrization of Γ by arclength (where L denotes the length of Γ), and $\bar{\gamma}$ be a lifting of γ to $\mathbb{R}^2$. Then $\bar{\gamma}$ traces a line which we may assume to be the x-axis. Let $g \in G$ be the unique element such that $g(\bar{\gamma}(0)) = \bar{\gamma}(L)$. Then g is either the "translation" $(x, y) \mapsto (x + L, y)$ or the "glide reflection" $(x, y) \mapsto (x + L, -y)$ and we define, respectively,

$$\tilde{\gamma}(t) := \left(t, \lambda \sin\left(\frac{2\pi t}{L}\right)\right) \quad \text{or} \quad \tilde{\gamma}(t) := \left(t, \lambda \sin\left(\frac{\pi t}{L}\right)\right),$$

for some $\lambda > 0$. Then the image of $\tilde{\gamma}$ is invariant under the action of g. Consequently, if $\pi\colon \mathbb{R}^2 \to M$ denotes the covering map, then $\pi \circ \tilde{\gamma}$ traces a smooth closed curve in M with only one or two vertices. Further note that, as $\lambda \to 0$, $\tilde{\gamma}$ converges to $\bar{\gamma}$, and consequently $\pi \circ \tilde{\gamma}$ converges to $\pi \circ \bar{\gamma} = \gamma$ with respect to the $\mathcal{C}^1$-norm. So since γ is simple, it follows that $\pi \circ \tilde{\gamma}$ will be simple as well for sufficiently small λ.

2.3. The hyperbolic case. If $K = -1$, or M is a hyperbolic surface, we need to establish a basic structure theorem first. Recall that an *end* of a surface is a nested sequence of subsets which eventually lie outside any given compact subset. Each element of this sequence is called an *end representative*, and two ends are equivalent if each representative of one end lies in a representative of the other. A *cusp* of a hyperbolic surface is an end with a representative which is isometric to a representative C of the "thin" end of the *parabolic cylinder* $\mathbb{H}^2/G$ where G is generated by the parabolic translations $(x, y) \mapsto (x + L, y)$; more explicitly, a cusp representative may be defined as

$$C = C(h) := \{(x, y) \in \mathbb{H}^2 \mid y > h\}/G,$$

for some $h > 0$. Alternatively, a cusp may be visualized as an end of the tapering surface of revolution in $\mathbb{R}^3$ known as the *pseudosphere*.

Proposition 2.3. *Every non-simply connected complete hyperbolic surface contains a simple closed geodesic or a cusp.*

In the special case where the surface is orientable and has finite Euler characteristic, the above proposition follows from a result in the recent book of Borthwick [6], Proposition 2.16. With the aid of this result, and some curve shortening, we give the more general proof which we seek. First we need to establish the following lemma. If M were orientable, this lemma would follow from basic results on curve shortening flow by curvature [17], [8]; however, when M is not orientable, the geodesic curvature is not well-defined along the entire curve; hence we use the "disk flow" method devised in [19] which does not require orientability.

Lemma 2.4. *Let M be a complete Riemannian surface and Γ be a simple closed curve of length L in M. Suppose that Γ is not isotopic to a point and the set of all curves in the isotopy class of Γ whose lengths are bounded above by L are confined to a compact region of M. Then Γ is isotopic to a simple closed geodesic.*

Proof. If M is compact, the lemma follows immediately from [19], Theorem 1.8. But the proof of Theorem 1.8 in [19] shows that compactness is needed only so that one can apply Ascoli's theorem to conclude that any length non-increasing homotopy Γ_t has a convergent subsequence. Of course one may draw the same conclusion as long as Γ_t is confined to a compact subset of M, which we assume is the case. □

Next we use the previous lemma to establish another basic fact:

Lemma 2.5. *Every complete non-orientable Riemannian surface contains a simple closed geodesic, which has a tubular neighborhood homeomorphic to a Möbius strip.*

Proof. If a complete Riemannian surface M is nonorientable then it must contain a Möbius band U. Let Γ be a smooth simple closed curve which is a retraction of U. By the isotopy extension lemma [22], if Γ' is any curve in M which is isotopic to Γ, then Γ' will also have a neighborhood U' which is homeomorphic to a Möbius band and retracts onto Γ'. This yields the following two observations. First, Γ is not isotopic to a point, because every surface is locally orientable. Second, no curve homotopic to Γ may be disjoint from Γ. This is due to the fact that there exists a smooth curve Γ' homotopic to Γ which intersects Γ only once, see Figure 1. Next note that any curve Γ'' homotopic to Γ will be homotopic to Γ' as well. Suppose now, towards a contradiction, that Γ'' is disjoint from Γ. Then, if # denotes the intersection number, we have

$$0 = \#(\Gamma'', \Gamma) = \#(\Gamma', \Gamma) = 1,$$

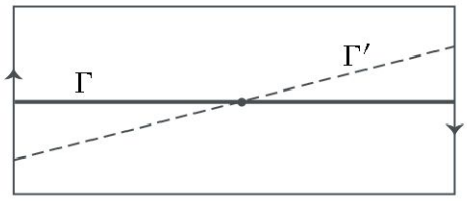

Figure 1

by the invariance of intersection number under homotopy [22]. Thus, $\Gamma'' \cap \Gamma \neq \emptyset$. This shows that any *length nonincreasing* isotopy of Γ must be confined within a compact region of M (if length of Γ is L, then this region would consist of points of M which are within a distance $L/2$ of Γ). It follows then, via Lemma 2.4, that Γ is isotopic to a simple closed geodesic. □

We need only one other basic result before proving Proposition 2.3. A *hyperbolic cylinder* is the quotient $\mathbb{H}^2/G$ where G is generated by the hyperbolic translations $(x, y) \mapsto (e^L x, e^L y)$. Thus a hyperbolic cylinder contains a simple closed geodesic, i.e., the image of the positive half of the y-axis, which cuts the cylinder in half. Following Borthwick [6] we define a *funnel*, as one of these halves. We say that a surface is *topologically finite* if it is homeomorphic to a compact surface minus a finite set of points.

Lemma 2.6 ([6]). *Every non-simply connected noncompact complete orientable hyperbolic surface with finite topology has a cusp or funnel end.*

Proof. If our hyperbolic surface $M = \mathbb{H}^2/G$ is homeomorphic to an annulus, or, equivalently, M is orientable and G has only one generator, then that generator is conjugate to either a parabolic or hyperbolic translation of $\mathbb{H}^2$, which we defined above. Consequently M is isometric to either the parabolic cylinder or hyperbolic cylinder, in which case M has a cusp or funnel end respectively, and we are done. On the other hand, if M is not homeomorphic to an annulus (and is not simply connected), then M will be "nonelementary" and the proof follows from Theorem 2.13 of [6] which classifies the ends of nonelementary geometrically finite orientable hyperbolic surfaces. To apply this classification result, we just need to note that since M is topologically finite, it is "geometrically finite" as well ([6], Theorem 2.10), i.e., G is finitely generated, or, equivalently, the fundamental region of M is a finite-sided convex polygon in $\mathbb{H}^2$. □

Now we are ready to prove Proposition 2.3:

Proof of Proposition 2.3. By Lemma 2.5 we may suppose that our surface, say M, is orientable. It is well-known that every compact non-simply connected orientable surface contains a simple closed geodesic, e.g., this follows from Lemma 2.4; or

see Section 9.6 in [38] for the hyperbolic case. So we may also suppose that M is noncompact. Then, if M has finite topology, by Lemma 2.6, one of the ends of M must be a "cusp" or a "funnel" [6]. In either case we would be done since, by definition, funnels are bounded by simple closed geodesics. So it remains to consider the case where M has infinite topology, although all we need is that M have at least four ends. Then we show that M must have a simple closed geodesic as follows.

By the generalized Jordan curve theorem, any simple closed curve Γ_0 divides M into two components. Choose Γ_0 so that each of the components of $M - \Gamma_0$ contains at least two ends of M, and Γ_0 is rectifiable. Then let Γ_t be a length nonincreasing isotopy of Γ_0, e.g., as defined by Hass and Scott [19], Theorem 1.8. Note that, by the isotopy extension lemma [22], the topology of $M - \Gamma_t$ does not depend on t. Let A_t be a continuous choice of a component of $M - \Gamma_t$, and $E \subset A_0$ be an end representative of M which does not contain all ends of M that are in A_0. Then Γ_t cannot be contained entirely in E for any time t, because then A_t would be disjoint from some of the ends of A_0. Now let $E' \subset E$ be another end representative such that the distance of $\partial E'$ from ∂E is bigger than $L/2$ were L is the length of Γ_0. Then $\Gamma_t \cap E' = \emptyset$ for all t.

Similarly, for each end of M we may choose an end representative which will always be disjoint from Γ_t. Then the complement of all these end representatives is a compact subset of M which contains Γ_t for all t, and so we may apply Lemma 2.4 to complete the proof. □

Having proved Proposition 2.3, we may now proceed with the rest of the proof of Theorem 1.1 by considering the following two cases:

2.3.1. If our hyperbolic surface $M = \mathbb{H}^2/G$ contains a cusp, then it contains a *horocycle*, i.e., a simple closed curve $\gamma : \mathbb{R}/L \to M$ which lifts to a horizontal line $\bar{\gamma}(t) = (t, h) \subset \mathbb{H}^2$. Then let $\tilde{\gamma}$ be the perturbation of $\bar{\gamma}$ given by

$$\tilde{\gamma}(t) := \left(t,\, h + \lambda \sin\left(\frac{2\pi t}{L}\right)\right),$$

see Figure 2. Recall that there is a unique element $g \in G$, given by $g(\bar{\gamma}(0)) = \bar{\gamma}(L)$,

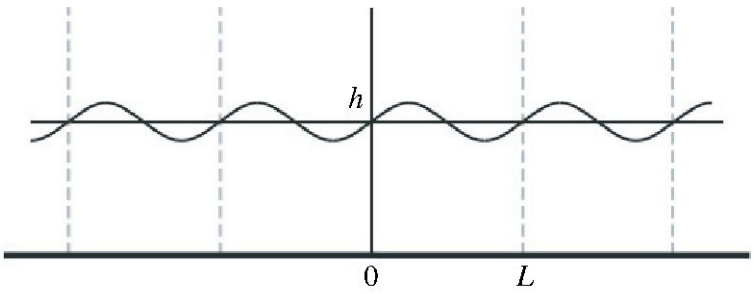

Figure 2

which in this case is the parabolic translation $(x, y) \mapsto (x + L, y)$. Further, as we argued in Section 2.2, since $\tilde{\gamma}$ is also invariant with respect to g, the projection $\pi \circ \tilde{\gamma}$ is a smooth closed curve in M with only two vertices for any $\lambda \neq 0$, by Lemma 2.2. Furthermore, again as $\lambda \to 0$, $\tilde{\gamma}$ converges to $\bar{\gamma}$ with respect to the $\mathcal{C}^1$ norm. Thus, since γ is simple, $\pi \circ \tilde{\gamma}$ will be simple as well.

2.3.2. If $M = \mathbb{H}^2/G$ contains a simple closed geodesic $\gamma : \mathbb{R}/L \to M$, then it lifts to a geodesic $\bar{\gamma} : \mathbb{R} \to \mathbb{H}^2$. We may assume that $\bar{\gamma}$ traces the upper half of the y-axis in its positive direction, and $\bar{\gamma}(0) = (0, 1)$. Then $\bar{\gamma}(L) = (0, e^L)$ (recall that the hyperbolic distance of $(0, 1)$ from $(0, y)$ is given by $\ln(y)$). As before, let $g \in G$ be the (unique) element such that $g(\gamma(0)) = \gamma(L)$. Then g is either the hyperbolic translation $(x, y) \mapsto (e^L x, e^L y)$ or the glide reflection $(x, y) \mapsto (-e^L x, e^L y)$, depending on wether or not small tubular neighborhoods of γ are orientable. So there are two cases to consider:

If g is the hyperbolic translation (i.e., a tubular neighborhood of γ in M is orientable), let

$$\tilde{\gamma}(t) := \left(\lambda t \sin\left(\frac{2\pi}{L}\ln(t)\right), t\right),$$

see Figure 3. Note that $\tilde{\gamma}(e^L t) = e^L \tilde{\gamma}(t)$, i.e., the image of $\tilde{\gamma}$ is invariant under the action of g. Thus $\pi \circ \tilde{\gamma}$ is a smooth closed curve in M, and choosing λ sufficiently

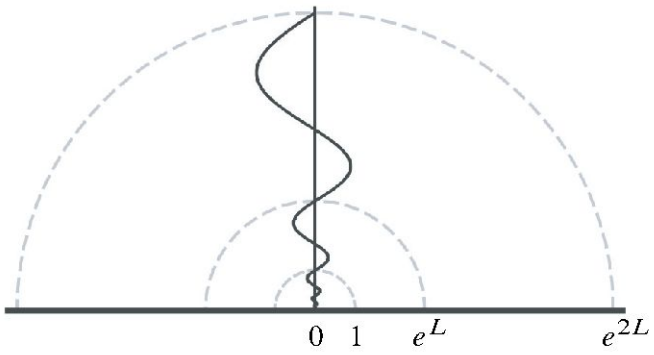

Figure 3

small we can make sure that $\pi \circ \tilde{\gamma}$ is simple as we argued in Section 2.2. Finally note that $\tilde{\gamma}$ has only two vertices on the interval $[1, e^L]$. Thus, the projection of $\pi \circ \tilde{\gamma}$ will have only two vertices, again by Lemma 2.2.

If g is the glide reflection (i.e., no neighborhood of γ in M is orientable) let

$$\tilde{\gamma}(t) := \left(\lambda t \sin\left(\frac{\pi}{L}\ln(t)\right), t\right).$$

Then the image of $\tilde{\gamma}$ is invariant under the action of g and $\pi \circ \tilde{\gamma}$ again yields the desired curve for small λ, which completes the proof of Theorem 1.1.

We record below the last part of the proof for future reference:

Lemma 2.7. *Any closed geodesic or horocycle in a hyperbolic surface is $\mathcal{C}^\infty$-close to a closed curve with no more than two vertices.* □

Note 2.8. The theorem of Möbius mentioned above is equivalent to the statement that every simple closed curve Γ in $\mathbb{S}^2$ which is symmetric with respect to the antipodal reflection ($\Gamma = -\Gamma$) has at least six inflection points. Since the antipodal reflection switches the sign of geodesic curvature, the number of inflection points of Γ must be $2m$ where m is odd. So, it is enough to show that Γ has more than two inflections. In this sense, the theorem of Möbius may be viewed as a special case of the "tennis ball theorem" [2], [3], [1] which proves the existence of at least four inflections for curves which bisect the area of $\mathbb{S}^2$. The latter result in turn follows from a theorem of Segre [42] who proved that any simple closed curve on $\mathbb{S}^2$ which contains the origin in its convex hull must have at least four inflection points. Another proof of Segre's theorem may be found in [51]. For other refinements or results related to the theorem of Möbius see, [40], [47], [35].

3. Proof of Theorem 1.2

Let M be a complete Riemannian surface satisfying the hypothesis of Theorem 1.2. By Lemma 2.1, and after a rescaling, we may again assume that M has constant curvature $K = 1, 0$, or -1 which result in the following three cases respectively:

3.1. The elliptic case. Recall that here M is the real projective plane $\mathbb{R}\mathbf{P}^2 = \mathbb{S}^2/\{\pm 1\}$. In this case we may construct an explicit example of a curve with only two vertices, which bounds an immersed compact surface, as shown in Figure 4. In this picture, $\mathbb{R}\mathbf{P}^2$ is represented as a hemisphere of $\mathbb{S}^2$, say the northern hemisphere, with

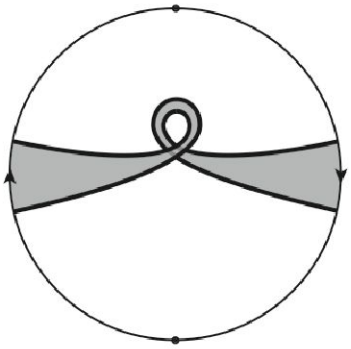

Figure 4

the antipodal points on its boundary, or the equator, identified, and we are looking at this hemisphere from "above" (i.e., in a direction orthogonal to the plane of the equator). Alternatively, the above picture may be regarded as that of a unit disk, and then

we may transfer the depicted region to a hemisphere via a stereographic projection, which preserves the number of vertices by Lemma 2.2. Note that the points where the curve intersects the boundary of the hemisphere are inflection points.

3.2. The parabolic case. In this case $M = \mathbb{R}^2/G$ where G has at most two generators one of which must be a translation or a glide reflection [44]. But the composition of two glide reflections is a translation. Thus G must contain a subgroup H generated by a translation. Then the projection $\mathbb{R}^2/H \to \mathbb{R}^2/G$ yields a covering of M by a cylinder. Thus it is enough to show that every flat cylinder contains a closed curve with only two vertices which bounds a compact immersed surface. Then the covering map yields the corresponding examples in all other topological types of flat surfaces (the twisted cylinder, the torus, and the Klein bottle).

So we may suppose that M is a flat cylinder, i.e., $M = \mathbb{R}^2/L\mathbb{Z}$ which is the quotient of $\mathbb{R}^2$ modulo the horizontal translations $(x, y) \mapsto (x + zL, y)$. Alternatively, we may think of M as the fundamental region $[0, L] \times \mathbb{R} \subset \mathbb{R}^2$ with its right and left hand sides identified. Then the curve we seek is depicted in the left hand side of Figure 5, and the picture on the right shows a lifting of that curve in $\mathbb{R}^2$. Note that

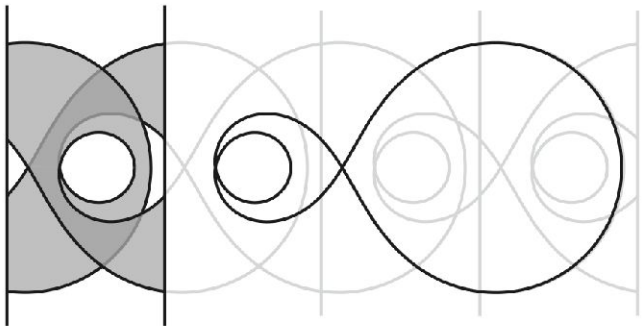

Figure 5

the curve on the cylinder bounds a compact immersed surface. Thus we only need to check that the given curve has only two vertices. Indeed the curve on the right is given by

$$\gamma(t) := \frac{1}{a^2 + 2a\cos\left(\frac{t}{5}\right)\cos(t) + \cos\left(\frac{t}{5}\right)^2}\left(a + \cos\left(\tfrac{t}{5}\right)\cos(t),\ \cos\left(\tfrac{t}{5}\right)\sin(t)\right),$$

where $a = 9/100$. To count the vertices of γ note that if we invert γ with respect to the unit circle and then translate it to the left by a distance of a we obtain the curve given by $r(\theta) = \cos(\theta/5)$ in polar coordinates, see Figure 6. A straight forward computation then shows that the derivative of the curvature of r is given by

$$\kappa'(\theta) := \frac{24\left(8 + 6\cos\left(\frac{2\theta}{5}\right)\right)\sin\left(\frac{2\theta}{5}\right)}{\left(13 + 12\cos\left(\frac{2\theta}{5}\right)\right)^{\frac{5}{2}}}.$$

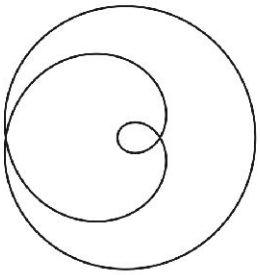

Figure 6

Thus r has only two vertices which occur at $\theta = 0$ and $\theta = 5\pi/2$, and one may easily check that both are nondegenerate critical points of curvature. Hence γ has only two vertices as well, since Möbius transformations preserve vertices [34], [20], as we had pointed out earlier. Finally, if γ has only two vertices as a curve in $\mathbb{R}^2$ then it has only two vertices in $\mathbb{H}^2$ as well, by Lemma 2.2.

Note 3.1. Another example of a closed curve on the torus, or the Klein bottle, which has only two vertices but bounds a compact immersed surface is depicted in Figure 7. Note that although this example is more simple than that of Figure 5, it does not work

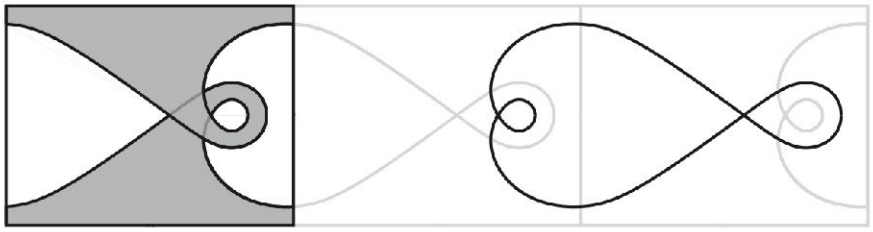

Figure 7

on the cylinder or twisted cylinder because the region that it bounds would not be compact in those surfaces.

Note 3.2. Although the example of Figure 5 bounds a compact immersed surface in the cylinder, the lifting of that curve does not bound any such surface in the plane. The conditions for a closed planar curve to bound a compact immersed disk were first described in the thesis of Blank [5]. See [27], [12], [4] for further refinements and generalizations of that result.

3.3. The hyperbolic case. Recall that in this case, by Proposition 2.3, M either has a cusp, or a simple closed geodesic.

3.3.1. Suppose first that M has a cusp, then we may construct an example similar to that of Figure 5 on M because cusps are asymptotic to cylinders. More precisely,

recall that a cusp is isometric to an end of a pseudosphere, which we may think of as a surface of revolution Σ about the z-axis in $\mathbb{R}^3$. Note that Σ contains a meridian m_ϵ of length ϵ for every sufficiently small ϵ. After a vertical translation we may assume that m_ϵ lies on the xy-plane. Further, after a homothety of $\mathbb{R}^3$, we may assume that m_ϵ has length 2π. Thus we obtain a sequence of surfaces Σ_ϵ which converge to the cylinder C given by $x^2 + y^2 = 1$, within any given compact ball B centered at the origin of $\mathbb{R}^3$, as $\epsilon \to 0$. In particular, for small ϵ, C and Σ_ϵ will be $\mathcal{C}^\infty$-close in B, and thus any curve in $B \cap C$ may be projected into Σ_ϵ by moving it along the normals to C. The new curve will be $\mathcal{C}^\infty$ close to the original one, and thus will have the same number of vertices as the original curve if all the vertices of the original curve are nondegenerate critical points of curvature, which, as we verified earlier, is the case in the example of Figure 5.

3.3.2. Now suppose that M has a simple closed geodesic Γ. Then Γ lifts to a geodesic in the upper half-plane $\mathbb{H}^2$, and after an isometry, we may suppose that this lifting traces the positive half of the y-axis. Also recall that homotheties of $\mathbb{R}^2$ are isometries of $\mathbb{H}^2$. We can use these homotheties by constructing an example similar to that of Figure 5 along the y-axis, see Figure 8. The curves depicted in this picture

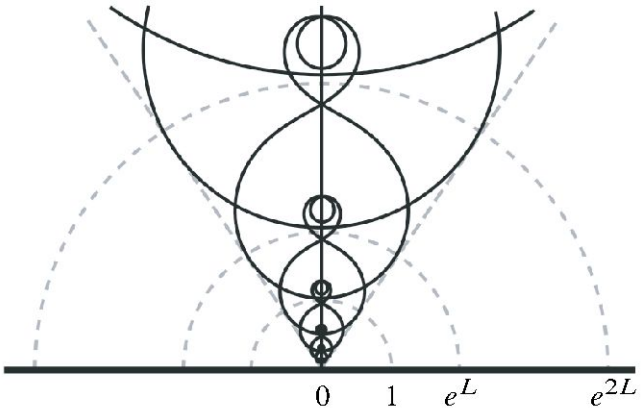

Figure 8

are rescalings of each other by a (Euclidean) factor of e^L where L is the length of Γ. This completes the proof once we note that example of Figure 5 may be constructed for any given L.

4. Proof of Theorem 1.3

Let M be an orientable space form of genus zero, flat torus, or a rescaling of $\mathbb{R}\mathbf{P}^2$, and $\Gamma \subset M$ be a simple closed curve which bounds an embedded surface. If $M = \mathbb{R}\mathbf{P}^2$, or a rescaling of $\mathbb{R}\mathbf{P}^2$, then Γ has at least three inflection points by Möbius's theorem,

as we pointed out in Section 2.1. If M is an orientable surface of genus zero or a torus, then Γ must bound a disk in M. Consequently Γ lifts to a simple closed curve in the universal cover of M, and then it follows from Kneser's theorem and Lemma 2.2 that Γ must have at least four vertices.

To complete the proof of Theorem 1.3 it remains then to show that if M is not an orientable space form of genus zero, flat torus, or a rescaling of $\mathbb{R}\mathbf{P}^2$, then it must contain a simple close curve Γ which bounds a compact surface but has no more than two vertices. Again, by Lemma 2.1, we may assume that M has constant curvature. Further, since we already know that M may not be $\mathbb{S}^2$ or $\mathbb{R}\mathbf{P}^2$, we may suppose that the curvature of M is nonpositive. The next result shows that we may assume that M is orientable as well:

Lemma 4.1. *Any nonorientable complete surface of nonpositive curvature contains a simple closed curve with only two vertices which bounds a compact embedded surface.*

Proof. First recall that, by Lemma 2.5, any nonorientable complete Riemannian surface M contains a simple closed geodesic $\gamma : \mathbb{R}/L \to M$ which has a tubular neighborhood homeomorphic to a Möbius strip. Let $\bar{\gamma}$ be a lifting of γ to the universal cover of M, which is $\mathbb{R}^2$ or $\mathbb{H}^2$.

If the universal cover of M is $\mathbb{R}^2$, i.e., M has zero curvature, then we may suppose, for convenience, that $\bar{\gamma}$ traces the x-axis. Now let

$$\tilde{\gamma}_{\pm}(t) := \left(t, \lambda\left(\sin\left(\frac{\pi t}{L}\right) \pm \epsilon\right)\right)$$

and note that these curves are invariant under the group of "glide reflections" $(x, y) \mapsto (x + L, -y)$, see Figure 9. Thus the covering map $\pi : \mathbb{R}^2 \to M$ sends these curves to

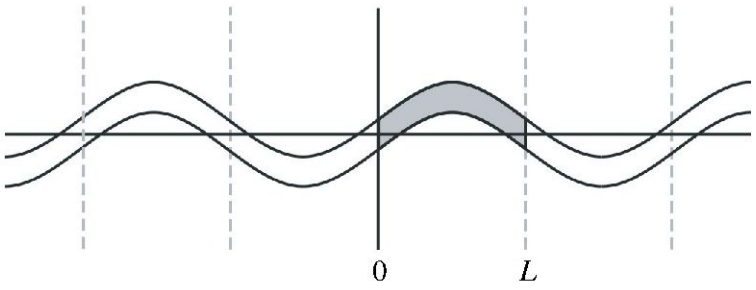

Figure 9

a single closed curve with only two vertices in M which converges to γ with respect to the $\mathcal{C}^1$-norm as $\lambda \to 0$. In particular, this curve will be embedded for small λ.

If the universal cover of M is $\mathbb{H}^2$, i.e., if M has negative curvature, then we may suppose that $\bar{\gamma}$ traces the positive half of y-axis in the positive direction and

$\bar{\gamma}(0) = (0, 1)$. Now let

$$\tilde{\gamma}_{\pm}(t) := \left(\lambda\, t\left(\sin\left(\frac{\pi \ln(t)}{L}\right) \pm \epsilon\right), t\right)$$

and again note that these curves are invariant under the group of "glide reflections" $(x, y) \mapsto (-e^L x, e^L y)$, see Figure 10. Thus the covering map $\pi : \mathbb{H}^2 \to M$ once

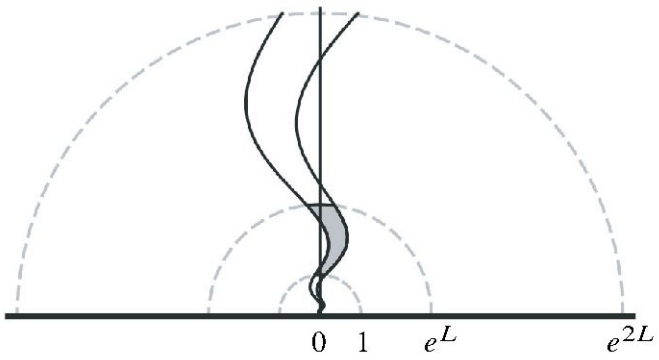

Figure 10

again sends these curves to the desired curve for small λ. □

Now recall that the only orientable parabolic 2-dimensional space forms other than the tori are the cylinders and the Euclidean plane, both of which have genus zero. Thus we may assume that M has negative curvature, and observe that

Lemma 4.2. *Every complete orientable hyperbolic surface M of nonzero genus contains a simple closed geodesic or a horocycle which bounds a compact subset of M.*

Proof. Since M has nonzero genus, and is not a torus, it is the connected sum of a torus T with a noncontractible surface M'; this follows from the normalization theorem of Richards [37] which states that any orientable surface is homeomorphic to the connected sum of $\mathbb{S}^2$ with a countable number of tori and minus a totally disconnected subset. Thus there exists a simple closed curve Γ which divides M into two components: one homeomorphic to T minus a disk and the other homeomorphic to M' minus a disk. Now let Γ_t be the curve shortening flow devised in [19]. Then as we argued in the proof of Proposition 2.3, Γ_t either converges to a simple closed geodesic or else eventually enters every representative of an end of M. The latter case may happen only if that end is a cusp and Γ_t is isotopic to a horocycle of that cusp. The proof is then complete once we recall that, by the isotopy extension lemma [22], the topology of $M - \Gamma_t$ is independent of t. In particular, the closure of one of the components of $M - \Gamma_t$ will be homeomorphic to T minus an open disk, which is a compact region of M. □

By Lemma 2.7, we may now perturb the curve given by the last result to obtain a simple closed curve with only two vertices, which completes the proof.

Appendix: More on perturbations of geodesics

In Section 2.3 we showed that any closed geodesic in a hyperbolic surface may be perturbed, in the $\mathcal{C}^\infty$ sense, to a closed curve with only two vertices. Here we include a more general result for orientable surfaces of constant curvature, which may not be complete.

Theorem 4.3. *Let Γ be a closed geodesic of length L in a Riemannian surface of constant curvature K, which is orientable in a neighborhood of Γ. Then, every neighborhood of Γ contains a closed curve which has only two vertices, and may be required to be arbitrarily $\mathcal{C}^\infty$-close to Γ, if, and only if, $K \neq (2\pi/L)^2$.*

The proof of this result follows from the following three lemmas. The basic idea here is again to perturb the geodesic in the direction of its normals according to a sine curve. To this end we first show that a neighborhood of Γ in M may be represented as a surface of revolution in $\mathbb{R}^3$. This observation, via the above theorem, shows that the only closed geodesics which cannot be perturbed to curves with only two vertices correspond to great circles in spheres (there are still many other examples of closed geodesics in noncomplete surfaces of constant positive curvature, which may be perturbed to curves with only two vertices, provided only that the length condition in the above theorem is satisfied.).

Lemma 4.4. *Let M and M' be Riemannian surfaces of constant curvature K which contain simple closed geodesics Γ and Γ' with orientable neighborhoods. Then there exist open neighborhoods U and U' of Γ and Γ', respectively, and an isometry $f : U' \to U$ which maps Γ to Γ'.*

Proof. Fix an orientation for M, and for each point p of Γ let $\{e_1(p), e_2(p)\}$ be a continuous choice of basis for T_pM such that e_1 is tangent to Γ and $(e_1(p), e_2(p))$ is in a fixed orientation class of T_pM. Similarly, let $\{e_1'(p'), e_2'(p')\}$ be a continuous choice of basis along Γ' such that $e_1'(p')$ is tangent to Γ' and $(e_1'(p'), e_2'(p'))$ is in a fixed orientation class of M' for all $p' \in \Gamma'$. Since M and M' have the same constant curvature, then, as is well known, they are locally isometric. Indeed, it follows from a theorem of Cartan, that for every $p \in \Gamma$, and $p' \in \Gamma'$, there is an open neighborhood U_p of p and an isometry $f := f_p : U_p \to M'$ such that $f(p) = p'$ and $df(e_i(p)) = e_i'(p)$, $i = 1, 2$, see [11], p. 158.

We may assume, without loss of generality, that the neighborhoods U_p mentioned above are *tubular*, which we define as follows. Let $N_\epsilon(\Gamma)$ be the set of all point of

M whose distance from Γ is less than ϵ, then by the tubular neighborhood theorem [43], for sufficiently small ϵ, $N_\epsilon(\Gamma)$ is fibrated by geodesic segments which meet Γ orthogonally at one point. This gives rise to a natural projection map $\pi : N_\epsilon(\Gamma) \to \Gamma$. We say that a neighborhood U of p is tubular provided that there exists a connected open neighborhood I of p in Γ such that $U = \pi^{-1}(I)$.

Now let $p(t)$ and $p'(t)$ be a pair of parametrizations of Γ and Γ' by arclength, and $U_t := U_{p(t)}$ be a tubular neighborhood of $p(t)$ such that there exists an isometry $f := f_t : U_t \to M'$ with $f(p(t)) = p'(t)$ and $df(e_i'(p)) = e_i'(p')$. By compactness of Γ, there will be a finite number of points $p_j = p(t_j)$ such that the tubular neighborhoods U_j cover Γ. Note that if $f_j : U_j \to M'$ are the corresponding local isometries, then $U_j' := f_j(U_j)$ also form an open covering of Γ'. Let $U := \cup U_j$, $U' := \cup U_j'$, and define $f : U \to U'$ by setting $f|_{U_j} := f_j$. It is simple to verify that f is well defined, i.e., whenever $U_j \cap U_k \neq \emptyset$, then $f_j = f_k$ on $U_j \cap U_k$, using the uniqueness of geodesics and the fact that the intersection of two tubular neighborhood is tubular, which shows that f is the desired isometry. □

Suppose we have a surface of revolution in $\mathbb{R}^3$ parametrized by

$$X(t, \theta) = \big(r(t)\cos(\theta), r(t)\sin(\theta), h(t)\big). \tag{1}$$

By a *neck* of X we mean a meridian, i.e., a curve $\theta \mapsto X(t_0, \theta)$, which occurs at a height where the tangent to the profile curve r is parallel to the axis of revolution, i.e., $r'(t_0) = 0$.

Lemma 4.5. *For any constant K and positive constant L, there exists a $\mathcal{C}^\infty$ surface of revolution in $\mathbb{R}^3$ with constant curvature K and a neck of length L.*

Proof. If $K = 0$, then it is obvious that we may let our surface to be a cylinder over a circle of length L. If $K > 0$, then, for $-\epsilon \leq t \leq \epsilon$, set

$$r(t) := \frac{L}{2\pi}\cos\left(\frac{t}{K}\right) \quad \text{and} \quad h(t) := \int_0^t \sqrt{1 - \left(\frac{L}{2\pi}\sin\left(\frac{t}{K}\right)\right)^2}\, dt.$$

Then the surface of revolution X given by (1) has constant curvature K, see [16], p. 483. Further X has a neck at $t = 0$ which has length L. Similarly, if $K < 0$, set

$$r(t) := \frac{L}{2\pi}\cosh\left(\frac{t}{K}\right) \quad \text{and} \quad h(t) := \int_0^t \sqrt{1 - \left(\frac{L}{2\pi}\sinh\left(\frac{t}{K}\right)\right)^2}\, dt.$$

Then X again has constant curvature K, see [16], p. 487, and it has a neck of length L at $t = 0$. □

Finally we need:

Lemma 4.6. *Let X be a $\mathcal{C}^\infty$ surface of revolution in $\mathbb{R}^3$ parametrized by*

$$X(t,\theta) := \big(r(t)\cos(\theta), r(t)\sin(\theta), t\big), \tag{2}$$

where $0 \le \theta \le 2\pi$ and $-\epsilon \le t \le \epsilon$. Suppose that X has a neck Γ at $t = 0$, and

$$1 + r(0)r''(0) \neq 0. \tag{3}$$

Then there exists an $\epsilon > 0$ such that for every $0 < \lambda \le \epsilon$ the curve

$$c_\lambda(\theta) := X\big(\lambda\cos(\theta), \theta\big)$$

has only two vertices.

Proof. First we recall that the geodesic curvature of c_λ is given by

$$k_\lambda(\theta) := \left\langle \frac{T_\lambda'(\theta)}{\|c'(\theta)\|}, \nu_\lambda(\theta) \right\rangle,$$

where $T_\lambda(\theta) := c_\lambda'(\theta)/\|c_\lambda'(\theta)\|$ is the unit tangent vector field of c_λ, and $\nu_\lambda(\theta)$ is a continuous normal vector field along c_λ which is tangent to X. To compute ν, we may let $n(t,\theta) := \partial_t X \times \partial_\theta X/\|\partial_t X \times \partial_\theta\|$ be a unit normal vector field of X, and set $\nu_\lambda(\theta) := n(\lambda\cos(\theta),\theta) \times T_\lambda(\theta)$. A straight forward calculation then shows that

$$k_\lambda(\theta) = \frac{\bar r'\bar r^2 + \lambda\bar r\big(-\lambda\bar r'\bar r''\sin^2(\theta) + \cos(\theta)(\bar r')^2 + \cos(\theta)\big) + 2\lambda^2\sin^2(\theta)\bar r'\big((\bar r')^2+1\big)}{\sqrt{(\bar r')^2+1}\big(\bar r^2 + \lambda^2\sin^2(\theta)\big((\bar r')^2+1\big)\big)^{3/2}},$$

where

$$\bar r := r\big(\lambda\cos(\theta)\big), \quad \bar r' := r'\big(\lambda\cos(\theta)\big), \quad \text{and} \quad \bar r'' := r''\big(\lambda\cos(\theta)\big).$$

Since by assumption $r'(0) = 0$, we have $k_0(\theta) = 0$ for all θ. So, fixing θ and applying Taylor's theorem, we obtain

$$k_\lambda(\theta) = \partial_\lambda k_0(\theta)\lambda + R_\lambda(\theta)\lambda^2, \tag{4}$$

where

$$\partial_\lambda k_0(\theta) = \frac{1 + r(0)r''(0)}{r^2(0)}\cos(\theta), \quad \text{and} \quad R_\lambda(\theta) = \int_0^\lambda \partial_u^2 k_u(\theta)(\lambda - u)\,du. \tag{5}$$

Note that (4) is valid for all (λ,θ) in $[0,1]\times[0,2\pi]$. Further, since $k_\lambda(\theta)$ depends smoothly on both λ and θ, so does $R_\lambda(\theta)$. Consequently, the $\mathcal{C}^2$-norm of $R_\lambda(\theta)$, as a function of θ, has a uniform upper bound A valid for all $\lambda \in [0,1]$:

$$A := \sup_{\lambda\in[0,1]} \|R_\lambda(\theta)\|_{\mathcal{C}^2} = \sup_{\lambda\in[0,1]}\sup_{\theta\in[0,2\pi]} \{R_\lambda(\theta), R_\lambda'(\theta), R_\lambda''(\theta)\} < \infty.$$

Furthermore, (4) shows that

$$\|k_\lambda(\theta) - \partial_\lambda k_0(\theta)\lambda\|_{\mathcal{C}^2} = \|R_\lambda(\theta)\|_{\mathcal{C}^2}\,\lambda^2 \leq A\lambda^2,$$

where all norms are with respect to θ. So

$$\lim_{\lambda\to 0^+} \|k_\lambda(\theta) - \partial_\lambda k_0(\theta)\lambda\|_{\mathcal{C}^2} = 0.$$

Combing the last equation with the computation for $\partial_\lambda k_0(\theta)$ in (5) we obtain

$$\lim_{\lambda\to 0^+} \left\| \frac{k_\lambda(\theta)}{\lambda} - \frac{1 + r(0)r''(0)}{r^2(0)}\cos(\theta) \right\|_{\mathcal{C}^2} = 0.$$

Now recall that $1 + r(0)r''(0) \neq 0$ by assumption. Thus, for small λ, $k_\lambda(\theta)/\lambda$ is $\mathcal{C}^2$-close to $C\cos(\theta)$ for some nonzero constant C, which yields that $k'_\lambda(\theta)/\lambda$ is $\mathcal{C}^1$-close to $-C\sin(\theta)$. It follows then that $k'_\lambda(\theta)/\lambda$, and consequently $k'_\lambda(\theta)$, has precisely two zeros, because $-C\sin(\theta)$ has only two zeros and at those points its derivative does not vanish. We conclude then that $k_\lambda(\theta)$ has only two local extrema, as claimed. □

The last three lemmas yield:

Proof of Theorem 4.3. Suppose that $K = (2\pi/L)^2$. Then, by Lemma 4.4, we may isometrically identify a neighborhood of Γ in M with a neighborhood of a great circle Γ' in a sphere of radius $L/(2\pi)$. But any $\mathcal{C}^1$ perturbation of Γ' will still be a simple closed curve, which by the classical four vertex theorem of Kneser on the sphere [25] must have four vertices (the spherical version of the classical four vertex theorem follows from the fact that the stereographic projection preserves vertices, as had already been observed by Kneser). This proves the "only if" part of the theorem.

Now suppose that $K \neq (2\pi/L)^2$. By Lemmas 4.4 and 4.5 we may identify a neighborhood of Γ with a surface of revolution in $\mathbb{R}^3$, which we may parametrize by X given in (2), so that Γ is identified with the neck Γ' of X at height $t = 0$. Then Lemma 4.6 completes the proof once we can verify that $1 + r(0)r''(0) \neq 0$. To see this note that a neck of a surface of revolution is a line of curvature, i.e., it is tangent to a principal direction field of the surface, because the Gauss map n sends any neck to a great circle of $\mathbb{S}^2$, whose plane is parallel to that of the neck, via a homothety of the neck. More precisely, if a neck has radius $r(0)$, then for any tangent vector v to that neck $dn(v) = v/r(0)$, where dn is the shape operator of X. So X has constant principal curvatures $1/r(0)$ in the direction of the tangent vectors of the neck Γ' (and with respect to the inward normal to X). Further, recall that a direction orthogonal to a principal direction is again a principal direction. Thus the other principal curvatures along Γ' are given by the curvatures of the profile curve which is $-r''(0)$ (with respect to the direction of the inward normal). So we conclude that the Gauss curvature (which is the product of principal curvatures) of X along

Γ' is $K = -r''(0)/r(0)$. Note further that $L = 2\pi r(0)$. Thus, by our assumption at the beginning of this paragraph, $4\pi^2 \neq KL^2 = -r''(0)r(0)4\pi^2$, which yields $1 \neq -r''(0)r(0)$ as desired. □

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References

[1] S. Angenent, Inflection points, extatic points and curve shortening. In *Hamiltonian systems with three or more degrees of freedom (S'Agaró, 1995)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 533, Kluwer Acad. Publishers, Dordrecht 1999, 3–10. Zbl 0960.53038 MR 1720878

[2] V. I. Arnold, On the number of flattening points on space curves. In *Sinaĭ's Moscow seminar on dynamical systems*, volume 171 of *Amer. Math. Soc. Transl. Ser. 2*, Amer. Math. Soc., Providence, RI, 1996, 11–22. Zbl 0973.53502 MR 1359089

[3] V. I. Arnold, *Topological invariants of plane curves and caustics*. University Lecture Series 5, Amer. Math. Soc., Providence, RI, 1994. Dean Jacqueline B. Lewis Memorial Lectures presented at Rutgers University, New Brunswick, New Jersey. MR 1286249

[4] K. D. Bailey, Extending closed plane curves to immersions of the disk with n handles. *Trans. Amer. Math. Soc.* **206** (1975), 1–24. Zbl 0301.57019 MR 0370621

[5] S. J. Blank, Extending immersions of the circle. Ph. D. Thesis, Brandeis University, Waltham, Mass., 1967.

[6] D. Borthwick, *Spectral theory of infinite-area hyperbolic surfaces*. Progr. Math. 256, Birkhäuser, Boston, Mass., 2007. Zbl 1130.58001 MR 2344504

[7] G. Cairns, M. Özdemir and E.-H. Tjaden, A counterexample to a conjecture of U. Pinkall. *Topology* **31**(1992), no. 3, 557–558. Zbl 0767.53001 MR 1174258

[8] K.-S. Chou and X.-P. Zhu, *The curve shortening problem*. Chapman & Hall/CRC, Boca Raton, FL, 2001. Zbl 1061.53045 MR 1888641

[9] S. I. R. Costa and M. Firer, Four-or-more-vertex theorems for constant curvature manifolds. In *Real and complex singularities (São Carlos, 1998)*, Chapman & Hall/CRC Res. Notes Math. 412, Chapman & Hall/CRC, Boca Raton, FL, 2000, 164–172. Zbl 0948.53003 MR 1715702

[10] B. E. J. Dahlberg, The converse of the four vertex theorem. *Proc. Amer. Math. Soc.* **133** (2005), no. 7, 2131–2135. Zbl 02150853 MR 2137880

[11] M. P. do Carmo, *Riemannian geometry*. Translated from the second Portuguese edition by Francis Flaherty, Math. Theory Appl., Birkhäuser, Boston, Mass., 1992. Zbl 0752.53001 MR 1138207

[12] G. K. Francis, Spherical curves that bound immersed discs. *Proc. Amer. Math. Soc.* **41** (1973), 87–93. Zbl 0281.57020 MR 0321112

[13] D. DeTurck, H. Gluck, D. Pomerleano and D. S. Vick, The four vertex theorem and its converse. *Notices Amer. Math. Soc.* **54**(2007), no. 2, 192–207. Zbl 1142.53004 MR 2285124

[14] M. Ghomi, A Riemannian four vertex theorem for surfaces with boundary. *Proc. Amer. Math. Soc.* **139** (2011), no. 1, 293–303. Zbl 1216.53040 MR 2729091

[15] H. Gluck, The converse to the four vertex theorem. *Enseignement Math.* (2) **17** (1971), 295–309. Zbl 0226.53001 MR 0344998

[16] A. Gray, *Modern differential geometry of curves and surfaces with Mathematica*. Second edition, CRC Press, Boca Raton, FL, 1998. Zbl 0942.53001 MR 1688379

[17] M. A. Grayson, Shortening embedded curves. *Ann. of Math.* (2) **129** (1989), no. 1, 71–111. Zbl 0686.53036 MR 0979601

[18] H. W. Guggenheimer, *Differential geometry*. Corrected reprint of the 1963 edition, Dover Books on Advanced Mathematics, Dover Publications Inc., New York 1977. Zbl 0357.53002 MR 0493768

[19] J. Hass and P. Scott, Shortening curves on surfaces. *Topology* **33** (1994), no. 1, 25–43. Zbl 0798.58019 MR 1259513

[20] E. Heil, Some vertex theorems proved by means of Moebius transformations. *Ann. Mat. Pura Appl.* (4) **85** (1970), 301–306. Zbl 0175.48103 MR 0266130

[21] E. Heil, A four-vertex theorem for space curves. *Math. Pannon.* **10** (1999), no. 1, 123–132. Zbl 0922.53001 MR 1678108

[22] M. W. Hirsch, *Differential topology*. Corrected reprint of the 1976 original, Springer-Verlag, New York 1994. Zbl 0356.57001 MR 1336822

[23] S. B. Jackson, Vertices for plane curves. *Bull. Amer. Math. Soc.* **50** (1944), 564–478. Zbl 0060.34909 MR 0010992

[24] S. B. Jackson, The four-vertex theorem for surfaces of constant curvature. *Amer. J. Math.* **67** (1945), 563–582. Zbl 0060.34910 MR 0014257

[25] A. Kneser, Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie. In *Festschrift Heinrich Weber*, B.G. Teubner, Leipzig 1912, 170–180. JFM 43.0463.01

[26] M. Maeda, The four-or-more vertex theorems in 2-dimensional space forms. *Nat. Sci. J. Fac. Educ. Hum. Sci. Yokohama Natl. Univ.* **1** (1998), 43–46. MR 1710269

[27] M. L. Marx, Extensions of normal immersions of S^1 into R^2. *Trans. Amer. Math. Soc.* **187** (1974), 309–326. Zbl 0284.30028 MR 0341505

[28] A. F. Möbius, Über die Grundformen der Linien der dritten Ordnung. In *Gesammelte Werke II*, Verlag von S. Hirzel, Leipzig 1886, 89–176.

[29] S. Montiel and A. Ros, *Curves and surfaces*. Translated and updated from the 1998 Spanish edition by the authors, Grad. Stud. in Math. 69, Amer. Math. Soc., Providence, RI, 2005. Zbl 1082.53003 MR 2165790

[30] S. Mukhopadhyaya, New methods in the geometry of a plane arc. *Bull. Calcutta Math. Soc.* **1** (1909), 31–37. JFM 40.0624.01

[31] R. Osserman, The four-or-more vertex theorem. *Amer. Math. Monthly* **92** (1985), no. 5, 332–337. Zbl 0579.53002 MR 0790188

[32] V. Ovsienko and S. Tabachnikov, *Projective differential geometry old and new. From the Schwarzian derivative to the cohomology of diffeomorphism groups*. Cambridge Tracts in Math. 165, Cambridge University Press, Cambridge 2005. Zbl 1073.53001 MR 2177471

[33] I. Pak, *Lectures on discrete and polyhedral geometry*. 2008. www.math.umn.edu/~pak/book.htm

[34] U. Pinkall, On the four-vertex theorem. *Aequationes Math.* **34** (1987), no. 2–3, 221–230. Zbl 0635.53003 MR 0921101

[35] W. F. Pohl, On a theorem related to the four-vertex theorem. *Ann. of Math.* (2) **84** (1966), 356–367. Zbl 0145.41803 MR 0198363

[36] E. Raphael, J.-M. di Meglio, M. Berger and E. Calabi, Convex Particles at Interfaces, *J. Phys. I France* **2** (1992), 571–579.

[37] I. Richards, On the classification of noncompact surfaces. *Trans. Amer. Math. Soc.* **106** (1963), 259–269. Zbl 0156.22203 MR 0143186

[38] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*. Second edition, Grad. Texts in Math. 149, Springer-Verlag, New York 2006. Zbl 1106.51009 MR 2249478

[39] S. Sasaki, The minimum number of points of inflexion of closed curves in the projective plane. *Tôhoku Math. J.* (2) **9** (1957), 113–117. Zbl 0088.14501 MR 0099700

[40] T. Sasaki, Inflection points and affine vertices of closed curves on 2-dimensional affine flat tori. Festschrift dedicated to Katsumi Nomizu on his 70th birthday (Leuven, 1994; Brussels, 1994), *Results Math.* **27** (1995), no. 1–2, 129–140. Zbl 0836.53007 MR 1317832

[41] V. D. Sedykh, Four vertices of a convex space curve. *Bull. London Math. Soc.* **26** (1994), no. 2, 177–180. Zbl 0807.53002 MR 1272305

[42] B. Segre, Alcune proprietà differenziali in grande delle curve chiuse sghembe. *Rend. Mat.* (6) **1** (1968), 237–297. Zbl 0185.24802 MR 0243466

[43] M. Spivak, *A comprehensive introduction to differential geometry. Vol. I*. Second edition, Publish or Perish Inc., Wilmington, Del., 1979. Zbl 0439.53001 MR 0532830

[44] J. Stillwell, *Geometry of surfaces*. Universitext, Springer-Verlag, New York 1992. Zbl 0752.53002 MR 1345154

[45] B. Süssmann, Curve shortening and the four-vertex theorem. *Port. Math.* (*N.S.*) **62** (2005), no. 3, 269–288. MR 2171673

[46] G. Thorbergsson and M. Umehara, A unified approach to the four vertex theorems. II. In *Differential and symplectic topology of knots and curves*, Amer. Math. Soc. Transl. (2) 190, Amer. Math. Soc., Providence, RI, 1999, 229–252. Zbl 1068.53004 MR 1738398

[47] G. Thorbergsson and M. Umehara, Inflection points and double tangents on anti-convex curves in the real projective plane. *Tohoku Math. J.* (2) **60** (2008), no. 2, 149–181. Zbl 1152.53009 MR 2428859

[48] M. Umehara, 6-vertex theorem for closed planar curve which bounds an immersed surface with nonzero genus. *Nagoya Math. J.* **134** (1994), 75–89. Zbl 0806.53002 MR 1280654

[49] M. Umehara, A unified approach to the four vertex theorems. I. In *Differential and symplectic topology of knots and curves*, Amer. Math. Soc. Transl. Ser. (2) 190, Amer. Math. Soc., Providence, RI, 1999, 185–228. Zbl 1068.53005 MR 1738397

[50] R. Uribe-Vargas, Four-vertex theorems, Sturm theory and Lagrangian singularities. *Math. Phys. Anal. Geom.* **7** (2004), no. 3, 223–237. Zbl 1068.51009 MR 2081714

[51] J. L. Weiner, Global properties of spherical curves. *J. Differential Geom.* **12** (1977), no. 3, 425–434. Zbl 0368.53001 MR 0514446

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