

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 89 (2014)  
  
**Artikel:** Finiteness of 3-manifolds associated with non-zero degree mappings  
**Autor:** Boileau, Michel / Rubinstein, J. Hyam / Wang, Shicheng  
**DOI:** <https://doi.org/10.5169/seals-515665>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 01.07.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Finiteness of 3-manifolds associated with non-zero degree mappings

Michel Boileau, J. Hyam Rubinstein and Shicheng Wang\*

**Abstract.** We prove a finiteness result for the  $\partial$ -patterned guts decomposition of all 3-manifolds obtained by splitting a given orientable, irreducible and  $\partial$ -irreducible 3-manifold along a closed incompressible surface. Then using the Thurston norm, we deduce that the JSJ-pieces of all 3-manifolds dominated by a given compact 3-manifold belong, up to homeomorphism, to a finite collection of compact 3-manifolds. We show also that any closed orientable 3-manifold dominates only finitely many integral homology spheres and any compact orientable 3-manifold dominates only finitely many exteriors of knots in  $S^3$ .

**Mathematics Subject Classification (2010).** 57M05, 57M25, 57M45.

**Keywords.** Degree one mappings, 3-manifolds, JSJ-decompositions, Thurston norm, Gromov volume, patterned guts.

### 1. Introduction

Maps between 3-manifolds have been studied for a long time, and have become an especially active subject after Thurston's revolutionary work on 3-manifold theory. The existence of non-zero degree proper maps between compact orientable 3-manifolds is a fundamental and difficult question in this area. We say that a compact, orientable 3-manifold  $M$  *dominates* a compact orientable 3-manifold  $N$  if there is a non-zero degree proper map  $f: M \rightarrow N$ . When the degree of  $f$  is one, we say that  $M$  *1-dominates*  $N$ .

The following simple and natural question was raised in the 1980s (and formally appeared in the 1990s, see ([Ki], Problem 3.100 (Y.Rong), and also [W2]).

**Question 1.** *Does a closed orientable 3-manifold 1-dominate at most finitely many closed, irreducible and orientable 3-manifolds?*

---

\***Acknowledgements** We thank a lot the referee for his careful reading of our paper, for providing many fine suggestions, and in particular for pointing out the incompleteness of our induction argument in Section 5. We thanks also Y. Liu and H. Sun for their comments on our drafts. The first and the last authors would like to acknowledge support by MOST of China, and Institut Universitaire de France; and the last two authors would like to acknowledge support by the Australian Research Council.

If we allow any degree, 3-manifolds supporting one of the geometries  $\mathbb{S}^3, \widetilde{\mathrm{PSL}_2(\mathbb{R})}, \mathrm{Nil}$  can dominate infinitely many 3-manifolds. Thus any closed orientable 3-manifold which dominates such closed 3-manifolds indeed dominates infinitely many 3-manifolds. At the moment these are the only known examples, so the following generalization of Question 1 makes sense:

**Question 2.** *Let  $M$  be a closed orientable 3-manifold. Does  $M$  dominate at most finitely many closed, irreducible, orientable 3-manifolds  $N$  not supporting the geometries of  $\mathbb{S}^3, \widetilde{\mathrm{PSL}_2(\mathbb{R})}, \mathrm{Nil}$ ?*

Many related partial answers to Questions 1 and 2 have already appeared in the literature, see for example [Ro1], [Ro2], [Ro3], [BW], [HWZ1], [HWZ2], [RW], [So1], [So2], [So3], [Re], [WZ], [De], [De1], [Gu], [BCG], [BBW].

Finiteness of closed irreducible targets implies finiteness of possibly non prime closed targets: the number of prime factors in the target is bounded by the number of closed, disjoint, non parallel, essential surfaces in the domain; furthermore a connected sum of finitely many closed 3-manifolds 1-dominates each of its prime summands, which are either irreducible or homeomorphic to  $S^1 \times S^2$ .

Since a non-zero degree proper map between two compact orientable 3-manifolds induces a non-zero degree map between their doubles, finiteness of closed targets implies finiteness of the targets in the setting of compact orientable 3-manifolds (see Remark 4.7 at the end of Section 4).

First we introduce some standard terminology in 3-manifold topology, see [Ja].

In this paper, all surfaces and 3-manifolds are compact and orientable. Also we will work in the piecewise linear category, so all spaces and maps will be PL. Suppose  $S$  (resp.  $P$ ) is a properly embedded surface (resp. an embedded 3-manifold) in a 3-manifold  $M$ . We use  $M \setminus S$  (resp.  $M \setminus P$ ) to denote the resulting manifold obtained by splitting  $M$  along  $S$  (resp. removing  $\mathrm{int} P$ , the interior of  $P$ ). Note that we allow the possibility that  $S$  is not connected, so that it has finitely many components.

A 3-manifold  $M$  is

- *prime* if it is not the connected sum of two 3-manifolds neither of which is  $S^3$ ;
- *irreducible* if every embedded sphere in  $M$  bounds a ball in  $M$ . A prime orientable 3-manifold which is not irreducible is homeomorphic to  $S^2 \times S^1$ ;
- *$\partial$ -irreducible* if for every properly embedded disc  $D$  in  $M$ , there is a ball  $B \subset M$  and a disc  $D' \subset \partial M$ , such that  $\partial D = \partial D'$  and  $\partial B = D \cup D'$ ;
- *atoroidal* if every  $\mathbb{Z} \oplus \mathbb{Z}$  subgroup in  $\pi_1 M$  is conjugate into  $\pi_1 \partial M$  and in addition  $\pi_1 M$  is not virtually abelian. An irreducible orientable 3-manifold such that every  $\mathbb{Z} \oplus \mathbb{Z}$  subgroup of  $\pi_1 M$  is conjugate into  $\pi_1 \partial M$  is either atoroidal,  $T^2 \times (0, 1)$ , or the twisted I-bundle over the Klein bottle.

The JSJ-decomposition ([JS], [Joh]) of a compact orientable irreducible 3-manifold  $M$  is the canonical splitting of  $M$  along a finite (possibly empty) collection  $\mathcal{T}$  of

disjoint and non-parallel, nor boundary-parallel, incompressible, embedded tori into maximal Seifert fibered or atoroidal compact submanifolds. We call the components of  $M \setminus \mathcal{T}$  the *JSJ-pieces* of  $M$ .

Thurston's geometrization conjecture stated that the atoroidal JSJ-pieces support a hyperbolic or a spherical metric on their interiors. W. Thurston proved his conjecture for any Haken 3-manifold (i.e. a compact, orientable, irreducible and  $\partial$ -irreducible 3-manifold which contains a properly embedded essential surface; for details see [Th1], [Th2]). The full conjecture has been settled recently by G. Perelman [Per] (see [KL], [MT], [CZ], [BBBMP]), hence every compact orientable, irreducible 3-manifold is *geometrizable* in the sense that it satisfies Thurston's geometrization conjecture.

The fact that the targets are geometrizable 3-manifolds is crucial when considering Questions 1 and 2. A consequence is that a positive answer to Question 2 implies a positive answer to Question 1. Furthermore, since the results of [Sol] (see also [Re], [Gu], [BCG]) and [WZ] show that a closed orientable 3-manifold dominates only finitely many 3-manifolds supporting either a hyperbolic structure with finite volume or a Seifert geometry  $\mathbb{H}^2 \times \mathbb{E}^1$ , Question 2 reduces to the following:

**Question 3.** *Let  $M$  be a closed orientable 3-manifold. Does  $M$  dominate at most finitely many, closed, orientable, irreducible 3-manifolds  $N$  with non-trivial JSJ-decomposition?*

There are some partial results for Question 3 in the case of sequences of degree 1 maps (see [Ro1], [So2]), or when the domain and the target have the same simplicial volume (see [So3], [De]). Question 3 is solved in [De] when  $M$  is a graph manifold.

A general approach to Question 3 can be divided into two steps:

- (1) *Finiteness of JSJ-pieces:* show that there is a finite set  $\mathcal{HS}(M)$  of complete hyperbolic 3-manifolds with finite volume and of Seifert manifolds such that each JSJ-piece of a 3-manifold  $N$  dominated by  $M$  belongs to  $\mathcal{HS}(M)$ .
- (2) *Finiteness of gluing:* For a given finite set  $\mathcal{HS}(M)$  of complete hyperbolic 3-manifolds with finite volume and of Seifert manifolds, there are only finitely many ways of gluing elements in  $\mathcal{HS}(M)$  to get closed 3-manifolds dominated by  $M$ .

Notice that with our terminology, a manifold supporting a Sol geometry has a non-trivial JSJ-decomposition with only one piece homeomorphic to a product  $T^2 \times I$  or two pieces homeomorphic to the twisted I-bundle over the Klein bottle. For such manifolds in the target, the finiteness of JSJ-pieces is trivially true, while the finiteness of gluing is much more subtle (see [WZ] for 1-domination and [BBW] if we allow arbitrary non-zero degree).

T. Soma proved the finiteness of hyperbolic JSJ-pieces in [So2]. One of the main results of this paper is to complete the proof of the first step by proving the finiteness of the Seifert fibered JSJ-pieces:

**Theorem 1.1** (Finiteness of JSJ-pieces). *Let  $M$  be a closed, orientable, 3-manifold. Then there is a finite set  $\mathcal{HS}(M)$  of complete hyperbolic 3-manifolds with finite volume and of Seifert fibered 3-manifolds, such that the JSJ-pieces of any closed, orientable, irreducible 3-manifold  $N$  dominated by  $M$  belong to  $\mathcal{HS}(M)$ , provided that  $N$  does not support the geometries of  $\mathbb{S}^3$ ,  $\overline{\mathrm{PSL}_2(\mathbb{R})}$ , Nil.*

The finiteness of the Seifert fibered JSJ-pieces follows from a finiteness result for the Thurston norm of all compact 3-manifolds  $M_S = M \setminus S$ , where  $S$  runs over all incompressible, orientable surfaces (not necessarily connected) in  $M$ , see Section 3.1. This latter result is derived from the finiteness of “patterned guts” of all the manifolds  $M_S = M \setminus S$ , where  $S$  runs over all incompressible, orientable surfaces in  $M$ , which we prove in Section 2.

We also prove the finiteness of gluing when the targets are irreducible, integral homology 3-spheres. Together with Theorem 1.1 (Theorem 4.1) this gives a positive answer to Question 3 when the targets are integral homology spheres.

**Theorem 1.2.** *Any closed orientable 3-manifold dominates only finitely many integral homology 3-spheres.*

Without any restriction on the possible degrees of the maps or on the geometry of the target, this is the best result one can expect, since any closed orientable 3-manifold dominates all 3-dimensional lens spaces, which are rational homology spheres.

Since a degree-one map induces an epimorphism at the level of homology groups, Theorem 1.2 gives a positive answer to Question 1 when the domain is an integral homology sphere:

**Corollary 1.3.** *An integral homology 3-sphere 1-dominates at most finitely many closed 3-manifolds.*

The argument for integral homology spheres can be modified to prove the following corollary.

**Corollary 1.4.** *Any compact orientable 3-manifold dominates at most finitely many knot complements in  $\mathbb{S}^3$ .*

This corollary is related to a question of J. Simon on epimorphisms between knot groups (see [Ki], Problem 1.12 (J. Simon), and Section 6).

The paper is organized as follow: The finiteness of patterned guts is discussed in Section 2; the finiteness of Thurston norm and Gromov volume is discussed in Section 3; the finiteness of JSJ-pieces is proved in Section 4. The last Sections 5 and 6, are devoted to finite domination results when the targets are integral homology 3-spheres or knot complements in  $\mathbb{S}^3$ .

We end the introduction by the following

**Remark 1.5.** We could define the notion of domination between 3-manifolds which are not necessarily orientable in terms of geometric degree [Ep]. But, then there are examples of non-orientable (hyperbolic) 3-manifolds which 1-dominate infinitely many orientable (hyperbolic) 3-manifolds (see [Ro3], [BW]). In Section 3 of [BW], by lifting the maps in those examples to the orientable double cover of the domain, maps between orientable hyperbolic 3-manifolds are produced whose degree is  $1 + (-1) = 0$  rather than 2, as wrongly claimed there. This error has been pointed out by T. Soma and many others.

## 2. Finiteness of patterned guts

In the Jaco–Shalen–Johannson decomposition of a compact orientable 3-manifold along essential tori and annuli, the guts consist of the pieces which are not  $I$ -bundles over surfaces with negative Euler characteristic. Finiteness of guts is a basic principle, which originated from H. Kneser’s work, see for example [A], [Ga2], [JR] for some recent applications of guts in 3-manifold theory. We first introduce the notion of patterned guts needed for our study of non-zero degree maps.

Suppose  $X$  is a  $\partial$ -irreducible and irreducible, compact, orientable 3-manifold. According to Jaco–Shalen–Johannson theory ([Ja], [JS], [Joh]), there is a unique, up to proper isotopy, characteristic 3-submanifold  $\Sigma \subset X$  which is an union of Seifert spaces and  $I$ -bundles.

This characteristic submanifold has a unique decomposition, up to proper isotopy:

$$\Sigma = (\Sigma \setminus IB_X^-) \cup IB_X^-,$$

where  $IB_X^-$  is formed by the components of the Seifert pairs which are  $I$ -bundles over surfaces  $F$ , where  $F$  has negative Euler characteristic  $\chi(F)$  if  $\partial F \neq \emptyset$ . We make the following convention in this paper: if a component of  $X$  is a Seifert manifold and also an  $I$ -bundle over a surface, we will always consider it as an  $I$ -bundle.

Therefore we have a decomposition

$$X = (X \setminus IB_X^-) \cup_{A_X} IB_X^- = G_X \cup_{A_X} IB_X^-,$$

where  $A_X$  is the collection of frontier annuli of  $IB_X^-$  in  $X$ . We call  $G_X = X \setminus IB_X^-$  the guts of  $X$ , and the decomposition above the  $GI$ -decomposition for  $X$ . The embeddings of  $G_X$ ,  $A_X$  and  $IB_X^-$  are unique up to proper isotopy in  $X$ .

Suppose  $S$  is a closed, incompressible surface in an irreducible 3-manifold  $M$ . Then  $M_S = M \setminus S$  is  $\partial$ -irreducible and irreducible. For such a surface  $S$ , we write the  $GI$  decomposition of  $M_S$  as

$$M_S = G_S \cup_{A_S} IB_S^-.$$

**Definition 2.1.** Suppose  $X$  is an orientable, irreducible and  $\partial$ -irreducible 3-manifold. A  $\partial$ -pattern for  $X$  is a finite collection of disjoint annuli  $A \subset \partial X$ , and given  $A$  we say that  $X$  is  $\partial$ -patterned.

**Example 2.2.** For each component  $G$  of  $G_S$ ,  $G \cap A_S$  is a  $\partial$ -pattern for  $G_S$ . We often call the pair  $(G, G \cap A_S)$  a *patterned guts component* for the surface  $S$ .

The main result of this section is the following finiteness result for patterned guts:

**Theorem 2.3** (Patterned guts finiteness). *Let  $M$  be a closed, orientable, irreducible 3-manifold. Then there is a finite set  $\mathcal{G}(M)$  of connected, compact, orientable,  $\partial$ -patterned 3-manifolds such that for each closed, incompressible (not necessarily connected) surface  $S \subset M$ , all patterned guts components of  $(G_S, G_S \cap A_S)$  belong to  $\mathcal{G}(M)$ .*

*Proof.* The proof of Theorem 2.3 consists of three steps.

Step 1. *Construct a first “approximation” to the GI-decomposition by applying a refined Kneser argument.*

Fix a triangulation  $K$  of  $M$ . Suppose that  $K$  has  $t$  tetrahedra. For simplicity, we also assume that  $K$  has only one vertex  $v$  (see [JR] for example). Let  $S_v$  be the normal sphere which is the boundary of a small regular neighborhood  $B_v$  of  $v$ . Suppose that  $S$  is a closed orientable incompressible surface in  $M$ . First deform  $S$  to be a normal surface in  $(M, K)$ . We can assume that  $S \cap S_v = \emptyset$ . Let  $S_* = S \cup S_v$ .

Each tetrahedron  $T$  has seven normal disc types, four triangular types and three quadrilateral types, see Figure 1. Since  $S_*$  contains  $S_v$  and  $S_*$  is embedded, for each tetrahedron  $T$  of  $K$ ,  $T \cap S_*$  contains all four triangular normal disc types but at most one quadrilateral normal disc type.

Let  $M_* = M \setminus B_v$ ,  $K_* = K \cap M_*$ , and  $T_* = T \cap M_*$  for each tetrahedron  $T$  in  $K$ . Then  $K_*$  is a truncated triangulation of  $M_*$ , and each  $T_*$  is a truncated tetrahedron. Now we consider  $S \subset M_* = |K_*|$ .

If  $S \cap T_*$  contains a quadrilateral normal disc, then  $T_* \setminus S$  contains two non-product regions, which are truncated prisms: they are truncated from  $T$  by using this quadrilateral normal disc and four non-parallel triangular normal discs  $S \cap T$ , see Figure 2. The boundary of each such a truncated prism component has seven faces:

- (1) two triangular normal discs which lie in  $S \cup S_v$ ;
- (2) one quadrilateral normal disc which lies in  $S$ ;
- (3) two hexagonal faces which lie in the boundary of  $T$ ;
- (4) two quadrilateral faces which lie in the boundary of  $T$ .

If  $S \cap T_*$  contains no quadrilateral normal disc, then  $T_* \setminus S$  contains just one non-product region, which is a truncated tetrahedron: it is truncated from  $T$  by using

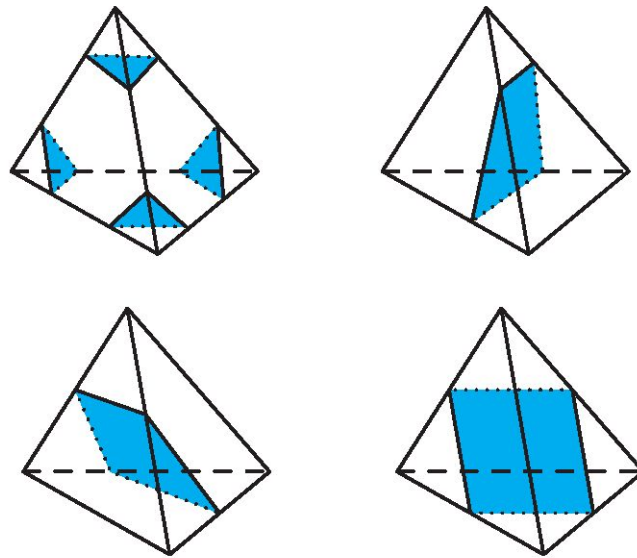


Figure 1

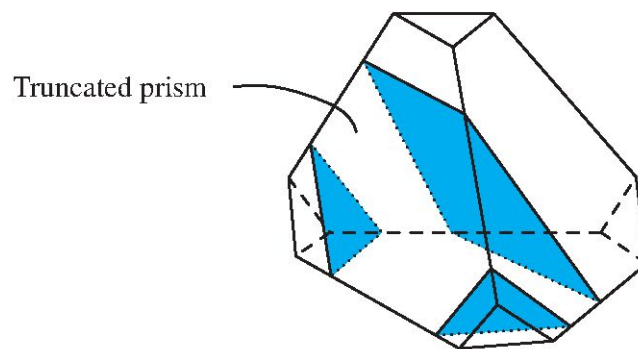


Figure 2

four non-parallel normal discs of triangular type. The boundary of such a truncated tetrahedron component has eight faces:

- (5) four normal discs of triangular type which lie in  $S \cup S_v$ ;
- (6) four hexagonal faces which lie in the boundary of  $T$ .

Note that each remaining component of  $T_* \setminus S$  is a product region, whose boundary is formed by two normal discs of the same triangular (resp. quadrilateral) type and three (resp. four) vertical quadrilateral faces which lie in  $\partial T$ , see Figure 3. Moreover in  $K_* \setminus S$ , each hexagonal face given in (3) or (6) is identified with a hexagonal face given in (3) or (6), and each quadrilateral face given in (4) is either identified with a quadrilateral face given in (4), or with a vertical quadrilateral face of a product region.

Let  $Q$  be a quadrilateral face given in (4). If in  $K_* \setminus S$ ,  $Q$  is identified with a vertical quadrilateral face of a product region, we call  $Q$  a *frontier quadrilateral face*.

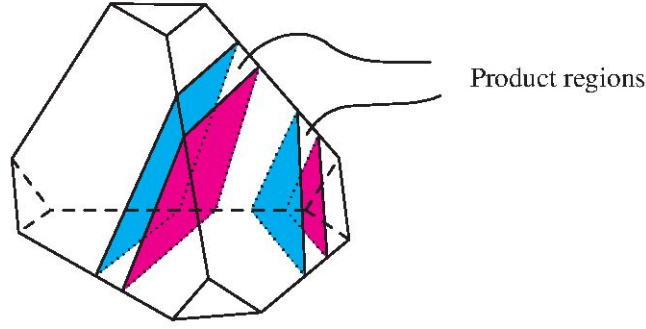


Figure 3

Otherwise we call  $Q$  a *non-frontier quadrilateral face*.

Now we glue together the truncated prism components and the truncated tetrahedron components of  $K_* \setminus S$  along their hexagonal faces and their non-frontier quadrilateral faces to get pieces  $P_1, \dots, P_k$ . Let  $G_S^1$  be the union of those pieces  $P_i$ ,  $i = 1, \dots, k$ .

Note that  $\partial M_{*S} = S_1 \cup S_2 \cup S_v$ , where  $S_1$  and  $S_2$  are two copies of  $S$ , and

$$(i) \quad \partial G_S^1 = (\partial M_{*S} \cap G_S^1) \cup (\text{union of frontier quadrilaterals}).$$

Now we have  $M_* \setminus S = G_S^1 \cup ((M_* \setminus S) \setminus G_S^1)$ . The components of  $(M_* \setminus S) \setminus G_S^1$  are obtained by gluing the product regions along their vertical quadrilateral faces, hence they are  $I$ -bundles, whose union is denoted by  $IB_S^1$ . The set  $IB_S^1$  is a product or a twisted  $I$ -bundle over a compact surface  $S'$ , denoted by  $N(S')$ . Let  $N(\partial S')$  denote the  $I$ -bundle structure restricted to  $\partial S'$ . Then

$$(ii) \quad \partial IB_S^1 = (\partial M_{*S} \cap \partial IB_S^1) \cup N(\partial S').$$

Clearly

$$(iii) \quad \partial M_{*S} = (\partial M_{*S} \cap G_S^1) \cup (\partial M_{*S} \cap \partial IB_S^1).$$

Combining the formulas (i), (ii) and (iii), it follows that the annuli  $N(\partial S')$  in (ii) are identified with the union of frontier quadrilaterals in (i). In conclusion, all those frontier quadrilaterals form the intersection  $IB_S^1 \cap G_S^1$ , which is a union of finitely many properly embedded annuli in  $M_* \setminus S$ , denoted by  $A_S^1$ . We call  $A_S^1$  the frontier annuli of  $G_S^1$  (of  $IB_S^1$ ). Now we get our first “approximation”  $GI$ -decomposition

$$M_* \setminus S = G_S^1 \cup_{A_S^1} IB_S^1.$$

For each  $S \subset M$ ,  $G_S^1$  is constructed from  $n \leq t$  truncated tetrahedra and  $m \leq 2t$  truncated prisms by gluing their hexagonal faces and non-frontier quadrilateral faces in pairs. It follows that there is a bound for the combinatorial (therefore the topological) types of the components of  $G_S^1$ . A very crude bound is  $5^t$ , obtained by

noting that there are 5 choices for each tetrahedron consisting of the empty set, the truncated tetrahedron or one of the 3 possible truncated prisms (note if we have any quadrilateral type we always get two truncated prisms in our guts). Moreover once  $G_S^1$  is formed, the position of the frontier annuli  $A_S^1 \subset G_S^1$  is fixed. Hence we reach the following conclusion:

**Conclusion 2.4.** There is a finite set  $\mathcal{G}^1(M)$  of compact, orientable, connected,  $\partial$ -patterned 3-manifolds such that for each closed, incompressible surface  $S \subset M$ , all patterned components of  $(G_S^1, G_S^1 \cap A_S^1)$  belong to  $\mathcal{G}^1(M)$ .

Step 2. Construct a second “approximation” to the GI-decomposition by absorption of “tiny” patterned 3-manifolds from  $G_S^1 \cup_{A_S^1} IB_S^1$ .

Suppose that a component  $A_i$  (or a pair of components  $A_i$  and  $A_j$ ) of  $A_S^1$  separates a component  $P$  from  $M_* \setminus S$  such that one of the following patterned 3-manifolds occurs:

- (i)  $(P, A_i) = (D^2 \times I, \partial D^2 \times I)$ , or  $((D^2 \times I) \setminus B_v, \partial D^2 \times I)$ ;
- (ii)  $(P, A_i) = (D^2 \times S^1, I \times S^1)$  or  $(P, A_i) = ((D^2 \times S^1) \setminus B_v, I \times S^1)$ , for some interval  $I \subset \partial D^2$ ;
- (iii)  $(P, A_i \cup A_j) = (A \times I, \partial A \times I)$  or  $((A \times I) \setminus B_v, \partial A \times I)$  for some annulus  $A$ .

We call any patterned 3-manifold of one of the types above *tiny*.

Note that a tiny patterned 3-manifold  $P$  may contain other tiny patterned 3-manifolds. Therefore  $P$  may contain (finitely) many components of  $A_S^1$ . But since  $A_S^1$  has finitely many components, there are only finitely many tiny patterned  $P$ .

Let  $P$  be a tiny patterned 3-manifold. We eliminate  $P$  by gluing it to its neighboring component(s) along  $A_i$  (and  $A_j$ ) and then delete from  $A_S^1$  all components of  $A_S^1$  in  $P$ . In this manner, we also eliminate all tiny patterned 3-manifolds contained in  $P$ . In such an absorption process, we get a new decomposition  $G_S^1(1) \cup_{A_S^1(1)} IB_S^1(1)$ : all components in  $G_S^1$  and  $IB_S^1$  which are contained in  $P$  or are adjacent to  $P$  become a new component of  $G_S^1(1) \cup IB_S^1(1)$ , all the remaining components in  $G_S^1$  and  $IB_S^1$  are preserved, and  $A_S^1(1)$  is obtained by removing from  $A_S^1$  all components of  $A_S^1$  in  $P$ . The new component in the new decomposition which contains  $P$  is considered as a “pseudo”  $I$ -bundle (respectively a “pseudo” guts) component if and only if the neighboring components of  $P$  are  $I$ -bundles (respectively guts components).

Now consider the tiny patterned 3-manifolds of the decomposition  $G_S^1(1) \cup_{A_S^1(1)} IB_S^1(1)$  defined as above (which indeed is a sub-collection of the tiny patterned 3-manifolds of  $G_S^1 \cup_{A_S^1} IB_S^1$ ). If there are some, we can continue this absorption process to get a new decomposition  $G_S^1(2) \cup_{A_S^1(2)} IB_S^1(2)$ . Repeating this process we get a sequence of decompositions  $G_S^1(n) \cup_{A_S^1(n)} IB_S^1(n)$ . Since  $A_S^1$  has only finitely many components and the number of components of  $A_S^1(n)$  is strictly decreasing,

this absorption process must stop for some  $n$ . Then we get our second “approximate”  $GI$ -decomposition without tiny patterned 3-manifolds, which is denoted by

$$M_* \setminus S = G_S^2 \cup_{A_S^2} IB_S^2.$$

Now we claim the following:

**Claim 2.5.** *Each annulus in  $A_S^2$  is incompressible and  $\partial$ -incompressible in  $M_* \setminus S$ .*

*Proof.* Suppose some annulus  $A_i \subset A_S^2$  is compressible in  $M_* \setminus S$ . Since  $M_* \setminus S$  is  $\partial$ -irreducible, each component of  $\partial A_i$  bounds a disk in  $\partial(M_* \setminus S)$ . Since  $M \setminus S$  is irreducible,  $A_i$  must separate from  $M_* \setminus S$  either a component homeomorphic to  $D^2 \times I$  or to  $(D^2 \times I) \setminus B_v$ . This contradicts the fact that no  $A_i$  meets the condition (i).

Suppose that some annulus  $A_i \subset A_S^2$  is  $\partial$ -compressible in  $M_* \setminus S$ . Since  $M_* \setminus S$  is irreducible and  $\partial$ -irreducible, it is not difficult to verify that  $A_i$  must separate from  $M_* \setminus S$  a component  $P$  homeomorphic to a solid torus or a punctured solid torus and which meets the condition (ii). This again gives a contradiction.  $\square$

Since  $A_S^2$  is incompressible and  $\partial$ -incompressible in  $M_* \setminus S$ ,  $A_S^2$  does not meet  $S_v$ , and we can plug the ball  $B_v$  back into  $M_* \setminus S$  to get a new “pseudo”  $GI$ -decomposition for  $M \setminus S$ , still denoted by

$$M \setminus S = G_S^2 \cup_{A_S^2} IB_S^2.$$

Let  $m$  be the number of pattern annuli in  $\mathcal{G}^1(M)$ . Let  $\mathcal{P}(M)$  be the set of patterned 3-manifolds consisting of  $m$  copies of a patterned 3-manifold of each type (i), (ii) and (iii), and of one 3-ball. Then the patterned 3-manifolds obtained from  $\mathcal{G}^1(M)$  and  $\mathcal{P}(M)$  by identifying some of their pattern annuli in pairs, and possibly plugging in the 3-ball, is a finite set  $\mathcal{G}^2(M)$  of patterned 3-manifolds. Since  $G_S^2$  is obtained from a subset of  $G_S^1 \subset \mathcal{G}^1(M)$  and a subset of  $\mathcal{P}(M)$  by identifying some of their pattern annuli in pairs, and possibly plugging in the 3-ball, it follows that, up to homeomorphism, the components of  $G_S^2$  belong to  $\mathcal{G}^2(M)$ . Hence we reach the following conclusion:

**Conclusion 2.6.** (1) There is a finite set  $\mathcal{G}^2(M)$  of compact, orientable, connected,  $\partial$ -patterned 3-manifolds such that for each closed, incompressible surface  $S \subset M$ , all patterned components of  $(G_S^2, G_S^2 \cap A_S^2)$  belong to  $\mathcal{G}^2(M)$ ;

(2) Each component of  $IB_S^2$  is an  $I$ -bundle over a surface  $F$  such that  $\chi(F) < 0$  if  $\partial F \neq \emptyset$ . Moreover  $A_S^2$  is incompressible and  $\partial$ -incompressible.

Step 3. *Comparing the decomposition  $G_S^2 \cup_{A_S^2} IB_S^2$  with the  $GI$ -decomposition.*

We recall that  $M_S = G_S \cup_{A_S} IB_S^-$  is the  $GI$ -decomposition. By the embedded version of the enclosing property of the JSJ-decomposition and Conclusion 2.6,  $IB_S^-$  is a sub-I-bundle of  $IB_S^-$  up to a proper isotopy of  $M \setminus S$ . Hence

$$G_S^2 = M_S \setminus IB_S^2 = G_S \cup_{A_S} (IB_S^- \setminus IB_S^2).$$

Suppose  $A_S^2$  has  $m_S$  components. Let  $T^*$  be the once punctured torus and define the patterned 3-manifold  $(P, A) = (T^* \times I, \partial T^* \times I)$ . Let  $M_S^{2*}$  be obtained from  $G_S^2$  and  $m_S$  copies of  $(P, A)$  by identifying each frontier annulus of  $G_S^2$  with a frontier annulus of  $P$ . Then  $M_S^{2*}$  is boundary irreducible and is uniquely determined by  $G_S^2$ . In particular there are finitely many topological types of  $M_S^{2*}$  for all incompressible surfaces  $S \subset M$  by Conclusion 2.6 (1).

Let  $M_S^{2*} = G_S^* \cup_{A_S^*} IB^{*-}$  be the  $GI$ -decomposition, which is unique up to isotopy. Hence there are finitely many topological types of  $G_S^*$  for all incompressible surfaces  $S \subset M$ . It is not difficult to see that  $(G_S^*, A_S^*) = (G_S, A_S)$  for each incompressible surface  $S \subset M$ . Hence Theorem 2.3 is proved.  $\square$

**Definition 2.7.** Let  $M$  be a closed orientable irreducible 3-manifold. Define:

$$\mathcal{M} = \{M_S, \tilde{M}_S, \text{ where } S \text{ runs over all incompressible surfaces in } M, \\ \text{and } \tilde{M}_S \text{ runs over all double coverings of } M_S\}.$$

Since each compact 3-manifold has only finitely many double coverings, the main results in Sections 2 and 3 and their proofs imply the following corollary:

**Corollary 2.8.** *Let  $M$  be a closed, irreducible 3-manifold. Then there is a finite set  $\tilde{\mathcal{G}}(M)$  of connected compact  $\partial$ -patterned 3-manifolds such that for any  $X \in \mathcal{M}$ , each component of the patterned guts  $(G_X, A_X)$  belongs to  $\tilde{\mathcal{G}}(M)$ .*

### 3. Thurston norm and Gromov volume

**3.1. Finiteness of the Thurston norm.** We first give a brief description of the Thurston norm on the second relative homology group  $H_2(X, Y; \mathbb{Z})$  of a compact, orientable 3-manifold  $X$ , where  $Y \subset \partial X$  is a subsurface.

Thurston [Th3] introduced a pseudo norm on  $H_2(X, Y; \mathbb{Z})$  using the fact that any homology class  $z \in H_2(X, Y; \mathbb{Z})$  can be represented by a properly embedded oriented surface  $(F, \partial F) \hookrightarrow (X, Y)$ . Set  $\chi_-(F) = \max\{0, -\chi(F)\}$  if  $F$  is connected, otherwise let  $\chi_-(F) = \sum \chi_-(F_i)$ , where  $F_i$  are the components of  $F$ . Then for an integral class  $z \in H_2(X, Y; \mathbb{Z})$ , the *Thurston norm*  $\|z\|$  of  $z$  is defined as

$$\|z\| = \inf \{ \chi_-(F) : F \text{ is an embedded closed orientable surface} \\ \text{representing the homology class } z \text{ in } H_2(X, Y; \mathbb{Z}) \}.$$

Thurston then shows that  $\|\cdot\|$  extends to a convex pseudo-norm on  $H_2(X, Y; \mathbb{R})$  which is linear on rays through the origin. The Thurston norm turned out to be very useful in the study of the topology of 3-dimensional manifolds.

In [Ga1] (see also [Pe]) Gabai shows that to define the Thurston norm, one can replace “embedded surfaces” by “singular surfaces” and still get the same norm.

**Definition 3.1.** Let  $X$  be a compact, orientable 3-manifold and  $Y \subset \partial X$  be a sub-surface. For a finite set of elements  $\alpha = \{a_1, \dots, a_k\}$  of  $H_2(X, Y; \mathbb{Z})$ , we define

$$TN(\alpha) = \max\{\|a_i\|, i = 1, \dots, k\}.$$

Then we define

$$TN(X, Y) = \min\{TN(\alpha) \mid \alpha \text{ runs over all finite sets of elements of } H_2(X, Y; \mathbb{Z}) \text{ which generate } H_2(X, Y; \mathbb{Q})\}$$

to be the Thurston norm of the pair  $(X, Y)$ .

**Lemma 3.2.** *Let  $p: (\tilde{X}, \tilde{Y}) \rightarrow (X, Y)$  be a proper non-zero degree map. Then  $TN(\tilde{X}, \tilde{Y}) \geq TN(X, Y)$ .*

*Proof.* Suppose  $\alpha = \{a_1, \dots, a_k\} \subset H_2(\tilde{X}, \tilde{Y}; \mathbb{Z})$  generates  $H_2(\tilde{X}, \tilde{Y}; \mathbb{Q})$ . Let  $(S_i, \partial S_i) \subset (\tilde{X}, \tilde{Y})$  be a proper surface which presents  $a_i$  and realizes its Thurston Norm.

Clearly  $p(\alpha) = \{p(a_1), \dots, p(a_k)\} \subset H_2(X, Y; \mathbb{Z})$ . Since non-zero degree maps induce surjections on rational homology,  $p(\alpha) = \{p(a_1), \dots, p(a_k)\}$  generates  $H_2(X, Y; \mathbb{Q})$ . Now each  $p(S_i)$  is a singular surface representing  $p(a_i)$ . By Gabai’s result [Ga1], it follows that  $\|p(\alpha)\| \leq \|\alpha\|$ , and therefore Lemma 3.2 is derived.  $\square$

Recall for each closed incompressible surface  $S \subset M$ , we have the  $GI$ -decomposition  $M_S = G_S \cup_{A_S} IB_S^-$ .

**Lemma 3.3.** *There is a double cover  $\tilde{M}_S = \tilde{G}_S \cup_{\tilde{A}_S} \widetilde{IB_S^-}$  of  $M_S = G_S \cup_{A_S} IB_S^-$  such that each component of  $\widetilde{IB_S^-}$  is a product of an orientable surface with the interval.*

*Proof.* An elementary fact is that each compact non-orientable surface  $F$  is doubly covered by an orientable surface  $\tilde{F}$  such that the restriction on each component of  $\partial \tilde{F}$  is a homeomorphism. It follows that each twisted  $I$ -bundle  $B$  over compact non-orientable surface  $F$  is doubly covered by a product  $I$ -bundle  $\tilde{B} = \tilde{F} \times I$  such that the restriction on each component of  $\partial \tilde{B} \times I$  is a homeomorphism.

For each twisted  $I$ -bundle component  $B$  of  $(IB_S^-, A_S)$ , pick a double covering given in the first paragraph, and for each remaining component of  $(IB_S^-, A_S)$  and

each component of  $(G_S, A_S)$ , pick two identical patterned copies of it. Obviously we can glue them together to get a double cover  $p: \tilde{M}_S \rightarrow M_S$ . Let  $\tilde{G}_S$ ,  $\tilde{A}_S$  and  $\widetilde{IB_S^-}$  be the pre-images of  $G_S$ ,  $A_S$  and  $IB_S^-$ , then one can verify from the definitions that  $\tilde{M}_S = \tilde{G}_S \cup_{\tilde{A}_S} \widetilde{IB_S^-}$  is the  $GI$ -decomposition of  $\tilde{M}_S$  and verifies the desired property.  $\square$

Now we are going to prove the main result of this section.

**Theorem 3.4** (Finiteness of the Thurston norm). *Let  $M$  be an irreducible, closed, orientable 3-manifold. Then  $TN(M_S, \partial M_S)$  takes at most finitely many values, when  $S$  runs over all closed, incompressible surfaces embedded in  $M$ .*

*Proof.* By Lemmas 3.2, we need only to prove Theorem 3.4 for double coverings  $(\tilde{M}_S, \partial \tilde{M}_S)$  provided by Lemma 3.3, for all incompressible surfaces  $S \subset M$ . For simplicity we still use  $M_S = G_S \cup_{A_S} IB_S^-$  to denote  $\tilde{M}_S = \tilde{G}_S \cup_{\tilde{A}_S} \widetilde{IB_S^-}$ . Then by Corollary 2.8 there are only finitely many topological types of patterned guts  $(G_S, A_S)$  for all incompressible surfaces  $S \subset M$ . Hence the number of components of  $A_S$  is uniformly bounded. Again by Lemma 3.3, each component of  $IB_S^-$  is a product of an orientable surface with the interval.

We first modify the decomposition so that the gluing annuli between the two parts become separating. For each component  $N(F)$  of  $IB_S^-$  we choose a curve in the interior of the base surface  $F$ , which co-bounds a planar subsurface  $Q$  together with all the boundary components of  $F$ . Since the number of boundary components of  $F$  is bounded by the number of components of  $A_S$ ,  $|\chi(Q)|$  is uniformly bounded above, for all incompressible surfaces  $S \subset M$ . Then we consider the new decomposition  $M = G'_S \cup_{A'_S} IB'_S$ , where  $G'_S$  is obtained by gluing to  $G_S$  the handlebodies  $N(Q)$  along the components of  $A_S$ , and  $IB'_S$  is the sub- $I$ -bundle of  $IB_S^-$  corresponding to the subsurfaces  $F - \text{int}(Q)$ . The gluing annuli  $A'_S$  are the separating annuli of  $N(\partial Q) - A_S$ , using our previous convention that  $N(Q)$  and  $N(\partial Q)$  are the  $I$ -bundle restricted to  $Q$  and  $\partial Q$  respectively.

For a given patterned guts  $(G_S, A_S)$ , there are only finitely many positive integer solutions  $\{m_1, \dots, m_k\}$  such that  $m_1 + \dots + m_k = m$  where  $m$  is the number of components of  $A_S$ , and for any such solution  $\{m_1, \dots, m_k\}$ , there are only finitely many ways to distribute  $m$  elements into  $k$  groups of cardinality  $m_1, \dots, m_k$  respectively. Hence by the construction and Theorem 2.3, there are only finitely many topological types of  $\partial$ -patterned 3-manifolds  $(G'_S, A'_S)$  for all incompressible surfaces  $S$  in  $M$ . Then the finiteness for the values of  $TN(M_S, \partial M_S)$  is a direct consequence of the following lemma.  $\square$

**Lemma 3.5.** *Let  $S \subset M$  be a closed incompressible surface, then  $TN(M_S, \partial M_S) \leq TN(G'_S, \partial G'_S \setminus \text{int} A'_S)$ .*

*Proof.* We consider the following natural homomorphisms induced by the inclusion maps:

$$\begin{aligned}\phi &: H_2(G'_S, \partial G'_S \setminus \text{int} A'_S; \mathbb{Z}) \rightarrow H_2(M_S, \partial M_S; \mathbb{Z}); \\ \psi &: H_2(IB'_S, \partial IB'_S \setminus \text{int} A'_S; \mathbb{Z}) \rightarrow H_2(M_S, \partial M_S; \mathbb{Z}).\end{aligned}$$

By applying the relative Mayer–Vietoris sequence (see [Do], page 52) to the pairs  $(G'_S, \partial G'_S \setminus \text{int} A'_S)$  and  $(IB'_S, \partial IB'_S \setminus \text{int} A'_S)$ , one gets the exact sequence

$$\begin{aligned}\cdots &\longrightarrow H_2(A'_S, \partial A'_S; \mathbb{Z}) \\ &\longrightarrow H_2(G'_S, \partial G'_S \setminus \text{int} A'_S; \mathbb{Z}) \oplus H_2(IB'_S, \partial IB'_S \setminus \text{int} A'_S; \mathbb{Z}) \\ &\longrightarrow H_2(M_S, \partial M_S; \mathbb{Z}) \longrightarrow H_1(A'_S, \partial A'_S; \mathbb{Z}) \\ &\longrightarrow H_1(G'_S, \partial G'_S \setminus \text{int} A'_S; \mathbb{Z}) \oplus H_1(IB'_S, \partial IB'_S \setminus \text{int} A'_S; \mathbb{Z}) \longrightarrow \cdots.\end{aligned}$$

We first show the injectivity of the homomorphism

$$H_1(A'_S, \partial A'_S; \mathbb{Z}) \rightarrow H_1(G'_S, \partial G'_S \setminus \text{int} A'_S; \mathbb{Z}) \oplus H_1(IB'_S, \partial IB'_S \setminus \text{int} A'_S; \mathbb{Z}).$$

To do this we need only to show the injectivity of each homomorphism

$$H_1(A, \partial A; \mathbb{Z}) \rightarrow H_1(IB'_A, \partial IB'_A \setminus \text{int} A; \mathbb{Z}),$$

where  $A$  is a component of  $A'_S$ , and  $IB'_A$  is the component of  $IB'_S$  containing  $A$ .

Note  $H_1(A, \partial A; \mathbb{Z}) = \mathbb{Z}$  is generated by any arc in  $A$  connecting the two components of  $\partial A$ , and  $IB'_A = F \times [0, 1]$ , where  $F$  is an orientable surface with  $\partial F \times [0, 1] = A$ . Let  $F^*$  be a proper oriented surface of  $\partial F \times [0, 1]$ . Then it is a direct geometric observation that the number of times that  $\partial F^*$  crosses  $A$  from  $\partial F \times \{0\}$  to  $\partial F \times \{1\}$  and from  $\partial F \times \{1\}$  to  $\partial F \times \{0\}$  must be the same. This shows the required injectivity.

Then by the exact sequence we have that

$$\partial_*: H_2(M_S, \partial M_S; \mathbb{Z}) \rightarrow H_1(A'_S, \partial A'_S; \mathbb{Z})$$

is null, and thus we get an epimorphism:

$$\phi + \psi: H_2(G'_S, \partial G'_S \setminus \text{int} A'_S; \mathbb{Z}) \oplus H_2(IB'_S, \partial IB'_S \setminus \text{int} A'_S; \mathbb{Z}) \rightarrow H_2(M_S, \partial M_S; \mathbb{Z}).$$

It is clear that  $H_2(IB'_S, \partial IB'_S \setminus \text{int} A'_S; \mathbb{Z})$  has a basis  $\gamma = \{c_1, \dots, c_m\}$  which is formed by a set of vertical annuli, whose Thurston norm vanishes. Hence for any generating set

$$\beta = \{b_1, \dots, b_n\}$$

of  $H_2(G'_S, \partial G'_S \setminus \text{int} A'_S; \mathbb{Z})$ ,

$$\alpha = \{\phi(b_1), \dots, \phi(b_n), \psi(c_1), \dots, \psi(c_m)\}$$

is a generating set of  $H_2(M_S, \partial M_S; \mathbb{Z})$ . It follows that  $TN(M_S, \partial M_S) \leq TN(\alpha) \leq TN(\beta)$ , since by the definition of Thurston norm  $\|\phi(b_i)\| \leq \|b_i\|$  and  $0 \leq \|\phi(c_j)\| \leq \|c_j\| = 0$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . Therefore  $TN(M_S, \partial M_S) \leq TN(G'_S, \partial G'_S \setminus \text{int} A'_S)$ .  $\square$

**3.2. Finiteness of absolute Gromov volume.** This section will not be used in the rest of the paper, but it provides a finiteness result for the absolute Gromov volumes of the compact manifolds  $M_S$ , analogous to the one for their Thurston norms.

First we recall the basic definitions about Gromov's simplicial volume (see [Gr]).

**Definition 3.6.** Let  $X$  be a compact orientable 3-manifold with boundary. Define the *relative Gromov volume*  $|X, \partial X|$  by

$$|X, \partial X| = \inf \left\{ \sum_{i=1}^n |\lambda_i| \mid \sum_{i=1}^n \lambda_i \sigma_i \text{ is a cycle representing a fundamental class in } H_3(X, \partial X; \mathbb{R}), \text{ where } \sigma_i: \Delta^3 \rightarrow X \text{ is a singular simplex and } \lambda_i \in \mathbb{R}, i = 1, \dots, n. \right\}.$$

A fundamental class in  $H_3(X, \partial X; \mathbb{R})$  is the image of any of the  $2^k$  fundamental classes in  $H_3(X, \partial X; \mathbb{Z})$  under the coefficient homomorphism, where  $k$  is the number of connected components of  $X$ .

For a manifold with non-empty boundary, there is another way of defining a simplicial volume, that we call the *absolute Gromov volume*.

**Definition 3.7.** Let  $X$  be a compact orientable 3-manifold with boundary and let  $D(X)$  be the double of  $X$ , obtained by identifying two copies of  $X$  along their boundary via the identity map. The *absolute Gromov volume* of  $X$ , denoted by  $|X|$ , is defined to be half of the Gromov volume of the closed manifold  $D(X)$ .

By the definitions of these two volumes, one has:  $|X| \leq |X, \partial X|$ . Moreover by [So5] and [Ku] they are equal if and only if  $\partial X = \emptyset$  or  $\chi(\partial X) \geq 0$ .

For example, let  $(X, A)$  be a patterned 3-manifold and let  $D_A(X)$  be the compact 3-manifold obtained by doubling  $X$  along the portion  $\partial X \setminus A$  of its boundary. Since  $\partial D_A(X)$  is a collection of tori, one has  $|D_A(X)| = |D_A(X), \partial D_A(X)|$ .

The following finiteness result holds for the absolute Gromov volume, while it is false for the relative Gromov volume.

**Proposition 3.8** (Finiteness of Gromov volume). *Let  $M$  be an irreducible, compact, orientable 3-manifold. Then the absolute Gromov volume  $|M \setminus S|$  takes only finitely many values for all incompressible surfaces  $S \subset M$ .*

*Proof.* For an incompressible surface  $S \subset M$ , we consider the *GI*-decomposition  $M_S = G_S \cup_{A_S} IB_S^-$ . By Conclusion 2.6 (2) in the proof of Theorem 2.3,  $\partial D(A_S)$  is a collection of incompressible tori in  $D(M_S)$ . Since the relative Gromov volume is additive under gluing along incompressible tori (see [So5]), we have:

$$|D(M_S)| = |D_A(G_S), \partial D_A(G_S)| + |D_A(IB_S^-), \partial D_A(IB_S^-)|.$$

Moreover the relative Gromov volume of  $D_A(IB_S^-)$  vanishes, because  $D_A(IB_S^-)$  is homeomorphic to an  $S^1$ -bundle (see [Gr], [Th1], Chapter 6) and the relative Gromov

volume of  $D_A(G_S)$  equals its absolute Gromov volume. Therefore we get  $|M_S| = \frac{1}{2}|D_A(G_S)|$ .

Now Theorem 2.3 shows that there are only finitely many possible topological type for  $D_A(G_S)$  when  $S$  runs over all incompressible surfaces  $S$  in  $M$ , and hence Proposition 3.8 follows.  $\square$

**Example 3.9.** We give an example of a closed orientable 3-manifold  $M$  such that the relative Gromov volume  $|M_S, \partial M_S|$  is unbounded when  $S$  runs over all incompressible surfaces  $S$  in  $M$ . Let  $X$  be a knot exterior in  $\mathbb{S}^3$  which contains incompressible Seifert surfaces of arbitrarily high genus (such examples exist and can even be hyperbolic, see [Gust]). Then the closed manifold  $M = D(X)$  contains non-separating incompressible closed surfaces  $S_n$ , with  $\chi_-(S_n)$  tending to infinity with  $n$ , formed by doubling the Seifert surfaces. When  $M$  is split open along such a surface, by definition of the relative Gromov volume, one has:  $|M_{S_n}, \partial M_{S_n}| \geq 2\chi_-(\partial M_{S_n}) = 4\chi_-(S_n)$ . Hence  $|M_{S_n}, \partial M_{S_n}|$  tends to infinity with  $n$ .

#### 4. Local domination

In this section we prove the finiteness of the JSJ-pieces for manifolds which are dominated by a given compact, orientable 3-manifold. We recall the statement that we are going to prove:

**Theorem 4.1** (Finiteness of JSJ-pieces). *Let  $M$  be a closed, orientable, 3-manifold. Then there is a finite set  $\mathcal{HS}(M)$  of complete hyperbolic 3-manifolds with finite volume and of Seifert fibered 3-manifolds, such that the JSJ-pieces of any closed, orientable, irreducible 3-manifold  $N$  dominated by  $M$  belong to  $\mathcal{HS}(M)$ , provided that  $N$  is not supporting the geometries of  $\mathbb{S}^3$ ,  $\overline{\text{PSL}_2(\mathbb{R})}$ , Nil.*

By [BW], Proposition 3.3, we can find an irreducible (even hyperbolic) closed, orientable 3-manifold which 1-dominates  $M$ . Hence in the remainder of the proof, we may assume that  $M$  is irreducible.

Let  $M$  be a closed orientable irreducible 3-manifold. By Haken's finiteness theorem, there is a maximum number  $h(M)$  of pairwise disjoint, non-parallel, closed, connected, incompressible surfaces embedded in  $M$ . The following elementary fact (see [W1] for example) will be used in this section and the next ones.

**Lemma 4.2.** *Let  $M$  and  $N$  be two closed, irreducible and orientable 3-manifolds. If  $M$  dominates  $N$ , then  $h(M) \geq h(N)$ .*

Let  $\Gamma(N)$  be the dual graph associated with the JSJ-decomposition of  $N$ . This graph has one vertex for each Seifert piece or piece with a hyperbolic metric of finite

volume and one edge for each incompressible torus boundary component of either type of piece. If  $M$  dominates  $N$ , then  $h(M)$  gives an upper bound for the number of edges of  $\Gamma(N)$ , by Lemma 4.2. Hence the number of JSJ-pieces of  $N$ , which is the number of vertices of  $\Gamma(N)$ , is bounded above by  $h(M) + 1$ . Therefore to prove Theorem 4.1, we need only show that the JSJ-pieces of all 3-manifolds  $N$  dominated by a closed, orientable 3-manifold  $M$  admit only finitely many topological types.

Recall the definition:  $\mathcal{M} = \{M_S, \tilde{M}_S\}$ , where  $S$  runs over all incompressible surfaces in  $M$ , and  $\tilde{M}_S$  runs over all double coverings of  $M_S$ .

By the proof of Theorem 3.4, we have:

**Corollary 4.3.**  $\sup\{TN(X, \partial X) \mid X \in \mathcal{M}\} \leq L(M)$  for some constant  $L(M) > 0$  depending only on  $M$ .  $\square$

**Proposition 4.4.** For a given integer  $L > 0$ , there is a finite set  $\mathcal{S}(L)$  of compact Seifert 3-manifolds such that if a Seifert manifold  $N$  with non-empty boundary and orientable base is dominated by a compact orientable 3-manifold  $P$  with  $TN(P, \partial P) \leq L$ , then  $N$  belongs to  $\mathcal{S}(L)$ .

*Proof.* Each homology class  $y$  of  $H_2(N, \partial N; \mathbb{Z})$  can be represented by an orientable incompressible and  $\partial$ -incompressible surface. Since  $N$  is an irreducible Seifert manifold, each incompressible surface is properly isotopic to either a vertical torus or annulus (foliated by Seifert circles), or a horizontal surface (transverse to all Seifert circles) (cf. [Ja], Chapter VI). Since  $\partial N \neq \emptyset$ ,  $N$  always admits horizontal surfaces.

Let  $\mathcal{O}$  be the orbifold base of  $N$  and  $h$  be a regular fiber of  $N$ . Suppose also that  $\mathcal{O}$ ,  $h$  and  $N$  are compatibly oriented. Let  $F$  be a horizontal surface of  $N$  and  $p: F \rightarrow \mathcal{O}$  the branched covering, induced by the restriction to  $F$  of the projection of  $N$  onto its base. Since  $\mathcal{O}$  is oriented, so is  $F$ . Note that the Euler characteristic  $\chi(\mathcal{O})$  is computed for an orbifold, so that each exceptional fiber of multiplicity  $n$  gives a term  $\frac{1}{n} - 1$ . Then we have  $\chi(F) = |d| \times \chi(\mathcal{O}) < 0$ , where  $d = \deg(p) \neq 0$  is equal to the algebraic intersection number  $[F] \cdot [h]$  of  $F$  and  $h$ . Up to reversing the orientation of  $F$ , one can always assume that  $d = [F] \cdot [h] > 0$ . Note that the geometric intersection number  $|F \cap h|$  of  $F$  and  $h$  (i.e. the minimal number of intersection points between  $F$  and  $h$  up to ambient isotopy) is precisely the absolute value of the algebraic intersection number  $[F] \cdot [h]$ .

Suppose further that  $F$  has minimal genus among all horizontal surfaces in  $N$ . If  $\chi(F) \geq 0$ ,  $F$  is a disc or an annulus, and thus  $N$  can be homeomorphic only to a solid torus, an  $S^1$ -bundle over the annulus or a twisted  $S^1$ -bundle over a Möbius band. Assuming that these three Seifert manifolds belong to  $\mathcal{S}(L)$ , we can suppose furthermore that  $\chi(F) < 0$ .

Let  $\|\cdot\|_N$  (resp.  $\|\cdot\|_P$ ) be the Thurston norm on  $H_2(N, \partial N; \mathbb{Z})$  (resp.  $H_2(P, \partial P; \mathbb{Z})$ ). Note that  $H_2(N, \partial N; \mathbb{Z})$  is torsion free and therefore it is precisely the integer lattice of  $H_2(N, \partial N; \mathbb{R})$ . Let  $V = \{y \in H_2(N, \partial N; \mathbb{Z}); \|y\|_N = 0\}$ . By the discussion above,  $V$  is the sublattice of  $H_2(N, \partial N; \mathbb{Z})$  generated by the vertical tori and annuli.

**Lemma 4.5.**  $H_2(N, \partial N; \mathbb{Z}) = \langle [F] \rangle \oplus V$ .

*Proof.* Pick any homology class  $y \in H_2(N, \partial N; \mathbb{Z})$ . If  $\|y\|_N = 0$ , then  $y \in V$ . Suppose  $\|y\|_N \neq 0$ . Let  $S$  be an orientable, incompressible and  $\partial$ -incompressible surface representing  $y = [S]$  with  $-\chi(S) = \|y\|_N$ . Since  $\chi(S) = -\|y\|_N < 0$ , after a proper isotopy we may assume that  $S$  is horizontal and that  $[S].[h] = |S \cap h| > 0$  (otherwise we replace  $y$  by  $-y$ ). Let  $\ell \geq 1$  be the integral part of  $[S].[h]/[F].[h]$ . Then  $(\ell + 1)([F].[h]) > [S].[h] \geq \ell([F].[h])$ , that is  $|F \cap h| = [F].[h] > [S - \ell F].[h] \geq 0$ .

If the homology class  $[S - \ell F]$  does not belong to  $V$ , then it can be represented by a horizontal surface  $S'$  such that

$$\|[S - \ell F]\|_N = -\chi(S') = -([S - \ell F].[h])\chi(\mathcal{O}) > -[F].[h]\chi(\mathcal{O}) = \|[F]\|_N.$$

This would contradict the minimality of the genus of  $F$  among all horizontal surfaces in  $N$ . Therefore  $[S - \ell F] \in V$  and  $y = [\ell F] + [S - \ell F]$ .  $\square$

By hypothesis, there is a compact, orientable 3-manifold  $P$  with  $TN(P, \partial P) \leq L$  and a non-zero degree map  $f: P \rightarrow N$ . Let further  $\alpha = \{z_1, \dots, z_m\}$  be a basis of  $H_2(P, \partial P; \mathbb{Z})$  realizing  $TN(P)$ :  $\max\{\|z_i\|_P; i = 1, \dots, m\} \leq L$ . For  $i = 1, \dots, m$ , let  $S_i$  be a properly embedded surface in  $P$  representing  $z_i$  with  $-\chi(S_i) = \|z_i\|_P$ .

For  $i = 1, \dots, m$ , we set  $y_i = [f(S_i)] = \ell_i[F] + v_i \in H_2(N, \partial N; \mathbb{Z})$ , where  $v_i \in V$ . By the triangle inequality and the fact that  $\|v_i\|_N = 0$ , we get

$$|\ell_i| \|[F]\|_N = \|\ell_i[F]\|_N = \|y_i - v_i\|_N \leq \|y_i\|_N + \|v_i\|_N = \|y_i\|_N.$$

By [Gal] (see also [Pe]) the Thurston norm  $\|y_i\|_N$  can be calculated using singular surfaces, therefore  $\|y_i\|_N \leq -\chi(S_i) = \|z_i\|_P \leq L$ . Combining the two inequalities, we have  $\ell_i \|[F]\|_N \leq L$  for  $i = 1, \dots, m$ .

Since  $f: P \rightarrow N$  has non-zero degree,  $f_*(H_2(P, \partial P; \mathbb{Z}))$  has finite index in  $H_2(N, \partial N; \mathbb{Z})$  and thus it cannot lie in  $V$ . Therefore there is some index  $i \in \{1, \dots, m\}$  with  $|\ell_i| \geq 1$ . It follows that  $\|[F]\|_N \leq L$ , hence the horizontal surface  $F$  can have only finitely many topological types, up to homeomorphism.

Cutting the Seifert manifold  $N$  along the horizontal surface  $F$ , we obtain a product  $F \times I$ , since the base  $\mathcal{O}$  and the surface  $F$  are orientable. Therefore  $N$  can be presented as a surface bundle over  $S^1$  with fiber  $F$  and orientation preserving monodromy  $g: F \rightarrow F$ . Since  $N$  is Seifert fibered,  $g$  must be a periodic map [Ja]. Chapter VI. However, up to conjugacy, a given compact surface admits only finitely many periodic homeomorphisms. Since any two conjugate monodromy maps define homeomorphic 3-manifolds, there are only finitely many possible homeomorphism types of Seifert manifolds  $N$  for a given compact surface  $F$ . Since  $F$  has only finitely many topological types, the proof of Proposition 4.4 is complete.  $\square$

*Proof of Theorem 4.1.* We assume first that  $N$  is not a Seifert manifold. By Soma's results ([So1], [So2]), we know that Theorem 4.1 holds for hyperbolic JSJ-pieces. Since  $N$  is not Seifert fibered, we may assume that  $N$  has a non-empty JSJ-family of tori  $\mathcal{T}$  and we have only to consider the Seifert fibered JSJ-pieces.

Let  $f: M \rightarrow N$  be a map of non-zero degree. After a homotopy of  $f$  we may assume that  $f^{-1}(\mathcal{T})$  is a non-empty collection of disjoint non-parallel closed incompressible surfaces in  $M$ . Let  $\mathcal{M}_f$  be the union of all components of  $M \setminus f^{-1}(\mathcal{T})$  and of all their double coverings. By definition we have  $\mathcal{M}_f \subset \mathcal{M}$ .

Let  $N_i \subset N$  be a Seifert fibered JSJ-piece. Then  $N_i$  is dominated by at least one component  $M_i$  of  $M \setminus f^{-1}(\mathcal{T})$ . Then the finiteness of such JSJ-pieces  $N_i$  with an orientable base follows immediately from Corollary 4.3 and Proposition 4.4.

If  $N_i$  has a non-orientable base orbifold, let  $\tilde{N}_i$  be the unique double cover of  $N_i$  which is Seifert fibered with an orientable base. Then a standard argument shows that a double cover  $\tilde{M}_i$  of  $M_i$  dominates  $\tilde{N}_i$ . Thus we get the finiteness of such 3-manifolds  $\tilde{N}_i$  as above from Corollary 4.3 and Proposition 4.4. Since any involution on such Seifert manifolds  $N_i$  is conjugate to a fiber preserving one by [MS], there are only finitely many conjugacy classes of involutions on each  $\tilde{N}_i$ . This implies the finiteness of the Seifert JSJ-pieces  $N_i$ .

The finiteness of Seifert manifolds  $N$  supporting a product geometry  $\mathbb{H}^2 \times \mathbb{R}$  follows also from Corollary 2.8 and Proposition 4.4 as above, and thus Theorem 4.1 is proved (see also [WZ]).  $\square$

Using a standard doubling construction, Theorem 4.1 can be extended to the following case where the 3-manifold targets have toric boundary.

**Corollary 4.6.** *Let  $M$  be a compact, orientable, 3-manifold. Then there is a finite set  $\mathcal{HS}(M)$  of complete hyperbolic 3-manifolds with finite volume and of Seifert fibered 3-manifolds, such that the JSJ-pieces of any compact, orientable, irreducible, 3-manifold  $N$  with non-empty toric boundary, dominated by  $M$  belong to  $\mathcal{HS}(M)$ .*

*Proof.* If  $N$  has non-empty boundary, so does  $M$ . Let  $D(N)$  be the double of  $N$ , obtained by gluing two copies of  $N$  along their boundaries via the identity map. Then the double  $D(M)$  of  $M$  dominates  $D(N)$ . Since the boundary of  $N$  is a collection of tori, the JSJ-pieces of  $D(N)$  are either exactly those of  $N$  and consist of two copies of hyperbolic and Seifert pieces in the JSJ-decomposition of  $D(N)$ , or there are new Seifert fibered pieces obtained by doubling some Seifert fibered pieces of  $N$  along some of their boundary tori. In any case, the finiteness of the JSJ-pieces of  $D(N)$  implies the finiteness of the JSJ-pieces of  $N$ .  $\square$

**Remark 4.7.** A similar double construction argument shows that finiteness of closed targets implies finiteness of the targets in the setting of compact orientable 3-manifolds. First finiteness of irreducible and  $\partial$ -irreducible compact targets implies finiteness of

compact targets. Since the double  $D(M)$  of an irreducible and  $\partial$ -irreducible compact 3-manifold  $M$  is Haken, there are, up to conjugacy, only finitely many involutions with 2-dimensional fixed point set on  $D(M)$  by [To] and the proof of the geometrization conjecture for Haken manifolds. Therefore only finitely many irreducible and  $\partial$ -irreducible compact 3-manifolds have homeomorphic doubles.

## 5. Integral homology spheres

The main result of this section gives a positive answer to Question 3 when the targets are integral homology spheres. It implies a positive answer to Question 2 when the targets are integral homology spheres and to Question 1 when the domain is an integral homology sphere.

**Theorem 1.2.** *Any closed orientable 3-manifold dominates at most finitely many integral homology spheres.*

Let us fix  $M$  as a closed orientable 3-manifold. As in the previous section, we may assume for the remainder of the proof that  $M$  is irreducible.

First we reduce the proof to the case where the target homology sphere  $N$  is irreducible. As in the previous section, the preimage of a collection of separating essential spheres associated with the prime decomposition of  $N$  can be assumed to be incompressible, disjoint and non-parallel surfaces in  $M$ . Hence there are at most  $h(M) + 1$  prime factors. Moreover by pinching all the prime factors except one to a point, it follows that each prime factor is dominated by  $M$ . Hence we have only to show the finiteness of the set  $\mathcal{D}(M)$  of homeomorphism classes of irreducible, integral homology 3-spheres  $N$  which are dominated by  $M$ .

A *slope* on a torus  $T$  is an isotopy class of essential simple closed curves. The set of slopes on  $T$  corresponds bijectively with  $\pm$ -classes of primitive elements of  $H_1(T; \mathbb{Z})$ .

Given a slope  $\alpha$  on a torus boundary component  $T$  of a 3-manifold  $Y$ , the  $\alpha$ -Dehn filling of  $Y$  with slope  $\alpha$  is the 3-manifold  $Y(\alpha) := (S^1 \times D^2) \cup_f Y$  where  $f$  is any homeomorphism  $\partial(S^1 \times D^2) \rightarrow T$  such that  $f(\{*\} \times \partial D^2)$  represents  $\alpha$ . It is well known that  $Y(\alpha)$  is independent of the choice of  $f$ .

First let us recall some definition and primary facts about Seifert manifolds.

Let  $F_{g,n}$  be an oriented  $n$ -punctured surface of genus  $g$  with boundary components  $c_1, \dots, c_n$  with  $n \geq 0$ . Then  $N' = F_{g,n} \times S^1$  is oriented if  $S^1$  is oriented. Let  $h_i$  be the oriented  $S^1$  fiber on the torus  $c_i \times h_i$  (call such pairs  $\{(c_i, h_i)\}$  a section-fiber coordinate system). Let  $0 \leq s \leq n$ , we attach  $s$  solid tori  $V_i$  to the boundary tori of  $N'$  such that the meridian of  $V_i$  is identified with the slope  $r_i = c_i^{a_i} h_i^{b_i}$  where  $a_i > 0$ ,  $(a_i, b_i) = 1$  for  $i = 1, \dots, s$ . We denote by  $N(g, n - s; \frac{b_1}{a_1}, \dots, \frac{b_s}{a_s})$  the resulting manifold which has the Seifert fibered structure extended from the circle

bundle structure of  $N'$ . Each orientable Seifert fibered space with orientable base  $F_g$  with  $n - s$  boundary components and  $s$  exceptional fibers is obtained in such a way.

**Lemma 5.1.** *Suppose  $N$  is a Seifert manifold given as above.*

(1) *Suppose  $N$  is an integer homology 3-sphere. Then  $N$  is closed and  $g = 0$ , furthermore  $(\prod_{i=1}^n a_i)(\sum_{i=1}^n \frac{b_i}{a_i}) = 1$ .*

(2) *Suppose  $n > s$  and  $N(\mu_{s+1}, \dots, \mu_n)$  is an integer homology 3-sphere, where  $\mu_j = (a_j, b_j)$ ,  $j = s + 1, \dots, n$ , then  $g = 0$  and moreover*

(i) *if each  $a_j > 0$  for  $j \in \{s + 1, \dots, n\}$ , then the Seifert fibration of  $N$  extends over  $N(\mu_{s+1}, \dots, \mu_n)$  and  $(\prod_{i=1}^n a_i)(\sum_{i=1}^n \frac{b_i}{a_i}) = 1$ ;*

(ii) *if some  $a_j = 0$  for some  $j \in \{s + 1, \dots, n\}$ , then  $b_j \prod_{i=1, i \neq j}^n a_i = 1$ .*

*Proof.* The proof of the lemma is an application of linear algebra. (1) is well known, see [HWZ1], 3.1, for example. (2) (i) mainly follows from (1).

For (2) (ii), if  $a_j = 0$  for some  $j \in \{s + 1, \dots, n\}$ , then  $b_j$  must be 1. For some  $i \neq j$ ,  $a_i = 0$  implies that  $N(\mu_{s+1}, \dots, \mu_n)$  has positive first Betti number, and  $a_i > 1$  implies that  $H_1(N(\mu_{s+1}, \dots, \mu_n), \mathbb{Z})$  contain a torsion element of order  $a_i$ . So  $a_i = 1$  for  $i \neq j$ .  $\square$

In the following we denote  $N(0, n - s; \frac{b_1}{a_1}, \dots, \frac{b_s}{a_s})$  by  $N(n - s; \frac{b_1}{a_1}, \dots, \frac{b_s}{a_s})$ , and  $N(0, 0; \frac{b_1}{a_1}, \dots, \frac{b_n}{a_n})$  by  $N(\frac{b_1}{a_1}, \dots, \frac{b_n}{a_n})$ .

**Lemma 5.2.** *Only finitely many Seifert fibered integral homology 3-spheres belong to  $\mathcal{D}(M)$ .*

*Proof.* A Seifert fibered integral homology 3-sphere must support the geometry of either  $S^3$  or  $\widetilde{\text{PSL}_2(\mathbb{R})}$ .

For Seifert manifolds supporting the geometry  $S^3$ , there are only two integral homology 3-spheres: the 3-sphere  $S^3$  and the Poincaré dodecahedral space.

Now suppose that  $N$  supports the geometry of  $\widetilde{\text{PSL}_2(\mathbb{R})}$ . Since  $N$  is an integral homology sphere, By [Sc],  $N = (\frac{b_1}{a_1}, \dots, \frac{b_n}{a_n})$  which satisfy:

- The rational Euler number  $e = -\sum_{i=1}^n \frac{b_i}{a_i}$  is non-zero.
- The Euler characteristic of the orbifold base  $B$  is  $\chi(B) = 2 - \sum_{i=1}^n (1 - \frac{1}{a_i}) < 0$ .
- $|e \prod_{i=1}^n a_i| = 1$  by Lemma 5.1.

Thus  $e = \frac{\pm 1}{\prod_{i=1}^n a_i}$  and  $b_i (\prod_{j \neq i} a_j) \equiv \pm 1$  modulo  $a_i$  for  $i = 1, \dots, n$ . Moreover the integers  $a_i$ ,  $i = 1, \dots, n$  are pairwise relatively prime. Therefore the unordered set  $\{a_1, \dots, a_n\}$  of integers determines the Seifert fibered homology sphere  $N$ , up to orientation. So we need only to show that if  $N$  is dominated by  $M$ , then  $n$

and the integers  $a_i, i = 1, \dots, n$ , take only finitely many values. In fact to do so, it is sufficient to get a uniform upper bound on  $\prod_{i=1}^n a_i$ , depending only on  $M$ .

We use the Seifert volume  $SV$  introduced by Brooks and Goldman [BG]. It has the following interesting properties:

- (1)  $SV(M) \geq dSV(N)$  if  $f: M \rightarrow N$  is a map of degree  $d \neq 0$ , for orientable 3-manifolds  $M$  and  $N$ .
- (2)  $SV(N) = \left| \frac{\chi(B)^2}{e(N)} \right|$  if  $N$  is a  $\widetilde{\text{PSL}_2(\mathbb{R})}$ -manifold with base orbifold  $B$ .

It is easy to see that the maximum of the Euler characteristic of the base  $B$  of  $N$  is obtained for the sphere with three cone points with orders  $\{2, 3, 7\}$ . Hence:

- (3)  $\chi(B) \leq -\frac{1}{42}$ .

Then by (1), (2) and (3) we have

$$SV(M) \geq dSV(N) = d \left| \frac{\chi(B)^2}{e(N)} \right| \geq \left| \frac{1}{42^2} \prod_{i=1}^n a_i \right|.$$

Therefore  $\prod_{i=1}^n a_i \leq 42^2 SV(M)$  and the proof of Lemma 5.2 is complete.  $\square$

The dual graph  $\Gamma(N)$  to the JSJ-decomposition of an irreducible homology sphere  $N$  is a tree. By Lemma 4.2, the number of edges of  $\Gamma(N)$  is  $\leq h(M)$ , the Haken number of  $M$ .

By the local domination theorem (Theorem 4.1) and Lemma 5.2, the geometric JSJ-pieces of the closed orientable 3-manifolds in  $\mathcal{D}(M)$  belong to a finite set  $\mathcal{HS}(M)$  of compact 3-manifolds with interiors admitting complete hyperbolic metrics with finite volume and of Seifert 3-manifolds.

For a given graph  $\Gamma$ , let  $\mathcal{D}(M, \Gamma) \subset \mathcal{D}(M)$  be the set of homeomorphism classes of closed orientable integer homology 3-spheres  $N$  such that:

- (1)  $N$  is dominated by  $M$ .
- (2) The JSJ-graph  $\Gamma(N)$  is abstractly isomorphic to  $\Gamma$ .
- (3) Each vertex manifold has a fixed topological type. Each torus boundary component of the vertex manifold is assigned to an edge on the vertex.

The local domination theorem and Lemma 4.2 reduce Theorem 1.2 to the following:

**Proposition 5.3.** *The set  $\mathcal{D}(M, \Gamma)$  is finite.*

Before starting the proof of this proposition, we need to introduce some notions, definitions and constructions which will be useful.

For each integral solid torus  $V$ , the kernel of the induced homomorphism

$$H_1(\partial V; \mathbb{Z}) \rightarrow H_1(V; \mathbb{Z})$$

is infinite cyclic, generated by an essential simple loop which bounds a properly embedded surface  $F_V$  in  $V$ . Such a surface  $F_V$  is called a Seifert surface for the integral homology solid torus  $V$ . Then the slope  $\lambda_V \in H_1(\partial V; \mathbb{Z})$  of  $\partial F_V$  does not depend of the Seifert surface and is uniquely determined by the topological type of  $V$ . We call  $\lambda_V$  the *longitudinal slope* of  $V$  on  $\partial V$ .

**The pinch construction.** Let  $Y$  be a compact, orientable 3-manifold with  $T \subset \partial Y$  a torus boundary component. Let  $Z$  be an integral homology solid torus,  $\phi: \partial Z \rightarrow T \subset \partial Y$  a gluing map and  $Y' = Z \cup_\phi Y$ . By pinching a Seifert surface  $F_Z$  onto a disk  $D^2$ , one can define a proper degree-one map  $p_Z: Z \rightarrow S^1 \times D^2$  such that  $p_Z^{-1}(\{x\} \times \partial D^2) = \lambda_Z$  for some point  $x \in S^1$ . Then one gets a degree-one map  $f_Z: Y' \rightarrow Y(\mu)$ , which is the identity on  $Y$  and where  $Y(\mu)$  is obtained by Dehn filling the component  $T$  of  $\partial Y$  with the filling slope  $\phi_*(\lambda_Z) = \mu$ .

Let  $e$  be an edge of  $\Gamma$  with vertices  $x$  and  $y$ . For each  $N \in \mathcal{D}(M, \Gamma)$  the edge  $e$  corresponds to an incompressible torus  $T_e \subset N$ , and its two vertices to two JSJ-pieces  $X$  and  $Y$  of  $N$ , adjacent to the torus  $T_e$ . Denote the component of  $\partial X$  (resp.  $\partial Y$ ) corresponding to  $T_e$  by  $\partial_e X$  (resp.  $\partial_e Y$ ). The embedded torus  $T_e$  splits  $N$  into two integral homology solid tori.

We call a slope on  $\partial_e X$  *longitudinal* if it is equal to the longitudinal slope  $\lambda_V$  of an integral homology solid torus  $V$  bounded by  $T_e$  in some  $N \in \mathcal{D}(M, \Gamma)$  and containing  $X$ . (See Figure 4.)

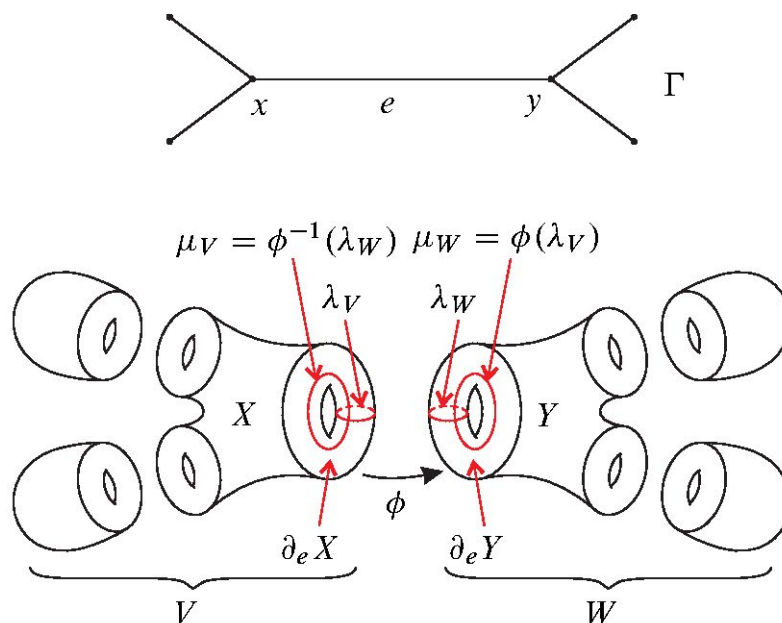


Figure 4

Let  $N \in \mathcal{D}(M, \Gamma)$ . The incompressible torus  $T_e$  splits  $N$  into two compact 3-manifolds  $V$  and  $W$  which are both integral homology solid tori with boundary  $T_e$ :

$N = V \cup_{T_e} W$ . The fact that  $N = V \cup_\phi W$  is an integral homology sphere forces the following:

**Lemma 5.4.** *The gluing map  $\phi: \partial V \rightarrow \partial W$  induces a map  $\phi_\star$  on the first homology group, such that  $\phi_\star(\lambda_V) \cdot \lambda_W = \pm 1$  and  $\phi_\star^{-1}(\lambda_W) \cdot \lambda_V = \pm 1$ .  $\square$*

**Definition 5.5.** (1) We call a gluing map  $\phi: \partial_e X \rightarrow \partial_e Y$  *allowable*, if there are two integral homology solid tori  $V$  and  $W$  such that  $(X, \partial_e X) \subset (V, \partial V)$ ,  $(Y, \partial_e Y) \subset (W, \partial W)$ , and  $N = V \cup_\phi W \in \mathcal{D}(M, \Gamma)$ .

(2) An allowable gluing map  $\phi: \partial_e X \rightarrow \partial_e Y$  is determined, up to isotopy, by the two pairs of slopes  $\{\lambda_V, \mu_V\} \in H_1(\partial_e X; \mathbb{Z}) \times H_1(\partial_e X; \mathbb{Z})$  and  $\{\lambda_W, \mu_W\} \in H_1(\partial_e Y; \mathbb{Z}) \times H_1(\partial_e Y; \mathbb{Z})$ , such that:

- (i)  $\lambda_V$  and  $\lambda_W$  are the longitudinal slopes of the integral homology solid tori  $V$  and  $W$  which define the gluing map  $\phi$  to be allowable.
- (ii)  $\mu_V = \phi_\star^{-1}(\lambda_W)$  and  $\mu_W = \phi_\star(\lambda_V)$ . They are called longitudinal-images or  $\ell$ -images for short.

By Lemma 5.4, the pair  $\{\lambda_V, \mu_V\}$  defines a basis of  $H_1(\partial_e X; \mathbb{Z})$  and the pair  $\{\lambda_W, \mu_W\}$  a basis of  $H_1(\partial_e Y; \mathbb{Z})$ . The pairs of slopes  $\{\lambda_V, \mu_V\}$  and  $\{\lambda_W, \mu_W\}$  are called *gluing patterns* for the tori  $\partial_e X$  and  $\partial_e Y$  (see Figure 4).

(3) Let  $Y$  a vertex manifold of  $\Gamma$  with  $k$  boundary components  $\partial_i Y, i = 1, \dots, k$ . A *gluing pattern* for  $\partial Y$  is a system of pairs of slopes  $\{(\lambda_1, \mu_1), \dots, (\lambda_k, \mu_k)\} \in (H_1(\partial_1 Y; \mathbb{Z}) \times H_1(\partial_1 Y; \mathbb{Z})) \times \dots \times (H_1(\partial_k Y; \mathbb{Z}) \times H_1(\partial_k Y; \mathbb{Z}))$  for which there are a collection  $Z_1, \dots, Z_k$  of integral homology solid tori and gluing maps  $\phi_i: \partial Z_i \rightarrow \partial_i Y, i = 1, \dots, k$  such that (see the top picture of Figure 5:

- (i)  $N = Y \cup_\phi \bigcup_{i=1}^k Z_i$  belongs to  $\mathcal{D}(M, \Gamma)$ , with  $\phi = \bigcup_{i=1}^k \phi_i: \bigcup_{i=1}^k \partial Z_i \rightarrow \partial Y$ .
- (ii)  $\lambda_i = \lambda_{W_i}$ , where  $W_i$  is the integral homology solid torus  $N \setminus \text{int}(Z_i)$ .
- (iii)  $\mu_i = \phi_\star(\lambda_{Z_i})$ , for  $i = 1, \dots, k$ .

Hence, each gluing map  $\phi_i: \partial Z_i \rightarrow \partial_i Y$  is allowable and each pair of slopes  $(\lambda_i, \mu_i)$  is a gluing pattern for the component  $\partial_i Y$ .

The slopes  $\{\mu_1, \dots, \mu_k\}$  are called a system of  $\ell$ -images for  $\partial Y$ .

(4) Two systems of slopes on  $\partial Y$  are  $A$ -equivalent if there is a homeomorphism  $\tau: (Y, \partial Y) \rightarrow (Y, \partial Y)$  which is a product of Dehn twists along properly embedded essential annuli in  $Y$  and sends one set to the other. In the same way two gluing patterns for  $\partial Y$  are  $A$ -equivalent, if there is such a homeomorphism of  $(Y, \partial Y)$  sending one to the other.

**Remark 5.6.** Let  $Y$  be a compact irreducible orientable 3-manifold with boundary an union of incompressible tori, and let  $\text{Aut}(Y)$  be the mapping class group of  $Y$ . By

[Joh] the subgroup  $\mathcal{A}(Y) \subset \text{Aut}(Y)$  generated by Dehn twists along essential tori and proper annuli in  $Y$  is of finite index in  $\text{Aut}(Y)$ . Therefore the finiteness, up to homeomorphisms of  $Y$ , of systems of slopes on  $\partial Y$  (or gluing patterns for  $\partial Y$ ) is equivalent to the finiteness of their  $A$ -equivalence classes, since Dehn twists along essential tori in  $Y$  do not affect the slopes on  $\partial Y$ .

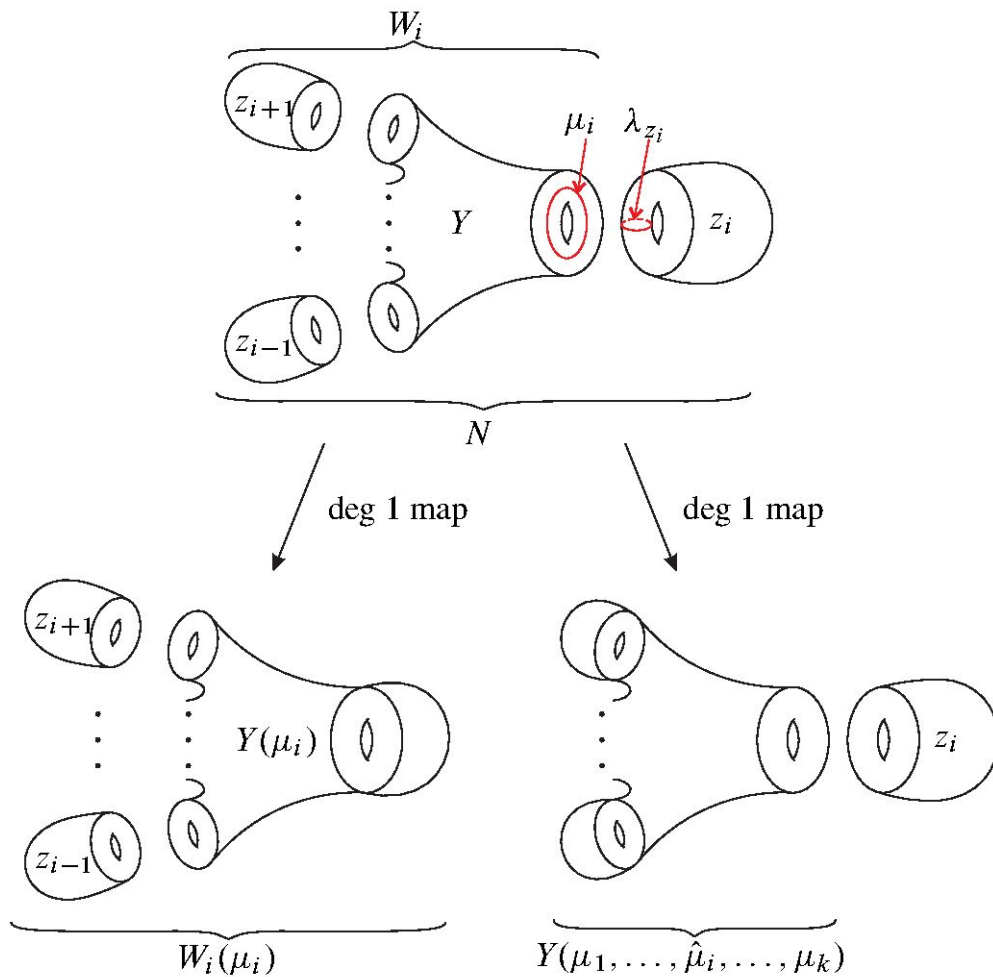


Figure 5

Now Proposition 5.3 follows from the following result:

**Proposition 5.7.** *When  $N$  runs over all elements in  $\mathcal{D}(M, \Gamma)$ , for each vertex manifold  $Y$  of  $\Gamma$  there are at most finitely many  $A$ -equivalence classes of gluing patterns for  $\partial Y$ , depending only on  $M$ .*

The first step of the proof of Proposition 5.7 is given by the following

**Proposition 5.8.** *When  $N$  runs over all elements in  $\mathcal{D}(M, \Gamma)$ , for each vertex manifold  $Y$  of  $\Gamma$ , there are at most finitely many  $A$ -equivalence classes of  $\ell$ -images  $(\mu_1, \dots, \mu_k)$  on  $\partial Y$ , depending only on  $M$ .*

*Proof.* Let  $Y$  be a vertex manifold of  $\Gamma$ . By definition, for each system of  $\ell$ -images  $(\mu_1, \dots, \mu_k)$  on  $\partial Y$ , there is a collection  $Z_1, \dots, Z_k$  of integral homology solid tori and gluing maps  $\phi_i: \partial Z_i \rightarrow \partial_i Y$ ,  $i = 1, \dots, k$  such that:

- (1)  $N = Y \cup_\phi \bigcup_{i=1}^k Z_i$  belongs to  $\mathcal{D}(M, \Gamma)$ , with  $\phi = \bigcup_{i=1}^k \phi_i: \bigcup_{i=1}^k \partial Z_i \rightarrow \partial Y$ .
- (2)  $\mu_i = \phi_\star(\lambda_{Z_i}), i = 1, \dots, k$ .

Let  $W_i$  be the integral homology solid torus  $N \setminus \text{int}(Z_i)$ , then  $Y$  is a JSJ-piece of  $W_i$ , for  $i = 1, \dots, k$ .

We distinguish two cases according to whether  $Y$  is hyperbolic or Seifert fibered. By definition of  $A$ -equivalence, we prove the finiteness of the systems of  $\ell$ -images in the hyperbolic case and the finiteness of the systems of  $\ell$ -images, up to Dehn-twists along properly embedded essential annuli, in the Seifert fibered case.

a)  *$Y$  is hyperbolic.* For each boundary component  $\partial_i Y, i \in \{1, \dots, k\}$ , Thurston's hyperbolic Dehn filling theorem [Th1] shows that the manifold  $Y(\mu_i)$  admits a complete hyperbolic metric, except for a finite set of slopes  $\mu_i \in H_1(\partial_i Y; \mathbb{Z})$ , depending only on  $Y$ . So we may assume that  $Y(\mu_i)$  is hyperbolic.

Therefore  $Y(\mu_i)$  is irreducible with incompressible boundary tori, and it is a hyperbolic piece in the JSJ-decomposition of the homology sphere  $W_i(\mu_i)$ . So  $W_i(\mu_i)$  is an irreducible homology sphere which is 1-dominated by  $N$  (see the top-left-down picture of Figure 5), and thus dominated by  $M$ . Since  $Y(\mu_i)$  is a geometric piece of a manifold dominated by  $M$ ,  $Y(\mu_i)$  can take only finitely many topological types, depending only on  $M$  by Theorem 4.1. Hence the hyperbolic volume of  $Y(\mu_i)$  takes finitely many values, depending only on  $M$ . Then Thurston's hyperbolic Dehn filling theorem shows that  $\mu_i$  belongs to a finite set of slopes in  $H_1(\partial_i Y; \mathbb{Z})$ , depending only on  $Y$  and  $M$ . Hence for each  $i \in \{1, \dots, k\}$  there are only finitely many possible  $\ell$ -images  $\mu_i \in H_1(\partial_i Y; \mathbb{Z})$ , depending only on  $M$ .

b)  *$Y$  is Seifert fibered.* Then the Seifert fibration is unique, up to isotopy.

Suppose  $Y = S(g, n - s; \frac{b_1}{a_1}, \dots, \frac{b_s}{a_s})$ , where  $n - s = k$ , and each  $a_i > 1$ ,  $i = 1, \dots, s$ . Let  $(c_i, h_i) \in H_1(\partial_i Y; \mathbb{Z}) \times H_1(\partial_i Y; \mathbb{Z})$  be a basis of  $H_1(\partial_i Y; \mathbb{Z})$ , where  $h_i$  represents the fiber of the circle fibration induced on  $\partial_i Y$  by the Seifert fibration of  $Y$  and  $c_i$  a section of this induced circle fibration. We set  $\mu_i = (a_{s+i}c_i + b_{s+i}h_i)$  in  $H_1(\partial_i Y; \mathbb{Z})$ . There is a degree one map from  $N$  to the manifold  $Y(\mu_1, \dots, \mu_k)$  obtained by pinching each homology solid torus  $Z_i$  to a solid torus, hence  $Y(\mu_1, \dots, \mu_k)$  is a Seifert fibered integral homology 3-sphere. It follows that  $g = 0$ , that is  $Y = S(n - s; \frac{b_1}{a_1}, \dots, \frac{b_s}{a_s})$ . Moreover  $n \geq 3$  since it is a JSJ-piece of  $N$ . We further divide the discussion into two cases:

Case (i):  $a_i \neq 0$  for each  $i \in \{s+1, \dots, s+k\}$ . Then by Lemma 5.1 (1) the Seifert fibration of  $Y$  extends over the integer homology 3-sphere  $Y(\mu_1, \dots, \mu_k)$  and

$$\left(\prod_{i=1}^n a_i\right) \left(\sum_1^n \frac{b_i}{a_i}\right) = 1. \quad (*)$$

Like in case a), for each  $i \in \{s+1, \dots, s+k\}$ ,  $W_i(\mu_i)$  is an integer homology 3-sphere dominated by  $M$ . Moreover if  $a_{s+i} \geq 2$ , the core of the filling solid torus becomes a singular fiber of  $Y(\mu_i)$  with index  $a_{s+i}$ , and thus  $Y(\mu_i)$  is a Seifert JSJ-piece of  $W_i(\mu_i)$ . In this case,  $Y(\mu_i)$  can take only finitely many topological types, depending only on  $M$ . Therefore  $a_{s+i} \leq C(M)$  for some integer  $C(M)$  depending only on  $M$ .

If  $\{(a_1, b_1), \dots, (a_i, b_i), \dots, (a_j, b_j), \dots, (a_n, b_n)\}$  is a solution of equation (\*), then for any integer  $k$ ,  $\{(a_1, b_1), \dots, (a_i, b_i + ka_i), \dots, (a_j, b_j - ka_j), \dots, (a_n, b_n)\}$  is also a solution of the equation (\*). Those two solutions represent two systems of  $\ell$ -images on  $\partial Y$  related by  $k$  full Dehn twist (with sign) along an essential vertical annulus in  $Y$  connecting  $\partial_i Y$  and  $\partial_j Y$ , so they are in the same  $A$ -equivalence class.

We say that two solutions of (\*) are in the same  $A$ -equivalence class, if one solution is obtained from another by finitely many Dehn twists along essential vertical annuli, like above. It is an elementary fact that there are only finitely many  $A$ -equivalence classes of solutions for the equation (\*) if each  $a_i$  is bounded by a constant  $C(M)$ . Hence there are only finitely many  $A$ -equivalence classes of systems of  $\ell$ -images on  $\partial Y$ , depending only on  $M$ .

Case (ii):  $a_j = 0$  for some  $j \in \{s+1, \dots, n\}$ . Then by Lemma 5.1 (2),

$$b_j \prod_{i=1, i \neq j}^n a_i = 1. \quad (**)$$

This implies that  $\mu_j = (0, 1)$  and  $\mu_i = (1, b_i)$  for  $i \neq j$ . By performing  $b_i$ -full Dehn twists along a vertical annulus connecting  $\partial_i Y$  and  $\partial_j Y$  for each  $i \neq j$ , we can transform the system of  $\ell$ -images  $\{(1, b_1), \dots, (1, b_{j-1}), (0, 1), (1, b_{j+1}), \dots, (1, b_n)\}$  to  $\{(1, 0), \dots, (1, 0), (0, 1), (1, 0), \dots, (1, 0)\}$ . That is, for a fixed  $j$ , the system of  $\ell$ -images on  $\partial Y$  is unique, up to  $A$ -equivalence. Since  $j$  picks only finitely many value, there are finitely many systems of  $\ell$ -images on  $\partial Y$ , up to  $A$ -equivalence.  $\square$

The next step of the proof of Proposition 5.7 is given by the following claim:

**Claim 5.9.** *A gluing pattern  $\{(\lambda_1, \mu_1), \dots, (\lambda_k, \mu_k)\}$  for  $\partial Y$  is uniquely determined by the system of  $\ell$ -images  $(\mu_1, \dots, \mu_k)$  on  $\partial Y$ .*

*Proof.* Fix a system of  $\ell$ -images  $(\mu_1, \dots, \mu_k)$  on  $\partial Y$ . Recall that for this system of slopes  $(\mu_1, \dots, \mu_k)$ , there is an homology sphere  $N \in M(\Gamma)$  such that  $N =$

$Y \cup \bigcup_{i=1}^k Z_i$ , and the image by gluing of each longitudinal slope  $\lambda_{Z_i}$  is  $\mu_i$ . As before, let  $W_i = N \setminus Z_i$ .

Then for each fixed  $i \in \{1, \dots, k\}$ , we can define a degree one map  $p: W_i \rightarrow Y(\mu_1, \dots, \hat{\mu}_i, \dots, \mu_k)$  which is the identity on the boundary  $\partial W_i = \partial_i Y$  by pinching each  $Z_j$  to a solid torus  $U_j$  whose meridian is matched with  $\mu_j$ ,  $j \in \{1, \dots, k, j \neq i\}$  (see the top-right-down picture of Figure 5). Then the Seifert surface  $F_{W_i}$  is pinched to a Seifert surface  $F_*$  of the integral homology solid torus  $Y(\mu_1, \dots, \hat{\mu}_i, \dots, \mu_k)$  bounded by  $\lambda_i$ . Since  $Y(\mu_1, \dots, \hat{\mu}_i, \dots, \mu_k)$  is a fixed integral homology solid torus, the longitudinal slope  $\lambda_i \in H_1(\partial_i Y; \mathbb{Z})$  is unique. This proves the claim.  $\square$

To finish the proof of Proposition 5.7, we distinguish two cases as usual:

*Y is hyperbolic.* If  $Y$  is hyperbolic, as we have seen in the proof of Proposition 5.8, there are only finitely many systems of  $\ell$ -images  $(\mu_1, \dots, \mu_k)$  on  $\partial Y$ . So by Claim 5.9 there are only finitely many possible choices of gluing patterns  $\{(\lambda_1, \mu_1), \dots, (\lambda_k, \mu_k)\}$  for  $\partial Y$ .

*Y is Seifert fibered.* Let  $(\mu'_1, \dots, \mu'_k) = \tau(\mu_1, \dots, \mu_k)$  be deduced from the system of  $\ell$ -images  $(\mu_1, \dots, \mu_k)$  by a homeomorphism  $\tau: Y \rightarrow Y$  which is a composition of Dehn twists along vertical annuli. Then, by the uniqueness in Claim 5.9, the system of  $\ell$ -images  $(\mu'_1, \dots, \mu'_k)$  determines the system  $(\lambda'_1, \dots, \lambda'_k) = \tau(\lambda_1, \dots, \lambda_k)$  of longitudinal slopes on  $\partial Y$ . Hence an  $A$ -equivalent class of systems of  $\ell$ -images on  $\partial Y$  determine a unique  $A$ -equivalent class of gluing pattern for  $\partial Y$ . Then by Proposition 5.8, there are only finitely many equivalent classes of gluing pattern for  $\partial Y$ .

This finishes the proof of Proposition 5.7.  $\square$

The following corollary of Proposition 5.7 implies Proposition 5.3.

For each  $N \in \mathcal{D}(M, \Gamma)$ , a submanifold  $L \subset N$  is called *canonical* if it is a component of  $N \setminus \mathcal{T}$ , where  $\mathcal{T}$  is a subfamily (may be empty) of JSJ-tori of  $N$ .

**Corollary 5.10.** *When  $N$  runs over all elements in  $\mathcal{D}(M, \Gamma)$ , the canonical submanifolds of  $N$  take at most finitely many topological types, depending only on  $M$ .*

*Proof.* The proof will be by induction on the number  $v(L)$  of JSJ-pieces of a canonical submanifold  $L$ .

Corollary 5.10 is valid for  $v(L) = 1$  since  $\mathcal{SH}(M)$  is finite. We suppose that it is valid for  $v(L) < m$  and we are going to verify it for  $v(L) = m$ .

Fix a connected subtree  $\Gamma_*$  of  $\Gamma$  with  $m$  vertices and let  $D(M, \Gamma_*)$  be the set of canonical submanifolds with dual JSJ-tree  $\Gamma_*$ . Choose a vertex  $y \in \Gamma_*$  with corresponding vertex manifold  $Y$ .

For each canonical submanifold  $L \in D(M, \Gamma_*)$ , we have  $L \setminus Y = \bigcup_{i=1}^p P_i$ , where  $P_i$  is a canonical submanifolds and  $v(P_i) < m$ , hence  $P_i$  can take only finitely many topological types by the induction hypothesis. So we may fix the topology of  $P_i$  for each  $i = 1, \dots, p$ .

We may suppose  $\partial Y = \{\partial_1 Y, \dots, \partial_k Y\}$  and  $\partial_0 P_i$  is the component of  $\partial P_i$  identified with  $\partial_i Y$  via a gluing map  $\phi_i$  (reordering the components of  $\partial Y$  if needed). So we can rewrite  $L = Y \cup_{\{\phi_i\}_{i=1,\dots,p}} \{P_i\}$ .

Fix a gluing pattern  $(\lambda_{P_i}, \mu_{P_i}) \subset \partial_0 P_i$  for each  $i = 1, \dots, p$ . Then each gluing map  $\phi_i$  is determined by the images  $(\phi_i(\lambda_{P_i}), \phi_i(\mu_{P_i}))$  on  $\partial_i Y$ . By definition  $(\lambda_i = \phi_i(\mu_{P_i}), \mu_i = \phi_i(\lambda_{P_i}))$  is a gluing pattern on  $\partial_i Y$ . Hence  $\{(\lambda_i, \mu_i), i = 1, \dots, p\}$  forms a subset of a gluing pattern  $\{(\lambda_i, \mu_i), i = 1, \dots, k\}$  for  $\partial Y$ . Any subset  $\{(\lambda'_i, \mu'_i), i = 1, \dots, p\}$  of a  $A$ -equivalent gluing pattern  $\{(\lambda'_i, \mu'_i), i = 1, \dots, k\}$  for  $\partial Y$  provides a canonical submanifold  $L' \in D(M, \Gamma_*)$  which is homeomorphic to  $L$ . By Proposition 5.7, there are only finitely many  $A$ -equivalent classes of gluing patterns for  $\partial Y$ , depending only on  $M$ . Hence a canonical submanifold  $L$  in  $D(M, \Gamma_*)$  can take at most finitely many topological types, depending only on  $M$ .  $\square$

## 6. Knot exteriors in $\mathbb{S}^3$

By Theorem 1.2 and an obvious twisted double construction, one gets the following straightforward corollary:

**Corollary 6.1.** *Each compact orientable 3-manifold with a torus boundary 1-dominates at most finitely many integral homology solid tori.*

A less direct and may be more interesting result is the following

**Theorem 6.2.** *A compact orientable 3-manifold  $M$  dominates at most finitely many exteriors of knots in  $\mathbb{S}^3$ .*

*Proof.* We call the exterior  $E(k) = S^3 \setminus N(k)$  of a knot  $k$  in  $\mathbb{S}^3$  a knot space, where  $N(k)$  is a tubular neighborhood of  $k$  in  $S^3$ . The dual graph  $\Gamma(k)$  to the JSJ-decomposition of  $E(k)$  is a rooted tree, where the root corresponds to the unique vertex manifold containing  $\partial E(k)$ .

Let  $\mathcal{K}(M)$  denote the set of homeomorphism classes of knot spaces  $E(k)$  dominated by  $M$ . By Lemma 4.2, there are only finitely many  $\Gamma(k)$  for all  $E(k)$  dominated by  $M$ . By the local domination theorem (Corollary 4.6) the JSJ-pieces of the knot spaces in  $\mathcal{K}(M)$  belong to a finite set  $\mathcal{HS}(M)$ .

For a given graph  $\Gamma$ , let  $\mathcal{K}(M, \Gamma) \subset \mathcal{K}(M)$  be the set of homeomorphism classes of knot space  $E(k)$ , such that:

- (1)  $E(k)$  is dominated by  $M$ .
- (2) The JSJ-graph  $\Gamma(k)$  is abstractly isomorphic to  $\Gamma$ .
- (3) Each vertex manifold has a fixed topological type. Each torus boundary component of the vertex manifold has assigned to an edge on the vertex.

Like in the case of integral homology spheres, the proof of Theorem 6.2 is reduced to the following result.  $\square$

**Proposition 6.3.** *The set  $\mathcal{K}(M, \Gamma)$  is finite.*

*Proof.* To apply the arguments in Section 5 to the present case, we will consider  $S^3 = E(k) \cup N(k)$  rather than just consider  $E(k)$ . Precisely  $\partial E(k)$  and the JSJ-tori of  $E(k)$  provide an extended JSJ-splitting of  $S^3 = E(k) \cup N(k)$  with one more torus  $\partial E(k)$  and one additional piece the solid torus  $N(k)$ . The dual graph  $\Gamma^*(k)$  of this extended decomposition is a tree obtained by adding one leaf on the root.

Now for each JSJ-piece  $Y$ , different from  $N(k)$ , of this extended JSJ-decomposition of  $S^3 = E(k) \cup N(k)$ , we can define  $A$ -equivalent classes of gluing patterns for  $\partial Y \setminus \partial E(k)$  like in Section 5. Similarly the proof of Proposition 6.3 is reduced to the following result.  $\square$

**Proposition 6.4.** *When  $N$  runs over all elements in  $\mathcal{K}(M, \Gamma)$ , for each vertex manifold  $Y$  of  $\Gamma$  there are at most finitely many  $A$ -equivalent classes of gluing patterns for  $\partial Y \setminus \partial E(k)$ , depending only on  $M$ .*

And Proposition 6.4 is reduced to the following

**Proposition 6.5.** *When  $N$  runs over all elements in  $\mathcal{K}(M, \Gamma)$ , for each vertex manifold  $Y$  of  $\Gamma$  there are at most finitely many  $A$ -equivalent classes of systems of  $\ell$ -images on  $\partial Y$ , depending only on  $M$ .*

**Remark 6.6.** When  $\partial E(k) \subset \partial Y$ , in order to show the finiteness of  $A$ -equivalent classes of gluing patterns on  $\partial Y \setminus \partial E(k)$ , we need the finiteness of  $A$ -equivalent classes of systems of  $\ell$ -images on  $\partial Y$ , including  $\partial E(k)$ .

*Proof of Proposition 6.5.* Let  $Y$  be the given vertex manifold with  $k + 1$  boundary components  $\partial_i Y, i = 0, 1, \dots, k$ . Now  $S^3 \setminus Y = \bigcup_{i=0}^k Z_i$ , where  $Z_0$  is a solid torus containing  $N(k)$  and bounded by  $\partial Y_0$ , and  $Z_i$  is a non-trivial knot space, bounded by  $\partial Y_i$  for  $i = 1, \dots, k$ . Recall that for a system of  $\ell$ -images  $(\mu_0, \mu_1, \dots, \mu_k)$  on  $(\partial_0 Y, \partial_1 Y, \dots, \partial_k Y)$ ,  $\mu_0$  is the image of the boundary  $\lambda_{Z_0}$  of a meridian disc of  $Z_0$  and  $\mu_i \subset \partial_i Y$  is the image of the longitudinal slope  $\lambda_{Z_i}, i = 1, \dots, k$ .

It is known that both the JSJ-pieces in knot spaces and their gluing are rather restrictive (see for example [Ja], IX.22, or [BS]), Chapter 2. A Seifert JSJ-piece of a knot space is either a torus knot space, a cable space or a composite space.

We may assume that  $k \geq 1$ , otherwise  $Y$  is a hyperbolic or a torus knot space and by [GL] the meridian  $\mu_0$  is unique. Below we distinguish three cases for the proof:

(i)  *$Y$  is hyperbolic.* Since the boundary tori  $\partial Y \setminus \partial_0 Y$  of the compact 3-manifold  $Y(\mu_0)$  is compressible, by Theorem 2.0.1 of [CGLS] the  $\ell$ -image  $\mu_0$  on  $\partial_0 Y$  can belong to at most three distinct slopes. The argument for the finiteness of the remaining

$\ell$ -images  $\mu_i, i = 1, \dots, k$ , is then the same as the corresponding part of the proof of Proposition 5.8.

(ii) *Y is a cable space.* Say  $Y$  is a  $(q, p)$ -cable space with  $p \geq 2$ . Then  $Y$  is a Seifert fiber space over annulus with a singular fiber of index  $p$ . Then  $\partial Y = \partial_0 Y \cup \partial_1 Y$  and we choose a basis on  $H_1(\partial Y_0; \mathbb{Z})$  and  $H_1(\partial Y_1; \mathbb{Z})$  represented by a section of the circle fibrations induced on  $\partial_0 Y$  and  $\partial_1 Y$  by the Seifert fibration of  $Y$  and the fiber of these induced circle fibrations.

The fact that  $Y(\mu_0)$  is a solid torus forces  $\mu_0$  to meet the fiber exactly once, that is  $\mu_0 = (1, q_0)$  in  $H_1(\partial Y_0; \mathbb{Z})$ . Moreover  $Y(\mu_1)$  must be a torus knot space  $E(T_{q+sp, p})$ , which falls into  $SH(M)$ . Hence it has only finitely many topological types. Therefore  $\mu_1 = (sp + q, q_1)$  and  $s$  takes only finitely many values. Then there are only finitely many  $A$ -equivalent classes of systems of  $\ell$ -images on  $\partial Y$  as in the corresponding part of the proof of Proposition 5.8.

(iii) *Y is a composite space.* It means that  $Y$  is homeomorphic to a product  $S^1 \times D_k$  where  $D_k$  is a disk with  $k$  holes. This corresponds to the case where the core  $k_0$  of the solid torus  $Z_0$  is not a prime knot. In this case the  $\ell$ -image  $\mu_0 \subset \partial_0 Y$  is isotopic to a fiber  $h = S^1 \times \{\star\}$ , whose slope is determined by the topological type of  $Y$ . Then the  $A$ -equivalence class of attaching patterns is unique as we shown in the corresponding part of the proof of Proposition 5.8.

This finishes the proof of Proposition 6.5. □

We call a homomorphism  $\phi: \pi_1(M) \rightarrow \pi_1(N)$  between 3-manifold groups *non-degenerate*, if  $\phi$  can be realized by a proper map  $f: M \rightarrow N$  of non-zero degree. The image of  $\pi_1(M)$  by such a non-degenerate homomorphism has finite index in  $\pi_1(N)$ .

Now we can translate Theorem 6.2 into the following

**Corollary 6.7.** *The fundamental group of a compact, orientable 3-manifold admits a non-degenerate homomorphism to only finitely many distinct knot groups.*

Corollary 6.7 is related to Simon's conjecture.

**Conjecture 6.8** ([Ki], Problem 1.12 (J. Simon)). *A knot group  $\pi_1(S^3 \setminus K)$  surjects onto at most finitely many distinct knot groups.*

This conjecture raised in 1970s has received recently a lot of attention (see for example [BBRW], [RW], [Si], [SW], [So4]). I. Agol and Y. Liu had confirmed Simon's conjecture in the summer of 2010 [AL] by proving that a finitely generated group  $G$  with the first Betti number  $\beta_1(G) = 1$  surjects onto finitely many knot groups. Our result holds with domain the fundamental group of any compact orientable 3-manifold and for non-surjective homomorphisms, but under the restrictive condition that the homomorphism is non-degenerate.

We give now a criterion for a homomorphism between knot groups to be non-degenerate:

**Lemma 6.9.** *A homomorphism  $\phi: \pi_1(E(k)) \rightarrow \pi_1(E(k'))$  is non-degenerate iff it sends the preferred longitude of  $k$  to a non-trivial peripheral element of  $\pi_1(\partial E(k'))$ .*

*Proof.* On the boundary tori  $\partial E(k)$  and  $\partial E(k')$ , let  $\{m, \ell\}$  and  $\{m', \ell'\}$  be meridian-preferred longitude pairs.

If  $\phi$  can be realized by a proper map  $f: E(k) \rightarrow E(k')$  of non-zero degree, then the restriction of  $\phi: \pi_1(\partial E(k)) \rightarrow \pi_1(\partial E(k'))$  is injective, and thus  $\phi(\ell)$  is a non-trivial element in  $\pi_1(\partial E(k'))$ .

Conversely, assume  $\phi(\ell)$  is a non-trivial element in  $\pi_1(\partial E(k'))$ . It will be null-homologous in  $H_1(E(k'))$ , hence  $\phi(\ell) = \ell'^n$  with  $n \in \mathbb{Z} \setminus \{0\}$ . Then  $\phi(m)$  belongs to the centralizer of  $\ell'^n$  in the knot group  $\pi_1(E(k'))$ . By [JS], Chapter VI, and the description of Seifert pieces in a knot complement, the centralizer of  $\ell'^n$  is the peripheral subgroup  $\pi_1(\partial E(k'))$ , so  $\phi(m)$  is a peripheral element which normally generates a finite index subgroup of the knot group  $\pi_1(E(k'))$ , and so generates a finite index subgroup of its first homology group. Hence  $\phi(m)$  must be equal to  $pm' + q\ell'$  for some integers  $p \neq 0, q \in \mathbb{Z}$ . This shows that  $\phi(\pi_1(\partial E(k))) \subset \pi_1(\partial E(k'))$  and that  $\phi$  is injective on  $\pi_1(\partial E(k))$ .

Then, since knot exteriors are  $K(\pi, 1)$ -spaces, a standard argument in algebraic topology and 3-manifold theory shows that the homomorphism  $\phi_i$  can be realized by a non-zero degree proper map  $f: E(k) \rightarrow E(k')$ .  $\square$

**Remark 6.10.** In [GR], [HKMS] many examples of degenerate epimorphisms between knot groups are given. There are epimorphisms between knot groups which do not send a meridian to a meridian: Suppose a knot  $k \subset \mathbb{S}^3$  whose group  $\pi_1(E(k))$  is normally generated by a non-peripheral element  $\mu$ , see Lemma 6.11. By [Gon] there exists a knot  $k' \subset \mathbb{S}^3$  and an epimorphism from  $\pi_1(E(k'))$  onto  $\pi_1(E(k))$  which sends a meridian of  $k'$  to  $\mu$ . The fact that knot groups are residually finite [He], hence hopfian, and Property P for knots in  $\mathbb{S}^3$  [KM] imply that the knots  $k$  and  $k'$  must be distinct. This construction has been pointed out to us by Cameron Gordon and Alan Reid.

**Lemma 6.11.** *Let  $k$  be a  $(1, 1)$ -knot in  $\mathbb{S}^3$  which is not a 2-bridge knot. Then  $\pi_1(E(k))$  is normally generated by a non-peripheral element.*

*Proof.* Recall that a  $(1, 1)$ -knot in  $\mathbb{S}^3$  is a knot which admits a 1-bridge presentation on a standard unknotted torus, therefore  $(1, 1)$ -knot is a tunnel number one knot and by construction the fundamental group  $\pi_1(E(k))$  is generated by two elements  $a, m$  with  $m$  a meridian. Let  $[a] = p[m] \in H_1(E(k); \mathbb{Z})$ , then  $\pi_1(E(k))$  is normally generated by the element  $b = am^{(1-p)}$ . By [BZ] this element cannot be peripheral since  $\pi_1(E(k))$  is generated by  $b$  and  $m$ , and the fact that  $k$  is not a 2-bridge knot.  $\square$

## References

- [A] I. Agol, Lower bounds on volumes of hyperbolic Haken 3-manifolds. *J. Amer. Math. Soc.* **20** (2007), no. 4, 1053–1077. [Zbl 1155.58016](#) [MR 2328715](#)
- [AL] I. Agol and Y. Liu. Presentation length and Simon’s conjecture. *J. Amer. Math. Soc.* **25** (2012), no. 1, 151–187. [Zbl 1237.57002](#) [MR 2833481](#)
- [BBBMP] L. Bessi eres, G. Besson, M. Boileau, S. Maillot, and J. Porti, *Geometrisation of 3-manifolds*. EMS Tracts Math. 13, European Mathematical Society (EMS), Z urich 2010. [Zbl 1244.57003](#) [MR 2683385](#)
- [BCG] G. Besson, G. Courtois, and S. Gallot, In egalit es de Milnor-Wood g eom etriques. *Comment. Math. Helv.* **82** (2007), no. 4, 753–803. [Zbl 1143.53040](#) [MR 2341839](#)
- [BBRW] M. Boileau, S. Boyer, A. W. Reid, and S. C. Wang, Simon’s conjecture for two-bridge knots. *Comm. Anal. Geom.* **18** (2010), no. 1, 121–143. [Zbl 1211.57004](#) [MR 2660460](#)
- [BBW] M. Boileau, S. Boyer, and S. C. Wang, Roots of torsion polynomials and dominations. In *The Zieschang Gedenkschrift*, Geom. Topol. Monogr. 14, Geom. Topol. Publ., Coventry 2008, 75–81. [Zbl 1146.57015](#) [MR 2484698](#)
- [BW] M. Boileau and S. C. Wang, Non-zero degree maps and surface bundles over  $S^1$ . *J. Differential. Geom.* **43** (1996), 789–806. [Zbl 0663.57006](#) [MR 0978294](#)
- [BZ] M. Boileau and B. Zimmermann, The  $\pi$ -orbifold group of a link. *Math. Z.* **200** (1989), 187–208. [Zbl 0663.57006](#) [MR 0978294](#)
- [BS] F. Bonahon and L. Siebenmann, *New geometric splittings of classical knots and the classification and symmetries of arborescent knots*, Geometry & Topology Monographs, to appear.  
<http://www-bcf.usc.edu/~fbonahon/Research/Preprints/BonSieb.pdf>.
- [BG] R. Brooks and W. Goldman, Volume in Seifert space. *Duke Math. J.* **51** (1984), 529–545. [Zbl 0546.57003](#) [MR 0757951](#)
- [CZ] H. Cao and X. Zhu, A complete proof of the Poincar e and geometrization conjectures – application of the Hamilton-Perelman theory of the Ricci flow. *Asian J. Math.* **10** (2006), no. 2, 165–492. [Zbl 1200.53057](#) [MR 2233789](#)
- [CGLS] M. Culler, C. McA. Gordon, J. Luecke, and P. B. Shalen, Dehn surgeries on knots. *Ann. of Math. (2)* **125** (1987), 237–300. [Zbl 0633.57006](#) [MR 0881270](#)
- [De] P. Derbez, Volume-convergent sequences of Haken 3-manifolds. *C. R. Math. Acad. Sci. Paris* **336** (2003), no. 10, 833–838. [Zbl 1037.57014](#) [MR 1990024](#)
- [De1] P. Derbez, Non-zero degree maps between closed orientable three-manifolds. *Trans. Amer. Math. Soc.* **359** (2007), no. 8, 3887–3911. [Zbl 1120.57009](#) [MR 2302517](#)
- [Do] A. Dold, *Lectures on algebraic topology*. Second edition, Grundlehren Math. Wiss. 200, Springer-Verlag, Berlin 1980. [Zbl 0434.55001](#) [MR 0606196](#)
- [Ep] D. Epstein, The degree of a map. *Proc. London Math. Soc. (3)* **16** (1966), 369–383. [Zbl 0148.43103](#) [MR 0192475](#)
- [Ga1] D. Gabai, Foliations and the topology of 3-manifolds. *J. Differential Geom.* **18** (1983), 479–536. [Zbl 0639.57008](#) [MR 0910018](#)
- [Ga2] D. Gabai, Essential laminations and Kneser normal form. *J. Differential Geom.* **53** (1999), 517–574. [Zbl 1028.57012](#) [MR 1806069](#)

- [Gon] F. Gonzalez-Acuña, Homomorphisms of knot groups. *Ann. of Math. (2)* **102** (1975), 373–377. [Zbl 0323.57010](#) [MR 0379671](#)
- [GR] F. Gonzalez-Acuña and A. Ramirez, Epimorphisms of knot groups onto free products. *Topology* **42** (2003), 1205–1227. [Zbl 1025.57010](#) [MR 1981354](#)
- [HKMS] K. Horie, T. Kitano, M. Matsumoto, and M. Suzuki, A partial order on the set of prime knots with up to 11 crossings. *J. Knot Theory Ramifications* **20** (2011), no. 2, 275–303. [Zbl 1218.57004](#) [MR 2782704](#)
- [Gu] L. Guth, The Hopf invariant and simplex straightening. [arXiv:0709.1247](#) [math.DG].
- [GL] C. McA. Gordon and J. Luecke, Knots are determined by their complements. *J. Amer. Math. Soc.* **2** (1989), no. 2, 371–415. [Zbl 0678.57005](#) [MR 0965210](#)
- [Gr] M. Gromov, Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.* **56** (1983), 5–99. [Zbl 0516.53046](#) [MR 0686042](#)
- [Gust] R. Gustafson, A simple genus one knot with incompressible spanning surfaces of arbitrarily high genus. *Pacific. J. Math.* **96** (1981), 81–98. [Zbl 0478.57003](#) [MR 0634764](#)
- [HWZ1] C. Hayat-Legend, S. C. Wang, and H. Zieschang, Minimal Seifert manifolds. *Math. Ann.* **308** (1997), 673–700. [Zbl 0877.57003](#) [MR 1464916](#)
- [HWZ2] C. Hayat-Legend, S. C. Wang, and H. Zieschang, Any 3-manifold 1-dominates at most finitely many 3-manifolds of  $S^3$ -geometry. *Proc. Amer. Math. Soc.* **130** (2002), no. 10, 3117–3121. [Zbl 1002.57042](#) [MR 1908938](#)
- [He] J. Hempel, Residual finiteness for 3-manifolds. In *Combinatorial group theory and topology*, Ann. of Math. Stud. 111, Princeton University Press, Princeton, NY, 1987, 379–396. [Zbl 0772.57002](#) [MR 0895623](#)
- [Ja] W. H. Jaco *Lectures on three-manifold topology*. CBMS Regional Conf. Ser. in Math. 43, Amer. Math. Soc., Providence, RI, 1980. [Zbl 0433.57001](#) [MR 0565450](#)
- [JR] W. Jaco and J. H. Rubinstein, 0-efficient triangulation of 3-manifolds. *J. Differential. Geom.* **65** (2003), 61–168. [Zbl 1068.57023](#) [MR 2057531](#)
- [JS] W. H. Jaco and P. B. Shalen, Seifert fibered spaces in 3-manifolds. *Mem. Amer. Math. Soc.* **21** (1979), No. 220. [Zbl 0415.57005](#) [MR 0539411](#)
- [Joh] K. Johansson, *Homotopy equivalence of 3-manifolds with boundary*. Lecture Notes in Math. 761, Springer-Verlag, Berlin 1979. [Zbl 0412.57007](#) [MR 0551744](#)
- [Ki] R. Kirby, Problems in low-dimensional topology. In *Geometric topology*, ed. by H. Kazez, AMS/IP Stud. Adv. Math. 2.2, International Press, Cambridge, Mass., 1997, 35–473. [Zbl](#) [MR 1470749](#)
- [KL] B. Kleiner and J. Lott, Notes on Perelman’s papers. *Geom. Topol.* **12** (2008), no. 5, 2587–2855. [Zbl 1204.53033](#) [MR 2460872](#)
- [KM] P. Kronheimer and T. Mrowka, Witten’s conjecture and property P. *Geom. Topol.* **8** (2004), 295–310. [Zbl 1072.57005](#) [MR 2023280](#)
- [Ku] T. Kuessner, Multicomplexes, bounded cohomology and additivity of simplicial volume. Preprint, [arXiv:math/0109057v2](#).
- [MP] S. Matveev and A. Perfil’ev, On the periodicity of the degrees of maps between Seifert manifolds. *Dokl. Akad. Nauk* **395** (2004), no. 4, 449–451. [MR 2115882](#)

- [MS]     W. Meeks and P. Scott, Finite group actions on 3-manifolds. *Invent. Math.* **86** (1986), 287–346. [Zbl 0626.57006](#) [MR 0856847](#)
- [MT]     J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*. Clay Math. Monogr. 3, Amer. Math. Soc., Providence, RI, 2007. [Zbl 1179.57045](#) [MR 2334563](#)
- [Per]     G. Perelman, The entropy formula for the Ricci flow and its geometric applications; Ricci flow with surgery on three-manifolds; Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. Preprints, [arXiv:math.DG/0211159](#); [arXiv:math.DG/0303109](#); [arXiv:math.DG/0307245](#).
- [Pe]     L. Person, A piecewise linear proof that the singular norm is the Thurston norm. *Topology Appl.* **51** (1993), 269–289. [Zbl 0794.57003](#) [MR 1237393](#)
- [RW]     A. Reid, and S. C. Wang, Non-Haken 3-manifolds are not large with respect to mappings of non-zero degree. *Comm. Ann. Geom.* **7** (1999), 105–132. [Zbl 0930.57012](#) [MR 1674109](#)
- [Re]     A. Reznikov, Analytic topology. In *European Congress of Mathematics*, Vol. I, Progr. Math. 201, Birkhäuser, Basel 2001, 519–532. [Zbl 1035.53042](#) [MR 1905340](#)
- [Ro1]     Y. Rong, Degree one maps between geometric 3-manifolds. *Trans. Amer. Math. Soc.* **332** (1992), 411–436. [Zbl 0758.57016](#) [MR 1052909](#)
- [Ro2]     Y. Rong, Maps between Seifert fibered spaces of infinite  $\pi_1$ . *Pacific J. Math.* **160** (1993), 143–154. [Zbl 0815.57002](#) [MR 1227509](#)
- [Ro3]     Y. Rong, Partial orders between Seifert manifolds. *Topology Appl.* **64** (1995), 191–200. [Zbl 0844.57001](#) [MR 1340870](#)
- [Sc]     P. Scott, The geometries of 3-manifolds. *Bull. London Math. Soc.* **15** (1983), 401–487. [Zbl 0561.57001](#) [MR 0705527](#)
- [Si]     D. Silver, Knot invariants from topological entropy. *Topology Appl.* **61** (1995), 159–177. [Zbl 0827.57004](#) [MR 1314616](#)
- [SW]     D. Silver and W. Whitten, Knot group epimorphisms. *J. Knot Theory Ramifications* **15** (2006), no. 2, 153–166. [Zbl 1096.57007](#) [MR 2207903](#)
- [So1]     T. Soma, Non-zero degree maps onto hyperbolic 3-manifolds. *J. Differential Geom.* **49** (1998), 517–546. [Zbl 0990.57005](#) [MR 1669645](#)
- [So2]     T. Soma, Sequences of degree-one maps between geometric 3-manifolds. *Math. Ann.* **316** (2000), 733–742. [Zbl 0955.57013](#) [MR 1758451](#)
- [So3]     T. Soma, Degree-one maps between hyperbolic 3-manifolds with the same volume limit. *Trans. Amer. Math. Soc.* **353** (2001), no. 7, 2753–2772. [Zbl 0969.57012](#) [MR 1828472](#)
- [So4]     T. Soma, Epimorphism sequences between hyperbolic 3-manifold groups. *Proc. Amer. Math. Soc.* **130** (2002), 1221–1223. [Zbl 0996.57010](#) [MR 1873800](#)
- [So5]     T. Soma, The Gromov invariant of links. *Invent. Math.* **64** (1981), 445–454. [Zbl 0478.57006](#) [MR 0632984](#)
- [Th1]     W. P. Thurston, *The geometry and topology of 3-manifolds*. Princeton University Lecture Notes, Princeton, NY, 1979.
- [Th2]     W. P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.* **316** (1982), 357–381. [Zbl 0496.57005](#) [MR 0648524](#)

- [Th3] W. P. Thurston, A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.* **339** (1986), 99–130. [Zbl 0585.57006](#) [MR 0823443](#)
- [To] J. L. Tollefson, Involutions of sufficiently large 3-manifold. *Topology* **20** (1981), 323–352. [Zbl 0469.57031](#) [MR 0617370](#)
- [W1] S. C. Wang, The existence of maps of non-zero degree between aspherical 3-manifolds. *Math. Z.* **208** (1991), 147–160. [Zbl 0737.57006](#) [MR 1125739](#)
- [W2] S. C. Wang, Non-zero degree one maps between 3-manifolds. In *Proceedings of the International Congress of Mathematicians* (Beijing, 2002), Vol. II, 457–468, Higher Ed. Press, Beijing 2002. [Zbl 1009.57025](#) [MR 1957056](#)
- [WZ] S. C. Wang and Q. Zhou, Any 3-manifold 1-dominates at most finitely many geometric 3-manifolds. *Math. Ann.* **322** (2002), 525–535. [Zbl 0992.57017](#) [MR 1895705](#)

Received November 25, 2008; revised July 6, 2011

Michel Boileau, Institut Mathématique de Toulouse, CNRS UMR 5219, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse Cedex 9, France

E-mail: [boileau@math.univ-toulouse.fr](mailto:boileau@math.univ-toulouse.fr)

J. Hyam Rubinstein, Department of Mathematics and Statistics, University of Melbourne, Parkville VIC. 3010, Australia

E-mail: [rubin@ms.unimelb.edu.au](mailto:rubin@ms.unimelb.edu.au)

Shicheng Wang, LAMA Department of Mathematics, Peking University, Beijing 100871, China

E-mail: [wangsc@math.pku.edu.cn](mailto:wangsc@math.pku.edu.cn)