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Removable and essential singular sets for higher dimensional conformal maps

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Abstract. In this article, we prove several results about the extension to the boundary of conformal immersions from an open subset Ω of a Riemannian manifold L into another Riemannian manifold N of the same dimension. In dimension $n \geq 3$, and when the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$ is zero, we completely classify the cases when $\partial\Omega$ contains essential singular points, showing that L and N are conformally flat and making the link with the theory of Kleinian groups.

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1. Introduction

The aim of this paper is to make progress toward the understanding of singular sets for conformal maps between Riemannian manifolds of dimension at least 3. The general problem we are considering can be stated very easily: assume that (L, g) and (N, h) are two smooth, connected, Riemannian manifolds of same dimension $n \geq 2$, and assume that we have a smooth immersion $s\colon L \setminus \Lambda \to N$, from the complement of a closed subset $\Lambda \subset L$, to the manifold N, which is conformal, namely $s^*h = e^{\varphi} g$ for some smooth function φ on $L \setminus \Lambda$. The set Λ is called a *singular set* for the conformal immersion s, and a data $s\colon L \setminus \Lambda \to N$ as above is referred to as a *conformal singularity*. A basic question is to understand under which conditions the singular set Λ is removable, namely it is possible to extend s "across" Λ.

The main contribution of the article is an almost complete understanding of the situation when the dimension n is at least 3, and the $(n-1)$-dimensional Hausdorff measure of Λ, denoted $\mathcal{H}^{n-1}(\Lambda)$, is zero. Under those assumptions, our principal result is Theorem 1.3, stated in Section 1.2 below, which yields a local classification of *essential* conformal singularities, namely those for which $s\colon L \setminus \Lambda \to N$ does not extend to a continuous map from L into the one-point compactification of N. Theorem 1.3 implies that such essential singular sets can only occur when L and N are

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conformally flat, and moreover N is a Kleinian manifold. As a consequence, except in very peculiar situations that are completely classified, singular sets with $\mathcal{H}^{n-1}(\Lambda) = 0$ are removable (maybe adding a point at infinity to N, when N is noncompact), and the extended map is still a conformal immersion (see Theorem 1.1). Finally, under the extra assumption that L is compact and the $(n-2)$-dimensional Hausdorff measure of Λ is zero, we also classify globally essential conformal singularities in Theorem 1.4: in this case L and N are both Kleinian manifolds.

Since conformal immersions are very peculiar instances in the much larger class of quasiregular mappings, it is natural, before describing our results into more details, to mention the existing theorems about removable sets and boundary behavior of quasiregular maps. Quasiregular mappings (see [IM], [R2], [V1] for comprehensive introductions to the subject) are usually presented as the "good" higher dimensional generalization of holomorphic functions of one complex variable. And indeed, classical theorems of function theory, such as Picard's theorem, or Painlevé's theorem on removable sets, find analogous statements in the framework of quasiregular mappings (see for instance [R1], [R3], [V2]). Most of those results, though, only deal with quasiregular mappings between domains of the extended space $\overline{\mathbb{R}}^n$. Although more recent works (for instance [BH], [HP], [P] and [Zo1], among others) aimed at some generalizations involving broader classes of target manifolds N, they do not help much for the problem we are considering, except in very peculiar cases. Moreover, let us stress that the tools used in the theory of quasiregular mappings involve elaborate analysis, while the very rigid behavior displayed by conformal immersions in higher dimension allow to settle the problem in the conformal framework by purely geometric arguments. Actually, we hope that the ideas introduced here will be helpful to study removable and essential singular sets for conformal structures which are not Riemannian, the Lorentz signature being of particular interest, and maybe for other geometric structures of the same kind, such as Cartan geometries.

1.1. Extension results. Throughout the paper, manifolds and maps between them are assumed to be smooth.

We consider as above a conformal immersion $s: L \setminus \Lambda \to N$, where (L, g) and (N, h) are two connected Riemannian manifolds of dimension $n \geq 3$. The conformal structure on $L \setminus \Lambda$ is that induced by (L, g). We will assume that $\mathcal{H}^{n-1}(\Lambda) = 0$, where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure on (M, g) (we refer to Chapter 4 of [Ma] for basic notions on Hausdorff measures). In particular, $L \setminus \Lambda$ is connected and dense in L. In the sequel, those sets satisfying the condition $\mathcal{H}^{n-1}(\Lambda) = 0$ will be referred to as *thin singular sets*. The points of a (thin) singular set Λ split naturally into three categories.

– The *removable singular points* are those $x_\infty \in \Lambda$ at which the map s extends continuously. In other words, there exists a point $y \in N$ so that for every sequence (x_k) of $L \setminus \Lambda$ converging to x_∞, the sequence $s(x_k)$ tends to y.

– *The poles* are those points $x_\infty \in \Lambda$ such that for every sequence (x_k) of $L \setminus \Lambda$ converging to x_∞, the sequence $s(x_k)$ leaves every compact subset of N.

– Finally, the points of Λ which are neither removable, nor poles are *essential singular points*.

One thus gets a partition $\Lambda = \Lambda_{\text{rem}} \cup \Lambda_{\text{pole}} \cup \Lambda_{\text{ess}}$ into removable singular points, poles and essential singular points. The results of this article will allow to determine the structure of those three sets for *thin singularities*. We begin with Λ_{rem}.

Theorem 1.1. *Let (L,g) and (N,h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$, and $s\colon L \setminus \Lambda \to N$ a conformal immersion. Then the set Λ_{rem} is open in Λ and s extends to a conformal immersion $s'\colon L \setminus (\Lambda_{\text{pole}} \cup \Lambda_{\text{ess}}) \to N$.*

In view of this result, it will be interesting to find criteria ensuring that Λ_{ess} is empty. We will prove in Theorem 3.7 that under the condition $\mathcal{H}^{n-1}(\Lambda) = 0$, an injectivity assumption on the immersion s is enough for that.

1.2. Local classification of thin essential singularities. Our next step will be to understand, when it is nonempty, the set Λ_{ess} of essential singular points. First, we introduce the following definition.

Definition 1.2 (Essential singular set). Let $s\colon L \setminus \Lambda \to N$ be a conformal singularity. We will say that Λ is an *essential singular set* as soon as $\Lambda_{\text{ess}} \neq \emptyset$. When $\Lambda_{\text{rem}} = \emptyset$ and $\Lambda_{\text{ess}} \neq \emptyset$, we will say that Λ is minimal essential.

The reader might like to see examples of conformal immersions admitting (minimal) essential singularities. That's what we do quickly now, referring to Section 4.1 for more details on the construction. Let Γ be an *infinite* Kleinian group, namely a discrete subgroup of the Möbius group $\mathrm{PO}(1, n+1)$ acting properly on a nonempty open subset $\Omega \subset \mathbf{S}^n$. We assume that the action of Γ on Ω is free and denote by $N := \Omega / \Gamma$ the corresponding Kleinian manifold. The conformal covering map $\pi\colon \mathbf{S}^n \setminus \Lambda \to N$, where Λ stands for the complement of Ω in $\mathbf{S}^n$, is an instance of conformal singularity which, under our assumption that Γ is infinite, turns out to be essential. Actually (see Section 4.1), $\Lambda = \Lambda_{\text{ess}} \cup \Lambda_{\text{pole}}$, and Λ_{ess} coincides with the limit set $\Lambda(\Gamma)$ of Γ. Such conformal singularities will be said to be *of Kleinian type*.

Our main result says basically that locally, all thin conformal singularities which are minimal essential (see Definition 1.2) are of Kleinian type. In particular, the existence of essential singular points imposes strong restrictions on the geometry: the source manifold must be conformally flat, and the target manifold has to be Kleinian. It is interesting to notice that this geometric restriction does not appear in dimension two, where all Riemannian manifolds are conformally flat.

Observe that in view of Theorem 1.1, studying thin singular sets which are essential reduces to studying minimal essential ones.

Theorem 1.3. *Let (L,g) and (N,h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$. Assume that $s\colon L \setminus \Lambda \to N$ is a conformal immersion for which Λ is a minimal essential singular set. Then:*

(1) *There exist an infinite Kleinian group $\Gamma \subset \mathrm{PO}(1,n+1)$, a connected open set $\Omega \subset \mathrm{S}^n$ on which Γ acts freely properly discontinuously, and a conformal diffeomorphism $\psi\colon N \to \Omega / \Gamma$.*

(2) *For each $x_\infty \in \Lambda$, there exist an open neighborhood $U \subset L$ containing x_∞, and a conformal diffeomorphism $\varphi\colon U \to V$, where V is an open subset of S^n, which makes the following diagram commute:*

$$\begin{tikzcd} U \setminus \Lambda \arrow[r, "\varphi"] \arrow[d, "s"] & V \setminus \partial\Omega \arrow[d, "\pi"] \\ N \arrow[r, "\psi"] & \Omega / \Gamma, \end{tikzcd}$$

where $\pi\colon \Omega \to \Omega / \Gamma$ is the covering map. In particular, $\varphi(U \cap \Lambda) = V \cap \partial\Omega$ and $\varphi(U \cap \Lambda_{\mathrm{ess}}) = V \cap \Lambda(\Gamma)$, where $\Lambda(\Gamma)$ denotes the limit set of the group Γ.

In Corollary 5.5, we will derive from Theorem 1.3 precise information about the behavior of a conformal immersion near an essential singular point. In particular, we will get an higher dimensional analogue of Picard's theorem.

1.3. Global classification of essential singularities. Theorem 1.3 describes completely the geometry of the target manifold N, for a thin essential conformal singularity $s\colon L \setminus \Lambda \to N$. The local geometry of L is also determined, but in full generality, we cannot expect to determine L globally. Now, if we assume that L is compact, and under the stronger assumption that the singular set has $(n-2)$-dimensional Hausdorff measure zero, the singularity $s\colon L \setminus \Lambda \to N$ can be described globally. In the statement below, for a Kleinian group Γ, we will denote by $\Lambda(\Gamma)$ the limit set of Γ, $\Omega(\Gamma) = \mathrm{S}^n \setminus \Lambda(\Gamma)$ its domain of discontinuity, and $M(\Gamma)$ the quotient $\Omega(\Gamma)/\Gamma$ (see Section 4.1 for the definitions).

Theorem 1.4. *Let (L,g) and (N,h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. We assume that L is compact. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-2}(\Lambda) = 0$. Assume that $s\colon L \setminus \Lambda \to N$ is a conformal immersion for which Λ is a minimal essential singular set. Then:*

(1) *There exists an infinite Kleinian group* $\Gamma \subset \mathrm{PO}(1, n+1)$*, a connected open subset* $\Omega \subset \mathrm{S}^n$ *on which* Γ *acts freely properly discontinuously, and a conformal diffeomorphism* $\psi : N \to \Omega / \Gamma$.

(2) *There exists a subgroup* $\Gamma' \subset \Gamma$ *with* $\Lambda(\Gamma') \subsetneq \Lambda(\Gamma)$ *such that* Γ' *acts freely properly discontinuously on* $\Omega(\Gamma')$*, and a conformal diffeomorphism* $\varphi : L \to M(\Gamma')$.

(3) *Let us call* $s' : \Omega / \Gamma' \to \Omega / \Gamma$ *the natural covering map, and let us define the closed subsets* Λ' *and* Λ'_{ess} *in* $M(\Gamma')$ *as the quotients* $(\partial\Omega \setminus \Lambda(\Gamma')) / \Gamma'$ *and* $(\Lambda(\Gamma) \setminus \Lambda(\Gamma')) / \Gamma'$*. Then the conformal diffeomorphism* φ *can be chosen such that* $\varphi(\Lambda) = \Lambda'$, $\varphi(\Lambda_{\mathrm{ess}}) = \Lambda'_{\mathrm{ess}}$*, and the following diagram commutes:*

$$\begin{tikzcd} L \setminus \Lambda \arrow[r, "\varphi"] \arrow[d, "s"] & M(\Gamma') \setminus \Lambda' \arrow[d, "s'"] \\ N \arrow[r, "\psi"] & \Omega / \Gamma. \end{tikzcd}$$

We will apply this theorem to get a full description of punctured essential singularities on compact manifolds in Theorem 7.1.

1.4. Organization of the paper. As we already mentioned it, the tools used in this paper are of geometric nature. Especially, the proofs heavily rely on the interpretation of conformal structures (in dimension ≥ 3) in terms of Cartan geometries. The necessary background on this topic, as well as the first technical results, are introduced in Section 2. They allow to begin the study of conformal singularities in Section 3. The main point is to understand the behavior of the 2-jet of a conformal immersion in the neighborhood of the singular set, as explained in Section 3.1. Theorem 1.1 is proved in Sections 3.2, and 3.3 contains another extension result for conformal embeddings, namely Theorem 3.7. In Section 3.4, we show that thin essential singular sets only occur on conformally flat manifolds, an important step toward Theorem 1.3.

Section 4 reviews some basic results about conformally flat structures. The reader familiar with this material may skip it, except maybe for Section 4.1 which gives more details about essential singularities of Kleinian types, and Section 4.3 which deals with the notion of Cauchy completion for conformally flat structures. This preparatory work allows to complete the proofs of Theorems 1.3 and 1.4 in Sections 5 and 6 respectively. We conclude the paper with Section 7, which provides a full description of punctured essential singularities on compact Riemannian manifolds.

2. Conformal structures and Cartan connections

Let (L, g) be a Riemannian manifold *of dimension* $n \geq 3$. Let $\hat{L}$ be the bundle of 2-jets of orthogonal frames on L, and $\pi_L : \hat{L} \to L$ the bundle map. The bundle $\hat{L}$ is a P-principal bundle over L, where P is the conformal group of the Euclidean space $\mathbb{R}^n$. The group P is a semi-direct product $(\mathbb{R}_+^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n$, where the factor $\mathbb{R}_+^*$ corresponds to homothetic transformations of positive ratio, $\mathrm{O}(n)$ is the group of linear orthogonal transformations, and $\mathbb{R}^n$ is identified with the subgroup of translations. Let S^n be the n-dimensional sphere, and $G := \mathrm{PO}(1, n+1)$ the *Möbius group*, namely the group of conformal transformations of the sphere. The group P is realized as the subgroup of G fixing a point $\nu \in \mathrm{S}^n$. We denote by $\mathfrak{g} := \mathfrak{o}(1, n+1)$ the Lie algebra of the Möbius group, and by $\mathfrak{p} \subset \mathfrak{o}(1, n+1)$ the Lie algebra of P.

2.1. Canonical Cartan connection associated to a conformal structure. Good references for the material presented in this section are Chapter IV of [Ko] and Chapter 7 of [Sh].

It is a fundamental fact, known since Elie Cartan, that under the assumption $n \geq 3$, the conformal class $[g]$ defines on the bundle $\hat{L}$ a *unique* normal Cartan connection ω^L with values in $\mathfrak{o}(1, n+1)$. The connection ω^L is a 1-form on $\hat{L}$ with values in the Lie algebra $\mathfrak{o}(1, n+1)$, and satisfying the following properties:

(1) For every $\hat{x} \in \hat{L}$, $\omega^L_{\hat{x}} : T_{\hat{x}}\hat{L} \to \mathfrak{o}(1, n+1)$ is an isomorphism of vector spaces.

(2) For every $X \in \mathfrak{p}$, the vector field $\widehat{X}$ on $\hat{L}$ defined by $\widehat{X}(\hat{x}) := \frac{d}{dt}_{|t=0}\, \hat{x}.e^{tX}$, where $Y \mapsto e^Y$ denotes the exponential map on $\mathrm{PO}(1, n+1)$, satisfies $\omega^L(\widehat{X}) = X$.

(3) For every $p \in P$, if R_p denotes the right action by p on $\hat{L}$, then $(R_p)^*\omega^L = \mathrm{Ad}\, p^{-1}\omega^L$.

The *normality condition* is put on the curvature of the connection to ensure uniqueness. The reader will find a precise statement of this condition in [Ko], Theorem 4.2, p. 135, or [Sh], Proposition 3.1, p. 285. The triple $(L, \hat{L}, \omega^L)$ will be referred to as the *normal Cartan bundle* associated to the conformal structure (L, g). For the conformally flat model $\mathrm{S}^n = \mathrm{PO}(1, n+1)/P$, the normal Cartan bundle is the Möbius group $G = \mathrm{PO}(1, n+1)$, and the Cartan connection is the Maurer–Cartan form ω^G.

Let us observe that if (L, g) and (N, h) are two connected n-dimensional Riemannian manifolds, $n \geq 3$, and if $s : (L, g) \to (N, h)$ is a smooth immersion, then s lifts to an immersion $\hat{s}$ between the bundles of 2-jets of frames of L and N respectively. If moreover s is conformal, $\hat{s}$ maps 2-jets of orthogonal frames to 2-jets of orthogonal frames. This yields a bundle map $\hat{s} : \hat{L} \to \widehat{N}$ lifting s. The 1-form $\hat{s}^*\omega^N$ is a Cartan connection on $\hat{L}$, with values in $\mathfrak{o}(1, n+1)$. Because ω^N is the normal Cartan connection associated to $[h]$, and because the normality condition is tensorial on the curvature of the connection, we get that $\hat{s}^*\omega^N$ also satisfies the normality condition.

By uniqueness of the normal Cartan connection, one must have $\hat{s}^*\omega^N = \omega^L$. We say that the lift $\hat{s}$ is a *geometric immersion* from $(\hat{L}, \omega^L)$ to $(\hat{N}, \omega^N)$.

2.2. Exponential map. On the bundle $\hat{L}$, the Cartan connection ω^L yields an exponential map in the following way. The data of u in $\mathfrak{o}(1, n+1)$ defines naturally a ω^L-constant vector field $\widehat{U}$ on $\hat{L}$ by the relation $\omega^L(\widehat{U}) = u$. We call ϕ_u^t the local flow generated on $\hat{L}$ by the field $\widehat{U}$. At each $\hat{x} \in \hat{L}$, let $\mathcal{W}_{\hat{x}} \subset \mathfrak{o}(1, n+1)$ be the set of vectors u such that ϕ_u^t is defined for $t \in [0, 1]$ at $\hat{x}$. Then one defines the exponential map at $\hat{x}$ as follows:

$$\exp(\hat{x}, u) := \phi_u^1.\hat{x} \quad \text{for all } u \in \mathcal{W}_{\hat{x}}.$$

Using the equivariance properties of the Cartan connection listed above, one shows easily the following important equivariance property for the exponential map

$$\exp(\hat{x}, u).p^{-1} = \exp(\hat{x}.p^{-1}, (\mathrm{Ad}\ p).u) \tag{1}$$

for every $u \in \mathcal{W}_{\hat{x}},\ \ p \in P$.

2.3. Injectivity radius. The Lie algebra $\mathfrak{o}(1, n+1)$ splits as a sum

$$\mathfrak{n}^- \oplus \mathbb{R} \oplus \mathfrak{o}(n) \oplus \mathfrak{n}^+$$

where $\mathfrak{p} = \mathbb{R} \oplus \mathfrak{o}(n) \oplus \mathfrak{n}^+$ is the Lie algebra of P. The algebra corresponding to the factor $\mathbb{R}$ is a Cartan subalgebra. The two abelian n-dimensional subalgebras $\mathfrak{n}^-$ and $\mathfrak{n}^+$ are the root spaces. They are left invariant by the adjoint action of $\mathbb{R} \oplus \mathfrak{o}(n)$. A detailed description of this material can be found in [Sh], Chapter 7. As we saw, the group P is a semi-direct product $P = (\mathbb{R}_+^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n$. We put on $\mathfrak{o}(1, n+1)$ a scalar product $\langle\, , \rangle$ which is $\mathrm{Ad}\, \mathrm{O}(n)$-invariant, and denote by $\|.\|$ the norm it induces on $\mathfrak{o}(1, n+1)$. For every $\lambda > 0$, we will denote $B_{\mathfrak{n}^-}(\lambda)$ (resp. $\overline{B}_{\mathfrak{n}^-}(\lambda)$) the open (resp. closed) ball of center 0 and radius λ in $\mathfrak{n}^-$, for the norm $\|.\|$. The map $u \mapsto \exp(\hat{x}, u)$ is a diffeomorphism from a sufficiently small neighborhood of $0 \in \mathfrak{o}(1, n+1)$ onto its image. Notice also that because $(\omega_{\hat{x}}^L)^{-1}(\mathfrak{n}^-)$ is transverse to $T_{\hat{x}}(\pi_L^{-1}(x)) = (\omega_{\hat{x}}^L)^{-1}(\mathfrak{p})$, the map $u \mapsto \pi_L \circ \exp(\hat{x}, u)$ is a diffeomorphism from a sufficiently small neighborhood of 0 in $\mathfrak{n}^-$ onto its image. We can then define the *injectivity radius at* $\hat{x}$ as

$$\mathrm{inj}_L(\hat{x}) := \inf\{\lambda > 0 \mid u \mapsto \pi_L \circ \exp(\hat{x}, u) \text{ defines an embedding on } B_{\mathfrak{n}^-}(\lambda)\}.$$

By the above remarks, $\mathrm{inj}_L(\hat{x}) > 0$, and actually $\mathrm{inj}_L(\hat{x})$ is bounded from below on compact subsets of $\hat{L}$.

2.4. Conformal balls, conformal cones. We stick to the notations introduced above. Let $S_{\mathfrak{n}^-}$ be the unit sphere of $\mathfrak{n}^-$, with respect to the norm $\|.\|$. Let $\mathcal{F}$ be a subset of $S_{\mathfrak{n}^-}$. In $\mathfrak{n}^-$, we define the cone over $\mathcal{F}$ of radius $\lambda > 0$ as

$$\mathcal{C}(\mathcal{F},\lambda) = \{v \in \mathfrak{n}^- \mid v = tw,\ t \in [0,\lambda],\ w \in \mathcal{F}\}.$$

For $x \in L$, $\hat{x} \in \hat{L}$ in the fiber of x, $0 < \lambda < \mathrm{inj}_L(\hat{x})$, and $\mathcal{F} \subset S_{\mathfrak{n}^-}$, we can define:

– $B_{\hat{x}}(\lambda) := \pi_L \circ \exp(\hat{x}, B_{\mathfrak{n}^-}(\lambda))$, a *conformal ball at* x;

– $C_{\hat{x}}(\mathcal{F},\lambda) := \pi_L \circ \exp(\hat{x}, \mathcal{C}(\mathcal{F},\lambda))$, a *conformal cone with vertex* x.

In the model space, namely the standard n-sphere $\mathrm{S}^n = \mathrm{PO}(1,n+1)/P$, we will simply consider conformal cones with vertex ν, defined by

$$C(\mathcal{F},\lambda) := \pi_G \circ \exp_G(\mathcal{C}(\mathcal{F},\lambda)),$$

where $\pi_G\colon \mathrm{PO}(1,n+1) \to \mathrm{S}^n$ is the bundle map and $\exp_G$ is the exponential map in $G = \mathrm{PO}(1,n+1)$.

Of course, a conformal immersion $s\colon L \to N$ maps conformal balls/cones of L to conformal balls/cones of N. Indeed, it is straightforward to check the relation

$$s(C_{\hat{x}_k}(\mathcal{F},\lambda)) = C_{\hat{s}(\hat{x}_k)}(\mathcal{F},\lambda). \tag{2}$$

2.5. Dynamics of Möbius maps on conformal cones of S^n

Lemma 2.1. *Let (p_k) be a sequence of P tending to infinity. Then, considering a subsequence of (p_k) if necessary, we are in one of the following cases:*

(1) *For every ball $\mathcal{B} \subset S_{\mathfrak{n}^-}$ (for the metric induced by $\|.\|$) with nonzero radius, there exists $\mathcal{B}' \subset \mathcal{B}$ a subball with nonzero radius and a real $r > 0$ such that for every $0 < \lambda \leq r$, $p_k.C(\mathcal{B}',\lambda) \to \nu$ for the Hausdorff topology as $k \to \infty$.*

(2) *There exists a sequence (l_k) of P converging to l_∞ such that $l_k p_k$ stays in the factor $\mathbb{R}_+^*$ of $P = (\mathbb{R}_+^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n$, and $(\mathrm{Ad}\, l_k p_k)(u) = \frac{1}{\lambda_k}u$ for every $u \in \mathfrak{n}^-$, with $\lim_{k\to\infty}\lambda_k = 0$. In particular, for every ball $\mathcal{B} \subset S_{\mathfrak{n}^-}$ with nonzero radius, $p_k.C(\mathcal{B},\lambda_k) \to l_\infty^{-1}.C(\mathcal{B},1)$ for the Hausdorff topology as $k \to \infty$.*

Proof. We keep the notation introduced in Section 2, especially Sections 2.2 and 2.4. In particular, recall the splitting

$$\mathfrak{o}(1,n+1) = \mathfrak{g} = \mathfrak{n}^- \oplus \mathbb{R} \oplus \mathfrak{o}(n) \oplus \mathfrak{n}^+,$$

where $\mathfrak{p}$ corresponds to $\mathbb{R} \oplus \mathfrak{o}(n) \oplus \mathfrak{n}^+$.

We introduce the map $\rho\colon \mathfrak{n}^- \to \mathrm{S}^n$ defined by $u \mapsto \exp_G(u).\nu$. It is a diffeomorphism between $\mathfrak{n}^-$ and the sphere minus a point o. Precomposing the stereographic projection with vertex ν with a suitable element of P, one gets a conformal diffeomorphism $j : \mathrm{S}^n \setminus \{\nu\} \to \mathbb{R}^n$ mapping the point o to the origin. The map j intertwines the action of P on $\mathrm{S}^n \setminus \{\nu\}$ and the affine action of $(\mathbb{R}_+^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n$ on $\mathbb{R}^n$. In the following, we will thus write the elements of P in the affine form $\lambda A + T$, with $\lambda \in \mathbb{R}_+^*$, $A \in \mathrm{O}(n)$, and $T \in \mathbb{R}^n$.

Let us denote by φ the map $j \circ \rho$. It is a diffeomorphism from $\mathfrak{n}^- \setminus \{0\}$ to $\mathbb{R}^n \setminus \{0\}$. For a suitable choice of the $(\mathrm{Ad}\,\mathrm{O}(n))$- invariant scalar product $\langle\, , \,\rangle$ (see Section 2.3), φ maps $S_{\mathfrak{n}^-}$ onto the Euclidean unit sphere. It is then not hard to check that every conformal cone $C(\mathcal{B}, \lambda)$, with ν removed, is mapped by j to the set

$$\widetilde{C}(\mathcal{B}, \lambda) = \left\{x = tu \in \mathbb{R}^n \mid t \in \left[\tfrac{1}{\lambda}; \infty\right[\ ,\ u \in \varphi(\mathcal{B})\right\}.$$

Let $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^n$ of Euclidean norm 1. Then we define the *half-line* $[x, u)$ as the set

$$[x, u) := \{x + tu \in \mathbb{R}^n \mid t \in \mathbb{R}_+\}.$$

The following lemma, the proof of which is left to the reader, gives a sufficient condition for a sequence of half-lines to leave every compact subset of $\mathbb{R}^n$. The notation $\|.\|$ stands for the Euclidean norm on $\mathbb{R}^n$.

Lemma 2.2. *Let $[x_k, v_k)$ be a sequence of half-lines in $\mathbb{R}^n$. Assume that whenever v_∞ is a cluster value of (v_k), then $-v_\infty$ is not a cluster value of $\frac{x_k}{\|x_k\|}$. Assume moreover that x_k leaves every compact subset of $\mathbb{R}^n$. Then $[x_k, v_k)$ leaves every compact subset of $\mathbb{R}^n$.*

We can now begin the proof of Lemma 2.1. Let us consider an unbounded sequence (p_k) in P. Thanks to the chart j, we see P as the conformal group of $\mathbb{R}^n$. Then the sequence (p_k) can be written as

$$p_k \colon x \mapsto \lambda_k A_k x + \mu_k u_k,$$

where $\lambda_k \in \mathbb{R}_+^*$, $\mu_k \in \mathbb{R}_+$, $A_k \in \mathrm{O}(n)$, and $\|u_k\| = 1$. Now, looking at a subsequence if necessary, we assume that $\lambda_k, \mu_k, \frac{\lambda_k}{\mu_k}$ all have limits in $\mathbb{R}_+^* \cup \{+\infty\}$, $u_k \to u_\infty$, and $A_k \to A_\infty$ in $\mathrm{O}(n)$. The conclusions of Lemma 2.1 won't be affected if we replace p_k by $(A_k)^{-1}.p_k$ so that we may assume $p_k = \lambda_k \mathrm{Id} + \mu_k u_k$.

- *First case*: μ_k *tends to* $a \in \mathbb{R}_+$. Let l_k be the translation of vector $-\mu_k u_k$. Clearly, $l_k \to l_\infty$ in P, where l_∞ is the translation of vector $-a u_\infty$, and $l_k p_k$ is just the homothetic transformation $x \mapsto \lambda_k x$, hence is in the factor $\mathbb{R}_+^*$ of P. It follows immediately that $(\mathrm{Ad}\, l_k p_k)(u) = \frac{1}{\lambda_k} u$ for every $u \in \mathfrak{n}^-$. Since (p_k) is unbounded, we can assume after taking a subsequence that $\lambda_k \to \infty$ or $\lambda_k \to 0$.

  If $\lambda_k \to \infty$, then for every $\lambda > 0$, $(\mathrm{Ad}\, l_k p_k)(\mathcal{C}(\mathcal{B}, \lambda)) \to 0_{\mathfrak{n}^-}$ as $k \to \infty$. Applying the map ρ, one gets

$$l_k p_k.C(\mathcal{B}, \lambda) \to \nu$$

and we are in the first case of Lemma 2.1.

If $\lambda_k \to 0$, we are in the second case of Lemma 2.1. Applying the map ρ to the equality $(\mathrm{Ad}\, l_k p_k)(\mathcal{C}(\mathcal{B}, \lambda_k)) = \mathcal{C}(\mathcal{B}, 1)$, we get

$$l_k p_k . C(\mathcal{B}, \lambda_k) = C(\mathcal{B}, 1) \quad \text{for every } k \in \mathbb{N},$$

hence

$$p_k . C(\mathcal{B}, \lambda_k) \to l_\infty^{-1} . C(\mathcal{B}, 1).$$

- *Second case*: $\mu_k \to \infty$ *and* $\frac{\lambda_k}{\mu_k} \to 0$. Let $\mathcal{B}' \subset \mathcal{B}$ be a closed subball with nonzero radius such that $-u_\infty \notin \varphi(\mathcal{B}')$. Let us consider $\lambda > 0$ and a sequence of half-lines $[\frac{1}{\lambda} v_k, v_k)$ in $\widetilde{C}(\mathcal{B}', \lambda)$. Here (v_k) is a sequence of $\varphi(\mathcal{B}')$. We observe that $p_k . [\frac{1}{\lambda} v_k, v_k) = [x_k, v_k)$, where $x_k = \frac{\lambda_k}{\lambda} v_k + \mu_k u_k$. Now, $\frac{x_k}{\|x_k\|} = \frac{\frac{\lambda_k}{\mu_k}\frac{1}{\lambda} v_k + u_k}{\|\frac{\lambda_k}{\mu_k}\frac{1}{\lambda} v_k + u_k\|}$ so that the only cluster value of $\frac{x_k}{\|x_k\|}$ is u_∞. We infer that if v_∞ is a cluster value of (v_k), then $-v_\infty$ cannot be a cluster value of $\frac{x_k}{\|x_k\|}$. Writing $x_k = \mu_k(\frac{\lambda_k}{\lambda \mu_k} v_k + u_k)$, we check that $x_k \to \infty$. Lemma 2.2 ensures that $p_k . [\frac{1}{\lambda} v_k, v_k) \to \infty$. Since it is true for every sequence $[\frac{1}{\lambda} v_k, v_k)$, we get $p_k . \widetilde{C}(\mathcal{B}', \lambda) \to \infty$. Hence $p_k . C(\mathcal{B}', \lambda) \to \nu$ and we are in the first case of Lemma 2.1.
- *Third case*: $\mu_k \to \infty$ *and* $\frac{\lambda_k}{\mu_k} \to b_\infty$, *with* $b_\infty \in \mathbb{R}_+^*$. We choose $\mathcal{B}' \subset \mathcal{B}$ a closed subball with nonzero radius such that $\varphi(\mathcal{B}') \cap -\varphi(\mathcal{B}') = \emptyset$ and $\varphi(\mathcal{B}') \cap \{u_\infty; -u_\infty\} = \emptyset$. For such a choice of $\mathcal{B}'$, there exist an open neighborhood $\mathcal{W}$ of $\varphi(\mathcal{B}')$ in the Euclidean unit sphere and β, η two positive reals such that

$$\inf_{(v,w) \in \varphi(\mathcal{B}') \times \mathcal{W}} \|v + w\| \geq \beta \tag{3}$$

and $\frac{x+z}{\|x+z\|} \in \mathcal{W}$ for every $x \in \varphi(\mathcal{B}')$ and every $z \in \mathbb{R}^n$ with $\|z\| \leq \eta$.

Let us put $r = \frac{b_\infty \eta}{2}$. For $\lambda \leq r$, let us consider a sequence of half-lines $[\frac{1}{\lambda} v_k, v_k)$ in $\widetilde{C}(\mathcal{B}', \lambda)$, where $v_k \in \varphi(\mathcal{B}')$. We observe that $p_k . [\frac{1}{\lambda} v_k, v_k) = [x_k, v_k)$, where $x_k = \frac{\lambda_k}{\lambda} v_k + \mu_k u_k$. Now, $\frac{x_k}{\|x_k\|} = \frac{v_k + \frac{\mu_k \lambda}{\lambda_k} u_k}{\|v_k + \frac{\mu_k \lambda}{\lambda_k} u_k\|}$, and for k large enough, $|\frac{\mu_k \lambda}{\lambda_k}| \leq \frac{2\lambda}{b_\infty} \leq \eta$ so that (3) implies

$$\left\| \frac{x_k}{\|x_k\|} + v_k \right\| \geq \beta.$$

It follows that if v_∞ is a cluster value of (v_k), then $-v_\infty$ cannot be a cluster value of $\frac{x_k}{\|x_k\|}$. Moreover, because $-u_\infty \notin \varphi(\mathcal{B}')$, 0 is not a cluster value of

$(\frac{\lambda_k}{\mu_k \lambda} v_k + u_k)$. Writing $x_k = \mu_k(\frac{\lambda_k}{\mu_k \lambda} v_k + u_k)$, we see that (x_k) tends to infinity. We conclude thanks to Lemma 2.2 that $p_k.[\frac{1}{\lambda} v_k, v_k) \to \infty$. Since it is true for every sequence (v_k) of $\varphi(\mathcal{B}')$, we get $p_k.\widetilde{C}(\mathcal{B}', \lambda) \to \infty$, and we are in the first case of Lemma 2.1.

- *Fourth case*: $\mu_k \to \infty$ *and* $\frac{\lambda_k}{\mu_k} \to \infty$. Let $\mathcal{B}' \subset \mathcal{B}$ be a closed subball with nonzero radius such that $\varphi(\mathcal{B}') \cap -\varphi(\mathcal{B}') = \emptyset$. Let us consider $\lambda > 0$, and $[\frac{1}{\lambda} v_k, v_k)$ a sequence of half-lines in $\widetilde{C}(\mathcal{B}', \lambda)$. For each integer k, $p_k.[\frac{1}{\lambda} v_k, v_k) = [x_k, v_k)$, with $x_k = \mu_k(\frac{\lambda_k}{\lambda \mu_k} v_k + u_k)$. It is clear that $x_k \to \infty$. The cluster values of $\frac{x_k}{\|x_k\|} = \frac{\frac{1}{\lambda} v_k + \frac{\mu_k}{\lambda_k} u_k}{\|\frac{1}{\lambda} v_k + \frac{\mu_k}{\lambda_k} u_k\|}$ are those of (v_k), hence are contained in $\varphi(\mathcal{B}')$. We use once more Lemma 2.2 and conclude
$$p_k.\widetilde{C}(\mathcal{B}', \lambda) \to \infty.$$
We are again in the first case of Lemma 2.1. □

2.6. Degeneration of conformal cones. We consider now a Riemannian manifold (L, g) of dimension $n \geq 3$. Our aim is to understand how the "shape" of a sequence of conformal cones $C_{\hat{z}_k}(\mathcal{F}, \lambda)$ evolves, as $\hat{z}_k$ leaves every compact subset in $\hat{L}$. The answer is partly contained in the lemma below.

Lemma 2.3. *Let (L, g) be a Riemannian manifold of dimension ≥ 3 and $(\hat{L}, \omega^L)$ the normal Cartan bundle associated to the conformal structure of g. Let (z_k) be a sequence of L converging to $z_\infty \in L$. Let $(\hat{z}_k)$ and $(\hat{z}'_k)$ be two lifts of (z_k) in $\hat{L}$. We assume that $\hat{z}_k$ converges in $\hat{L}$, while $\hat{z}'_k = \hat{z}_k.p_k$ for a sequence (p_k) of P tending to infinity. Assume that* $\inf_{k \in \mathbb{N}}(\mathrm{inj}_L(\hat{z}'_k)) > 0$. *Then for every* $0 < \lambda < \inf_{k \in \mathbb{N}}(\mathrm{inj}_L(\hat{z}_k), \mathrm{inj}_L(\hat{z}'_k))$, *and every $\mathcal{F} \subset S_{\mathfrak{n}^-}$ such that $p_k.C(\mathcal{F}, \lambda) \to \nu$, as $k \to \infty$, for the Hausdorff topology on* S^n, *we must have $C_{\hat{z}'_k}(\mathcal{F}, \lambda) \to z_\infty$ for the Hausdorff topology on L.*

Proof. This lemma is a particular case of Lemma 7 in [Fr1] (see also [Fr2], Corollary 3.3), and the reader will find a complete proof there. The proof involves the notion of development of curves, that we don't introduce here. The upshot is that a conformal cone is a union of conformal geodesics, namely curves of the form $t \mapsto \pi_L \circ \exp(\hat{x}, tu)$, for $u \in \mathfrak{n}^-$. A point $\hat{x}$ in the fiber of x being chosen, one can develop any conformal geodesic passing through x into the sphere S^n, and thus any conformal cone can be developed. For instance, in the situation of Lemma 2.3, the developmental of $C_{\hat{z}'_k}(\mathcal{F}, \lambda)$ with respect to $\hat{z}_k$ is $p_k.C(\mathcal{F}, \lambda)$. Now, the lemma follows from the fact that conformal geodesics developing on short curves in S^n are themselves short ([Fr2], Lemma 3.1), and that conformal geodesics of S^n which are Hausdorff-close to ν must be short ([Fr2], Proposition 3.2). □

3. Extension results

We consider (L, g) and (N, h) two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subsetneq L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$, and $s\colon L \setminus \Lambda \to N$ a conformal immersion. We denote by $(L, \hat{L}, \omega^L)$ and $(N, \widehat{N}, \omega^N)$ the normal Cartan bundles associated to the respective conformal structures, as introduced in Section 2.1. If $\widehat{\Lambda} = \pi_L^{-1}(\Lambda)$ is the inverse image of Λ in $\hat{L}$, then $(\hat{L} \setminus \widehat{\Lambda}, \omega^L)$ is the normal Cartan bundle of $(L \setminus \Lambda, g)$. As we saw in 2.1, we can lift s to a bundle map $\hat{s}\colon (\hat{L} \setminus \widehat{\Lambda}, \omega^L) \to (\widehat{N}, \omega^N)$ satisfying $\hat{s}^* \omega^N = \omega^L$.

3.1. Holonomy sequences at a boundary point. Let us consider $x_\infty \in \Lambda$ which is not a pole for s. It means that there exists (x_k) a sequence of $L \setminus \Lambda$ which converges to x_∞, and such that $s(x_k)$ converges to $y_\infty \in N$. We will actually get more information working in the bundle $\hat{L} \setminus \widehat{\Lambda}$. Let $\hat{x}_\infty \in \widehat{\Lambda}$ in the fiber above x_∞, and let $(\hat{x}_k)$ be a sequence of $\hat{L} \setminus \widehat{\Lambda}$ projecting on (x_k) and converging to $\hat{x}_\infty$. The point is that $\hat{s}(\hat{x}_k)$ may not converge in $\widehat{N}$, but there always exists a sequence (p_k) such that $\hat{s}(\hat{x}_k).p_k^{-1}$ does converge to a point $\hat{y}_\infty \in \widehat{N}$ in the fiber of y_∞.

Definition 3.1 (holonomy sequence at x_∞). A sequence (p_k) as above will be called a *holonomy sequence at* x_∞ (associated to (x_k)).

Let us stress the fact that a holonomy sequence involves the choice of a sequence (x_k) tending to x_∞ such that $s(x_k)$ converges in N. In particular, the concept of holonomy sequence only makes sense when $x_\infty \in \Lambda_{\mathrm{rem}} \cup \Lambda_{\mathrm{ess}}$. The holonomy sequence (p_k) just encodes the behavior of the 2-jets of s along the sequence (x_k). If we already know for instance that s is the restriction of a conformal immersion from L to N, then the sequence (p_k) can be chosen constant to the identity. The projection of (p_k) on the factor $\mathbb{R}_+^* \times \mathrm{O}(n) \subset (\mathbb{R}_+^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n$ represents the sequence of tangent maps $D_{x_k} s$, read in local trivializations of the bundle of orthonormal frames. The study of the holonomy sequence will be, as we shall see, a major tool in understanding the dynamical behavior of s along (x_k). In particular, we will see that for thin singular sets Λ, removable singularities are characterized by bounded holonomy sequences, while essential ones appear together with unbounded holonomy sequences.

3.2. Characterization of removable points by holonomy, and proof of Theorem 1.1. Our aim now is to characterize the removable and essential singular points in terms of holonomy sequences. This will be done in several steps, leading to Theorem 3.6 at the end of the section, which clearly implies Theorem 1.1.

We will first need a technical lemma saying that it is possible to include "thick" conformal cones in the complement of closed sets of $(n-1)$-dimensional Hausdorff measure zero.

Lemma 3.2. *Let (L, g) be a Riemannian manifold of dimension $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$. For every $x \in L \setminus \Lambda$, every $\hat{x} \in \hat{L}$ in the fiber of x, and every $0 < \lambda < \mathrm{inj}_L(\hat{x})$, there exists a dense G_δ-set $\mathcal{U}_{\hat{x}} \subset S_{\mathfrak{n}^-}$ such that $C_{\hat{x}}(\mathcal{U}_{\hat{x}}, \lambda) \subset L \setminus \Lambda$.*

Proof. Let $\widehat{\Lambda}$ be the inverse image of Λ by the bundle map $\pi_L \colon \hat{L} \to L$. Let us call F the subset of $\bar{B}_{\mathfrak{n}^-}(\lambda)$ such that $\exp(\hat{x}, F) = \exp(\hat{x}, \bar{B}_{\mathfrak{n}^-}(\lambda)) \cap \widehat{\Lambda}$. By assumption, this set F has $(n-1)$-dimensional Hausdorff measure zero. Let m_0 be an integer such that $\frac{1}{m_0} \leq \lambda$. For every $m \geq m_0$, we call $\pi_m \colon u \mapsto \frac{u}{\|u\|}$ the radial projection from $A_m = \bar{B}_{\mathfrak{n}^-}(\lambda) \setminus B_{\mathfrak{n}^-}(\frac{1}{m})$ to $S_{\mathfrak{n}^-}$. This is a Lipschitz map, which is moreover closed. Hence, the set $\pi_m(F \cap A_m)$ is a closed subset of $S_{\mathfrak{n}^-}$, the $(n-1)$-dimensional Hausdorff measure of which is zero. In particular, its complement $\mathcal{U}_m$ is open and dense in $S_{\mathfrak{n}^-}$. Thus $\bigcap_{m \geq m_0} \mathcal{U}_m$ is a dense G_δ-set of $S_{\mathfrak{n}^-}$ that we call $\mathcal{U}_{\hat{x}}$. It is now clear by construction that $C_{\hat{x}}(\mathcal{U}_{\hat{x}}, \lambda) \subset L \setminus \Lambda$. □

Let us now give a sufficient condition, in terms of holonomy sequences, for a singular point to be removable.

Proposition 3.3. *Let (L, g) and (N, h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$, and $s \colon L \setminus \Lambda \to N$ a conformal immersion. Let x_∞ be a point of $\Lambda_{\mathrm{rem}} \cup \Lambda_{\mathrm{ess}}$. If there is a holonomy sequence of s at x_∞ which is bounded in P, then here exists U_{x_∞} an open subset of L containing x_∞ such that s extends to a conformal immersion $s_{x_\infty} \colon U_{x_\infty} \cup (L \setminus \Lambda) \to N$. In particular $x_\infty \in \Lambda_{\mathrm{rem}}$.*

Proof. Our hypothesis is that there is $\hat{x}_\infty \in \hat{L}$ in the fiber of x_∞, a sequence $(\hat{x}_k)$ in $\hat{L} \setminus \widehat{\Lambda}$ converging to $\hat{x}_\infty$, and a bounded sequence (p_k) in P such that $\hat{s}(\hat{x}_k).p_k^{-1}$ is converging in $\widehat{N}$. Considering subsequences, we may assume that (p_k) has a limit $p_\infty \in P$. Because $\hat{s}(\hat{x}_k.p_k^{-1}) = \hat{s}(\hat{x}_k).p_k^{-1}$, we can assume, replacing $\hat{x}_\infty$ by $\hat{x}_\infty.p_\infty^{-1}$ and $(\hat{x}_k)$ by $(\hat{x}_k.p_k^{-1})$, that $\hat{y}_k := \hat{s}(\hat{x}_k)$ is converging to $\hat{y}_\infty \in \widehat{N}$. Because $(\hat{y}_k)$ stays in a compact subset of $\widehat{N}$, we can find $k_0 \geq 0$, and $0 < \lambda < \min(\mathrm{inj}_L(\hat{x}_{k_0}), \mathrm{inj}_N(\hat{y}_{k_0}))$, such that $B_{\hat{x}_{k_0}}(\lambda)$ and $B_{\hat{y}_{k_0}}(\lambda)$ contain x_∞ and y_∞ respectively.

Lemma 3.2 yields a dense G_δ set $\mathcal{U} \subset S_{\mathfrak{n}^-}$ such that $C_{x_{k_0}}(\mathcal{U}, \lambda) \subset L \setminus \Lambda$. Let us define $s'_{x_\infty} \colon B_{\hat{x}_{k_0}}(\lambda) \to N$ by the formula

$$s'_{x_\infty}(\pi_L \circ \exp(\hat{x}_{k_0}, u)) := \pi_N \circ \exp(\hat{y}_{k_0}, u) \quad \text{for all } u \in B_{\mathfrak{n}^-}(\lambda).$$

This is a smooth diffeomorphism from $B_{\hat{x}_{k_0}}(\lambda)$ onto its image. On the other hand, because $\hat{s}$ is a lift of s satisfying $\hat{s}^*\omega^N = \omega^L$, we get for every $u \in \mathcal{C}(\mathcal{U}, \lambda)$,

$$s(\pi_L \circ \exp(\hat{x}_{k_0}, u)) = \pi_N \circ \exp(\hat{s}(\hat{x}_{k_0}), u) = \pi_N \circ \exp(\hat{y}_{k_0}, u).$$

In other words, s and s'_{x_∞} coincide on $C_{x_{k_0}}(\mathcal{U},\lambda)$, which is dense in $B_{\hat{x}_{k_0}}(\lambda)\setminus\Lambda$, hence they coincide on $B_{\hat{x}_{k_0}}(\lambda)\setminus\Lambda$. But because $\mathcal{H}^{n-1}(\Lambda)=0$, $B_{\hat{x}_{k_0}}(\lambda)\setminus\Lambda$ is dense in $B_{\hat{x}_{k_0}}(\lambda)$. As a consequence s'_{x_∞} is a conformal immersion on $B_{\hat{x}_{k_0}}(\lambda)$. Finally, the map $s_{x_\infty}\colon B_{\hat{x}_{k_0}}(\lambda)\cup(L\setminus\Lambda)\to N$ defined by s'_{x_∞} on $B_{\hat{x}_{k_0}}(\lambda)$, and s on $L\setminus\Lambda$ is well defined, and is a smooth conformal immersion extending s. □

In the same way, we have the following sufficient condition for a singular point to be essential.

Proposition 3.4. *Let (L,g) and (N,h) be two connected n-dimensional Riemannian manifolds, $n\geq 3$. Let $\Lambda\subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda)=0$, and $s\colon L\setminus\Lambda\to N$ a conformal immersion. Let x_∞ be a point of $\Lambda_{\mathrm{rem}}\cup\Lambda_{\mathrm{ess}}$. If there is a holonomy sequence of s at x_∞ which is unbounded in P, then $x_\infty\in\Lambda_{\mathrm{ess}}$.*

Proof. The key step for proving the proposition will be the following technical lemma, which will also be useful later on in other proofs. The lemma says that the existence of an unbounded holonomy sequence at x_∞ provides some non-equicontinuity phenomena which forbid x_∞ to be in Λ_{rem}.

Lemma 3.5. *Let x_∞ be a point of $\Lambda_{\mathrm{rem}}\cup\Lambda_{\mathrm{ess}}$. Assume that (x_k) is a sequence of $L\setminus\Lambda$ converging to x_∞, such that $s(x_k)$ converges to $y_\infty\in N$. If (p_k) is an unbounded holonomy sequence of s at x_∞ associated to (x_k), then:*

(1) *There exists a sequence (l_k) of P converging to l_∞ such that $l_k p_k$ stays in the factor $\mathbb{R}^*_+$ of $P=(\mathbb{R}^*_+\times \mathrm{O}(n))\ltimes\mathbb{R}^n$, and $(\mathrm{Ad}\, l_k p_k)(u)=\frac{1}{\lambda_k}u$ for every $u\in\mathfrak{n}^-$, with $\lim_{k\to\infty}\lambda_k=0$.*

(2) *If $t_0>0$, and $\gamma\colon[0,t_0[\to L\setminus\Lambda$ is a smooth curve satisfying $\gamma(t_k)=x_k$ for some sequence (t_k) of $[0,t_0[$ converging to t_0. Then, there exists (t'_k) a sequence of $[0,t_0[$ tending to t_0 such that $\gamma(t'_k)$ converges to x_∞, and $s(\gamma(t'_k))$ converges to $y'_\infty\in N$, with $y'_\infty\neq y_\infty$.*

Proof. By hypothesis, there exist $\hat{x}_\infty\in\hat{L}$ in the fiber of x_∞, $\hat{y}_\infty\in\hat{N}$ in the fiber of y_∞, $(\hat{x}_k)$ a sequence of $\hat{L}\setminus\hat{\Lambda}$ converging to $\hat{x}_\infty$, and an unbounded sequence (p_k) in P such that $\hat{s}(\hat{x}_k).p_k^{-1}$ converges to $\hat{y}_\infty\in\hat{N}$.

To show the first point of the lemma, we have to check that (p_k) does not satisfy the first point of Lemma 2.1. Assume, for a contradiction, that it is the case. We get a ball $\mathcal{B}\subset S_{\mathfrak{n}^-}$ with nonzero radius, and $\lambda>0$ such that $p_k.C(\mathcal{B},\lambda)\to\nu$. We can assume $0<\lambda_0<\inf_{k\geq 0}\mathrm{inj}_L(\hat{x}_k)$. Lemma 3.2 implies the existence of a dense G_δ-set $\mathcal{U}\subset\mathcal{B}$ such that for every $k\geq 0$, the cone $C_{\hat{x}_k}(\mathcal{U},\lambda_0)$ is contained in $L\setminus\Lambda$. Because $\mathrm{inj}_N(\hat{s}(\hat{x}_k))=\mathrm{inj}_L(\hat{x}_k)$ is bounded from below by a positive number independent of k, and because $p_k.C(\mathcal{U},\lambda)\to\nu$, we can apply Lemma 2.3

for $z_k := s(x_k)$, $\hat{z}'_k := \hat{s}(\hat{x}_k)$, and $\hat{z}_k := \hat{s}(\hat{x}_k).p_k^{-1}$. Together with relation (2), this yields

$$s(C_{\hat{x}_k}(\mathcal{U}, \lambda_0)) \to y_\infty \quad \text{as } k \to \infty. \tag{4}$$

This is actually impossible. Indeed, because $\lambda_0 < \inf_{k\geq 0} \operatorname{inj}_L(\hat{x}_k)$, we get that for every $k \geq 0$, the map $u \mapsto \pi_L \circ \exp(\hat{x}_k, u)$ is a diffeomorphism from $B_{\mathfrak{n}^-}(\lambda_0)$ onto its image. We deduce that any conformal cone $C_{\hat{x}_k}(\mathcal{B}, \lambda_0)$ has nonempty interior, and actually, all the sets $C_{\hat{x}_k}(\mathcal{B}, \lambda_0)$ contain a common open subset $U \subset L$ for $k \geq k_0$ large enough. Then, for every $k \geq 0$, $U_k := U \cap C_{\hat{x}_k}(\mathcal{U}, \lambda_0)$ is a dense G_δ-set of $U \setminus \Lambda$, and the same is true for $U_\infty = \bigcap_{k\geq k_0} U_k$. From relation (4), we get $s(U_\infty) = y_\infty$, which contradicts the fact that s is an immersion, hence locally injective on $U \setminus \Lambda$.

We now prove the second point of the lemma. By assumption $x_k = \gamma(t_k)$ for some smooth $\gamma\colon [0, t_0[\to L \setminus \Lambda$. The first point of the lemma tells us that replacing (t_k) by a subsequence if necessary (which amounts to consider a subsequence of $(\hat{x}_k)$, and the corresponding subsequence of (p_k)), and replacing $\hat{y}_k$ by $\hat{y}_k.l_k^{-1}$ for a sequence (l_k) of P tending to l_∞, we may assume that the sequence (p_k) satisfies $(\operatorname{Ad} p_k)(u) = \frac{1}{\lambda_k} u$ for every $u \in \mathfrak{n}^-$ with $\lim_{k\to\infty} \lambda_k = 0$.

We choose $0 < r_0 < \frac{1}{2} \min_{k\in\mathbb{N}\cup\{\infty\}}(\operatorname{inj}_L(\hat{x}_k), \operatorname{inj}_N(\hat{y}_k))$ so that for every $k \in \mathbb{N} \cup \{\infty\}$, the maps $\varphi_k\colon u \mapsto \pi_L \circ \exp(\hat{x}_k, u)$ and $\psi_k\colon u \mapsto \pi_N \circ \exp(\hat{y}_k, u)$ are well defined, and are diffeomorphisms from $B_{\mathfrak{n}^-}(2r_0)$ to open subsets U_k and V_k of L and N respectively. For every $k \geq 0$, we define $F_k := \varphi_k^{-1}(U_k \cap \Lambda)$.

Lemma 3.2 ensures the existence of a dense G_δ-set $\mathcal{U} \subset S_{\mathfrak{n}^-}$ such that for every $k \geq 0$, $C_{\gamma(t_k)}(\mathcal{U}, 2r_0) \subset L \setminus \Lambda$. For $k \geq k_0$ big enough, we will have $2\lambda_k r_0 < 2r_0$, and then, Lemma 3.2 amounts to say that $\mathcal{C}(\mathcal{U}, 2\lambda_k r_0) \subset B_{\mathfrak{n}^-}(2\lambda_k r_0) \setminus F_k$. Then, from relation (1), we infer that for every $u \in \mathcal{C}(\mathcal{U}, 2\lambda_k r_0)$

$$\hat{s}(\exp(\hat{x}_k, u)).p_k^{-1} = \exp\Big(\hat{y}_k, \frac{1}{\lambda_k} u\Big). \tag{5}$$

Observing that for each k, $\mathcal{C}(\mathcal{U}, 2\lambda_k r_0)$ is dense in $B_{\mathfrak{n}^-}(2\lambda_k r_0) \setminus F_k$, we deduce that formula (5) holds actually for every $u \in B_{\mathfrak{n}^-}(2\lambda_k r_0) \setminus F_k$.

As $\lambda_k \to 0$, the sequence of conformal balls $B_{\hat{x}_k}(2\lambda_k r_0) = \varphi_k(B_{\mathfrak{n}^-}(2\lambda_k r_0))$ tends to x_∞ for the Hausdorff topology on L. This means that choosing $k_0 \geq 0$ large enough, we are sure that for $k \geq k_0$, $\gamma([0, t_0[)$ is not contained in $B_{\hat{x}_k}(2\lambda_k r_0)$. In particular, for every $k \geq k_0$, there exists $u_k \in \mathfrak{n}^-$ with $\|u_k\| = r_0 \lambda_k$, and $t'_k \in [0, t_0[$ such that $\varphi_k(u_k) = \gamma(t'_k)$. Considering a subsequence, we may assume that $(\frac{u_k}{\lambda_k})$ converges to v_∞. Because $\gamma([0, t_0[)$ is contained in $L \setminus \Lambda$, we have $u_k \in B_{\mathfrak{n}^-}(2\lambda_k r_0) \setminus F_k$ for every $k \geq k_0$. Formula (5) then holds, and projecting on L and N, we get

$$s(\varphi_k(u_k)) = \psi_k\Big(\frac{u_k}{\lambda_k}\Big).$$

Making $k \to \infty$ yields

$$\lim_{k\to\infty} s(\gamma(t'_k)) = \psi_\infty(v_\infty).$$

Because $\|v_\infty\| = r_0$ and ψ_∞ is a diffeomorphism from $B_{\mathrm{n}}-(2r_0)$ onto its image, we get that $y'_\infty = \psi_\infty(v_\infty)$ is different from $y_\infty = \psi_\infty(0)$. Finally, because $\gamma(t'_k)$ tends to x_∞, and $\gamma([0,t_0[) \subset L \setminus \Lambda$, we see that the only cluster value of (t'_k) in $[0,t_0]$ is t_0. Hence $t'_k \to t_0$, as desired. □

We can now prove Proposition 3.4. Our hypothesis is that $x_\infty \in \Lambda_{\mathrm{rem}} \cup \Lambda_{\mathrm{ess}}$ admits an unbounded holonomy sequence. This holonomy sequence is associated to some sequence (x_k) of $L \setminus \Lambda$ converging to x_∞. Let $\gamma\colon [0,1[\to L \setminus \Lambda$ be a smooth curve such that $\gamma(1-\frac{1}{k}) = x_k$ for every $k \geq 1$, where $x_k := \pi_L(\hat{x}_k)$. The second point of Lemma 3.5 ensures the existence of a sequence (t'_k) tending to 1 such that $\gamma(t'_k)$ tends to x_∞, and $s(\gamma(t'_k))$ tends to $y'_\infty \neq y_\infty$. This forbids x_∞ to be in Λ_{rem}, and we deduce that the existence of an unbounded holonomy sequence implies $x_\infty \in \Lambda_{\mathrm{ess}}$. □

Let us collect the results of this section into a single statement.

Theorem 3.6. *Let (L,g) and (N,h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$, and $s\colon L \setminus \Lambda \to N$ a conformal immersion. Let x_∞ be a point of $\Lambda_{\mathrm{ess}} \cup \Lambda_{\mathrm{rem}}$. Then the following statements are equivalent:*

(1) *The point x_∞ is in Λ_{rem}.*

(2) *There exists U_{x_∞} an open subset of L containing x_∞ such that s extends to a conformal immersion $s_{x_\infty}\colon U_{x_\infty} \cup (L \setminus \Lambda) \to N$.*

(3) *There is a holonomy sequence of s at x_∞ which is bounded in P.*

(4) *All the holonomy sequences of s at x_∞ are bounded in P.*

Proof. It is obvious that point (2) implies point (1), and that point (4) implies point (3). Proposition 3.3 shows that (3) implies (2). Proposition 3.4 shows that (1) implies (4). □

3.3. An extension theorem for conformal embeddings. In view of Theorem 3.6, we will get interesting extension results when the set Λ_{ess} is empty. As the following theorem shows, this is actually the case as soon as the map s is injective (compare with the result proved in [V2] for quasiconformal maps).

Theorem 3.7. *Let (L,g) and (N,h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$, and $s\colon L \setminus \Lambda \to N$ a conformal embedding. Then:*

(1) *The set Λ_{ess} is empty and s extends to a conformal embedding*

$$s' : L \setminus \Lambda_{\mathrm{pole}} \to N.$$

(2) *When L is compact, then $s' : L \setminus \Lambda_{\mathrm{pole}} \to N$ is a conformal diffeomorphism.*

(3) *When both L and N are compact, Λ_{pole} is empty so that (L, g) and (N, h) are conformally diffeomorphic.*

Assuming that L is a compact manifold, Theorem 3.7 classifies, *all possible conformal embeddings* of the Riemannian manifold $(L \setminus \Lambda, g)$ into Riemannian manifolds of the same dimension. It also gives a uniqueness result for the conformal compactification of $(L \setminus \Lambda, g)$: *the only compact Riemannian manifold in which $(L \setminus \Lambda, g)$ can be embedded as an open subset is (L, g).*

The end of this section is devoted to the proof of Theorem 3.7. The first step is to show that near an essential singular point, a conformal immersion is highly noninjective. To formalize this, it is convenient to use the notion of cluster set. If x_∞ is a point of the singular set Λ, the cluster set of x_∞ is defined as

$$\mathrm{Clust}(x_\infty) := \{y \in N \mid \exists\, (x_k) \text{ a sequence in } L \setminus \Lambda,\ x_k \to x_\infty, \text{ and } s(x_k) \to y\}.$$

The following proposition identifies the cluster set of an essential singular point.

Proposition 3.8. *Let (L, g) and (N, h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$, and $s : L \setminus \Lambda \to N$ a conformal immersion. Assume that Λ_{ess} is not empty. Then for every $x_\infty \in \Lambda_{\mathrm{ess}}$, $\mathrm{Clust}(x_\infty) = N$. In particular, for every neighborhood U of x_∞ in L, $s(U \setminus \Lambda)$ is a dense open subset of N.*

Proposition 3.8 will be improved later, since we will deduce from Theorem 1.3 that if $x_\infty \in \Lambda_{\mathrm{ess}}$, and if U is a neighborhood of x_∞ in L, we actually have $s(U \setminus \Lambda) = N$ (see Corollary 5.5).

Proof. Let $y_\infty \in \mathrm{Clust}(x_\infty)$. Let us pick $\hat{x}_\infty$ in the fiber of x_∞, $(\hat{x}_k)$ a sequence of $\hat{L} \setminus \hat{\Lambda}$ converging to $\hat{x}_\infty$, and (p_k) a sequence of P such that $\hat{y}_k := \hat{s}(\hat{x}_k).p_k^{-1}$ tends to a point $\hat{y}_\infty$ in the fiber above y_∞. By Theorem 3.6, the sequence (p_k) is unbounded, and the first point of Lemma 3.5 ensures that considering subsequences, we may assume that (p_k) is contained in the factor $\mathbb{R}_+^*$ of $P = (\mathbb{R}_+^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n$. Moreover, always by Lemma 3.5, there exists (λ_k) a sequence of $\mathbb{R}_+^*$ converging to 0 such that for every $\mu > 0$,

$$(\mathrm{Ad}\ p_k).B_{\mathfrak{n}^-}(\mu\lambda_k) = B_{\mathfrak{n}^-}(\mu). \tag{6}$$

If μ is chosen smaller than $\min_{k \in \mathbb{N} \cup \{\infty\}}(\mathrm{inj}_L(\hat{x}_k), \mathrm{inj}_N(\hat{y}_k))$, the maps $u \mapsto \pi_N \circ \exp(\hat{y}_k, u)$ and $u \mapsto \pi_L \circ \exp(\hat{x}_k, u)$ are well defined and diffeomorphisms from

$B_{\mathfrak{n}^-}(\mu\lambda_k)$ on their images for every $k \in \mathbb{N} \cup \{\infty\}$. Lemma 3.2 implies the existence of a dense G_δ-set $\mathcal{U} \subset S_{\mathfrak{n}^-}$ such that $C_{\hat{x}_k}(\mathcal{U}, \mu\lambda_k) \subset L \setminus \Lambda$ for every $k \geq 0$. Relations (1) and (6) then yield

$$s(C_{\hat{x}_k}(\mathcal{U}, \mu\lambda_k)) = C_{\hat{y}_k}(\mathcal{U}, \mu).$$

In particular, one has $s(C_{\hat{x}_k}(\mathcal{U}, \mu\lambda_k)) \to C_{\hat{y}_\infty}(\mathcal{U}, \mu)$ as $k \to \infty$. We infer that $C_{\hat{y}_\infty}(\mathcal{U}, \mu) \subset \mathrm{Clust}(x_\infty)$, and finally $B_{\hat{y}_\infty}(\mu) \subset \mathrm{Clust}(x_\infty)$ because $\mathrm{Clust}(x_\infty)$ is a closed set. Since $B_{\hat{y}_\infty}(\mu)$ is a neighborhood of y_∞, we just showed that $\mathrm{Clust}(x_\infty)$ is an open set. We assumed that N is connected so that we get $\mathrm{Clust}(x_\infty) = N$. In particular, for every neighborhood U of x_∞ in L, we must have $\overline{s(U \setminus \Lambda)} = N$, hence $s(U \setminus \Lambda)$ is a dense open subset of N. □

We can now prove the first point of Theorem 3.7. Proposition 3.8 above ensures that if $s\colon L \setminus \Lambda \to N$ admits essential singular points, then s cannot be injective. We infer that Λ_{ess} is empty and $\Lambda = \Lambda_{\mathrm{rem}} \cup \Lambda_{\mathrm{pole}}$. By Theorem 1.1, we know that $L \setminus \Lambda_{\mathrm{pole}}$ is an open subset of L, and that s extends to a conformal immersion $s'\colon L \setminus \Lambda_{\mathrm{pole}} \to N$. Actually s' is injective, hence an embedding. Indeed, if s' is not injective, we can find two disjoint open sets U and V in $L \setminus \Lambda_{\mathrm{pole}}$ such that s' maps U and V diffeomorphically on the same open set W. Because $s'(U \cap (L \setminus \Lambda))$ and $s'(U \cap (L \setminus \Lambda))$ are two dense open subsets of W, they intersect, contradicting the injectivity of s on $L \setminus \Lambda$.

Let us proceed with the second point of Theorem 3.7. Assuming that L is compact, the definition of poles implies that the immersion $s'\colon L \setminus \Lambda_{\mathrm{pole}} \to N$ is a proper map. By connectedness of N, it has to be onto. Finally s' is a conformal diffeomorphism between $(L \setminus \Lambda_{\mathrm{pole}}, g)$ and (N, h).

If moreover N is also assumed to be compact, then Λ_{pole} is empty, and we get that (L, g) and (N, h) are conformally diffeomorphic. This shows the third point of the theorem.

3.4. Essential singular points imply conformal flatness. We are now going to make an important step toward Theorem 1.3, proving that the existence of thin essential singular sets is only possible on conformally flat manifolds. Thus, generically, by Theorem 1.1, if a thin singular set contains no poles (for instance if N is compact), it is always possible to extend a conformal immersion across it. In the following, by *conformal curvature* on a Riemannian manifold, we will mean the Weyl curvature tensor when the dimension is ≥ 4, and the Cotton tensor when the dimension is 3 (see [AG], p. 131).

Proposition 3.9. *Let (L, g) and (N, h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$, and $s\colon L \setminus \Lambda \to N$ a conformal immersion. Assume that Λ_{ess} is not empty. Then for* *every* $x_\infty \in \Lambda_{\mathrm{ess}}$, *and every* y_∞ *in* $\mathrm{Clust}(x_\infty)$, *the conformal curvature vanishes at* y_∞. *In particular, the manifolds* (L, g) *and* (N, h) *are both conformally flat.*

Proof. We pick $y_\infty \in \mathrm{Clust}(x_\infty)$, and we consider $\hat{x}_\infty$, $\hat{x}_k$, $\hat{y}_k$, $\hat{y}_\infty$, p_k, μ and $\mathcal{U}$ as at the begining of the proof of Proposition 3.8. On $\hat{L}$, there is, associated to the normal Cartan connection ω^L, a *curvature function* κ (we don't give details here, and refer the reader to [Sh], Chapters 5.3 and 7). This is a map $\kappa\colon \hat{L} \to \mathrm{Hom}(\Lambda^2(\mathfrak{o}(1, n+1)\,/\,\mathfrak{p}), \mathfrak{p})$, satisfying the equivariance relation:

$$\kappa_{\hat{x}}(v, w) = (\mathrm{Ad}\ p^{-1}).\kappa_{\hat{x}.p^{-1}}((\mathrm{Ad}\ p).v, (\mathrm{Ad}\ p).w). \tag{7}$$

The vanishing of the Cartan curvature κ at $\hat{x}$ implies the vanishing of κ on the fiber of $\hat{x}$. It thus makes sense to say that κ vanishes at a point $x \in L$, and this is equivalent to the vanishing of the conformal curvature at x (see Chapter 7 of [Sh]). Hence, to get the lemma, it is enough to show that κ vanishes at y_∞.

For convenience, we will see κ as a map from $\hat{L}$ to $\mathrm{Hom}(\Lambda^2(\mathfrak{n}^-), \mathfrak{p})$. Then, relation (7) still holds, provided $p \in \mathbb{R}_+^* \times \mathrm{O}(n) \subset P$. Now, since $\hat{s}$ satisfies $\hat{s}^*\omega^N = \omega^L$, we have for every $v, w \in \mathfrak{n}^-$, and every $k \in \mathbb{N}$

$$\kappa_{\hat{x}_k}(v, w) = \kappa_{\hat{s}(\hat{x}_k)}(v, w).$$

By relation (7), we also get

$$\kappa_{\hat{s}(\hat{x}_k)}(v, w) = (\mathrm{Ad}\ p_k^{-1}).\kappa_{\hat{y}_k}((\mathrm{Ad}\ p_k).v, (\mathrm{Ad}\ p_k).w).$$

Recall that $\mathrm{Ad}\ p_k^{-1}$ (resp. $\mathrm{Ad}\ p_k$) acts trivially on $\mathbb{R} \oplus \mathfrak{o}(n)$, and by multiplication by $\frac{1}{\lambda_k}$ on $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$). Writing $\kappa^{(1)}_{\hat{y}_k}(v, w)$ and $\kappa^{(2)}_{\hat{y}_k}(v, w)$ for the components of $\kappa_{\hat{y}_k}(v, w)$ on $\mathbb{R} \oplus \mathfrak{o}(n)$ and $\mathfrak{n}^+$ respectively, the last two equalities yield

$$\kappa_{\hat{x}_k}(v, w) = \frac{1}{\lambda_k^2}\kappa^{(1)}_{\hat{y}_k}(v, w) + \frac{1}{\lambda_k^3}\kappa^{(2)}_{\hat{y}_k}(v, w).$$

Since $\lambda_k \to 0$, making $k \to \infty$ gives $\kappa_{\hat{y}_\infty}(v, w) = 0$, and finally $\kappa_{\hat{y}_\infty} = 0$. The conformal curvature vanishes on $\mathrm{Clust}(x_\infty)$, and by Proposition 3.8, $\mathrm{Clust}(x_\infty) = N$ so that (N, h) is conformally flat. The manifold $(L \setminus \Lambda, g)$ is mapped into (N, h) by a conformal immersion, hence $(L \setminus \Lambda, g)$ is itself conformally flat. Finally, because $\mathcal{H}^{n-1}(\Lambda) = 0$, $L \setminus \Lambda$ is dense in L, and we get that (L, g) is also conformally flat. □

4. Background on conformally flat manifolds

By Proposition 3.9, conformal singularities $s\colon L \setminus \Lambda \to N$ such that $\mathcal{H}^{n-1}(\Lambda) = 0$ and $\Lambda_{\mathrm{ess}} \neq \emptyset$ only occur when L and N are conformally flat. To go further and prove Theorem 1.3, we will need basic notions about conformally flat manifolds that we gather in this section. Good general references on the subject are [Go], [M], Section 3, and [Th], Chapter 3, p. 139. All manifolds in the sequel are still assumed to have dimension ≥ 3.

4.1. Kleinian manifolds and essential singular sets of Kleinian type. One calls *Kleinian group* a discrete subgroup Γ of the Möbius group $\mathrm{PO}(1, n+1)$ which acts freely properly and discontinuously on some nonempty open subset $\Omega \subset \mathrm{S}^n$ (we refer the reader to Chapter 2 of [A], Sections 3.6, 4.6 and 4.7 in [Ka] and Section 5 in [M] for details on the material below).

Given a Kleinian group Γ, there exists a maximal open set $\Omega(\Gamma) \subset \mathrm{S}^n$ on which the action of Γ is proper. This open set $\Omega(\Gamma)$ is called the *domain of discontinuity* of Γ, and its complement in S^n, denoted $\Lambda(\Gamma)$, is called the *limit set of* Γ. There are several characterizations of the limit set $\Lambda(\Gamma)$, but two of them will be of particular interest for our purpose. Let us consider any point $x \in \Omega(\Gamma)$, and denote $\overline{\Gamma.x}$ the closure of the orbit $\Gamma.x$ into S^n. Then the limit set $\Lambda(\Gamma)$ coincides with $\overline{\Gamma.x} \setminus \Gamma.x$ (see for instance [A], Lemma 2.2, p. 42).

Another useful characterization is as follows: the limit set $\Lambda(\Gamma)$ comprises exactly those points $x \in \mathrm{S}^n$ at which the family $\{\gamma\}_{\gamma\in\Gamma}$ fails to be equicontinuous (see [M], Chapter 5). The group Γ being assumed to be discrete, we observe that its limit set is empty if and only if Γ is finite.

If $\Gamma \subset \mathrm{PO}(1, n+1)$ is a Kleinian group, and $\Omega \subset \mathrm{S}^n$ is a Γ-invariant open set on which the action of Γ is free and properly discontinuous, then the quotient manifold $N := \Omega/\Gamma$ is naturally endowed with a conformally flat structure, and the covering map $\pi\colon \Omega \to N$ is conformal. Such a quotient Ω/Γ is called a *Kleinian manifold*. When the action of Γ is free on $\Omega(\Gamma)$, the Kleinian manifold $\Omega(\Gamma)/\Gamma$ will be denoted $M(\Gamma)$. It is then the maximal Kleinian manifold that one can build up thanks to the group Γ.

Let us now consider $\Gamma \subset \mathrm{PO}(1, n+1)$ an *infinite* Kleinian group, and Ω an open subset of S^n on which Γ acts freely properly discontinuously. Let $N := \Omega / \Gamma$ be the associated Kleinian manifold. Observe that because we assumed Γ infinite, Ω is a proper open subset of S^n. Denoting by Λ the complement of Ω in S^n, the covering map $\pi\colon \mathrm{S}^n \setminus \Lambda \to N$ yields a conformal singularity. The set Λ turns out to be an essential singular set for π. To see this, we first observe that because Γ acts freely properly discontinuously on Ω, we have $\Lambda(\Gamma) \subset \Lambda$. Actually, $\Lambda(\Gamma) \subset \Lambda_{\mathrm{ess}}$. Indeed, let $x_\infty \in \Lambda(\Gamma)$, and let y and y' be two distinct points of N. Let z and z' in Ω satisfying $\pi(z) = y$ and $\pi(z') = y'$. By the characterization of the limit set described above, there exist two sequences (γ_n) and (γ'_n) in Γ such that $x_n := \gamma_n.z$ and $x'_n := \gamma'_n.z'$ converge to x_∞ (actually, we can choose $\gamma_n = \gamma'_n$). Because $\pi(x_n) = y$ while $\pi(x'_n) = y'$, the point x_∞ is neither removable, nor a pole, hence is an essential singular point. On the other hand, let us consider $x_\infty \in \Lambda$ which is not a pole. It is easily checked that there must be a sequence (γ_n) in Γ which is not equicontinuous at x_∞ so that $x_\infty \in \Lambda(\Gamma)$. In particular, x_∞ is an essential singular point. The previous discussion shows that $\Lambda_{\text{ess}} = \Lambda(\Gamma)$ is not empty, and $\Lambda = \Lambda_{\text{ess}} \bigcup \Lambda_{\text{pole}}$. In other words, we have built a conformal singularity $\pi\colon \mathrm{S}^n \setminus \Lambda \to N$ with an essential singular set Λ, which is minimal essential in the sense of Definition 1.2. We say that such a conformal singularity is *of Kleinian type*.

4.2. Holonomy coverings. Among conformally flat manifolds, a nice subset comprises those who admit conformal immersions into the standard sphere. Such immersions are called *developing maps*. When it exists, a developing map is essentially unique.

Fact 4.1. *If (M, g) is a connected conformally flat manifold of dimension $n \geq 3$, and if δ_1, δ_2 are two conformal immersions from M to S^n, then there exists an element g of the Möbius group such that $\delta_2 = g \circ \delta_1$.*

The key point to get the fact above is Liouville's theorem (see for instance [Sp], p. 310): *a conformal immersion between two connected open subsets U and V of S^n, $n \geq 3$, is the restriction of a Möbius transformation.*

One thus get a Möbius transformation g such that the set where $\delta_2 = g \circ \delta_1$ is nonempty and has empty boundary.

Fact 4.1 easily implies that if $\delta\colon M \to \mathrm{S}^n$ is a developing map, there exists a group homomorphism

$$\rho\colon \mathrm{Conf}(M, [g]) \to \mathrm{PO}(1, n+1),$$

called the *holonomy morphism* associated to δ such that for every $\varphi \in \mathrm{Conf}(M, [g])$

$$\delta \circ \varphi = \rho(\varphi) \circ \delta. \tag{8}$$

Let us now consider a conformally flat structure $(M, [g])$. It is a classical result, which already appears in [Ku] (see also [M], Section 3) that the universal covering $(\widetilde{M}, [\tilde{g}])$, endowed with the lift $[\tilde{g}]$ of the conformal structure $[g]$, admits a developing map $\tilde{\delta}\colon \widetilde{M} \to \mathrm{S}^n$. Let us identify $\pi_1(M)$ with a discrete subgroup $\Gamma \subset \mathrm{Conf}(\widetilde{M}, [\tilde{g}])$, and call $\Gamma_{\tilde{\rho}}$ the kernel of the holonomy morphism $\tilde{\rho}\colon \Gamma \to \mathrm{PO}(1, n+1)$. The developing map $\tilde{\delta}$ induces a conformal immersion δ from the quotient manifold $\mathcal{M} := \widetilde{M} / \Gamma_{\tilde{\rho}}$ to S^n. This manifold $\mathcal{M}$ is called the *holonomy covering of M*. It is in some sense the "smallest" conformal covering of M admitting a conformal immersion to the sphere. This is the meaning of the following lemma.

Lemma 4.2. *Let M be a connected n-dimensional conformally flat Riemannian manifold, $n \geq 3$, and $\mathcal{M}$ the holonomy covering of M. Assume that $\mathcal{M}'$ is another connected n-dimensional conformally flat Riemannian manifold such that:*

(1) *There exists a conformal immersion $\delta'\colon \mathcal{M}' \to \mathrm{S}^n$.*

(2) *There exists a conformal covering map* $\pi : \mathcal{M}' \to M$.

Then there exists a conformal covering map from $\mathcal{M}'$ *onto* $\mathcal{M}$.

Proof. Let us call $\widetilde{M}$ the conformal universal covering of M, and identify $\pi_1(M)$ with a discrete group Γ of conformal transformations of $\widetilde{M}$ so that M is conformally diffeomorphic to $\widetilde{M} / \Gamma$. Because $\mathcal{M}'$ is a covering of M, there exists Γ' a subgroup of Γ such that $\mathcal{M}'$ is conformally equivalent to $\widetilde{M} / \Gamma'$. The immersion δ' lifts to a conformal immersion $\tilde{\delta}' : \widetilde{M} \to \mathrm{S}^n$. Let $\tilde{\delta} : \widetilde{M} \to \mathrm{S}^n$ be a developing map, and $\tilde{\rho} : \Gamma \to \mathrm{PO}(1, n+1)$ the associated holonomy morphism. By Fact 4.1, there exists $g \in \mathrm{PO}(1, n+1)$ such that $\tilde{\delta}' = g \circ \tilde{\delta}$. Now, for every $\gamma \in \Gamma'$, one has $\tilde{\delta}' \circ \gamma = \tilde{\delta}'$ so that $g \circ \tilde{\delta} \circ \gamma = g \circ \tilde{\delta}$. Finally, we get $\Gamma' \subset \Gamma_{\tilde{\rho}} = \operatorname{Ker} \tilde{\rho}$. Hence, there is a conformal covering map from $\mathcal{M}' = \widetilde{M} / \Gamma'$ onto $\mathcal{M} = \widetilde{M} / \Gamma_{\tilde{\rho}}$. □

Lemma 4.3. *Let* M *and* N *be two connected* n*-dimensional conformally flat manifolds,* $n \geq 3$. *Let* $\mathcal{N}$ *be the holonomy covering of* N. *Assume there exists a conformal immersion* $\delta : M \to \mathrm{S}^n$. *Then any conformal immersion* $s : M \to N$ *can be lifted to a conformal immersion* $\sigma : M \to \mathcal{N}$.

Proof. Let $\widetilde{M}$ and $\widetilde{N}$ be the conformal universal coverings of M and N respectively, and $\pi_M : \widetilde{M} \to M$, $\pi_N : \widetilde{N} \to N$ the associated covering maps. We denote by Γ_M and Γ_N the fundamental groups of M and N, seen as discrete subgroups of conformal transformations of $\widetilde{M}$ and $\widetilde{N}$. The conformal immersion δ lifts to a developing map $\delta_M : \widetilde{M} \to \mathrm{S}^n$, satisfying $\delta_M \circ \gamma = \delta_M$ for every $\gamma \in \Gamma_M$. We also introduce δ_N a developing map on $\widetilde{N}$, and denote by $\rho_N : \Gamma_N \to \mathrm{PO}(1, n+1)$ the associated holonomy morphism. The conformal immersion s lifts to a conformal immersion $\tilde{s} : \widetilde{M} \to \widetilde{N}$, and there is a morphism $\rho : \Gamma_M \to \Gamma_N$ such that for every $\gamma \in \Gamma_M$, $\tilde{s} \circ \gamma = \rho(\gamma) \circ \tilde{s}$. Thanks to Fact 4.1, there exists an element $g \in \mathrm{PO}(1, n+1)$ such that $\delta_N \circ \tilde{s} = g \circ \delta_M$. For every $x \in \widetilde{M}$ and every $\gamma \in \Gamma_M$, we have on the one hand

$$\delta_N(\rho(\gamma).\tilde{s}(x)) = \rho_N(\rho(\gamma)).\delta_N(\tilde{s}(x))$$

and on the other hand

$$\delta_N(\rho(\gamma).\tilde{s}(x)) = \delta_N(\tilde{s}(\gamma.x)) = g.\delta_M(\gamma.x)) = g.\delta_M(x) = \delta_N(\tilde{s}(x)).$$

We thus get that $\rho_N(\rho(\gamma))$ fixes pointwise an open subset of S^n, hence is the identical transformation. We conclude that $\rho(\Gamma_M) \subset \operatorname{Ker} \rho_N$, hence the map $\tilde{s}$ induces a conformal immersion $\sigma : M \to \mathcal{N}$, where $\mathcal{N}$ is the holonomy covering of N. By construction, σ is a lift of s. □

4.3. Cauchy completion of a conformally flat structure. The normal Cartan connection associated to a conformal structure allows to define an abstract notion of "conformal boundary", derived from the b-boundary construction introduced in [S2].

We sketch the construction of this boundary below. More details are available in Sections 2 and 4 of [Fr3]. Fix once for all a basis $X_1, \dots, X_s$ of the Lie algebra $\mathfrak{g} := \mathfrak{o}(1, n+1)$. Given a Riemannian manifold (M, g), with $\dim M \geq 3$, let us call $(M, \widehat{M}, \omega^M)$ the normal Cartan bundle associated to the conformal structure defined by g. Denote by $\mathcal{R}$ the frame field on $\widehat{M}$ defined by $\mathcal{R}(\hat{x}) = ((\omega^M_{\hat{x}})^{-1}(X_1), \dots, (\omega^M_{\hat{x}})^{-1}(X_s))$. This determines uniquely a Riemannian metric ρ^M on $\widehat{M}$ having the property that $\mathcal{R}(\hat{x})$ is $\rho^M_{\hat{x}}$-orthonormal for every $\hat{x} \in \widehat{M}$. The Riemannian metric ρ^M defines a distance d_M on $\widehat{M}$ by the formula

$$d_M(\hat{x}, \hat{y}) = \frac{\delta_M(\hat{x}, \hat{y})}{1 + \delta_M(\hat{x}, \hat{y})},$$

where

– $\delta_M(\hat{x}, \hat{y})$ is the infimum of the ρ^M-lengths of piecewise C^1 curves joining $\hat{x}$ and $\hat{y}$ if $\hat{x}$ and $\hat{y}$ are in the same connected component of $\widehat{M}$,

– $\delta_M(\hat{x}, \hat{y}) = -2$ otherwise.

One can look at the Cauchy completion $\widehat{M}_c$ of the metric space $(\widehat{M}, d_M)$, and define the Cauchy boundary $\partial_c \widehat{M}$ as $\partial_c \widehat{M} := \widehat{M}_c \setminus \widehat{M}$. Recall that $\widehat{M}$ is a P-principal bundle over M, where P is the stabilizer of a point $\nu \in \mathrm{S}^n$ in the Möbius group $\mathrm{PO}(1, n+1)$. Given $p \in P$, the right multiplication R_p is Lipschitz with respect to d_M, and the right action of P extends continuously to $\widehat{M}_c$. The *conformal Cauchy completion* of (M, g) is defined as the quotient space $M_c := \widehat{M}_c / P$.

Let us illustrate the construction in the case of the standard sphere S^n, where the conformal Cartan bundle is identified with the Lie group $G = \mathrm{PO}(1, n+1)$, and the Cartan connection is merely the Maurer–Cartan form ω^G. The Riemannian metric ρ^G constructed as above is left-invariant on G so that (G, ρ^G) is a homogeneous Riemannian manifold, hence complete. We infer that $G_c = \emptyset$, and the conformal Cauchy boundary of S^n is empty as well.

Generally, the action of P on $\widehat{M}_c$ is very bad behaved near points of $\partial_c \widehat{M}$ so that the space M_c may not be Hausdorff. It is thus quite remarkable that M_c is Hausdorff when (M, g) admits a conformal immersion in the standard sphere S^n, as the following proposition shows.

Proposition 4.4. *Let M be a n-dimensional conformally flat manifold, $n \geq 3$. Assume there exists a conformal immersion $\delta \colon M \to \mathrm{S}^n$. Then:*

(1) *The conformal Cauchy completion M_c is a Hausdorff space, in which M is a dense open subset.*

(2) *The conformal immersion δ extends to a continuous map $\delta \colon M_c \to \mathrm{S}^n$.*

(3) *Every conformal diffeomorphism φ of M extends to a homeomorphism of M_c.*

Proof. We call ρ^M and ρ^G the Riemannian metrics constructed on $\widehat{M}$ and G as explained above, *using a same basis* $X_1, \dots, X_s$ *of* $\mathfrak{o}(1, n+1)$. The conformal immersion $\delta \colon M \to \mathbf{S}^n$ lifts to an isometric immersion $\hat{\delta} \colon (\widehat{M}, \rho^M) \to (G, \rho^G)$. As a consequence, $\hat{\delta} \colon (\widehat{M}, d_M) \to (G, d_G)$ is 1-Lipschitz. Because (G, d_G) is a complete metric space, $\hat{\delta}$ extends to a 1-Lipschitz map $\hat{\delta} \colon (\widehat{M}_c, d_M) \to (G, d_G)$. This extended map $\hat{\delta}$ is still P-equivariant for the (extended) action of P on $\widehat{M}_c$ and on G. Every conformal diffeomorphism $\varphi \in \mathrm{Conf}(M)$ lifts to an isometry $\hat{\varphi}$ of $(\widehat{M}, \rho^M)$, hence extends to an isometry, still denoted $\hat{\varphi}$ on $(\widehat{M}_c, d_M)$. The action of P is free and proper on $\widehat{M}_c$ because the right action of P on G is free and proper, and $\hat{\delta}$ maps $\widehat{M}_c$ continuously and P-equivariantly on G. As a consequence, $M_c = \widehat{M}_c / P$ is Hausdorff. The map $\hat{\delta} \colon \widehat{M}_c \to G$ induces a continuous $\delta \colon M_c \to G / P = \mathbf{S}^n$, extending δ. Finally, for every $\varphi \in \mathrm{Conf}(M)$, the homeomorphism $\hat{\varphi} \colon \widehat{M}_c \to \widehat{M}_c$ commutes with the right action of P, hence induces a homeomorphism $\varphi \colon M_c \to M_c$. □

5. Proof of the local classification theorem

In this section, we prove Theorem 1.3. Let $s \colon L \setminus \Lambda \to N$ be a conformal immersion, where Λ is an essential singular set satisfying $\mathcal{H}^{n-1}(\Lambda) = 0$. We assume also that the singular set is essential and minimal in the sense that $\Lambda = \Lambda_{\mathrm{pole}} \cup \Lambda_{\mathrm{ess}}$, with $\Lambda_{\mathrm{ess}} \neq \emptyset$. As explained in the introduction, because of Theorem 1.1, this hypothesis $\Lambda_{\mathrm{rem}} = \emptyset$ is harmless. By Proposition 3.9 we know that both L and N are conformally flat manifolds.

5.1. The target manifold N is Kleinian. We call $\mathcal{N}$ the holonomy covering of N. There is a discrete subgroup Γ of conformal transformations of $\mathcal{N}$, acting freely properly discontinuously on $\mathcal{N}$ such that N is conformally diffeomorphic to $\mathcal{N} / \Gamma$. Showing that N is Kleinian amounts to show that $\mathcal{N}$ is conformally diffeomorphic to an open subset of $\mathbf{S}^n$. The upshot of the proof is as follows: we are going to construct a bigger n-dimensional conformal manifold $\mathcal{N}'$, in which $\mathcal{N}$ embeds conformally as an open subset, and such that the action of Γ extends conformally to $\mathcal{N}'$. The point is that the extended action of Γ on $\mathcal{N}'$ is no longer proper, what forces $\mathcal{N}'$ to be conformally equivalent to $\mathbf{S}^n$ or the Euclidean space (see Theorem 5.1 below). Because $\mathcal{N}$ embeds conformally into $\mathcal{N}'$, it is conformally diffeomorphic to an open subset of the sphere, as desired.

Theorem 5.1 ([Fe], [Sch], [Fr1]). *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. The three following assertions are equivalent:*

(1) *The group of conformal transformations $\mathrm{Conf}(M)$ does not act properly on M.*

(2) *The group of conformal transformations Conf(M) does not preserve any Riemannian metric g' in the conformal class $[g]$.*

(3) *The manifold (M, g) is conformally diffeomorphic to the standard sphere* $\mathbb{S}^n$, *or to the Euclidean space* $\mathbb{R}^n$.

A version of the theorem for the identity component of the conformal group, and for compact manifolds, originally appeared in [Ob].

We are now explaining how one can construct a manifold $\mathcal{N}'$ with the properties listed above.

In the remaining of this section, we pick $x_\infty \in \Lambda_{\mathrm{ess}}$, and U a connected neighborhood of x_∞ in L such that there exists a conformal embedding $\varphi\colon U \to \mathbb{S}^n$. Lemma 4.3 ensures that the conformal immersion $s\colon U \setminus \Lambda \to N$ lifts to a conformal immersion $\sigma\colon U \setminus \Lambda \to \mathcal{N}$. By definition of the holonomy covering, there exists a conformal immersion $\delta\colon \mathcal{N} \to \mathbb{S}^n$. Then the map $\delta \circ \sigma \circ \varphi^{-1}\colon \varphi(U \setminus \Lambda) \to \mathbb{S}^n$ is a conformal immersion from $\varphi(U \setminus \Lambda)$ to an open subset of the sphere. Because $\mathcal{H}^{n-1}(\Lambda) = 0$, $\varphi(U \setminus \Lambda)$ is a connected open subset of $\mathbb{S}^n$ and Liouville's theorem ensures that $\delta \circ \sigma \circ \varphi^{-1}$ is the restriction of a Möbius transformation. In particular, it is injective and so is σ. We thus get that $\sigma\colon U \setminus \Lambda \to \mathcal{N}$ is a conformal embedding.

In the following, we denote by $(\mathcal{N}, \widehat{\mathcal{N}}, \omega^{\mathcal{N}})$ the normal Cartan bundle associated to the conformal structure on $\mathcal{N}$. As in Section 4.3, we define the Riemannian metric $\rho^{\mathcal{N}}$ on $\widehat{\mathcal{N}}$, the associated distance $d_{\mathcal{N}}$, $\widehat{\mathcal{N}}_c$ the Cauchy completion of $(\widehat{\mathcal{N}}, d_{\mathcal{N}})$, and $\mathcal{N}_c$ the conformal Cauchy completion of $\mathcal{N}$. The distance on $\widehat{\mathcal{N}}_c$ is still denoted $d_{\mathcal{N}}$.

Lemma 5.2. *The conformal embedding $\sigma\colon U \setminus \Lambda \to \mathcal{N}$ extends to a continuous map $\sigma\colon U \to \mathcal{N}_c$, which is a homeomorphism from U onto an open subset $W \subset \mathcal{N}_c$. The extended map σ sends $\Lambda \cap U$ into $\partial_c \mathcal{N} := \mathcal{N}_c \setminus \mathcal{N}$.*

Proof. Let us call $\widehat{U}$ and $\widehat{\Lambda}$ the inverse images of U and Λ in $\hat{L}$. The conformal immersion σ lifts to an isometric immersion $\hat{\sigma}\colon (\widehat{U} \setminus \widehat{\Lambda}, \rho^L) \to (\widehat{\mathcal{N}}, \rho^{\mathcal{N}})$. Call d_U (resp. $d_{U\setminus\Lambda}$) the distance induced by the Riemannian metric ρ^L on the open set $\widehat{U}$ (resp. $\widehat{U} \setminus \widehat{\Lambda}$). Because $\widehat{\Lambda} \cap \widehat{U}$ has $(\dim(\widehat{U})-1)$-dimensional Hausdorff measure zero, we get that $d_U = d_{U\setminus\Lambda}$ (this fact is probably standard; the reader can find a proof in [Fr3], Lemma 3.3). As a consequence, the map $\hat{\sigma}\colon (\widehat{U} \setminus \widehat{\Lambda}, d_{U\setminus\Lambda}) \to (\widehat{\mathcal{N}}, d_{\mathcal{N}})$, which is 1-Lipschitz, is also 1-Lipschitz if we put the metric d_U on $\widehat{U} \setminus \widehat{\Lambda}$. Hence, it extends to a 1-Lipschitz map $\hat{\sigma}\colon (\widehat{U}, d_U) \to (\widehat{\mathcal{N}}_c, d_{\mathcal{N}})$. This map is P-equivariant on the dense open subset $\widehat{U} \setminus \widehat{\Lambda}$, hence on $\widehat{U}$, and defines an extension of σ to a continuous map $\sigma\colon U \to \mathcal{N}_c$.

We are now going to show that the map $\sigma\colon U \to \mathcal{N}_c$ is open.

Because $\sigma\colon U \setminus \Lambda \to \mathcal{N}$ is an embedding, it is open on $U \setminus \Lambda$. It is thus enough to check that whenever $x \in \Lambda \cap U$, and $V \subset U$ is an open set containing x, the image $\sigma(V)$ is a neighborhood of $z := \sigma(x)$. Let $\hat{x} \in \widehat{U}$ be a point in the fiber of x, let $\hat{z} = \hat{\sigma}(\hat{x}) \in \widehat{\mathcal{N}}_c$, and let $r > 0$ be very small so that $\overline{B(\hat{x},r)}$, the closure of the ball of radius r for ρ^L, is compact and contained in $\widehat{V} := \pi_L^{-1}(V)$. We claim that if $B(\hat{z},\frac{r}{5})$ denotes the metric ball centered at $\hat{z}$ and of radius $\frac{r}{5}$ in $(\widehat{\mathcal{N}}_c, d_{\mathcal{N}})$, we have the inclusion $B(\hat{z},\frac{r}{5}) \subset \hat{\sigma}(\overline{B(\hat{x},r)})$, what will be enough to conclude, because the projections $\widehat{V} \to V$ and $\widehat{\mathcal{N}}_c \to \mathcal{N}_c$ are open maps. Let us consider $\hat{z}' \in \widehat{\mathcal{N}}_c$ such that $d_{\mathcal{N}}(\hat{z}',\hat{z}) < \frac{r}{4}$. Let us consider $(\hat{x}_k)$ a sequence of $\widehat{U} \setminus \widehat{\Lambda}$ converging to $\hat{x}$, and $(\hat{z}'_k)$ a sequence of $\widehat{\mathcal{N}}$ converging to $\hat{z}'$. We consider indices k large enough so that the points $\hat{z}_k := \hat{\sigma}(\hat{x}_k)$ and $\hat{z}'_k$ satisfy

$$d_{\mathcal{N}}(\hat{z}_k,\hat{z}'_k) \leq \frac{r}{2}$$

and

$$d_U(\hat{x}_k,\hat{x}) < \frac{r}{5}.$$

There is a curve $\beta_k\colon [0,1] \to \widehat{\mathcal{N}}$ joining $\hat{z}_k$ to $\hat{z}'_k$, and having $\rho^{\mathcal{N}}$-length smaller than $\frac{3r}{4}$. The key point is that there exist a lift $\alpha_k\colon [0,1] \to B(\hat{x},r) \setminus \widehat{\Lambda}$ such that $\alpha_k(0) = \hat{x}_k$ and $\hat{\sigma} \circ \alpha_k = \beta_k$. Let us see why it is true. Let $t_\infty := \sup\{t \in [0,1]$, the lift α_k exists on $[0,t[\}$. Because $\hat{\sigma}\colon (\widehat{U} \setminus \widehat{\Lambda}, \rho^L) \to (\widehat{\mathcal{N}}, \rho^{\mathcal{N}})$ is an isometric immersion, $\alpha_{k|[0,t_\infty[}$ has finite length so that $\hat{y}_\infty := \lim_{t\to t_\infty} \alpha_k(t)$ exists. Moreover, the ρ^L length of $\alpha_{k|[0,t_\infty[}$ is smaller than $\frac{3r}{4}$, so we get $d_U(\hat{x}, y_\infty) < r$, and $\hat{y}_\infty \in B(\hat{x},r)$. If we prove that $\hat{y}_\infty \notin \widehat{\Lambda} \cap B(\hat{x},r)$, we will get that α_k exists on $[0,1]$. As we saw, the immersion $\sigma\colon U \setminus \Lambda \to \mathcal{N}$ is an embedding, so Theorem 3.7 ensures that all points of $\Lambda \cap U$ are either removable or poles with respect to σ. Since σ is a lift of s, any point of Λ which is removable for σ is removable for s, and the minimality assumption on Λ precisely says that there are no such points. We conclude that every point of $\Lambda \cap U$ is a pole for σ. Hence, if we had $\hat{y}_\infty \in \widehat{\Lambda} \cap B(\hat{x},r)$, then $\hat{\sigma}(\alpha_k(t))$ should leave every compact subset of $\widehat{\mathcal{N}}$ as $t \to t_\infty$, a contradiction with $\beta_k([0,1]) \subset \widehat{\mathcal{N}}$.

The end point $\hat{x}'_k$ of α_k is mapped to $\hat{z}'_k$ by $\hat{\sigma}$. By compactness of $\overline{B(\hat{x},r)}$, we get a point $\hat{x}' \in \overline{B(\hat{x},r)}$ such that $\hat{\sigma}(\hat{x}') = \hat{z}'$, what concludes the proof that $\sigma\colon U \to \mathcal{N}_c$ is open. It remains to check that it is injective to get that σ maps U homeomorphically onto its image W. Let us assume for a contradiction that there are $x_1 \neq x_2$ in U such that $\sigma(x_1) = \sigma(x_2) = y$. Because σ is open, there are U_1 and U_2 two disjoint open subsets of U such that $\sigma(U_1) \cap \sigma(U_2)$ contains an open set V. Now $\sigma(U_1 \setminus \Lambda) \cap V$ and $\sigma(U_2 \setminus \Lambda) \cap V$ being two dense open subsets of V, they must intersect, contradicting the injectivity of σ on $U \setminus \Lambda$.

We showed above that all points of $\Lambda \cap U$ are poles for the embedding $\sigma\colon U \setminus \Lambda \to \mathcal{N}$, which implies $\sigma(\Lambda) \subset \partial_c \mathcal{N}$. □

Corollary 5.3. *The holonomy covering $\mathcal{N}$ is conformally diffeomorphic to an open subset of S^n, and N is a Kleinian manifold.*

Proof. We saw in Section 4.2 that associated to the conformal immersion $\delta\colon \mathcal{N} \to \mathrm{S}^n$, there is a group homomorphism $\rho\colon \Gamma \to \mathrm{PO}(1, n+1)$ satisfying the equivariance relation

$$\delta \circ \gamma = \rho(\gamma) \circ \delta \tag{9}$$

for every $\gamma \in \Gamma$. Proposition 4.4 shows that the action of Γ extends to an action by homeomorphisms on $\mathcal{N}_c$, and that δ extends to a continuous map $\delta\colon \mathcal{N}_c \to \mathrm{S}^n$. In particular, by density of $\mathcal{N}$ in $\mathcal{N}_c$, the equivariance relation (9) still holds on $\mathcal{N}_c$. Let us define $\mathcal{N}' := \mathcal{N} \cup \bigcup_{\gamma\in\Gamma} \gamma(W)$. It is an open subset of $\mathcal{N}_c$, and in particular it is Hausdorff by Proposition 4.4. By the previous proposition, the map $\delta \circ \sigma\colon U \to \mathrm{S}^n$ is continuous and coincides with the restriction of a Möbius transformation on the dense open set $U \setminus \Lambda$. Hence it is the restriction of a Möbius transformation. In particular $\delta\colon W \to \mathrm{S}^n$ is a homeomorphism onto its image. By relation (9), for every $\gamma \in \Gamma$, $\delta\colon \gamma(W) \to \mathrm{S}^n$ is a homeomorphism onto its image as well. From those remarks, we infer that $\mathcal{N}'$ is a second countable Hausdorff space. The topological immersion $\delta\colon \mathcal{N}' \to \mathrm{S}^n$ yields an atlas which endows $\mathcal{N}'$ with a structure of smooth conformally flat manifold, the conformal structure $\mathcal{C}$ on $\mathcal{N}'$ extending that of $\mathcal{N}$. The equivariance relation (9), available on $\mathcal{N}'$, tells that in the charts of this atlas, the action of $\gamma \in \Gamma$ reads as the restriction of the action of $\rho(\gamma) \in \mathrm{PO}(1, n+1)$. In particular, Γ acts as a group of smooth conformal transformations of $(\mathcal{N}', \mathcal{C})$.

We claim that the group $\mathrm{Conf}(\mathcal{N}')$ cannot preserve any Riemannian metric g' on $\mathcal{N}'$. Indeed, assuming it is the case, we can consider the function $\mu\colon \mathcal{N}' \to \mathbb{R}_+$, which to each $z \in \mathcal{N}'$ associates the distance (measured thanks to g') from z to the closed set $\partial_c\mathcal{N} \cap \mathcal{N}'$. It is continuous and Γ-invariant. Now, Proposition 3.8 implies that there exists a dense G_δ-set $\mathcal{G} \subset N$ such that for every $y \in \mathcal{G}$, the fiber $s^{-1}\{y\}$ accumulates on our point $x_\infty \in \Lambda_{\mathrm{ess}}$. Because σ is a lift of s, we get a sequence (γ_k) of Γ, and a point $z_0 \in \mathcal{N}$ such that $\gamma_k.z_0$ converges to $\sigma(x_\infty)$. This is a contradiction because on the one hand $\mu(z_0) > 0$, and on the other hand $\mu(\gamma_k.z_0)$ tend to $\mu(\sigma(x_\infty)) = 0$ as $k \to \infty$.

The previous claim, together with Theorem 5.1 ensures that $(\mathcal{N}', \mathcal{C})$ is conformally equivalent to the standard n-sphere or the Euclidean n-space. We infer that $\delta\colon \mathcal{N} \to \mathrm{S}^n$ is injective (Liouville's theorem), and N is a Kleinian manifold. □

Remark 5.4. Actually, because the manifold $\mathcal{N}'$ is conformally flat, we just need the conclusions of Theorem 5.1 for conformally flat manifolds, and this result is actually much easier to prove than the general case.

5.2. End of the proof of Theorem 1.3. We keep the notations of Section 5.1. Thanks to the work done there, we know that the developing map $\delta\colon \mathcal{N} \to \mathrm{S}^n$ is injective so that δ is a conformal diffeomorphism between $\mathcal{N}$ and a connected open subset $\Omega \subset \mathrm{S}^n$. Identifying Γ with $\rho(\Gamma)$, we see Γ as a Kleinian group in $\mathrm{PO}(1, n+1)$ and get a commutative diagram

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\delta"] \arrow[d, "\pi_{\mathcal{N}}"] & \Omega \arrow[d, "\pi"] \\ N \arrow[r, "\psi"] & \Omega / \Gamma, \end{tikzcd}$$

where ψ is a conformal diffeomorphism. We already noticed that Γ does not act properly on $\mathcal{N}'$ so that Γ is infinite.

Let us pick $x_\infty \in \Lambda$, and a connected neighborhood U of x_∞ in L, which is conformally diffeomorphic to an open subset of the sphere. By Lemma 4.3, the conformal immersion $s\colon U \setminus \Lambda \to N$ lifts to a conformal immersion $\sigma\colon U \setminus \Lambda \to \mathcal{N}$. Liouville's theorem ensures that $\varphi := \delta \circ \sigma$ extends to a conformal immersion $\varphi\colon U \to \mathrm{S}^n$. Let us call $V := \varphi(U)$. On $U \setminus \Lambda$, the relation $\pi \circ \varphi = \psi \circ s$ holds so that φ yields a one-to-one correspondence between points of $\Lambda \cap U$ which are essential (resp. poles) for s to points of $\overline{\Omega} \cap V$ which are essential (resp. poles) for π. By the discussion of Section 4.1, φ maps $U \cap \Lambda$ to $V \cap \partial\Omega$, and $U \cap \Lambda_{\mathrm{ess}}$ to $V \cap \Lambda(\Gamma)$. This completes the proof of Theorem 1.3.

5.3. Consequences of the local classification theorem. Because Theorem 1.3 classifies locally all thin conformal singularities admitting essential points, the study of a conformal immersion near an essential singular point reduces to understanding what is going on for singularities of Kleinian type. We can summarize the results in the following corollary.

Corollary 5.5. *Let (L, g) and (N, h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$. Assume that $s\colon L \setminus \Lambda \to N$ is a conformal immersion. Then:*

(1) *The set Λ_{ess} is closed. If it is nonempty, it is either discrete, or perfect.*

(2) *If Λ_{pole} is nonempty, its closure in Λ is the set $\Lambda_{\mathrm{pole}} \cup \Lambda_{\mathrm{ess}}$.*

(3) *Assume that Λ is minimal essential. Then for every $x_\infty \in \Lambda_{\mathrm{ess}}$ and any neighborhood U of x_∞ in L, $s(U \setminus \Lambda) = N$.*

(4) *If Λ is discrete and contains at least one essential singular point, then $\Lambda_{\mathrm{pole}} = \emptyset$ and (N, h) is conformally diffeomorphic to a Euclidean manifold, or a generalized Hopf manifold.*

We define generalized Hopf manifolds as quotients of $\mathbb{R}^n \setminus \{0\}$ by an infinite discrete subgroup of conformal transformations. Topologically, those manifolds are finite quotients of $\mathrm{S}^1 \times \mathrm{S}^{n-1}$ (see Section 7.2 for a complete description of those manifolds).

When the singular set Λ is reduced to a point, the third and fourth points of the corollary can be compared to Picard's theorem about the behavior of a meromorphic function in the neighborhood of an isolated essential singularity. Let us also mention that when $s\colon L\setminus\Lambda \to N$ is merely a quasiconformal immersion, and when $\Lambda = \{p\}$ is an isolated essential singularity, then V. A. Zorich proved in [Zo1] and [Zo2] that $s(U \setminus p) = N$ for every neighborhood U, and that up to finite quotient, N is homeomorphic to a product $\mathbb{R}^k \times \mathrm{T}^{n-k}$ or $\mathrm{S}^1 \times \mathrm{S}^{n-1}$. Its proof does not imply Corollary 5.5 in the conformal framework, though (see also [R2], Theorem 2.1, p. 81, and [HP] for other generalizations of Picard's theorem in the quasiregular setting).

Proof of Corollary 5.5

First point. We first explain why Λ_{ess} is closed. Let us consider (x_k) a sequence of Λ_{ess} which converges to $x_\infty \in \Lambda$. From Proposition 3.8, we know that $\mathrm{Clust}(x_k) = N$ for all $k \in \mathbb{N}$. Hence, if we fix y and y' two distinct points of N, one can build two sequences (y_k) and (z_k) in $L \setminus \Lambda$ which converge to x_∞ such that $s(y_k) \to y$ and $s(z_k) \to y'$. It follows that $x_\infty \in \Lambda_{\mathrm{ess}}$. Now, thanks to Theorem 1.1, we extend s to a conformal immersion $s'\colon L \setminus (\Lambda_{\mathrm{ess}} \cup \Lambda_{\mathrm{pole}}) \to N$. Theorem 1.3 implies that $N = \Omega/\Gamma$, for an infinite Kleinian group Γ. It is a classical fact that the limit set $\Lambda(\Gamma)$ is either a perfect set, or has at most two points ([A], Theorem 2.3, p. 43). If we are in the former case, Theorem 1.3 ensures that Λ_{ess} is perfect. If $\Lambda(\Gamma)$ has one or two points, then again by Theorem 1.3, all the points of Λ_{ess} are isolated.

Second point. Assume that Λ_{pole} is nonempty, and let us show that the closure $\overline{\Lambda_{\mathrm{pole}}}$ is $\Lambda_{\mathrm{pole}} \cup \Lambda_{\mathrm{ess}}$. By Theorem 1.1, there is no harm assuming that $\Lambda = \Lambda_{\mathrm{pole}} \cup \Lambda_{\mathrm{ess}}$. If Λ_{ess} is empty, the claim is clear. Assume now that Λ_{ess} is nonempty. It is enough to check that every point of Λ_{ess} is in the closure of Λ_{pole}. Recall that by Theorem 1.3, for each x_∞, there is a neighborhood U of x_∞ and a commutative diagram

$$\begin{tikzcd} U \setminus \Lambda \arrow[r, "\varphi"] \arrow[d, "s"] & V \setminus \partial\Omega \arrow[d, "\pi"] \\ N \arrow[r, "\psi"] & \Omega / \Gamma. \end{tikzcd}$$

Moreover, $\varphi(U \cap \Lambda) = V \cap \partial\Omega$ and $\varphi(U \cap \Lambda_{\mathrm{ess}}) = V \cap \Lambda(\Gamma)$. We infer that $\partial\Omega \setminus \Lambda(\Gamma)$ is nonempty, and we are reduced to show that every point in $\Lambda(\Gamma)$ is accumulated by points in $\partial\Omega \setminus \Lambda(\Gamma)$. But this is clear, because if $z \in \partial\Omega \setminus \Lambda(\Gamma)$, we will have $\Gamma.z \subset \partial\Omega \setminus \Lambda(\Gamma)$ and $\overline{\Gamma.z} = \Lambda(\Gamma) \cup \Gamma.z$.

Third point. We assume now that Λ is minimal essential. We want to show that if $x_\infty \in \Lambda_{\mathrm{ess}}$ and if U is any neighborhood of x_∞ in L, then $s(U \setminus \Lambda) = N$. By Theorem 1.3, the manifold N is Kleinian, conformally diffeomorphic to Ω/Γ, for some infinite discrete Γ. For any $z \in \Omega$, the closure of $\Gamma.z$ contains $\Lambda(\Gamma)$. In particular, if $z_\infty \in \Lambda(\Gamma)$ and if V is a neighborhood of z_∞ in S^n, then $\pi(V \setminus \Lambda(\Gamma)) = \Omega/\Gamma$. Theorem 1.3 implies directly that $s(U \setminus \Lambda) = N$.

Fourth point. Let us assume that Λ is a discrete set containing at least one essential singular point. Thanks to Theorem 1.1, we can assume that $\Lambda = \Lambda_{\mathrm{pole}} \cup \Lambda_{\mathrm{ess}}$. The second point of the corollary implies that in the presence of essential singular points, Λ_{pole} is not closed as soon as it is nonempty. Because Λ is discrete, we infer that Λ_{pole} must be empty. If Γ is the infinite Kleinian group such that $N = \Omega/\Gamma$, then $\Lambda(\Gamma)$ has one or two points (if not, Λ_{ess} would be perfect), and Theorem 1.3 actually implies that $\Omega = \mathrm{S}^n \setminus \Lambda(\Gamma)$, otherwise Λ would contain poles. We infer that if $\Lambda(\Gamma)$ has one point, Γ is a discrete subgroup of conformal transformations of $\mathbb{R}^n$ acting freely properly discontinuously on $\mathbb{R}^n$. Then, one checks easily that Γ is a discrete subgroup of Euclidean motions, and N is a Euclidean manifold. If $\Lambda(\Gamma)$ has two points, then N is conformally diffeomorphic to a quotient of $\mathbb{R}^n \setminus \{0\}$ by an infinite discrete group of conformal transformations, namely N is a generalized Hopf manifold. □

6. Proof of Theorem 1.4

We are now considering thin essential conformal singular sets on a compact manifold L. This compactness assumption on L allows us to prove:

Proposition 6.1. *Let (L, g) and (N, h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$. Let $\Lambda \subset L$ be a closed subset such that $\mathcal{H}^{n-1}(\Lambda) = 0$, and $s\colon L \setminus \Lambda \to N$ a conformal immersion. If L is compact, and $\Lambda = \Lambda_{\mathrm{pole}} \cup \Lambda_{\mathrm{ess}}$, then $s\colon L \setminus \Lambda \to N$ is a covering map onto N.*

Proof. Let $\alpha\colon [0,1] \to N$ be a continuous path, let $x_0 \in L \setminus \Lambda$ such that $s(x_0) = \alpha(0)$. We want to show the existence of $\gamma\colon [0,1] \to L \setminus \Lambda$, a lift of α satisfying $\gamma(0) = x_0$. If we cannot lift α, there exists $t_\infty \in [0,1[$, and $\gamma\colon [0, t_\infty[\to L \setminus \Lambda$ a lift of $\alpha\colon [0, t_\infty[\to N$ such that $\gamma(0) = x_0$ and $\gamma(t)$ leaves every compact subset of $L \setminus \Lambda$ as t tends to t_∞. By compactness of L, for every sequence (t_k) tending to t_∞, the set A of cluster values of $\gamma(t_k)$ in L is nonempty and contained in Λ. Let x_∞ be a point of A. Since $s(\gamma(t_k))$ tends to $\alpha(t_\infty)$, we get $x_\infty \notin \Lambda_{\mathrm{pole}}$. Hence we should have $x_\infty \in \Lambda_{\mathrm{ess}}$. But this is not possible. Indeed, if $x_\infty \in \Lambda_{\mathrm{ess}}$, we first assume, considering a subsequence of (t_k), that $\gamma(t_k)$ tends to x_∞. Then we use the second point of Lemma 3.5, and get the existence of a sequence (t'_k) in $[0, t_\infty[$, which converges to t_∞ such that $\gamma(t'_k)$ converges to x_∞, and such that $s(\gamma(t'_k))$ converges to $y' \in N$, with $y' \neq \alpha(t_\infty)$. This contradicts the fact that γ is a lift of $\alpha_{|[0,t_\infty[}$. □

We are now under the hypotheses of Theorem 1.4: the manifold L is compact and the singular set is minimal essential, *i.e* $\Lambda = \Lambda_{\mathrm{ess}} \cup \Lambda_{\mathrm{pole}}$. Moreover, we do the assumption $\mathcal{H}^{n-2}(\Lambda) = 0$. Theorem 1.3 ensures that (L, g) and (N, h) are conformally flat, and that N is actually conformally diffeomorphic, via a diffeomorphism ψ, to a Kleinian manifold Ω / Γ. From Theorem 1.3, we also get that the boundary $\partial\Omega$ satisfies $\mathcal{H}^{n-2}(\partial\Omega) = 0$, hence Ω is simply connected as the following lemma shows.

Lemma 6.2 ([LV], Theorem 6.13). *Let M be a connected, simply connected, n-dimensional Riemannian manifold, $n \geq 3$. Assume that E is a closed subset of M satisfying $\mathcal{H}^{n-2}(E) = 0$. Then $M \setminus E$ is still simply connected.*

Let us call $\tilde{L}$ the conformal universal covering of L and denote by $\pi_L \colon \tilde{L} \to L$ the associated covering map. We call $\widetilde{\Lambda}$ the inverse image of Λ by π_L. Observe that $\tilde{L} \setminus \widetilde{\Lambda}$ is simply connected by Lemma 6.2. By Proposition 6.1, our conformal immersion $s \colon L \setminus \Lambda \to N$ is a covering, hence it lifts to a conformal diffeomorphism $\sigma \colon \tilde{L} \setminus \widetilde{\Lambda} \to \Omega$. In particular

$$\pi \circ \sigma = \psi \circ s \circ \pi_L. \tag{10}$$

Apply Theorem 3.7 to get that $\sigma^{-1} \colon \Omega \to \tilde{L}$ extends to a conformal diffeomorphism $\sigma^{-1} \colon \Omega' \to \tilde{L}$, where $\Omega' \subset \mathrm{S}^n$ is an open subset containing Ω. We denote again by $\sigma \colon \tilde{L} \to \Omega'$ the inverse map. Observe that $\sigma(\widetilde{\Lambda}) = \Omega' \cap \partial\Omega$. The map σ induces a homomorphism $\rho \colon \pi_1(L) \to \mathrm{PO}(1, n+1)$ such that for every $\gamma \in \pi_1(L)$, the equivariance relation $\sigma \circ \gamma = \rho(\gamma) \circ \sigma$ holds. The group $\Gamma' := \rho(\pi_1(L))$ is a discrete subgroup of $\mathrm{PO}(1, n+1)$ acting freely properly discontinuously on Ω'. Let us call $\pi' \colon \Omega' \to \Omega'/\Gamma'$ the conformal covering map. There is a conformal diffeomorphism $\varphi \colon L \to \Omega'/\Gamma'$ such that

$$\pi' \circ \sigma = \varphi \circ \pi_L. \tag{11}$$

Let us check that $\Omega' = \Omega(\Gamma')$. If Γ' is finite, the compactness of L leads to $\Omega' = \mathrm{S}^n$. If Γ' is infinite, one has $\Omega' \subset \Omega(\Gamma')$, since the action of Γ' is proper on Ω'. On the other hand, the compactness of L forces the action of Γ' to be nonequicontinuous at each point of $\partial\Omega'$, yielding the inclusion $\partial\Omega' \subset \Lambda(\Gamma')$. In any case, we get that $\Omega' = \Omega(\Gamma')$, as claimed in Theorem 1.4.

For every $\gamma \in \pi_1(L)$, relation (10) leads to the identity $\pi \circ \rho(\gamma) = \pi$ on Ω so that one has the inclusion $\Gamma' \subset \Gamma$. Hence, the identity map of Ω induces a covering map $s' \colon \Omega/\Gamma' \to \Omega/\Gamma$, satisfying for every $y \in \Omega$

$$s' \circ \pi'(y) = \pi(y). \tag{12}$$

Observe that if we define $\Lambda' = \pi'(\Omega' \cap \partial\Omega)$, then Ω/Γ' is merely $M(\Gamma') \setminus \Lambda'$.

Relations (10), (11) and (12) lead to the commutative diagram

$$\begin{tikzcd} L \setminus \Lambda \arrow[r, "\varphi"] \arrow[d, "s"] & M(\Gamma') \setminus \Lambda' \arrow[d, "s'"] \\ N \arrow[r, "\psi"] & \Omega / \Gamma. \end{tikzcd}$$

The diffeomorphism φ maps Λ to Λ' because σ maps $\tilde{\Lambda}$ to $\partial\Omega \cap \Omega'$. Finally, it is easily checked that the essential singular points of Λ' for s' are the π'-images of the essential singular points of $\partial\Omega \cap \Omega'$ for π, namely the points of $\partial\Omega \cap \Omega'$ which are in $\Lambda(\Gamma)$. This means $\Lambda(\Gamma) \cap \Omega' \neq \emptyset$, hence $\Lambda(\Gamma') \subsetneq \Lambda(\Gamma)$.

7. Isolated essential singularities on compact manifolds

Our aim in this section is to understand completely the conformal singularities $s\colon L \setminus \Lambda \to N$, where N is a compact manifold and Λ is a finite number of essential singular points. It turns out that very few possibilities arise, and they are listed in Theorem 7.1 below. First of all, let us enumerate some examples.

7.1. Euclidean singularities on the sphere. Let us consider an *infinite* discrete subgroup $\Gamma \subset (\mathbb{R}_+^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n$, acting freely properly discontinuously on $\mathbb{R}^n$. One checks that for the action to be free, Γ must actually be a subgroup of $\mathrm{O}(n) \ltimes \mathbb{R}^n$. The quotient manifold $N = \mathbb{R}^n / \Gamma$ is then a Euclidean manifold. We see Γ as acting conformally on $\mathrm{S}^n \setminus \{\nu\}$, fixing ν, and consider the covering map $s\colon \mathrm{S}^n \setminus \{\nu\} \to N$. It is a conformal immersion, and because Γ is infinite, we have $\Lambda(\Gamma) = \{\nu\}$. Hence, as we already saw, ν is an essential singular point for s. A conformal singularity $s\colon \mathrm{S}^n \setminus \{\nu\} \to N$ as described above will be referred to as *Euclidean singularity on the sphere.*

7.2. Singularities of Hopf type on the sphere. Let us now fix o a second point on the sphere S^n, distinct from the point ν. There is a conformal diffeomorphism mapping $\mathrm{S}^n \setminus \{o; \nu\}$ onto $\mathbb{R}^n \setminus \{0\}$. The group G of conformal transformations of $\mathbb{R}^n \setminus \{0\}$ is generated by the inversion $\iota\colon x \mapsto -\frac{x}{\|x\|^2}$, and the group $\mathbb{R}_+^* \times \mathrm{O}(n)$ of linear conformal transformations on $\mathbb{R}^n$. Let us choose an *infinite* discrete group $\Gamma \subset G$ acting freely, properly and discontinuously on $\mathbb{R}^n \setminus \{0\}$. It is not hard to check that Γ has a finite index subgroup generated by a linear conformal contraction. As previously, the quotient $N = (\mathbb{R}^n \setminus \{0\}) / \Gamma$ is called a *generalized Hopf manifold.* The covering map $s\colon \mathrm{S}^n \setminus \{o; \nu\} \to N$ is conformal, and because Γ is infinite, both ν and o are essential punctured singularities. Conformal singularities $s\colon \mathrm{S}^n \setminus \{o; \nu\} \to N$ constructed as above will be referred to as *singularities of Hopf type on the sphere.*

7.3. Singularities of Hopf type on the projective space. Let us go back to the previous construction, and assume that our infinite discrete subgroup $\Gamma \subset G$ contains the inversion ι. Then, the subgroup $\Gamma_o \subset \Gamma$ of transformations fixing individually the points ν and o is normal in Γ. Let us call N_o the quotient manifold $(\mathbb{R}^n \setminus \{0\}) / \Gamma_o$. Because ι normalizes Γ_o, and because Γ acts freely on $\mathbb{R}^n \setminus \{0\}$, ι induces a conformal involution $\bar{\iota}$ without fixed points on N_o. The quotient $N_o/\langle\bar{\iota}\rangle$ is actually conformally diffeomorphic to $N := (\mathbb{R}^n \setminus \{0\}) / \Gamma$. The quotient of $\mathrm{S}^n \setminus \{o; \nu\}$ by $\langle\iota\rangle$ is conformally diffeomorphic to $\mathbb{R}\boldsymbol{P}^n$ with a point ν removed. The natural covering map $\pi : \mathrm{S}^n \setminus \{o; \nu\} \to N_o$ induces a conformal immersion $s : \mathbb{R}\boldsymbol{P}^n \setminus \{\nu\} \to N$, for which ν is an essential singular point. Conformal singularities constructed in this way will be referred to as *singularities of Hopf type on the projective space*.

7.4. Classification result. We are now investigating essential singular sets on compact manifolds, comprising only a finite number of points. By Theorem 1.1, and the fourth point of Corollary 5.5, we just have to focus on the case where all the points are essential. Then, it turns out that the three kinds of singularities described in the previous section are the only possible.

Theorem 7.1. *Let (L, g) and (N, h) be two connected n-dimensional Riemannian manifolds, $n \geq 3$, with L compact. Let $\Lambda := \{p_1, \dots, p_m\}$ be a finite number of points on L. Assume that $s : L \setminus \Lambda \to N$ is a conformal immersion such that each p_i is an essential singular point for s. Then $m = 1$ or $m = 2$ and:*

(1) *If $m = 1$, either there exists a Euclidean singularity on the sphere $s' : \mathrm{S}^n \setminus \{\nu\} \to N'$, a conformal diffeomorphism $\varphi : L \to \mathrm{S}^n$ sending p_1 to ν and a conformal diffeomorphism $\psi : N \to N'$ making the diagram*

$$\begin{array}{ccc} L \setminus \{p_1\} & \xrightarrow{\varphi} & \mathrm{S}^n \setminus \{\nu\} \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ N & \xrightarrow{\psi} & N' \end{array}$$

commute.

Or there exists a singularity of Hopf type on the projective space $s' : \mathbb{R}\boldsymbol{P}^n \setminus \{\nu\} \to N'$, a conformal diffeomorphism $\varphi : L \to \mathbb{R}\boldsymbol{P}^n$ sending p_1 to ν and a conformal diffeomorphism $\psi : N \to N'$ making the diagram

$$\begin{array}{ccc} L \setminus \{p_1\} & \xrightarrow{\varphi} & \mathbb{R}\boldsymbol{P}^n \setminus \{\nu\} \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ N & \xrightarrow{\psi} & N' \end{array}$$

commute.

(2) *If $m = 2$, there exists a singularity of Hopf type on the sphere $s' : \mathrm{S}^n \setminus \{o; \nu\} \to N'$, a conformal diffeomorphism $\varphi : L \to \mathrm{S}^n$ sending $\{p_1; p_2\}$ to $\{o; \nu\}$ and a conformal diffeomorphism $\psi : N \to N'$ making the diagram*

$$\begin{tikzcd} L \setminus \{p_1; p_2\} \arrow[r, "\varphi"] \arrow[d, "s"] & \mathrm{S}^n \setminus \{o; \nu\} \arrow[d, "s'"] \\ N \arrow[r, "\psi"] & N' \end{tikzcd}$$

commute.

Proof. We first apply Theorem 1.3 in a neighborhood of any of the p_i's. We get that N is conformally diffeomorphic to a Kleinian manifold Ω/Γ, where the limit set $\Lambda(\Gamma)$ has one or two points (otherwise Λ_{ess} would be a perfect set), and $\Omega = \Omega(\Gamma)$ (otherwise Λ_{pole} would be nonempty).

Assume first that $\Lambda(\Gamma)$ is made of a single point ν. The group Γ is a discrete group of conformal transformations of $\mathrm{S}^n \setminus \{\nu\}$, namely $\mathbb{R}^n$, which acts freely properly discontinuously on $\mathbb{R}^n$. As a consequence, Γ is a discrete group of Euclidean motions, and N is conformally diffeomorphic to a Euclidean manifold $N' = \mathbb{R}^n/\Gamma$. Theorem 1.4 makes the structure of L and Λ explicit: there must be a subgroup $\Gamma' \subset \Gamma$, with $\Lambda(\Gamma') \subsetneq \Lambda(\Gamma)$, as well as an open subset Ω' properly containing Ω such that L is conformally diffeomorphic to Ω'/Γ', and Λ_{ess} is obtained as the quotient $(\Omega' \cap \Lambda(\Gamma))/\Gamma'$. This implies in particular $\Lambda(\Gamma') = \emptyset$, hence Γ' is finite, and because Γ' acts cocompactly on Ω', we must have $\Omega' = \mathrm{S}^n$. Since the action of Γ' on S^n must be free, and Γ' fixes ν, we infer that Γ' is trivial. We get that $m = 1$, L is conformally diffeomorphic to S^n, and we are in the first case of the theorem.

Assume now that $\Lambda(\Gamma)$ comprises two points o and ν. Applying Theorem 1.4, and with the same notations as above, we get that Γ is a discrete group in the conformal group of $\mathbb{R}^n \setminus \{0\}$. The limit set of the subgroup Γ' has two points or is empty, but because $\Lambda(\Gamma') \subsetneq \Lambda(\Gamma)$, we are in the second alternative: Γ' is once again finite, and $\Omega' = \mathrm{S}^n$. Because Γ' acts freely on S^n, and leaves $\{o; \nu\}$ invariant, it is either trivial, or generated by a conformal involution of S^n, without fixed point, and switching o and ν.

It is not hard to check that such a fixed-point free involution switching o and ν is conjugated, in the conformal group of $\mathbb{R}^n \setminus \{0\}$, to the inversion $\iota : x \mapsto -\frac{x}{\|x\|^2}$, so if Γ' is nontrivial, there is no harm in assuming $\Gamma' = \langle \iota \rangle$. Then $m = 1$, L conformally diffeomorphic to $\mathbb{R}\boldsymbol{P}^n$, and we are in the second case of the theorem.

Finally, if Γ' is trivial, then $m = 2$, L is conformally diffeomorphic to S^n and we are in the third case of the theorem. □

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