Quantitative properties of convex representations

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Quantitative properties of convex representations

Andrés Sambarino

Abstract. Let Γ be a discrete subgroup of $\mathrm{PGL}(d,\mathbb{R})$. Fix a norm $\|\ \|$ on $\mathbb{R}^d$ and let $N_\Gamma(t)$ be the number of elements in Γ whose operator norm is $\leq t$. In this article we prove an asymptotic for the growth of $N_\Gamma(t)$ when $t \to \infty$ for a class of Γ's which contains, in particular, Hitchin representations of surface groups and groups dividing a convex set of $\mathbb{P}(\mathbb{R}^d)$. We also prove analogue counting theorems for the growth of the spectral radii. More precise information is given for Hitchin representations.

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1. Introduction

Let $\widetilde{M}$ be a simply connected complete manifold of sectional curvature $K \leq -1$ and Γ be a torsion-free discrete co-compact group of isometries of $\widetilde{M}$. This work consists in studying specific quantitative properties of certain representations $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$.

Recall that Γ is a hyperbolic group, its boundary $\partial\Gamma$ is identified with $\widetilde{M}$'s geometric boundary, and that $\partial\Gamma$ has a natural structure of compact metrizable space coming from some Gromov distance (see Ghys–de la Harpe [11]).

Definition 1.1. We say that an irreducible representation $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ is *strictly convex* if there exists a ρ-equivariant Hölder continuous map

$$(\xi,\eta)\colon \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)\times \mathrm{Gr}_{d-1}(\mathbb{R}^d),$$

where $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ is the Grassmannian of hyperplanes of $\mathbb{R}^d$, such that $\mathbb{R}^d = \xi(x)\oplus \eta(y)$ whenever $x \neq y$.

We show in Lemma 5.1 that strictly convex representations are proximal, that is, every element $\rho(\gamma)$ is a proximal matrix. This implies (cf. Corollary 5.2) that for each $x \in \partial\Gamma$ one has $\xi(x) \subset \eta(x)$, and that the equivariant map (ξ,η) is necessarily unique.

Among strictly convex representations we find the following.

Deformations of hyperbolic manifolds in projective structures. A consequence of the work of Koszul [14] and Benoist [4] is that if Γ is the fundamental group of a closed hyperbolic manifold of dimension $d-1$ and $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ is a deformation of the embedding $\Gamma \subset \mathrm{PSO}(d-1,1) \hookrightarrow \mathrm{PGL}(d,\mathbb{R})$, then $\rho(\Gamma)$ leaves invariant an open convex set Ω of $\mathbb{P}(\mathbb{R}^d)$, and the quotient $\rho(\Gamma)\backslash\Omega$ is a compact manifold. This gives an identification $\xi\colon \partial\Gamma \to \partial\Omega \subset \mathbb{P}(\mathbb{R}^d)$.

Benoist [3] has shown that Ω is strictly convex and its boundary $\partial\Omega$ is of class $\mathrm{C}^{1+\alpha}$. The identification ξ and the tangent space of $\partial\Omega$ at $\xi(x)$,

$$\eta\colon \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d),$$

are thus ρ-equivariant and Hölder. Since $\partial\Omega$ is strictly convex we have $\mathbb{R}^d = \xi(x)\oplus \eta(y)$ if $x \neq y$. These deformations are always irreducible and Zariski-dense when the deformation is non-trivial. Hence we have that $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ is a strictly convex representation.

Groups dividing a convex set of $\mathbb{P}(\mathbb{R}^d)$. These examples contain the former but we treat them separately because they do not fall exactly in our terminology. Nevertheless, the methods of this article apply directly to this setting.

Consider some open convex set Ω of $\mathbb{P}(\mathbb{R}^d)$ and $\overline{\Omega}$ its closure. Suppose that $\mathbb{P}(V)\cap\overline{\Omega} = \emptyset$ for some hyperplane V of $\mathbb{R}^d$. Assume there exists a discrete subgroup Γ of $\mathrm{PGL}(d,\mathbb{R})$ that leaves Ω invariant. The action of Γ on Ω is necessarily properly discontinuous, and we assume it is also co-compact.

Benoist [3] has shown that if Ω is strictly convex then $\partial\Omega$ is $\mathrm{C}^{1+\alpha}$ and the group Γ is hyperbolic in the sense of Gromov. Following the last example one finds that $\Gamma \subset \mathrm{PGL}(d,\mathbb{R})$ is strictly convex.

Hitchin representations of surface groups. Let Σ be a closed orientable hyperbolic surface and let $\pi_1(\Sigma) \subset \mathrm{PGL}(2,\mathbb{R})$ be its fundamental group. Labourie [15] has shown that if $\rho\colon \pi_1(\Sigma) \to \mathrm{PGL}(d,\mathbb{R})$ is a deformation of the unique irreducible morphism (up to conjugacy) $\mathrm{PGL}(2,\mathbb{R}) \to \mathrm{PGL}(d,\mathbb{R})$, then ρ is irreducible and

there exists a ρ-equivariant Hölder map $\zeta\colon \partial\pi_1(\Sigma) \to \mathscr{F}$ where $\mathscr{F}$ is the space of complete flags of $\mathbb{R}^d$. He shows that this curve is a *Frenet curve*: for $x \in \partial\pi_1(\Sigma)$ set $\zeta_i(x)$ to be the i-th space of the flag $\zeta(x)$, then if $d = d_1 + \cdots + d_k$ and $x_1, \ldots, x_k$ are pairwise distinct, then

$$\mathbb{R}^d = \bigoplus_1^k \zeta_{d_i}(x_i),$$

and if $n = n_1 + \cdots + n_k \leq d$, then

$$\lim_{(x_i)\to x} \bigoplus_1^k \zeta_{n_i}(x_i) = \zeta_n(x).$$

The first condition implies that, by considering the first and last coordinate of ζ, $\xi := \zeta_1$ and $\eta := \zeta_d$, one obtains a strictly convex representation.

Composition. If $\rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is a Zariski-dense Hitchin representation then the composition of ρ with some irreducible representation $\Lambda\colon \mathrm{PGL}(d, \mathbb{R}) \to \mathrm{PGL}(k, \mathbb{R})$ is strictly convex.

Fix a norm $\| \ \|$ in $\mathbb{R}^d$ (not necessarily Euclidean). For an element $g \in \mathrm{PGL}(d, \mathbb{R})$ we define its norm $\|g\|$ as the operator norm of some lift $\tilde{g} \in \mathrm{GL}(d, \mathbb{R})$ such that $\det \tilde{g} \in \{-1, +1\}$. In the same way one can define the spectral radius of g, since these quantities do not depend on the choice of the lift.

The main goal of this article is to prove the following result, a direct consequence of Theorem A below.

Corollary (of Theorem A). *Let $\rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation. Then there exist positive real numbers h and c such that*

$$cR^{-h}\#\{\gamma \in \Gamma\colon \|\rho(\gamma)\| \leq R\} \to 1$$

when R goes to infinity.

The constant h is independent of the norm chosen and is thus invariant under conjugation of ρ by elements of $\mathrm{PGL}(d, \mathbb{R})$. This follows from the fact that any two norms in $\mathbb{R}^d$ are equivalent.

We shall now state the stronger result from which the corollary is deduced. The dynamics of each $\gamma \in \Gamma$ on $\partial\Gamma$ is of type north-south, i.e., γ has exactly two fixed points, γ_+ and γ_-, and the basin of attraction of γ_+ is $\partial\Gamma - \{\gamma_-\}$.

Theorem A shows that these fixed points are well distributed on $\partial\Gamma$. Denote by $C(X)$ the space of continuous real functions over some space X and by $C^*(X)$ its dual space.

Theorem A. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation. Then there exist positive real numbers h and c and two probabilities μ and $\bar{\mu}$ on $\partial\Gamma$ such that*

$$ce^{-ht} \sum_{\gamma\in\Gamma:\ \log\|\rho(\gamma)\|\le t} \delta_{\gamma_-}\otimes\delta_{\gamma_+} \to \bar{\mu}\otimes\mu$$

when $t\to\infty$, in $C^*(\partial\Gamma\times\partial\Gamma)$.

The previous corollary is deduced from Theorem A by considering the constant function equal to 1 and the change of parameter $t=\log R$.

For a matrix $g\in\mathrm{PGL}(d,\mathbb{R})$ denote by $\lambda_1(g)$ the logarithm of the spectral radius of g. An element g of a given subgroup G is *primitive* if it can not be written as a positive power of another element of G.

Theorem B. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation. Then there exists a positive real number h such that*

$$hte^{-ht}\#\{[\gamma]\in[\Gamma]\ \textit{primitive}\colon \lambda_1(\rho(\gamma))\le t\}\to 1$$

when $t\to\infty$, where $[\Gamma]$ is the set of conjugacy classes of Γ.

The constant h is the same for both Theorems A and B. Theorem A, inspired by Roblin's work [24], implies the following corollary which explains how attractive lines of $\rho(\Gamma)$ are distributed in $\mathbb{P}(\mathbb{R}^d)$. Write g_+ for the attractive line of a proximal matrix in $\mathrm{PGL}(d,\mathbb{R})$.

Corollary. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation. Then there exist positive real numbers h and c and a probability ν on $\mathbb{P}(\mathbb{R}^d)$ such that*

$$che^{-ht} \sum_{\gamma\in\Gamma:\ \log\|\rho(\gamma)\|\le t} \delta_{\rho(\gamma)_+} \to \nu$$

when $t\to\infty$.

The probability ν in the above corollary is non-atomic, ergodic for the action of $\rho(\Gamma)$ and its support generates $\mathbb{R}^d$. Moreover ν verifies a Patterson–Sullivan property, namely: for every $\gamma\in\Gamma$ one has

$$\frac{d(\rho(\gamma)_*\nu)}{d\nu}(v)=\left(\frac{\|\rho(\gamma)v\|}{\|v\|}\right)^{-h}=e^{-h\log\frac{\|\rho(\gamma)v\|}{\|v\|}}.$$

We now turn our attention to *hyperconvex* representations introduced by Labourie in [15]. Fix some real semi-simple algebraic non-compact group G and denote by P a minimal parabolic subgroup. Write $\mathscr{F}=G/P$; the set $\mathscr{F}$ is called the *Furstenberg*

boundary of the symmetric space of G. The product $\mathscr{F} \times \mathscr{F}$ has a unique open G-orbit, which we shall call $\partial^2\mathscr{F}$.

For example, when $G = \mathrm{PGL}(d, \mathbb{R})$ the set $\mathscr{F}$ is the set of complete flags of $\mathbb{R}^d$, i.e., families of subspaces $\{V_i\}_{i=0}^d$ such that $V_i \subset V_{i+1}$ and $\dim V_i = i$; and the set $\partial^2\mathscr{F}$ is the set of flags in general position, i.e., pairs $\{V_i\}$ and $\{W_i\}$ such that for every i one has

$$V_i \oplus W_{d-i} = \mathbb{R}^d.$$

Definition 1.2. We say that a representation $\rho\colon \Gamma \to G$ is *hyperconvex* if it admits a Hölder continuous equivariant map $\zeta\colon \partial\Gamma \to \mathscr{F}$ such that whenever $x \neq y$ in $\partial\Gamma$, the pair $(\zeta(x), \zeta(y))$ belongs to $\partial^2\mathscr{F}$.

As mentioned before, Labourie [15] has shown that Hitchin representations of surface groups into $\mathrm{PGL}(d, \mathbb{R})$ provide examples of hyperconvex representations.

The same method for proving Theorems A and B yields the following result. Denote by $\mathfrak{a}$ a Cartan subalgebra of G's Lie algebra $\mathfrak{g}$, and let $a\colon G \to \mathfrak{a}$ be the Cartan projection. Fix some Weyl chamber $\mathfrak{a}^+$ and denote by $\lambda\colon \Gamma \to \mathfrak{a}^+$ the Jordan projection.

Benoist [2] introduced the *limit cone* $\mathscr{L}_\Lambda$ of a Zariski-dense subgroup Λ of G as the closed cone containing $\{\lambda(g) : g \in \Lambda\}$. He has shown that this cone is convex and has non-empty interior. We shall consider also the *dual cone*

$$\mathscr{L}_\Lambda^* := \{\varphi \in \mathfrak{a}^* : \varphi|\mathscr{L}_\Lambda \geq 0\}.$$

For a hyperconvex representation write $\mathscr{L}_\rho$ for its limit cone and $\mathscr{L}_\rho^*$ for its dual cone.

Theorem C (Theorems 7.8 and 7.13). *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation and consider φ in the interior of $\mathscr{L}_\rho^*$. Then there exists $h_\varphi > 0$ such that*

$$h_\varphi t e^{h_\varphi t} \#\{[\gamma] \in [\Gamma] \text{ primitive}\colon \varphi(\lambda(\rho\gamma)) \leq t\} \to 1.$$

Moreover there exists $c_\varphi > 0$ such that

$$c_\varphi e^{h_\varphi t} \#\{\gamma \in \Gamma\colon \varphi(a(\rho\gamma)) \leq t\} \to 1.$$

Of particular interest is the following corollary for $\mathrm{PGL}(d, \mathbb{R})$. The Cartan algebra is

$$\mathfrak{v} = \{(v_1, \ldots, v_d) \in \mathbb{R}^d : v_1 + \cdots + v_d = 0\}$$

and the Weyl chamber $\mathfrak{a}^+ = \{v \in \mathfrak{a} : v_1 \geq \cdots \geq v_d\}$. The linear form $\varphi\colon \mathfrak{v} \to \mathbb{R}$,

$$\varphi(v_1, \ldots, v_d) = v_1 - v_d,$$

is strictly positive on the Weyl chamber (except at $\{0\}$) and thus with the change of parameter $t = \log R$ one obtains the following.

Corollary. *Let Σ be a closed orientable surface of genus ≥ 2 and let $\rho\colon \pi_1(\Sigma) \to \mathrm{PGL}(d,\mathbb{R})$ be a Hitchin representation. Then there exists $h_1 > 0$ such that*

$$h_1 R^{-h_1} \log R \#\{[\gamma] \in [\pi_1(\Sigma)] \text{ primitive} : \tfrac{\lambda_{\max}(\rho\gamma)}{\lambda_{\min}(\rho\gamma)} \leq R\} \to 1,$$

when $R \to \infty$ where $\lambda_{\max}(g)$ (resp. $\lambda_{\min}(g)$) denotes the modulus of g's maximal (resp. minimal) eigenvalue. Moreover, for some fixed Euclidean norm $\| \ \|$ on $\mathbb{R}^d$, there exists $c > 0$ such that

$$cR^{-h_1} \#\{\gamma \in \pi_1(\Sigma)\colon \|\rho(\gamma)\|\|\rho(\gamma^{-1})\| \leq R\} \to 1$$

when $R \to \infty$.

Remark. If one assumes that $\rho(\pi_1(\Sigma))$ is Zariski-dense in the above corollary, then the result is a direct consequence of Theorem C, if not, it is a consequence of our method since Hitchin representations are irreducible (Labourie [15]). In fact, the corollary holds for any strictly convex representation.

Theorem C has a non-trivial consequence for the orbital counting problem on the symmetric space of G. Let X be this symmetric space, o some point on X and d_X the induced distance for a G-invariant Riemannian metric on X. For a subgroup Λ of G set

$$h_\Lambda = \limsup_{s\to\infty} \frac{\log \#\{g \in \Lambda : d_X(o, go) \leq s\}}{s}.$$

We then show the following:

Corollary (Corollary 7.15). *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation. Then there exists $C > 0$ such that*

$$e^{-h_{\rho(\Gamma)}t} \#\{\gamma \in \Gamma : d_X(o, \rho(\gamma)o) \leq t\} \leq C$$

for every t large enough.

In [25] we give an asymptotic for the orbital counting problem for hyperconvex representations.

Counting problems in higher rank geometry have been studied for lattices and for Schottky groups. In the case of lattices Eskin–McMullen [10] find an asymptotic for the growth of $\#\{g \in \Lambda : d_X(o, go) \leq t\}$. This asymptotic is (up to a constant) the volume of the ball of radius t in X (for the Haar measure) and thus contains a polynomial term. Similar results to those of Eskin–McMullen [10] have been obtained independently by Duke–Rudnick–Sarnak [9]. Gorodnik–Oh [12] prove a distribution theorem (in the spirit of Theorem A) for lattices for an orbit on the symmetric space.

For Schottky groups the asymptotic equivalence of $\#\{g \in \Lambda : d_X(o, go) \leq t\}$ is shown by Quint [23] to be exponential with no polynomial term. For these groups there is also a distribution theorem due to Thirion [29].

In Margulis' thesis [18] the following principle appeared: one should prove a mixing property for an appropriate dynamical system to obtain a counting result. In Eskin–McMullen's work [10] it is the mixing property of the action of the Cartan group that is needed. This principle is also applied by Roblin [24] and by Thirion [29].

In this work we still exploit the relation with dynamical systems but in a slightly different manner. We find a symbolic flow and apply counting theorems for periodic orbits due to Parry–Pollicott [19] and the spatial distribution of these due to Bowen [7].

Method and techniques. Recall that we have identified the boundary of the group Γ with the geometric boundary of $\widetilde{M}$. Set $B\colon \partial\Gamma \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ to be the Busemann function of $\widetilde{M}$, i.e., if $x \in \partial\Gamma$ and $p, q \in \widetilde{M}$ then

$$B_x(p,q) = \lim_{z \to x} d(p,z) - d(q,z).$$

Using the Busemann function one constructs a homeomorphism between $T^1\widetilde{M}$, the unitary tangent bundle of $\widetilde{M}$, and $\partial^2\Gamma \times \mathbb{R}$, where

$$\partial^2\Gamma = \{(x,y) \in \partial\Gamma \times \partial\Gamma : x \neq y\}.$$

In order to do so one fixes some point $o \in \widetilde{M}$ and to $(p,v) \in T^1\widetilde{M}$ one associates

$$(p,v) \mapsto (v_{-\infty}, v_\infty, B_{v_\infty}(p,o))$$

where $v_{-\infty}$ and v_∞ are the origin and end points in $\partial\Gamma$ of the geodesic through p with speed v. This is called the *Hopf parametrization* of the unitary tangent bundle.

Some key facts are that the action of an isometry g of $\widetilde{M}$ is read via this parametrization as

$$g(x,y,t) = (gx, gy, t - B_y(o, g^{-1}o)),$$

and that the geodesic flow is now the translation flow on $\partial^2\Gamma \times \mathbb{R}$.

Hopf's parametrization also shows that invariant measures of the geodesic flow are in correspondence with Γ-invariant measures on $\partial^2\Gamma$.

Patterson–Sullivan's measure on $\partial\Gamma$ induces a Γ-invariant measure on $\partial^2\Gamma$ whose corresponding measure in $\Gamma\backslash T^1\widetilde{M}$ is the measure of maximal entropy of the geodesic flow. This fact is of particular importance in Roblin's work [24] where he obtains counting theorems in the negative curvature case.

The main idea of this work is then to construct a flow in a similar fashion to Hopf's parametrization, considering an appropriate cocycle, and give a description of its measure of maximal entropy. We now explain how this flow is built.

Consider some Hölder cocycle $c\colon \Gamma \times \partial\Gamma \to \mathbb{R}$, i.e., c verifies

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1 x) + c(\gamma_1, x)$$

for every pair $\gamma_0, \gamma_1 \in \Gamma$; and $c(\gamma, \cdot)$ is Hölder continuous for every $\gamma \in \Gamma$ (the same exponent is assumed for every γ).

The basic example of a Hölder cocycle is the Busemann cocycle

$$(\gamma, x) \mapsto B_x(o, \gamma^{-1}o).$$

However, when one has a strictly convex representation $\rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ with equivariant map $\xi\colon \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$, for each norm $\| \ \|$ on $\mathbb{R}^d$ one has the following cocycle of particular interest to us:

$$\beta_1(\gamma, x) = \log \frac{\|\rho(\gamma)v\|}{\|v\|}$$

where $v \in \xi(x) - \{0\}$.

We are interested in understanding Γ's action on $\partial^2\Gamma \times \mathbb{R}$ via any cocycle c,

$$\gamma(x, y, t) = (\gamma x, \gamma y, t - c(\gamma, y)),$$

and give conditions to obtain the translation flow in the quotient $\Gamma\backslash\partial^2\Gamma \times \mathbb{R}$.

The periods of a Hölder cocycle c are defined as $\ell_c(\gamma) = c(\gamma, \gamma_+)$, where γ_+ is the attractive fixed point of γ, and the *exponential growth rate* of c is defined as

$$h_c := \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \ell_c(\gamma) \le s\}}{s}$$

where $[\gamma]$ is the conjugacy class of γ.

For example, the period $|\gamma|$ of $\gamma \in \Gamma$ for Busemann's cocycle is the length of the closed geodesic associated to γ, and its exponential growth rate coincides with the topological entropy of the geodesic flow.

We then show the following:

Theorem (The reparametrizing Theorem 3.2). *Consider a Hölder cocycle $c\colon \Gamma \times \partial\Gamma \to \mathbb{R}$ such that $h_c \in (0, \infty)$. Then the action of Γ on $\partial^2\Gamma \times \mathbb{R}$ via c is proper and co-compact. Moreover, the translation flow*

$$\psi_t\colon \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft, \quad \psi_t(x, y, s) = (x, y, s - t),$$

is conjugated to a Hölder reparametrization of the geodesic flow on $\Gamma\backslash T^1\widetilde{M}$. It is topologically mixing and its topological entropy is h_c.

This result is cohomology invariant, that is, if one adds to c a cocycle of the form $(\gamma, x) \mapsto U(\gamma x) - U(x)$ for some function $U : \partial\Gamma \to \mathbb{R}$, the statement of the theorem does not change. Namely because with the function U one constructs Γ-equivariant homeomorphisms from one space to the other.

Analysis on the cocycle β_1 defined before and this theorem will imply Theorem B and the first item of Theorem C. In order to prove Theorem A (and the second item of Theorem C) further analysis of the flow ψ_t is needed. Mainly because it is a fixed cocycle we are interested in, and not only its cohomology class.

When a Hölder cocycle has finite and positive exponential growth rate, Ledrappier [16] has shown the existence of a Patterson–Sullivan measure associated to it, i.e., a probability μ on $\partial\Gamma$ such that

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-h_c c(\gamma^{-1},x)}.$$

A *dual cocycle* of c is a Hölder cocycle $\bar{c}\colon \Gamma \times \partial\Gamma \to \mathbb{R}$ such that the periods $\ell_{\bar{c}}(\gamma) = \ell_c(\gamma^{-1})$. A *Gromov product* for a pair of dual cocycles $\{c, \bar{c}\}$ is a function $[\cdot,\cdot]_{\{c,\bar{c}\}}\colon \partial^2\Gamma \to \mathbb{R}$ such that for every $\gamma \in \Gamma$ and $(x, y) \in \partial^2\Gamma$ one has

$$[\gamma x, \gamma y]_{\{c,\bar{c}\}} - [x, y]_{\{c,\bar{c}\}} = -(\bar{c}(\gamma, x) + c(\gamma, y)).$$

It is a consequence of the work by Ledrappier [16] (and we shall explain this below) that given a Hölder cocycle c there exists a dual cocycle $\bar{c}$ and a Gromov product for the pair $\{c, \bar{c}\}$.

We can now describe the measure of maximal entropy of the translation flow ψ_t.

Theorem (The reparametrizing Theorem 3.2). *Consider a Hölder cocycle c with $h_c \in (0, \infty)$, a dual cocycle $\bar{c}$ and a Gromov product $[\cdot,\cdot]$ for the pair $\{c, \bar{c}\}$. Denote by μ and $\bar{\mu}$ the Patterson–Sullivan's probabilities for the cocycles c and $\bar{c}$ respectively. Then the measure*

$$e^{-h_c[\cdot,\cdot]}\bar{\mu} \otimes \mu \otimes ds$$

is Γ-invariant on $\partial^2\Gamma \times \mathbb{R}$ (for the action $\Gamma \curvearrowright \partial^2\Gamma \times \mathbb{R}$ via c) and induces (up to a constant) ψ_t's probability of maximal entropy on the quotient $\Gamma\backslash\partial^2\Gamma \times \mathbb{R}$.

Theorems A and the second item of Theorem C are consequences of these two theorems by using appropriate cocycles, the key point is the following proposition which sets strictly convex (and hyperconvex) representations in the context of Hölder cocycles with finite and positive exponential growth.

Let $\rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation with equivariant map $\xi\colon \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$. Fix some norm $\|\ \|$ on $\mathbb{R}^d$ and consider the cocycle

$$\beta_1(\gamma, x) = \log \frac{\|\rho(\gamma)v\|}{\|v\|}$$

for some $v \in \xi(x) - \{0\}$.

Proposition (Proposition 5.4). *Let $\rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation, then the period $\beta_1(\gamma, \gamma_+)$ is $\lambda_1(\rho\gamma)$ i.e., the logarithm of the spectral radius*

of $\rho(\gamma)$, and the exponential growth rate of the cocycle β_1 is finite and positive, that is to say,

$$\limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho(\gamma)) \leq s\}}{s}$$

belongs to $(0, \infty)$.

Section 2 is devoted to the study of reparametrizations of Anosov flows. This allows us to apply counting theorems for hyperbolic flows in our setting. In Section 3 we study Hölder cocycles with finite positive exponential growth rate. We prove there the reparametrizing Theorem 3.2. In Section 4 we study consequences of Parry–Pollicott's prime orbit theorem and Bowen's spatial distribution result for a general cocycle. In Section 5 we show the above proposition. Section 6 is devoted to the proof of Theorems A and B. Finally, in the last section we study hyperconvex representations and prove Theorem C.

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2. Cross sections and arithmeticity of periods

The main objectives of this section are Lemma 2.4 and Corollary 2.10. The former explains how measures of maximal entropy of reparametrizations arise and the latter deals with reparametrizations of geodesic flows on closed negatively curved manifolds.

Let X be a compact metric space and $\phi_t : X \circlearrowleft$ a continuous flow on X without fixed points.

Definition 2.1. We will say that $\phi_t : X \circlearrowleft$ is *topologically weakly mixing* if the only solution to the equation

$$w\phi_t = e^{2\pi i a t} w,$$

for $w : X \to S^1$ continuous and $a \in \mathbb{R}$, is $a = 0$ and $w =$constant.

Remark 2.1. Consider some periodic orbit τ of period $p(\tau)$ of the flow $\phi_t : X \circlearrowleft$. If $\phi_t : X \circlearrowleft$ is not weak mixing let $w : X \to S^1$ and $a \in \mathbb{R} - \{0\}$ verify $w\phi_t = e^{2\pi i a t} w$. Since $\phi_{p(\tau)} x = x$ for any $x \in \tau$ one then finds $\exp\{2\pi i a p(\tau)\} = 1$ which implies that $p(\tau)$ belongs to the discrete group $a^{-1}\mathbb{Z}$. That is, the periods of a non weak mixing flow generate a discrete group of $\mathbb{R}$.

A closed subset K of X is a *cross section* for ϕ_t if the function $T_\phi\colon K \times \mathbb{R} \to X$ given by $T_\phi(x,t) = \phi_t(x)$ is a surjective local homeomorphism.

Remark 2.2. If $\phi_t\colon X \circlearrowleft$ admits a cross section then X fibers over the circle and the projection of a periodic orbit (seen as map from $S^1 \to S^1$) has non-zero index.

Remark 2.2 admits a converse due to Schwartzman [27]:

Lemma 2.1 (Schwartzman [27], p. 280). *There exists a continuous function $w\colon X \to S^1$ differentiable in the flow's direction such that its derivative in the flow's direction w' is nowhere zero if and only if the flow admits a cross section.*

We now turn our attention to reparametrizations of flows. Let $F\colon X \to \mathbb{R}$ be a positive continuous function. Set $\kappa\colon X \times \mathbb{R} \to \mathbb{R}$ as

$$\kappa(x,t) = \int_0^t F\phi_s(x)ds, \tag{1}$$

if t is positive, and $\kappa(x,t) := -\kappa(\phi_t x, -t)$ for t negative. Thus, κ verifies de cocycle property $\kappa(x,t+s) = \kappa(\phi_t x, s) + \kappa(x,t)$ for every $t,s \in \mathbb{R}$ and $x \in X$.

Since $F > 0$ and X is compact F has a positive minimum and $\kappa(x,\cdot)$ is an increasing homeomorphism of $\mathbb{R}$. We then have an inverse $\alpha\colon X \times \mathbb{R} \to \mathbb{R}$ that verifies

$$\alpha(x,\kappa(x,t)) = \kappa(x,\alpha(x,t)) = t \tag{2}$$

for every $(x,t) \in X \times \mathbb{R}$.

Definition 2.2. The *reparametrization* of ϕ_t by F is the flow $\psi_t\colon X \circlearrowleft$ defined as $\psi_t(x) := \phi_{\alpha(x,t)}(x)$. If F is Hölder continuous we shall say that ψ_t is a Hölder reparametrization of ϕ_t.

Remark 2.3. The cocycle property for κ and equation (2) imply that ψ_t is in fact a flow.

The advantage of cross sections is that the definition is invariant via reparametrizations.

Lemma 2.2. *Let ψ_t be a reparametrization of ϕ_t. Then ϕ_t admits a cross section if and only if ψ_t does.*

Proof. Let K be a cross section for ϕ_t, we need to show that the map $T_\psi\colon X \times \mathbb{R} \to X$, $(x,t) \mapsto \psi_t(x)$, is a surjective local homeomorphism. But this is evident in view of the relation

$$T_\psi = T_\phi \circ \varphi,$$

where φ is the homeomorphism $\varphi\colon K \times \mathbb{R} \circlearrowleft$, $(x,t) \mapsto (x,\alpha(x,t))$. □

One then finds the following corollary.

Corollary 2.3. *A flow $\phi_t : X \circlearrowleft$ does not admit a cross section if and only if every reparametrization of ϕ_t is topologically weakly mixing.*

Proof. Consider some reparametrization ψ_t of ϕ_t and assume ψ_t is not weak mixing, that is, there exists $w\colon X \to S^1$ such that $w\psi_t(x) = e^{2\pi i a t} w(x)$ for some $a \neq 0$. Such w is differentiable in the direction of the flow and

$$\frac{w'(x)}{2\pi i w(x)} = a \neq 0.$$

Applying Schwartzman's Lemma 2.1 one obtains a cross section for ψ_t and thus a cross section for ϕ_t.

If ϕ_t admits a cross section one applies Schwartzman's Lemma 2.1 and finds a continuous function $w\colon X \to S^1$ whose derivative in the flow's direction is never zero. Set

$$F(x) = \frac{w'(x)}{2\pi i w(x)}$$

and consider ψ_t, the reparametrization of ϕ_t by F. One easily verifies that $w\psi_t = e^{2\pi i t} w$ and thus ψ_t is not topologically weakly mixing. □

If m is a ϕ_t-invariant probability on X then the probability m' defined by

$$dm'/dm(\cdot) = F(\cdot)/m(F)$$

is ψ_t-invariant. In particular, if τ is a periodic orbit of ϕ_t then it is also periodic for ψ_t and the new period is

$$\int_\tau F.$$

This relation between invariant probabilities induces a bijection and Abramov [1] relates the corresponding metric entropies:

$$h(\psi_t, m') = h(\phi_t, m) \Big/ \int F dm. \tag{3}$$

Denote by $\mathcal{M}^{\phi_t}$ the set of ϕ_t-invariant probabilities. The *pressure* of a continuous function $F\colon X \to \mathbb{R}$ is defined as

$$P(\phi_t, F) = \sup_{m \in \mathcal{M}^{\phi_t}} h(\phi_t, m) + \int_X F dm.$$

A probability m such that the supremum is attained is called an *equilibrium state* of F.

Lemma 2.4. *Let $\psi_t : X \circlearrowleft$ be the reparametrization of ϕ_t by $F : X \to \mathbb{R}_+^*$. Assume that the equation*

$$P(\phi_t, -sF) = 0, \quad s \in \mathbb{R},$$

has a finite positive solution h. Then h is the topological entropy of ψ_t. In particular the solution is unique. Conversely, if $h_{\text{top}}(\psi_t)$ is finite then it is a solution to the above equation. If this is the case, the bijection $m \mapsto m'$ induces a bijection between equilibrium states of $-hF$ and probabilities of maximal entropy for ψ_t.

Proof. Abramov's formula (3) directly implies

$$h(\phi_t, m) - s \int F dm = (h(\psi_t, m') - s) \int F dm,$$

for any ϕ_t-invariant probability m. If $P(\phi_t, -hF) = 0$, the last equation together with the fact that F is strictly positive, imply

$$0 = \sup_{m \in \mathcal{M}^{\phi_t}} h(\psi_t, m') - h.$$

Applying the variational principle one has $h = h_{\text{top}}(\psi_t)$.

Conversely, if $h_{\text{top}}(\psi_t)$ is finite the result follows directly form Abramov's formula and $F > 0$.

If m_F is an equilibrium state of $-h_{\text{top}}(\psi_t)F$ then, since $P(\phi_t, -h_{\text{top}}(\psi_t)F) = 0$ one has that the metric entropy $h(\psi_t, m'_F) = h_{\text{top}}(\psi_t)$. The bijection $m \mapsto m'$ induces thus a bijection between equilibrium states of $-h_{\text{top}}(\psi_t)F$ and probabilities of maximal entropy for ψ_t. □

We now restrict our study to hyperbolic flows: Assume from now on that X is a compact manifold and that the flow $\phi_t : X \circlearrowleft$ is C^1. We say that ϕ_t is *Anosov* if the tangent bundle of X splits as a sum of three $d\phi_t$-invariant bundles

$$TX = E^s \oplus E^0 \oplus E^u,$$

and there exist positive constants C and c such that E^0 is the direction of the flow and for every $t \geq 0$ one has that for every $v \in E^s$,

$$\|d\phi_t v\| \leq Ce^{-ct} \|v\|,$$

and for every $v \in E^u$, $\|d\phi_{-t} v\| \leq Ce^{-ct} \|v\|$.

In this setting there is an extra equivalence for the existence of cross sections:

Proposition 2.5. *Let $\phi_t : X \circlearrowleft$ be an Anosov flow. Then ϕ_t admits a cross section if and only if there exists $F : X \to \mathbb{R}_+^*$ Hölder such that the subgroup of $\mathbb{R}$ spanned by*

$$\left\{ \int_\tau F : \tau \text{ periodic} \right\}$$

is discrete.

Proof. Assume such F exists, and assume (without loss of generality) that $\langle\{\int_\tau F : \tau \text{ periodic}\}\rangle = \mathbb{Z}$. Recall that we have defined

$$\kappa(x,t) = \int_0^t F(\phi_s x)ds.$$

The cocycle $\Theta\colon \mathbb{R} \times X \to S^1$ given by $\Theta(x,t) = e^{2\pi i \kappa(x,t)}$ is, by Livšic's theorem [17], cohomologically trivial and thus there exists $w\colon X \to S^1$ Hölder continuous such that

$$\frac{w\phi_t(x)}{w(x)} = \exp\{2\pi i \int_0^t F(\phi_s x)ds\},$$

and one finds a cross section applying Schwartzman's Lemma 2.1.

Assume now that ϕ_t admits a cross section. Applying Schwartzman's Lemma 2.1 one finds a continuous function $w\colon X \to S^1$ such that its derivative in the flow's direction is never zero. One can assume that such w is in fact differentiable (by considering another function close to w) and thus the function $F(x) = w'(x)/2\pi i w(x)$ is differentiable with integer periods. □

The following proposition together with Lemma 2.4 implies that a Hölder reparametrization of an Anosov flow has a unique probability of maximal entropy.

Proposition 2.6 (Bowen–Ruelle [6]). *Let $\phi_t : X \circlearrowleft$ be an Anosov flow. Then given a Hölder potential $G : X \to \mathbb{R}$ there exists a unique equilibrium state for G. Equilibrium states are thus ergodic.*

Corollary 2.7. *Let $\phi_t : X \circlearrowleft$ be an Anosov flow and ψ_t be a Hölder reparametrization of ϕ_t. Then ψ_t has a unique probability of maximal entropy and it is ergodic with respect to this measure.*

We are interested in finding Markov partitions for reparametrizations of Anosov flows.

Definition 2.3. Let $\varphi_t : X \circlearrowleft$ be a flow. We shall say that the triplet (Σ, π, r) is a *Markov coding* for φ_t if Σ is a sub-shift of finite type, $\pi : \Sigma \to X$ and $r : \Sigma \to \mathbb{R}_+^*$ are Hölder continuous and the function $\pi_r : \Sigma \times \mathbb{R} \to X$ defined by

$$\pi_r(x,t) = \varphi_t \pi(x)$$

verifies the following conditions:

i) π_r is surjective and Hölder.

ii) Let $\sigma\colon \Sigma \circlearrowleft$ be the shift and let $\hat{r}\colon \Sigma \times \mathbb{R} \circlearrowleft$ be defined by

$$\hat{r}(x,t) = (\sigma x, t - r(x)).$$

Then π_r is $\hat{r}$-invariant.

iii) $\pi_r\colon \Sigma \times \mathbb{R}/\hat{r} \to X$ is bounded-to-one and injective on a residual set which is of full measure for every ergodic invariant measure of total support (for σ_t^f).

iv) Consider the translation flow $\sigma_t^r\colon \Sigma \times \mathbb{R}/\hat{r} \circlearrowleft$. Then $\pi_r \sigma_t^r = \varphi_t \pi_r$.

Remark 2.4. If a flow $\varphi_t\colon X \circlearrowleft$ admits a Markov coding, then it has a unique probability of maximal entropy and the function $\pi_r\colon \Sigma \times \mathbb{R}/\hat{r} \to X$ is an isomorphism between the probabilities of maximal entropy of σ_t^r and that of φ_t. In particular the topological entropy of φ_t coincides with that of σ_t^r.

Theorem 2.8 (Bowen [7], [8]). *A transitive Anosov flow admits a Markov coding.*

Lemma 2.9. *Let (Σ, π, r) be a Markov coding for a transitive Anosov flow $\phi_t\colon X \circlearrowleft$. Set $\psi_t\colon X \circlearrowleft$ to be a Hölder reparametrization of ϕ_t by $F\colon X \to \mathbb{R}_+^*$ and define $f\colon \Sigma \to \mathbb{R}_+^*$ as*

$$f(z) = \int_0^{r(z)} F\phi_s(\pi(z))ds.$$

Then (Σ, π, f) is a Markov coding for ψ_t. If, moreover, ϕ_t does not admit a cross section, then the translation flow $\sigma_t^f\colon \Sigma \times \mathbb{R}/\hat{f} \circlearrowleft$ is topologically weakly mixing.

We remark that every Markov coding for ψ_t can be obtained in this manner.

Proof. We need to check that the function $\pi_f\colon \Sigma \times \mathbb{R} \to X$ defined by $\pi_f(z,s) := \psi_s(\pi(z))$ is $\hat{f}$-invariant and conjugates the translation flow on $\Sigma \times \mathbb{R}/\hat{f}$ with the flow ψ_t. To prove invariance by $\hat{f}$ we will prove that for every $(z,s) \in \Sigma \times \mathbb{R}$ one has

$$\pi_f(z, s + f(z)) = \pi_f(\sigma z, s).$$

The computation is intricate but direct. Recall that by definition we have $f(z) = \kappa(\pi(z), r(z))$ (see equation (1)). This immediately implies $\alpha(\pi(z), f(z)) = r(z)$. We then have

$$\begin{aligned}\pi_f(z, s + f(z)) &= \psi_{s+f(z)}(\pi z) = \psi_s \circ \psi_{f(z)}(\pi z)\\ &= \psi_s \circ \phi_{\alpha(\pi(z), f(z))}(\pi z) = \psi_s \circ \phi_{r(z)}(\pi z)\\ &= \psi_s(\pi(\sigma z))\end{aligned}$$

since (Σ, π, r) is a Markov coding for ϕ_t. This proves invariance.

The remaining properties of Markov coding then follow.

Suppose now that $\phi_t|X$ does not admit a cross section. We must then show that σ_t^f is weak mixing. Applying Proposition 2.5 one has that the periods $\int_\tau F$ generate a dense subgroup of $\mathbb{R}$. Since $\hat{\pi}: \Sigma \times \mathbb{R} \to X$ is surjective, the periods of σ_t^f-periodic orbits also generate a dense subgroup of $\mathbb{R}$ and Remark 2.1 implies that σ_t^f is weak mixing. □

We now find the following corollary:

Corollary 2.10. *Let Γ be a co-compact group of isometries of a complete simply connected manifold of negative curvature $\widetilde{M}$. Let $\phi_t: \Gamma\backslash T^1\widetilde{M} \circlearrowleft$ be the geodesic flow and $\psi_t: \Gamma\backslash T^1\widetilde{M} \circlearrowleft$ be a Hölder reparametrization of ϕ_t. Consider a Markov coding (Σ, π, f) for ψ_t. Then the flow σ_t^f is weak mixing.*

Proof. Since the geodesic flow is a transitive Anosov flow, Lemma 2.9 applies. It remains to prove that the geodesic flow on a compact manifold of negative curvature does not admit a cross section. As observed before (Remark 2.2) we only need to find a homologically trivial periodic orbit (since such orbit will always have zero index as map $S^1 \to S^1$).

In negative curvature we can find two elements in Γ, a and b, that do not commute, the closed geodesic associated to the commutator $aba^{-1}b^{-1}$ is then the required periodic orbit. □

3. Cocycles with finite exponential growth rate

Let Γ be a torsion-free discrete co-compact isometry group of a complete simply connected manifold with negative curvature $\widetilde{M}$. We identify the boundary of the group Γ with the geometric boundary of $\widetilde{M}$.

Definition 3.1. A *Hölder cocycle* is a function $c: \Gamma \times \partial\Gamma \to \mathbb{R}$ such that

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1 x) + c(\gamma_1, x)$$

for any $\gamma_0, \gamma_1 \in \Gamma$ and $x \in \partial\Gamma$, and where $c(\gamma, \cdot)$ is a Hölder map for every $\gamma \in \Gamma$ (the same exponent is assumed for every $\gamma \in \Gamma$).

Given a Hölder cocycle c we define the *periods* of c as the numbers

$$\ell_c(\gamma) := c(\gamma, \gamma_+)$$

where γ_+ is the attractive fixed point of γ in $\Gamma - \{e\}$. The cocycle property implies that the period of an element γ only depends on its conjugacy class $[\gamma] \in [\Gamma]$.

Two cocycles c and c' are said to be cohomologous if there exists a Hölder function $U\colon \partial\Gamma \to \mathbb{R}$ such that for all $\gamma \in \Gamma$ one has

$$c(\gamma, x) - c'(\gamma, x) = U(\gamma x) - U(x).$$

One easily deduces from the definition that the set of periods of a cocycle is a cohomological invariant.

We shall be interested in cocycles whose periods are positive, that is, $\ell_c(\gamma) > 0$ for every $\gamma \in \Gamma$. The *exponential growth rate* for such cocycle c is defined as

$$h_c := \limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] : \ell_c(\gamma) \le t\} \in \mathbb{R}_+ \cup \{\infty\}.$$

It is a consequence of Ledrappier's work (cf. Corollary 3.6) that a Hölder cocycle c with positive periods verifies $h_c > 0$. If, moreover, c has finite exponential growth rate then, following Patterson's construction, Ledrappier [16] shows the existence of a *Patterson–Sullivan* probability μ over $\partial\Gamma$ for the cocycle $h_c c$, that is to say, μ verifies

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-h_c c(\gamma^{-1}, x)}.$$

Theorem 3.1 (Ledrappier [16], p. 102). *Let c be a Hölder cocycle with positive periods. Then c has finite positive exponential growth rate h_c if and only if there exists a Patterson–Sullivan probability for the cocycle $h_c c$. If this is the case, the Patterson–Sullivan probability is unique.*

Let $\bar{c}$ be a cocycle such that $\ell_{\bar{c}}(\gamma) = \ell_c(\gamma^{-1})$ (this always exists as shown in the next section); such $\bar{c}$ is called *a dual cocycle of* c.

Set $\partial^2\Gamma$ to be the set of pairs $(x, y) \in \partial\Gamma \times \partial\Gamma$ such that $x \neq y$. We shall say that a function $[\cdot,\cdot]\colon \partial^2\Gamma \to \mathbb{R}$ is a *Gromov product* for a pair of dual cocycles $\{c, \bar{c}\}$ if for every $\gamma \in \Gamma$ and $(x, y) \in \partial^2\Gamma$ one has

$$[\gamma x, \gamma y] - [x, y] = -(\bar{c}(\gamma, x) + c(\gamma, y)).$$

Denote by μ and $\bar{\mu}$ the Patterson–Sullivan probabilities associated to c and $\bar{c}$ respectively. The main theorem of this section is the following:

Theorem 3.2 (The reparametrizing theorem). *Let c be a Hölder cocycle with positive periods such that h_c is finite and positive. Then the following holds.*

(1) *The action of Γ in $\partial^2\Gamma \times \mathbb{R}$,*

$$\gamma(x, y, s) = (\gamma x, \gamma y, s - c(\gamma, y)),$$

is proper and co-compact. Moreover, the translation flow $\psi_t\colon \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$,

$$\psi_t(x, y, s) = (x, y, s - t),$$

is conjugated to a Hölder reparametrization of the geodesic flow on $\Gamma\backslash T^1\widetilde{M}$. *The conjugating map is also Hölder continuous. The topological entropy of* ψ_t *is* h_c.

(2) *The measure*

$$e^{-h_c[\cdot,\cdot]}\bar{\mu}\otimes\mu\otimes ds$$

on $\partial^2\Gamma\times\mathbb{R}$ *induces on the quotient* $\Gamma\backslash\partial^2\Gamma\times\mathbb{R}$ *the measure of maximal entropy of* ψ_t.

Remark 3.1. The first item of the theorem is cohomology invariant. That is, a change in the choice of the cocycle (in c's cohomology class) does not change the statement of Theorem 3.2. For the second item, it is the class of zero sets of the measure $e^{-h_c[\cdot,\cdot]}\bar{\mu}\otimes\mu\otimes ds$ that is cohomology invariant as the following result of Ledrappier [16] shows.

Theorem 3.3 (Ledrappier [16], p. 101). *Let* c *and* c' *be Hölder cocycles with positive periods and finite exponential growth rate. Let* μ *and* μ' *be two quasi-invariant measures of Hölder cocycle* $h_c c$ *and* $h_{c'}c'$ *respectively. Then* μ *and* μ' *have the same zero sets if and only if* $h_c c$ *and* $h_{c'}c'$ *are cohomologous.*

To prove Theorem 3.2 we shall find an appropriate cocycle: following Ledrappier [16] we associate to the cocycle c a Γ-invariant Hölder function $F: T^1M\to\mathbb{R}$. The fact that the cocycle is of finite exponential growth rate together with a Livšic-type lemma will allow us to choose such F to be positive.

One then finishes copying the Hopf parametrization of $T^1\widetilde{M}$. Namely we construct a homeomorphism $T^1\widetilde{M}\to\partial^2\Gamma\times\mathbb{R}$ such that the action of Γ on $T^1\widetilde{M}$ is sent to the action we need (this implies properness of the action) and the action of the geodesic flow will be reparametrized on the right side.

Concerning the proof of the second item: Since the measure of maximal entropy of a reparametrization has the same zero sets as an equilibrium state (Lemma 2.4), we will conclude giving a description of the induced measure by this equilibrium state on $\partial^2\Gamma$.

Proof of the first item of Theorem 3.2. Identify the unit tangent bundle of $\widetilde{M}$ with $\widetilde{M}\times\partial\Gamma$ and denote by ϕ_t the geodesic flow on M. For a given Γ-invariant Hölder function $H: T^1\widetilde{M}\to\mathbb{R}$, Schapira [26] introduced the following geometric cocycle: for $z\in\partial\Gamma$ define $B_z^H:\widetilde{M}\times\widetilde{M}\to\mathbb{R}$ by

$$B_z^H(p,q)=\lim_{s\to\infty}\int_0^{s+B_z(p,q)}H(\phi_t(p,z))dt-\int_0^s H(\phi_t(q,z))dt, \tag{4}$$

where $B_z:\widetilde{M}\times\widetilde{M}\to\mathbb{R}$ is the Busemann function (when $H\equiv 1$, $B_z^1(p,q)$ is exactly $B_z(p,q)$). The expression is convergent since H is Hölder continuous and the geodesic flow is Anosov.

One finds the following properties:

Lemma 3.4. *Let $o, p, q \in \widetilde{M}$ and $z \in \partial\Gamma$. Then*

i) $B_z^H(p,q) = B_{\gamma z}^H(\gamma p, \gamma q)$ *for every* $\gamma \in \Gamma$,

ii) $B_z^H(p,q) = B_z^H(p,o) + B_z^H(o,q)$,

iii) *if q belongs to the geodesic line from p to z one has*

$$B_z^H(p,q) = \int_p^q H,$$

where $\int_p^q H$ is the integral of H over the unique oriented geodesic segment from p to q.

Proof. Property i) follows directly from the Γ-invariance of H and the Busemann function. Property iii) is a direct consequence of the definition. We now prove property ii). By definition

$$B_z^H(p,o) = \lim_{s\to\infty} \int_0^{s+B_z(p,o)} H(\phi_t(p,z))dt - \int_0^s H(\phi_t(o,z))dt.$$

If we consider the change of parameter $s \mapsto s + B_z(o,q)$ the last limit becomes

$$B_z^H(p,o) = \lim_{s\to\infty} \int_0^{s+B_z(p,o)+B_z(o,q)} H(\phi_t(p,z))dt - \int_0^{s+B_z(o,q)} H(\phi_t(o,z))dt$$

and thus, since $B_z(p,o) + B_z(o,q) = B_z(p,q)$ we have

$$\begin{aligned} &B_z^H(p,o) + B_z^H(o,q) \\ &\quad = \lim_{s\to\infty} \int_0^{s+B_z(p,q)} H(\phi_t(p,z))dt - \int_0^{s+B_z(o,q)} H(\phi_t(o,z))dt \\ &\qquad + \lim_{s\to\infty} \int_0^{s+B_z(o,q)} H(\phi_t(o,z))dt - \int_0^s H(\phi_t(q,z))dt \\ &\quad = B_z^H(p,q). \end{aligned}$$

□

Given a Γ-invariant Hölder function $H : T^1\widetilde{M} \to \mathbb{R}$ one can associate to H a Hölder cocycle over the group Γ:

$$c_H(\gamma, z) = B_z^H(\gamma^{-1}o, o), \tag{5}$$

where o is some point on $\widetilde{M}$ fixed from now on.

Two Γ-invariant Hölder functions $H, H' \colon T^1\widetilde{M} \to \mathbb{R}$ are said to be cohomologous (according to Livšic) if there exists a Hölder Γ-invariant function $V \colon T^1\widetilde{M} \to \mathbb{R}$, differentiable in the direction of the geodesic flow, such that

$$H(p,z) - H'(p,z) = \frac{\partial V \circ \phi_t}{\partial t}(p,z).$$

The conjugacy class $[\gamma]$, of an element $\gamma \in \Gamma$, is naturally identified with the closed geodesic on $\Gamma\backslash T^1\widetilde{M}$ associated to γ. The *periods* of the function H are defined to be the numbers

$$\int_{[\gamma]} H.$$

One easily sees that the periods of H are exactly the periods of c_H; the periods of H are a Livšic-cohomology invariant. We can now state a theorem of Ledrappier.

Theorem 3.5 (Ledrappier [16], p. 105). *The map $H \mapsto c_H$ induces a bijection between cohomology classes of Γ-invariant Hölder functions and cohomology classes of Hölder cocycles. The corresponding classes have the same periods.*

Recall that $|\gamma|$ denotes the length of the closed geodesic on $\Gamma\backslash\widetilde{M}$ associated to γ.

Corollary 3.6. *Let c be a Hölder cocycle with positive periods. Then the exponential growth rate h_c is positive.*

Proof. Let $F \colon T^1\widetilde{M} \to \mathbb{R}$ be such that the Hölder cocycles c_F and c are cohomologous. Since c_F has positive periods F must have a positive maximum K and thus $\ell_c(\gamma) \leq K|\gamma|$, which implies

$$\#\{[\gamma] \in [\Gamma] : \ell_c(\gamma) \leq t\} \geq \#\{[\gamma] \in [\Gamma] : |\gamma| \leq t/K\}.$$

The exponential growth rate of the quantity on the right is known to be strictly positive and the corollary is proved. □

We will need the following lemma.

Lemma 3.7 (Ledrappier [16], p. 106). *Let c be a Hölder cocycle with positive periods. Then the exponential growth rate of c is finite if and only if*

$$\inf_{[\gamma]} \frac{\ell_c(\gamma)}{|\gamma|} > 0.$$

We shall now state the positive Livšic-type lemma.

Lemma 3.8. *Let X be a compact metric space equipped with a flow $\phi_t : X \circlearrowleft$. Consider a Hölder continuous function $f : X \to \mathbb{R}$, differentiable in the flow's direction, such that*

$$\int_X f dm > 0$$

for every ϕ_t-invariant probability m. Then f is cohomologous to a strictly positive Hölder continuous function.

We thank François Labourie for the following argument:

Proof. Note that for every $t \in \mathbb{R}$ the function f is cohomologous to its Birkhoff integral

$$x \mapsto \frac{1}{t} \int_0^t f(\phi_s x) ds.$$

It then suffices to show that there exists t such that for every x one has

$$\frac{1}{t} \int_0^t f(\phi_s x) ds > 0.$$

If this is not the case for every $\varepsilon > 0$ there exists $t_n \to \infty$ and $x_n \in X$ such that

$$\frac{1}{t_n} \int_0^{t_n} f(\phi_s x_n) ds < \varepsilon.$$

Since the set of invariant probabilities is compact we can find $k > 0$ such that $\int_X f dm > k$ for all $m \in \mathcal{M}^{\phi_t}$. Consider an accumulation point m_0 of the sequence of probabilities m_n defined as

$$m_n(g) = \frac{1}{t_n} \int_0^{t_n} g(\phi_s x_n) ds.$$

Then m_0 is a ϕ_t-invariant probability for which one has

$$\int_X f dm_0 \leq \varepsilon < k.$$

This finishes the proof. □

Our last tool is Anosov's closing lemma.

Theorem 3.9 (Anosov's closing lemma, cf. [28]). *Let $\phi_t : X \circlearrowleft$ be a transitive Anosov flow. Then convex combinations of periodic orbits are dense in the set $\mathcal{M}^{\phi_t}$ of invariant probabilities of ϕ_t.*

Proof of first item of Theorem 3.2. We begin with a Hölder cocycle c with positive periods and finite exponential growth rate. By Ledrappier's Theorem 3.5 we find a Γ-invariant Hölder function $H: T^1\widetilde{M} \to \mathbb{R}$ whose periods coincide with those of c.

Ledrappier's Lemma 3.7 then implies that

$$\inf_{[\gamma]} \frac{\int_{[\gamma]} H}{|\gamma|} > 0.$$

From Anosov's closing lemma we get $\int H dm > 0$ for every ϕ_t-invariant probability m. Applying Lemma 3.8 we find that H is cohomologous to a strictly positive Hölder function F, and its cocycle c_F (defined by the formula (5)) is cohomologous to c.

We shall prove the statement for the cocycle c_F. The idea is to construct a parametrization of $T^1\widetilde{M}$ using B_z^F, the geometric cocycle of F (equation (4)), as follows:

Fix some point $o \in \widetilde{M}$ and for a geodesic through (p, v) denote by $v_{-\infty}$ and v_∞ its origin and end points in $\partial\Gamma$, then define

$$E: (p, v) \mapsto (v_{-\infty}, v_\infty, B^F_{v_\infty}(p, o)).$$

Consider some geodesic $a(t)$ in $T^1\widetilde{M}$ with endpoints $a(-\infty) = v_{-\infty}$ and $a(\infty) = v_\infty$. Applying Lemma 3.4 we have, for every $t \in \mathbb{R}$, that

$$E(a(t)) = (v_{-\infty}, v_\infty, B^F_{v_\infty}(a(0), o) - \int_0^t F(a(s))ds).$$

Since $F > 0$ we deduce that E is injective when restricted to the geodesic $\{a(t) : t \in \mathbb{R}\}$, and since F has a positive minimum it is surjective over the set $\{(v_{-\infty}, v_\infty)\} \times \mathbb{R}$. This implies that E is a homeomorphism from $T^1\widetilde{M}$ to $\partial^2\Gamma \times \mathbb{R}$.

E is Γ-equivariant: Write $E(p, v) = (x, y, B^F_y(p, o))$ and consider some $\gamma \in \Gamma$, then by definition

$$E(\gamma(p, v)) = (\gamma x, \gamma y, B^F_{\gamma y}(\gamma p, o)).$$

Applying Lemma 3.4 one has

$$B^F_{\gamma y}(\gamma p, o) = B^F_{\gamma y}(\gamma p, \gamma o) + B^F_{\gamma y}(\gamma o, o) = B^F_y(p, o) - c_F(\gamma, y).$$

One concludes that E is a Γ-equivariant homeomorphism between $\Gamma \curvearrowright T^1\widetilde{M}$ and the action $\Gamma \curvearrowright \partial^2\Gamma \times \mathbb{R}$ via c_F. Since $\Gamma \curvearrowright T^1\widetilde{M}$ is proper (and co-compact), so is the action on $\partial^2\Gamma \times \mathbb{R}$ via c_F.

The geodesic flow is reparametrized: If $(p, v) \mapsto (v_{-\infty}, v_\infty, B^F_{v_\infty}(p, o))$ and $q \in \widetilde{M}$ is the base point of $\phi_t(p, v)$, then by definition

$$E(\phi_t(p, v)) = (v_{-\infty}, v_\infty, B^F_{v_\infty}(q, o)).$$

Applying again Lemma 3.4 yields

$$B^F_{v_\infty}(q,o) = B^F_{v_\infty}(p,o) - \int_0^t F\phi_t(p,v)dt.$$

This means exactly

$$E(\phi_t(p,v)) = \psi_{\int_0^t F\phi_s(p,v)ds} E(p,v),$$

in other words, the flow $E^{-1}\psi_t E$ is the reparametrization of the geodesic flow by F (see Definition 2.2). □

Proof of the second item of Theorem 3.2. In the last subsection we showed that the flow $\psi_t : \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$ is Hölder conjugated to a Hölder reparametrization of the geodesic flow.

We will thus prove the second statement of Theorem 3.2 in the following situation: $F : T^1\widetilde{M} \to \mathbb{R}^*_+$ is a Γ-invariant positive Hölder function and $\psi_t : T^1\widetilde{M} \circlearrowleft$ is the reparametrization of the geodesic flow by F. From now on we fix the cocycle c_F associated to F.

Remark 3.2. To retrieve Theorem 3.2 for a general cocycle c it suffices to remark that the class of zero sets of the measure of maximal entropy is invariant under cohomology and to observe that the measure

$$e^{-h_c[\cdot,\cdot]}\bar{\mu} \otimes \mu \otimes ds$$

is Γ-invariant for the action of Γ on $\partial^2\Gamma \times \mathbb{R}$ via c.

We need the following lemma of Ledrappier.

Lemma 3.10 (Ledrappier [16], p. 106). *If there exists h such that $P(-hF) = 0$ then h is the exponential growth rate of c_F. Conversely, if the exponential growth rate h of c_F is finite and positive then $P(-hF) = 0$.*

Corollary 3.11. *The topological entropy of the flow ψ_t is the exponential growth rate of the cocycle c_F.*

Proof. Let h be c_F's exponential growth rate. Then Ledrappier's Lemma 3.10 implies $P(-hF) = 0$. Lemma 2.4 states that this condition determines ψ_t's topological entropy $h_{\text{top}}(\psi_t)$, and thus $h = h_{\text{top}}(\psi_t)$. □

Recall that $h = h_{c_F}$. We now give a precise description of the measure induced on $\partial^2\Gamma$ by the equilibrium state of $-hF$. Denote by $a : T^1M \to T^1M$ the antipodal map. The periods of the function $\bar{F} : (p,w) \mapsto F(a(p,w))$ are the numbers $\ell_{c_F}(\gamma^{-1})$ and thus $\overline{c_F} := c_{\bar{F}}$ is a dual cocycle of c_F.

Lemma 3.12. *The cocycles c_F and $\overline{c_F}$ have the same exponential growth rate.*

Proof. The function $\gamma \mapsto \gamma^{-1}$ induces a bijection between the sets $\{\gamma \in \Gamma : \ell_{c_F}(\gamma) \leq t\}$ and $\{\gamma \in \Gamma : \ell_{c_F}(\gamma^{-1}) \leq t\}$. □

Define $[\cdot,\cdot]_F : \partial^2\Gamma \to \mathbb{R}$ as

$$[x, y]_F = B_x^{\bar{F}}(o, u) + B_y^F(o, u)$$

for any point u in the geodesic determined by x and y, where

$$B^F, B^{\bar{F}} : \partial\widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$$

are the geometric cocycles defined by (4) for the functions F and $\bar{F}$ respectively. Notice that $[\cdot,\cdot]_F$ is a Gromov product for the pair $\{c_F, \overline{c_F}\}$.

Denote by μ_F and $\bar{\mu}_F$ the quasi-invariant measures whose cocycles are c_F and $\overline{c_F}$ respectively.The second item of the theorem is then deduced from the following result of Schapira (Proposition 2.4 of [26]).

Proposition 3.13. *Identify $T^1\widetilde{M}$ with $\partial^2\widetilde{M} \times \mathbb{R}$ via the Hopf parametrization. Then the measure*

$$m_F := e^{-h[x,y]_F} d\bar{\mu}_F(x) d\mu_F(y) ds$$

induces in the quotient $\Gamma\backslash T^1\widetilde{M}$ the Gibbs state of $-hF$.

In order to finish the proof of the second item of Theorem 3.2 we remark that, as observed in Section 2 (Proposition 2.7), ψ_t has a unique probability of maximal entropy ν. According to Lemma 2.4, ν has the same zero sets as the equilibrium state of $-hF$, and thus, according to the above proposition, the lift of ν to $\partial^2\Gamma \times \mathbb{R}$ has the same zero sets as $\bar{\mu} \otimes \mu \otimes ds$. This finishes the proof.

4. Counting periods and distribution of fixed points

In this section we extract as much counting information as we can for a general cocycle. We study the consequences of Parry–Pollicott's prime orbit theorem and Bowen's spatial distribution result via the reparametrizing Theorem 3.2.

From the first item of Theorem 3.2 we deduce the following counting result. Recall that $\gamma \in \Gamma$ is *primitive* if it can not be written as a (positive) power of another element of Γ.

Corollary 4.1. *Let $c : \Gamma \times \partial\Gamma \to \mathbb{R}$ be a Hölder cocycle with non-negative periods and such that $h_c \in (0, \infty)$. Then*

$$h_c t e^{-h_c t} \#\{[g] \in [\Gamma] \text{ primitive} : \ell_c(\gamma) \leq t\} \to 1$$

as $t \to \infty$.

In other words, in order to obtain a precise counting result for the periods of a Hölder cocycle c, it is sufficient to prove that it has finite exponential growth rate.

Proof. Following Theorem 3.2 the translation flow $\psi_t : \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$ (Γ acting on $\partial^2\Gamma \times \mathbb{R}$ via c) is well defined and is a reparametrization of the geodesic flow. If τ is a periodic orbit of ψ_t, then any lift to $\partial^2\Gamma \times \mathbb{R}$ is of the form (γ_-, γ_+, s) for some primitive $\gamma \in \Gamma$ and $s \in \mathbb{R}$. One checks that

$$\gamma(\gamma_-, \gamma_+, s) = (\gamma_-, \gamma_+, s - \ell_c(\gamma))$$

which implies that the period $p(\tau)$ of τ is $\ell_c(\gamma)$ since γ was chosen primitive. One then has

$$\#\{\gamma \in [\Gamma] \text{ primitive} : \ell_c(\gamma) \leq t\} = \#\{\tau \text{ periodic} : p(\tau) \leq t\}.$$

We are led to count the number of periodic orbits of period $\leq t$ for the flow ψ_t. Since ψ_t is a reparametrization of the geodesic flow, Corollary 2.10 implies that we have a weak mixing Markov coding (Σ, π, f) associated to ψ_t. Recall that ψ_t's topological entropy coincides with the topological entropy of σ_t^f. One finishes by applying the following theorem of Parry–Pollicott [19] (see also [20]). This completes the proof. □

Theorem 4.2 (Prime orbit theorem [19]). *Let Σ be a sub-shift of finite type and let $f : \Sigma \to \mathbb{R}_+^*$ be Hölder continuous. Suppose that the suspension flow $\sigma_t^f : \Sigma \times \mathbb{R}/\hat{f} \circlearrowleft$ is weak mixing, and set $p(\tau)$ the period of a σ_t^f-periodic orbit. Then*

$$hte^{-ht}\#\{\tau \text{ periodic}: p(\tau) \leq t\} \to 1$$

when $t \to \infty$, where h is the topological entropy of the suspension flow σ_t^f.

We prove now a distribution property of fixed points on $\partial\Gamma$ for a Hölder cocycle c with non-negative periods and $h_c \in (0, \infty)$.

Consider a dual cocycle $\bar{c}$, a Gromov product $[\cdot,\cdot] : \partial^2\Gamma \to \mathbb{R}$ and denote by μ and $\bar{\mu}$ the Patterson–Sullivan probabilities of c and $\bar{c}$ respectively. Finally write $C_c(\partial^2\Gamma)$ for the space of real continuous functions on $\partial^2\Gamma$ with compact support.

The following proposition is inspired in Roblin [24].

Proposition 4.3. *Let $\|m_c\|$ denote the total mass of the measure $m_c = e^{-h_c[\cdot,\cdot]}\bar{\mu} \otimes \mu \otimes ds$ on the compact quotient $\Gamma\backslash\partial^2\Gamma \times \mathbb{R}$. Then we have the convergence*

$$\nu_t := \|m_c\| h_c e^{-h_c t} \sum_{\gamma\in\Gamma:\ell_c(\gamma)\leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to e^{-h_c[\cdot,\cdot]}\bar{\mu} \otimes \mu$$

in $C_c^*(\partial^2\Gamma)$ when $t \to \infty$.

We shall use the following distribution result due to Bowen [7], [8].

Theorem 4.4 (Bowen [7], [8]). *Let Σ be a sub-shift of finite type and $f : \Sigma \to \mathbb{R}_+^*$ be Hölder continuous. Then*

$$\#\{\tau\ \sigma_t^f\text{-periodic} : p(\tau) \le t\}^{-1} \sum_{\tau : p(\tau) \le t} \frac{1}{p(\tau)} \mathrm{Leb}_\tau$$

converges to the probability of maximal entropy of σ_t^f, where Leb_τ is the Lebesgue measure on τ of length $p(\tau)$.

Proof of Proposition 4.3. As observed before we have a weak mixing Markov coding for the flow $\psi_t : \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$ (Corollary 2.10). Applying Parry–Pollicott's prime orbit theorem (Theorem 4.2) together with Bowen's result we find the convergence of

$$h_c t e^{-h_c t} \sum_{\tau : p(\tau) \le t} \frac{1}{p(\tau)} \mathrm{Leb}_\tau$$

to the probability of maximal entropy of ψ_t on $\Gamma\backslash\partial^2\Gamma \times \mathbb{R}$, when $t \to \infty$. The reparametrizing Theorem 3.2 states that this measure is lifted to $\partial^2\Gamma \times \mathbb{R}$ as

$$\frac{e^{-h_c[\cdot,\cdot]}\bar{\mu} \otimes \mu \otimes ds}{\|m_c\|}.$$

Since periodic orbits of ψ_t are of the form (γ_-, γ_+, s) for some $\gamma \in \Gamma$ primitive, and since the period of such orbit is $\ell_c(\gamma)$ we have the convergence

$$h_c t e^{-h_c t} \sum_{\gamma \text{ primitive: } \ell_c(\gamma) \le t} \frac{1}{\ell_c(\gamma)} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \otimes ds \to \frac{e^{-h_c[\cdot,\cdot]}\bar{\mu} \otimes \mu \otimes ds}{\|m_c\|}$$

in $\partial^2\Gamma \times \mathbb{R}$.

We can delete the $\mathbb{R}$-component by comparing the measures of sets of the form $A \times B \times I$ for some interval I, and we find

$$\sigma_t := \|m_c\| h_c t e^{-h_c t} \sum_{\gamma \text{ primitive: } \ell_c(\gamma) \le t} \frac{1}{\ell_c(\gamma)} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to e^{-h_c[\cdot,\cdot]}\bar{\mu} \otimes \mu$$

when $t \to \infty$.

In order to finish the proof of the proposition we shall delete the terms $t/\ell_c(\gamma)$ with the restriction "γ primitive". We will follow a method of Roblin ([24], p. 71)

The integer part of $t\ell_c(\gamma)^{-1}$ is the number of powers of γ such that $\ell_c(\gamma^n) \le t$, that is,

$$\left[\frac{t}{\ell_c(\gamma)}\right] = \#\{n \in \mathbb{N} : \ell_c(\gamma^n) = n\ell_c(\gamma) \le t\}.$$

We then have that ν_t equals

$$\|m_c\| h_c e^{-h_c t} \sum_{\substack{\gamma \text{ primitive} \\ \ell_c(\gamma) \le t}} \left[\frac{t}{\ell_c(\gamma)}\right] \delta_{\gamma_-} \otimes \delta_{\gamma_+}$$

and we find $\nu_t(f) \le \sigma_t(f)$ for every measurable $f \ge 0$.

For a complementary inequality fix some $\kappa > 0$. Now, if $e^{-\kappa} t < \ell_c(\gamma) \le t$ we have $[t/\ell_c(\gamma)] \ge e^{-\kappa} t/\ell_c(\gamma)$ and

$$\begin{aligned} \nu_t &\ge \|m_c\| e^{-\kappa} h_c t e^{-h_c t} \sum_{\substack{\gamma \text{ primitive} \\ e^{-\kappa} t < \ell_c(\gamma) \le t}} \frac{1}{\ell_c(\gamma)} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \\ &= e^{-\kappa} \sigma_t - e^{-\kappa} h_c t e^{-h_c t} \sum_{\substack{\gamma \text{ primitive} \\ \ell_c(\gamma) \le e^{-\kappa} t}} \frac{1}{\ell_c(\gamma)} \delta_{\gamma_-} \otimes \delta_{\gamma_+}. \end{aligned}$$

Since the second term goes to zero when $t \to \infty$ we find that for every measurable $f \ge 0$,

$$\limsup \nu_t(f) \ge e^{-k} \limsup \sigma_t(f).$$

Since κ is arbitrary, these two inequalities show the proposition. □

5. Exponential growth of convex representations

Let $\rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation with ρ-equivariant Hölder map $\xi\colon \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$. For a fixed norm $\| \ \|$ on $\mathbb{R}^d$ define the Hölder cocycle

$$\beta_1(\gamma, x) = \log \frac{\|\rho(\gamma) v\|}{\|v\|}$$

for $v \in \xi(x) - \{0\}$.

In order to apply Theorem 3.2 we need to prove that the cocycle β_1 is of positive periods and has finite exponential growth rate. The main purpose of this section is Proposition 5.4.

We shall first show that the period $\beta_1(\gamma, \gamma_+)$ is exactly $\lambda_1(\rho(\gamma))$, the logarithm of the spectral radius of some lift of $\rho(\gamma)$ with determinant $\in \{-1, 1\}$.

We say that $g \in \mathrm{PGL}(d, \mathbb{R})$ is *proximal* if it has a unique complex eigenvalue of maximal modulus, and its generalized eigenspace is one-dimensional. This eigenvalue is necessarily real and its modulus is equal to $\exp \lambda_1(g)$. We will denote by g_+ the g-fixed line of $\mathbb{R}^d$ consisting of eigenvectors of this eigenvalue and by g_- the g-invariant complement of g_+ (that is, $\mathbb{R}^d = g_+ \oplus g_-$). g_+ is an attractor on $\mathbb{P}(\mathbb{R}^d)$ for the action of g and g_- is a repelling hyperplane.

Lemma 5.1. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation. Then for every $\gamma \in \Gamma$, $\rho(\gamma)$ is proximal and $\xi(\gamma_+)$ is its attractive fixed line.*

Proof. Consider $\gamma_0 \in \Gamma$ and write, to simplify the notation, $a = \exp \lambda_1(\rho(\gamma_0))$. We consider a lift of $\rho(\gamma_0)$ to $\mathrm{SL}(d,\mathbb{R})_\pm$ which we still call $\rho(\gamma_0)$.

Let V_0 be the sum of all generalized $\rho(\gamma_0)$-eigenspaces of eigenvalues with modulus equal to a. We will show that $V_0 = \xi(\gamma_{0+})$. Set $V = V_0 \cap \eta(\gamma_{0-})$, and recall that $\mathbb{R}^d = \xi(\gamma_{0+}) \oplus \eta(\gamma_{0-})$.

Since V is a sum of generalized eigenspaces in $\eta(\gamma_{0-})$, it has a $\rho(\gamma_0)$-invariant complement $W \subset \eta(\gamma_{0-})$ and thus $\mathbb{R}^d = \xi(\gamma_{0+}) \oplus W \oplus V$.

We claim that $\xi(\gamma_{0+}) \oplus W$ contains a $\rho(\Gamma)$-invariant subspace. Since ρ is irreducible we obtain $V = \{0\}$ and $V_0 = \xi(\gamma_{0+})$, which implies the lemma. For this we will show that

$$\xi(\partial\Gamma) \subset \mathbb{P}(\xi(\gamma_{0+}) \oplus W).$$

Let $x \in \partial\Gamma - \{\gamma_{0-}\}$. Since $\gamma_0^n x \to \gamma_{0+}$ the same occurs via ξ, that is,

$$\rho({\gamma_0}^n)\xi(x) \to \xi(\gamma_{0+}) \tag{6}$$

in $\mathbb{P}(\mathbb{R}^d)$. Take some u_x in the line $\xi(x)$ and write, following the decomposition $\mathbb{R}^d = \xi(\gamma_{0+}) \oplus V \oplus W$,

$$u_x = u_+ + v + w$$

for some $u_+ \in \xi(\gamma_{0+})$, $v \in V$ and $w \in W$. We now consider the sequence

$$\frac{\rho(\gamma_0^n)u_x}{a^n} = \frac{\rho(\gamma_0^n)(u_+ + v + w)}{a^n}.$$

Since the spectral radius of $\rho(\gamma_0)|W$ is strictly smaller than a (by definition of V) we have

$$\rho(\gamma_0^n)w/a^n \to 0.$$

Also, since u_+ is an eigenvector of $\rho(\gamma_0)$ we must have either $\rho(\gamma_0)u_+/a = \pm u_+$ or $\rho(\gamma_0^n)u_+/a^n \to 0$.

On the other hand, since $\rho(\gamma_0)|V$ consists of Jordan blocks of eigenvalues with modulus a we have $a^n \le c\|\rho(\gamma_0^n)v\|$ for some $c > 0$ and all n sufficiently large. This implies that the sequence

$$\frac{\rho(\gamma_0^n)v}{a^n}$$

is far from zero (when $v \neq 0$).

Consequently: if $\rho(\gamma_0^n)u_+/a^n \to 0$, then the limit line of $\rho(\gamma_0^n)\xi(x)$ is contained in $\mathbb{P}(V)$. This contradicts equation (6) and the convexity of ρ. We then have that $\rho(\gamma_0)u_+/a = \pm u_+$ and, since $\rho(\gamma_0^n)v/a^n$ is far from zero, in order that (6) holds we must have $v = 0$. Thus $\xi(\partial\Gamma) \subset \mathbb{P}(\xi(\gamma_{0+}) \oplus W)$ which implies $V = 0$. This finishes the proof. $\square$

We then find the following corollaries.

Corollary 5.2. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation. Then the equivariant maps $\xi\colon \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\eta\colon \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ are unique and for every $x \in \partial\Gamma$ one has $\xi(x) \subset \eta(x)$.*

Proof. The fact that $\xi(\gamma_+)$ is the attractive line of $\rho(\gamma)$ and the fact that attractors $\{\gamma_+ : \gamma \in \Gamma\}$ form a dense subset of $\partial\Gamma$ prove uniqueness of ξ and, by analogue reasoning, uniqueness of η.

Since $\eta(\gamma_-)$ is the repeller hyperplane of $\rho(\gamma)$ and $\xi(\gamma_-)$ is $\rho(\gamma^{-1})$'s attractive line we must have $\xi(\gamma_-) \subset \eta(\gamma_-)$. Again, density of repellers implies that $\xi(x) \subset \eta(x)$ for every $x \in \partial\Gamma$. □

Corollary 5.3. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation. Then the period $\beta_1(\gamma,\gamma_+)$ is $\lambda_1(\rho(\gamma))$. Moreover $\lambda_1(\rho(\gamma)) > 0$ for every $\gamma \in \Gamma$.*

Proof. By Lemma 5.1 we have that $\xi(\gamma_+)$ is the fixed attractive line of $\rho(\gamma)$. We then have

$$\beta_1(\gamma,\gamma_+) = \log \frac{\|\rho(\gamma)u_+\|}{\|u_+\|} = \lambda_1(\rho(\gamma))$$

were $u_+ \in \xi(\gamma_+)$.

The fact that the periods are positive is also a consequence of the fact that $\rho(\gamma)$ is proximal. If $\lambda_1(\rho(\gamma)) = 0$ then considering some lift of $\rho(\gamma)$ with determinant in $\{-1,1\}$ one sees that every eigenvalue of this lift would be of modulus 1 and thus $\rho(\gamma)$ would not be proximal. □

Since β_1 is a cocycle with positive periods, Corollary 3.6 implies that the exponential growth rate of β_1 is positive. The objective now is the proof of the following proposition:

Proposition 5.4. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation. Then*

$$\limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho(\gamma)) \le s\}}{s}$$

is finite.

The following lemma provides a general property of hyperbolic groups for which we refer the reader to Tukia [31]. For the second assertion of the lemma one can apply explicitly Lemma 1.6 of Bowditch [5].

Lemma 5.5. *Let Γ be a hyperbolic group and let $\{\gamma_n\}$ be a sequence in Γ going to infinity. Then there exists a subsequence $\{\gamma_{n_k}\}$ and two points $x_0, y_0 \in \partial\Gamma$ (not necessarily distinct) such that $\gamma_{n_k}x \to x_0$ uniformly on compact sets of $\partial\Gamma - \{y_0\}$. Moreover one can assume that $\gamma_{n_k+} \to x_0$ and $\gamma_{n_k-} \to y_0$.*

In order to prove Lemma 5.4 we need some quantified version of proximality. Define Gromov's product $\mathscr{G}\colon \mathbb{P}(\mathbb{R}^{d\,*}) \times \mathbb{P}(\mathbb{R}^d) - \Delta \to \mathbb{R}$ as

$$\mathscr{G}(\theta, v) = \log \frac{|\theta(v)|}{\|\theta\|\|v\|},$$

where $\Delta = \{(\theta, v) : \theta(v) = 0\}$. We say that a linear transformation g is (r, ε)-proximal for some $r \in \mathbb{R}_+$ and $\varepsilon > 0$ if it is proximal,

$$\exp \mathscr{G}(g_-, g_+) > r,$$

and the complement of an ε-neighborhood of g_- is sent by g to an ε-neighborhood of g_+. The following lemmas (5.6 and 5.7) will also be used in the proof of Theorem A.

Lemma 5.6 (Benoist [2]). *Let r and δ be positive numbers. Then there exists ε such that for every (r, ε)-proximal transformation g one has*

$$|\log \|g\| - \lambda_1(g) + \mathscr{G}(g_-, g_+)| < \delta.$$

Proof. Consider the compact sets

$$P_{r,\varepsilon} = \{(r, \varepsilon)\text{-proximal linear transformations with norm } 1\}.$$

For a fixed r consider $P_r = \bigcap_\varepsilon P_{r,\varepsilon}$. An element $T \in P_r$ is a rank-one operator with the constraint $\|T\| = 1$ and such that $\operatorname{im} T \cap \ker T = \{0\}$. One explicitly writes

$$Tw = \frac{\theta(w)}{\|\theta\|\|v\|} v$$

where $v \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^{d\,*}$ are such that $\theta(v) \neq 0$. It is easy to verify that the above formula for T gives a rank-one operator with norm equal to 1.

One finishes with the remark that the function $g \mapsto \lambda_1(g)$ is continuous and $\lambda_1(T) = \mathscr{G}(\theta, v)$ for $T \in P_r$. □

Define $[\cdot, \cdot]\colon \partial^2\Gamma \to \mathbb{R}$ as

$$[x, y] = \mathscr{G}(\eta(x), \xi(y))$$

for $x, y \in \partial\Gamma$ distinct.

Lemma 5.7. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation. Fix $r \in \mathbb{R}_+$ and $\varepsilon > 0$. Then the set*

$$\{\gamma \in \Gamma : \exp([\gamma_-, \gamma_+]) > r \text{ and } \rho(\gamma) \text{ is not } (r, \varepsilon)\text{-proximal}\}$$

is finite.

Proof. Let $\gamma_n \to \infty$ be a sequence in Γ such that $\exp[\gamma_{n-}, \gamma_{n+}] > r$. Since ξ and η are uniformly continuous we have that $d_o(\gamma_{n-}, \gamma_{n+}) > \kappa$ for some $\kappa > 0$ and some Gromov distance d_o in $\partial\Gamma$. By applying Lemma 5.5 we find a subsequence (still called γ_n) and two points x_0, y_0 such that γ_{n-} and γ_{n+} converge to y_0 and x_0 respectively, and such that $\gamma_n x \to x_0$ for every $x \neq y_0$.

We have $x_0 \neq y_0$ since $d_o(\gamma_{n-}, \gamma_{n+}) > \kappa$.

By considering again a subsequence we assume that

$$\frac{\rho(\gamma_n)}{\|\rho(\gamma_n)\|} \to T$$

for some linear transformation T of $\mathbb{R}^d$. We will prove that T is a proximal rank-one operator, which implies that for sufficiently large n, $\rho(\gamma_n)$ is (r, ε)-proximal.

Since $\xi(\gamma_{n+})$ is $\rho(\gamma_n)$-invariant for all n we have that $\xi(x_0)$ is T-invariant and by analogue reasoning we also have that $\eta(y_0)$ is T-invariant (recall that we also have $\mathbb{R}^d = \xi(x_0) \oplus \eta(y_0)$ since ρ is strictly convex and $x_0 \neq y_0$).

Consider now a point $x \in \partial\Gamma - \{y_0\}$ and u_x a vector in the line $\xi(x)$. Write $u_x = u + v$ for some $u \in \xi(x_0)$ and $v \in \eta(y_0)$. Since $\rho(\gamma_n)\xi(x) \to \xi(x_0)$ we must have $Tu_x \in \xi(x_0)$ and thus $Tv = 0$ (since $\eta(y_0)$ is T-invariant).

Consequently, $\xi(\partial\Gamma) \subset \mathbb{P}(\xi(x_0) + \ker T)$. Irreducibility of ρ implies that $\mathbb{R}^d = \xi(x_0) + \ker T$. In order to finish we remark that since $\|T\| = 1$ we must have $T|\xi(x_0) \neq 0$ and thus $T \neq 0$. We then have a rank-one operator whose image is not contained in its kernel. □

The following lemma states that strictly convex representations are discrete and, using the fact that the fundamental group of a negatively curved manifold is torsion-free, they are also injective.

Lemma 5.8. *Let Γ be a non-elementary hyperbolic group and $\rho_0 \colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be an irreducible representation such that there exists a ρ_0-equivariant continuous map $\xi_0 \colon \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$. Then ρ_0 is discrete with finite kernel. In particular, strictly convex representations are discrete and injective.*

Proof. Assume there exists a divergent sequence $\gamma_n \to \infty$ in Γ such that $\rho_0(\gamma_n)$ converges to $g \in \mathrm{PGL}(d, \mathbb{R})$. Consider a subsequence (which we still call γ_n) and the points $x_0, y_0 \in \partial\Gamma$ given by Lemma 5.5. Then for any $x \in \partial\Gamma$ different from y_0 one has $\gamma_n x \to x_0$.

Since ξ_0 is ρ_0-equivariant we have that $\rho_0(\gamma_n)\xi_0(x) \to \xi_0(x_0)$ and thus

$$g\xi_0(x) = \xi_0(x_0)$$

for every $x \neq y_0$. Since g is injective one obtains that ξ_0 is constant and thus ρ_0 fixes a line in $\mathbb{R}^d$. This contradicts irreducibility.

We proved that ρ_0 is proper and thus has finite kernel. This finishes the proof of the lemma. □

We can now prove that the exponential growth rate of the cocycle β_1 is finite.

Proof of Lemma 5.4. Since the action of Γ on $T^1\widetilde{M}$ is co-compact, one has a compact fundamental domain D. Since a conjugacy class $[\gamma] \in [\Gamma]$ is identified with a closed geodesic, one can always find a representative $\gamma_0 \in [\gamma]$ such that the γ_0-invariant geodesic on $T^1\widetilde{M}$ intersects D. The fact that D is compact implies that the fixed points of γ_0 on $\partial\Gamma$ are necessarily far away by some constant independent of $[\gamma]$.

In other words, there exists some constant $k > 0$ such that every conjugacy class of $[\gamma]$ has a representative γ_0 with $d_o(\gamma_{0-}, \gamma_{0+}) > k$ for some Gromov distance d_o on $\partial\Gamma$.

Since the equivariant maps ξ and η are uniformly continuous one has that every conjugacy class $[\gamma]$ has a representative γ_0 such that $\exp[\gamma_{0-}, \gamma_{0+}] > r$ for some r independent of $[\gamma]$.

From now on we fix some number $\delta > 0$ and consider $\varepsilon > 0$ given by Lemma 5.6. Thus, applying Lemma 5.7, all γ's with $\exp[\gamma_-, \gamma_+] > r$ (but a finite number depending only on r and ε) are (r, ε)-proximal and thus verify, according to Benoist's Lemma 5.6,

$$\log \|\rho(\gamma)\| + \log r - \delta \leq \lambda_1(\rho(\gamma)).$$

One concludes, by choosing for each conjugacy class $[\gamma]$ a representative γ_0 with $\exp[\gamma_{0-}, \gamma_{0+}] > r$, that

$$\begin{aligned}
&\#\{[\gamma] \in [\Gamma] : \lambda_1(\rho(\gamma)) \leq t\} \\
&\qquad \leq \#\{\gamma : [\gamma_-, \gamma_+] > \log r \text{ and } \log \|\rho(\gamma)\| \leq t + \delta - \log r\} \\
&\qquad\qquad + \#\{\text{finite set independent of } t\} \\
&\qquad \leq \#\{\gamma \in \Gamma : \log \|\rho(\gamma)\| \leq t + \delta - \log r\} \\
&\qquad\qquad + \#\{\text{finite set independent of } t\}.
\end{aligned}$$

Since the cardinal of the finite set is negligible when computing the exponential growth rate, one is led to study the exponential growth rate of the quantity $\#\{\gamma \in \Gamma : \log \|\rho(\gamma)\| \leq t\}$ when $t \to \infty$.

Lemma 5.8 states that $\rho(\Gamma)$ is discrete and injective and thus the fact that the exponential growth rate of $\#\{\gamma \in \Gamma : \log \|\rho(\gamma)\| \leq t\}$ is finite when $t \to \infty$ is implied by the following general fact. □

We remark that the statement of the following lemma is independent of the norm $\| \ \|$ chosen in $\mathbb{R}^d$.

Lemma 5.9. *Let Λ be a discrete subgroup of* $\mathrm{PGL}(d, \mathbb{R})$. *Then*

$$\limsup_{t \to \infty} \frac{\log \#\{g \in \Lambda : \log \|g\| \leq t\}}{t} < \infty.$$

Proof. This is a consequence of the following estimation of the Haar measure of $\mathrm{PGL}(d,\mathbb{R})$ which can be found in Helgason [13]:

$$\limsup_{R\to\infty} \frac{\log \mathrm{Haar}\{g \in \mathrm{PGL}(d,\mathbb{R}) : \|g\| \le R\}}{\log R} < \infty. \qquad \square$$

6. Theorems A and B

Counting the growth of the spectral radii. We prove now Theorem B.

Theorem 6.1. *Let $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation. Then there exists $h > 0$ such that*

$$hte^{-ht}\#\{[\gamma] \in [\Gamma] \text{ primitive}\colon \lambda_1(\rho\gamma) \le t\} \to 1$$

when $t \to \infty$.

Proof. Recall that according to Corollary 5.3 the cocycle β_1 has periods $\beta_1(\gamma,\gamma_+) = \lambda_1(\rho\gamma)$. Proposition 5.4 states that β_1 has finite and positive exponential growth rate and thus Corollary 4.1 applies. The result then follows. $\square$

Dual cocycle of β_1 and Gromov product. In order to prove Theorem A we introduce a natural dual cocycle of β_1 and the Gromov product associated to this pair.

First recall that we have two ρ-equivariant Hölder maps $\xi\colon \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\eta\colon \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ such that $\xi(x) \not\subset \eta(y)$ if $x \ne y$. We have defined the cocycle

$$\beta_1(\gamma,x) = \log \frac{\|\rho(\gamma)v\|}{\|v\|}$$

for any $v \in \xi(x) - \{0\}$. Define then $\overline{\beta}_1\colon \Gamma \times \partial\Gamma \to \mathbb{R}$ as

$$\overline{\beta}_1(\gamma,x) = \log \frac{\|\rho(\gamma)\theta\|}{\|\theta\|}$$

for any $\theta \in \mathbb{R}^{d*}$ such that $\ker\theta = \eta(x)$.

Lemma 6.2. *Let $g \in \mathrm{GL}(d,\mathbb{R})$ be proximal with maximal eigenvalue a, and let $\theta \in \mathbb{R}^{d*}$ such that $\ker\theta = g_-$. Then $g\theta = a^{-1}\theta$.*

Proof. Since $\ker\theta = g_-$ one has $g\theta = b\theta$ for some real b. Consider now some $u_+ \in g_+$. One has

$$b\theta(u_+) = g\theta(u_+) = \frac{1}{a}\theta(u_+)$$

and, since $\theta(u_+) \ne 0$, we have $b = a^{-1}$ $\square$

One trivially deduces the following lemma.

Lemma 6.3. *The period* $\overline{\beta_1}(\gamma, \gamma_+)$ *is* $\lambda_1(\rho\gamma^{-1})$. *One obtains thus that the pair* $\{\beta_1, \overline{\beta_1}\}$ *is a pair of dual cocycles.*

Recall that we have defined the Gromov product $\mathscr{G}\colon \mathbb{P}(\mathbb{R}^{d*}) \times \mathbb{P}(\mathbb{R}^d) - \Delta \to \mathbb{R}$ as

$$\mathscr{G}(\theta, v) = \log \frac{|\theta(v)|}{\|\theta\|\|v\|},$$

where $\Delta = \{(\theta, v) : \theta(v) = 0\}$, and $[x, y] = \mathscr{G}(\eta(x), \xi(y))$ for $x, y \in \partial\Gamma$ distinct.

Lemma 6.4. *The function* $[\cdot,\cdot]\colon \partial^2\Gamma \to \mathbb{R}$ *is a Gromov product for the pair* $\{\beta_1, \overline{\beta_1}\}$.

Proof. One easily verifies that for every $g \in \mathrm{PGL}(d, \mathbb{R})$ one has

$$\mathscr{G}(g\theta, gv) - \mathscr{G}(\theta, v) = -\left(\log \frac{\|g\theta\|}{\|\theta\|} + \log \frac{\|gv\|}{\|v\|}\right),$$

(recall that the action of $\mathrm{PGL}(d, \mathbb{R})$ on $\mathbb{P}(\mathbb{R}^{d*})$ coherent with the identification $\mathbb{P}(\mathbb{R}^{d*}) \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ is $\theta \mapsto \theta \circ g^{-1}$) this means exactly that for every $\gamma \in \Gamma$ one has

$$[\gamma x, \gamma y] - [x, y] = -(\overline{\beta_1}(\gamma, x) + \beta_1(\gamma, y)).$$

This finishes the proof. □

The proof of Theorem A. We can now prove Theorem A. To simplify notation write $\ell(\gamma)$ for the periods

$$\ell(\gamma) = \beta_1(\gamma, \gamma_+) = \lambda_1(\rho\gamma),$$

and h for the exponential growth rate of β_1,

$$h := \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \ell(\gamma) \le t\}}{s}.$$

Let μ and $\bar{\mu}$ be the Patterson–Sullivan measures of β_1 and $\overline{\beta_1}$ respectively. Write $\|m_\rho\|$ for the total mass of the measure $e^{-h[\cdot,\cdot]}\bar{\mu} \otimes \mu \otimes ds$ on the compact quotient $\Gamma\backslash\partial^2\Gamma \times \mathbb{R}$.

Theorem 6.5. *One has*

$$\|m_\rho\| h e^{-ht} \sum_{\log\|\rho\gamma\| \le t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to \bar{\mu} \otimes \mu$$

as $t \to \infty$ *on* $C^*(\partial\Gamma \times \partial\Gamma)$.

Proof. Since $h \in (0, \infty)$ Proposition 4.3 applies and thus

$$\|m_\rho\| h e^{-ht} \sum_{\ell(\gamma) \le t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to e^{-h[\cdot,\cdot]} \bar{\mu} \otimes \mu$$

on $C^*(\partial^2 \Gamma)$.

Choose some positive δ and let $A, B \subset \partial\Gamma$ be two disjoint open subsets small enough such that $[\cdot,\cdot]\colon A \times B \to \mathbb{R}$ is constant r up to δ, that is $|[x, y] - r| \le \delta$ for every $(x, y) \in A \times B$.

Lemma 5.7 allows us to assume (excluding a finite set of Γ that depends on r and δ) that if $\gamma_- \in A$ and $\gamma_+ \in B$ then $\rho(\gamma)$ is $(\exp r, \varepsilon)$-proximal, where ε comes from Lemma 5.6 for $\exp r$ and δ.

We then have, by Benoist's Lemma 5.6, that $|\log \|\rho(\gamma)\| - \ell(\gamma) + r| \le 2\delta$. That is,

$$\ell(\gamma) - r - 2\delta \le \log \|\rho(\gamma)\| \le \ell(\gamma) - r + 2\delta.$$

Set

$$\theta_t := \|m_\rho\| h e^{-ht} \sum_{\log \|\rho(\gamma)\| \le t} \delta_{\gamma_-} \otimes \delta_{\gamma_+}.$$

The last inequalities imply that for all $t > 0$,

$$\begin{aligned} e^{-2h\delta} e^{hr} \|m_\rho\| h e^{-h(t+r-2\delta)} & \sum_{\ell(\gamma) \le t+r-2\delta} \delta_{\gamma_-}(A) \delta_{\gamma_+}(B) \\ &\le \theta_t(A \times B) \\ &\le e^{2h\delta} e^{hr} \|m_\rho\| h e^{-h(t+r+2\delta)} \sum_{\ell(\gamma) \le t+r+2\delta} \delta_{\gamma_-}(A) \delta_{\gamma_+}(B). \end{aligned}$$

Applying Proposition 4.3 we find, when $t \to \infty$, that

$$\begin{aligned} e^{-2h\delta} e^{h(r-[\cdot,\cdot])} \bar{\mu} \otimes \mu(A \times B) &\le \liminf_{t \to \infty} \theta_t(A \times B) \\ &\le \limsup_{t \to \infty} \theta_t(A \times B) \\ &\le e^{2h\delta} e^{h(r-[\cdot,\cdot])} \bar{\mu} \otimes \mu(A \times B), \end{aligned}$$

hence one has, since $|r - [x, y]| \le \delta$ for every $(x, y) \in A \times B$, that

$$e^{-3h\delta} \bar{\mu}(A) \mu(B) \le \liminf_{t \to \infty} \theta_t(A \times B) \le \limsup_{t \to \infty} \theta_t(A \times B) \le e^{3h\delta} \bar{\mu}(A) \mu(B).$$

Since δ is arbitrary this argument proves the convergence of $\theta_t \to \bar{\mu} \otimes \mu$ outside the diagonal, that is to say, subsets of $\partial\Gamma \times \partial\Gamma - \{(x, x) : x \in \partial\Gamma\}$. In order to finish we will prove the following: Given ε_0 there exists an open covering $\mathcal{U}$ of $\partial\Gamma$ such that $\sum_{U \in \mathcal{U}} \theta_t(U \times U) \le \varepsilon_0$ for all t large enough. The following argument was personally communicated by Thomas Roblin.

Since $\bar{\mu}$ and μ have no atoms and $\gamma_*\mu \ll \mu$ for every $\gamma \in \Gamma$, one has that the diagonal has measure zero for $\bar{\mu} \otimes \gamma_*\mu$ (for every $\gamma \in \Gamma$).

Fix two elements γ_0 and γ_1 in Γ and fix some $\varepsilon_0 > 0$. We can assume that γ_0 and γ_1 have no common fixed point in $\partial\Gamma$. Choose an open covering $\mathcal{U}$ of $\partial\Gamma$ such that for every $i = 0, 1$ one has

$$\sum_{U \in \mathcal{U}} \bar{\mu}(U) \times \mu(\gamma_i(U)) < \varepsilon_0.$$

By refining $\mathcal{U}$ we can assume that for every $U \in \mathcal{U}$ there exists $i \in \{0, 1\}$ such that $\gamma_i \bar{U} \cap \bar{U} = \emptyset$ where $\bar{U}$ is U's closure.

Since $\partial\Gamma$ is compact we may assume that the covering $\mathcal{U}$ is finite and thus, by enlarging the U's, we can consider a new covering $\mathcal{V}$ verifying the following:

(1) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $\bar{U} \subset V$, and for each $V \in \mathcal{V}$ there exists a unique U verifying this condition;

(2) if $\gamma_i \bar{U} \cap \bar{U} = \emptyset$ for some $i \in \{0, 1\}$ then $\gamma_i \bar{V} \cap \bar{V} = \emptyset$ for the unique V such that $\bar{U} \subset V$;

(3) $\sum_{V \in \mathcal{V}} \mu \otimes \gamma_{i*}\mu(V \times V) < \varepsilon_0$ for every $i \in \{0, 1\}$.

Consider some $U \in \mathcal{U}$ and suppose that $\gamma_0 \bar{U} \cap \bar{U} = \emptyset$. We study the set $\Gamma_U := \{\gamma \in \Gamma : (\gamma_-, \gamma_+) \in U \times U\}$.

Lemma 6.6. *Consider $V \in \mathcal{V}$ such that $\bar{U} \subset V$. Except for a finite number of $\gamma \in \Gamma_U$, the repeller $(\gamma_0\gamma)_-$ of $\gamma_0\gamma$ belongs to V and the attractor $(\gamma_0\gamma)_+ \in \gamma_0 V$.*

Proof. Consider a sequence $\gamma_n \in \Gamma_U$ and the points x_0, y_0 given by Lemma 5.5. Since $\gamma_{n-} \to y_0$ and $\gamma_{n+} \to x_0$ we have that x_0 and y_0 belong to $\bar{U} \subset V$, for a unique $V \in \mathcal{V}$. Thus, since $y_0 \notin \gamma_0 \bar{V}$, one has $\gamma_n(\gamma_0 V) \to x_0$ uniformly.

This implies that the set

$$F_U = \{\gamma \in \Gamma_U : \gamma(\gamma_0 V) \not\subseteq V\}$$

is finite.

Consider now some $\gamma \in \Gamma_U - F_U$. The sequence $(\gamma_0\gamma)^n \gamma_+$ is contained in $\gamma_0 V$ and thus (since γ_+ is not the repeller of $\gamma_0\gamma$) the attractor of $\gamma_0\gamma$ also belongs to $\gamma_0 V$.

Analogue reasoning gives the remaining statement of the lemma. □

It follows from the lemma that

$$\theta_t(U \times U) \leq \|m_\rho\| h e^{-ht} \sum_{\gamma : \log\|\rho(\gamma)\| \leq \log\|\rho(\gamma_0)\| + t} \delta_{\gamma_-}(V) \otimes \delta_{\gamma_+}(\gamma_0 V)$$
$$+ \|m_\rho\| h e^{-ht} \#\{\text{finite set independent of } t\},$$

where $V \in \mathcal{V}$ is such that $\bar{U} \subset V$. Since $V \times \gamma_0 V$ is far from the diagonal and the cardinality of the finite set does not depend on t, the right-hand side of the formula converges to $\|\gamma_0\|\bar{\mu}(V) \times \mu(\gamma_0 V)$ when $t \to \infty$.

One then has, since V is unique for each given $U \in \mathcal{U}$, that

$$\sum_{U \in \mathcal{U}} \theta_t(U \times U) \leq \sum_{i \in \{0,1\}} \sum_{V \in \mathcal{V}} \|\gamma_i\| \bar{\mu}(V) \mu(\gamma_i V) \leq 2\varepsilon_0 \max\{\|\gamma_0\|, \|\gamma_1\|\}.$$

Since γ_0 and γ_1 are fixed and ε_0 is arbitrarily small the theorem is proved. □

7. Hyperconvex representations: Theorem C

We are now interested in studying hyperconvex representations on some real algebraic non-compact semi-simple Lie group G. The purpose of this section is to prove Theorem C. In order to do so we must find an appropriate pair of dual Hölder cocycles and the Gromov product associated to them.

Denote by P a minimal parabolic subgroup of G and write $\mathscr{F} = G/P$. The set $\mathscr{F}$ is called the *Furstenberg boundary* of the symmetric space of G. The product $\mathscr{F} \times \mathscr{F}$ has a unique open G-orbit, denoted by $\partial^2 \mathscr{F}$.

Recall that Γ is the fundamental group of some closed negatively curved manifold M.

Definition 7.1. We say that a representation $\rho\colon \Gamma \to G$ is *hyperconvex* if it admits a Hölder continuous equivariant map $\zeta\colon \partial\Gamma \to \mathscr{F}$ such that whenever $x \neq y$ in $\partial\Gamma$ one has that the pair $(\zeta(x), \zeta(y))$ belongs to $\partial^2 \mathscr{F}$.

The relation between hyperconvex representations and strictly convex ones is given by the following lemma:

Lemma 7.1. *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation and let $\Lambda\colon G \to \mathrm{PGL}(d, \mathbb{R})$ be a proximal irreducible representation. Then the composition $\Lambda \circ \rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is strictly convex.*

Proof. Consider the highest weight χ of Λ. Since Λ is proximal the weight space of χ is one-dimensional, one thus obtains that $\Lambda(P)$ stabilizes a line in $\mathbb{R}^d$. Considering the dual representation one obtains an equivariant mapping into hyperplanes and one has that $\partial^2 \mathscr{F}$ is mapped to $\mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{d^*}) - \Delta$ where

$$\Delta = \{(v, \theta) \in \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{d^*}) : \theta(v) = 0\}.$$

One then obtains the equivariant mappings

$$\partial^2 \Gamma \to \partial^2 \mathscr{F} \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{d^*}).$$

Irreducibility of $\Lambda \circ \rho$ follows from Zariski density of $\rho(\Gamma)$ and irreducibility of Λ. $\square$

Fix a maximal compact subgroup K of G, consider a Cartan subalgebra $\mathfrak{a}$ of G's Lie algebra $\mathfrak{g}$ and fix some Weyl chamber $\mathfrak{a}^+$. Denote by $a\colon G \to \mathfrak{a}$ the Cartan projection following the Cartan decomposition $G = K\exp(\mathfrak{a}^+)K$. Consider also the Jordan projection $\lambda\colon G \to \mathfrak{a}^+$. These two projections are related by

$$\frac{1}{n}a(g^n) \to \lambda(g)$$

for every $g \in G$, (cf. Benoist [2]).

One says that $g \in G$ is *purely loxodromic* if $\lambda(g)$ belongs to the interior of the Weyl chamber $\mathfrak{a}^+$. A purely loxodromic element $g \in G$ has two remarkable fixed points in $\mathscr{F}$, namely g_+ and g_-. These points verify the following property: for every $z \in \mathscr{F}$ such that $(z, g_-) \in \partial^2\mathscr{F}$ one has that $g^n z \to g_+$ when $n \to \infty$. One then says that g_+ is the attractor of g and g_- is the repeller.

The existence of sufficiently many irreducible representations of G implies that Zariski-dense hyperconvex representations are purely loxodromic:

Consider the set Π of simple roots of $\mathfrak{a}$ on $\mathfrak{g}$ such that

$$\mathfrak{a}^+ = \{v \in \mathfrak{a} : \alpha(v) \geq 0 \text{ for all } \alpha \in \Pi\}$$

and consider the set $\{\omega_\alpha\}_{\alpha\in\Pi}$ of fundamental weights of Π.

Proposition 7.2 (Tits [30]). *For each $\alpha \in \Pi$ there exists a finite-dimensional proximal irreducible representation $\Lambda_\alpha\colon G \to \mathrm{PGL}(V_\alpha)$ such that the highest weight χ_α of Λ_α is an integer multiple of the fundamental weight ω_α. Moreover, any other weight of Λ_α is of the form*

$$\chi_\alpha - \alpha - \sum_{\beta\in\Pi} n_\beta \beta$$

with $n_\beta \in \mathbb{N}$.

Corollary 7.3. *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation with equivariant mapping $\zeta\colon \partial\Gamma \to \mathscr{F}$. Then for every $\gamma \in \Gamma$ the image $\rho(\gamma)$ is purely loxodromic; moreover, $\zeta(\gamma_+)$ is the attractor of $\rho(\gamma)$ in $\mathscr{F}$ and $\zeta(\gamma_-)$ is the repeller.*

Proof. We will show that for every $\alpha \in \Pi$ and $\gamma \in \Gamma$ one has $\alpha(\lambda(\rho\gamma)) > 0$.

Fix some $\alpha \in \Pi$ and consider Tits' representation $\Lambda_\alpha\colon G \to \mathrm{PGL}(V_\alpha)$ ([30]). Recall that for every $g \in G$ one has that $\chi_\alpha(\lambda(g)) = \lambda_1(\Lambda_\alpha g)$ and $\alpha(\lambda(g)) = \lambda_1(\Lambda_\alpha(g)) - \lambda_2(\Lambda_\alpha(g))$.

Since ρ is Zariski-dense and hyperconvex, Lemma 7.1 implies that $\Lambda_\alpha \circ \rho$ is strictly convex and thus (following Lemma 5.1) proximal. One concludes that

$$\alpha(\lambda(\rho\gamma)) = \lambda_1(\Lambda_\alpha(\rho\gamma)) - \lambda_2(\Lambda_\alpha\rho\gamma) > 0.$$

The last statement follows from Lemma 5.1. This completes the proof. □

The equivariant function ζ of the definition is then unique since attracting points γ_+ are dense in $\partial\Gamma$.

Busemann cocycle. Let us recall Busemann's cocycle on G introduced by Quint in [22]. The set $\mathscr{F}$ is K-homogeneous with stabilizer M. Quint defines $\sigma\colon G\times\mathscr{F}\to\mathfrak{a}$ to verify the following equation:

$$gk = l\exp(\sigma(g,kM))n$$

following Iwasawa's decomposition of $G = Ke^{\mathfrak{a}}N$, where N is the unipotent radical of G.

Note that $\sigma\colon G\times\mathscr{F}\to\mathfrak{a}$ verifies the cocycle relation

$$\sigma(gh,x) = \sigma(g,hx) + \sigma(h,x).$$

We need the following lemma of Quint [22]: Recall that for a given proximal irreducible representation $\Lambda\colon G\to \mathrm{PGL}(d,\mathbb{R})$ we have an equivariant map $\xi_\Lambda\colon\mathscr{F}\to\mathbb{P}(\mathbb{R}^d)$.

Lemma 7.4 (Lemma 6.4 of Quint [22]). *Consider some proximal irreducible representation $\Lambda\colon G\to\mathrm{PGL}(d,\mathbb{R})$. Then there exists a norm $\|\ \|$ on $\mathbb{R}^d$ such that for every $x\in\mathscr{F}$ and $g\in G$ one has*

$$\log\frac{\|\Lambda(g)v\|}{\|v\|} = \chi(\sigma(g,x)),$$

where $v\in\xi_\Lambda(x)-\{0\}$ and χ is the maximal weight of Λ.

Consider now a hyperconvex representation $\rho\colon\Gamma\to G$ with equivariant map $\zeta\colon\partial\Gamma\to\mathscr{F}$. Define the ($\mathfrak{a}$-valued) cocycle $\beta\colon\Gamma\times\partial\Gamma\to\mathfrak{a}$ by

$$\beta(\gamma,x) = \sigma(\rho(\gamma),\zeta(x)).$$

Lemma 7.5. *The periods of β are $\beta(\gamma,\gamma_+) = \lambda(\rho\gamma)$.*

Proof. The lemma follows directly from Quint [22], Lemma 7.4, Corollary 7.3 and Corollary 5.3 for strictly convex representations. □

Recall that for a Zariski-dense subgroup Λ of G, Benoist [2] has introduced the *limit cone* $\mathscr{L}_\Lambda$ as the closed cone containing $\{\lambda(g) : g\in\Lambda\}$. He has proved that this cone is convex and has non-empty interior. Consider its dual cone

$$\mathscr{L}_\Lambda{}^* := \{\varphi\in\mathfrak{a}^* : \varphi|\mathscr{L}_\Lambda\geq 0\}.$$

Ledrappier [16] (Theorem 3.5) implies the following corollary: Let $\mathscr{L}_\rho$ denote the limit cone of a hyperconvex representation $\rho\colon\Gamma\to G$ and $\mathscr{L}_\rho^*$ its dual cone.

Corollary 7.6. *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation. Then there exists a Γ-invariant Hölder continuous function $F : T^1\widetilde{M} \to \mathfrak{a}$ such that*

$$\int_{[\gamma]} F = \lambda(\rho\gamma)$$

for every conjugacy class $[\gamma] \in [\Gamma]$. The closure of the set

$$\left\{\frac{\lambda(\rho\gamma)}{|\gamma|} : \gamma \in \Gamma\right\}$$

is compact and generates the limit cone $\mathscr{L}_\rho$.

Proof. The first statement is a consequence of Ledrappier [16] (Theorem 3.5) for the vector cocycle $\beta : \Gamma \times \partial\Gamma \to \mathfrak{a}$ together with Lemma 7.5. Fix some norm $\| \ \|_{\mathfrak{a}}$ on $\mathfrak{a}$. Since for every $\gamma \in \Gamma$ one has

$$\frac{\|\lambda(\rho\gamma)\|_{\mathfrak{a}}}{|\gamma|} \leq \max \|F\|_{\mathfrak{a}},$$

one finds that the set

$$\left\{\frac{\lambda(\rho\gamma)}{|\gamma|} : \gamma \in \Gamma\right\}$$

is bounded and thus has compact closure. □

We remark that *a priori* the closure of $\{\lambda(\rho\gamma)/|\gamma| : \gamma \in \Gamma\}$ may contain zero, nevertheless the following lemma forbids this to happen.

Lemma 7.7. *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation and consider some φ in the dual cone $\mathscr{L}_\rho^*$. Then the cocycle $\varphi \circ \beta\colon \Gamma \times \partial\Gamma \to \mathbb{R}$ has finite and positive exponential growth rate if and only if φ belongs to the interior of $\mathscr{L}_\rho^*$.*

Proof. We will first show that for every simple root α, the weight χ_α for Tits's representation $\Lambda_\alpha\colon G \to \mathrm{PGL}(V_\alpha)$, is strictly positive on the limit cone $\mathscr{L}_\rho$. Recall that $\Lambda_\alpha\rho\colon \Gamma \to \mathrm{PGL}(V_\alpha)$ is strictly convex and that $\chi_\alpha(\lambda(\rho\gamma)) = \lambda_1(\Lambda_\alpha\rho\gamma)$. Proposition 5.4 states that the exponential growth rate of

$$\#\{[\gamma] \in [\Gamma] : \lambda_1(\Lambda_\alpha\rho\gamma) \leq t\}$$

is finite and thus applying Ledrappier's Lemma 3.7 we obtain

$$\inf_{[\gamma]\in[\Gamma]} \frac{\chi_\alpha(\lambda(\rho\gamma))}{|\gamma|} > 0.$$

The fundamental weight χ_α is then strictly positive on the closure of $\{\lambda(\rho\gamma)/|\gamma| : [\gamma] \in [\Gamma]\}$. Since this closure is compact and generates the limit cone (Corollary 7.6), χ_α is strictly positive on $\mathscr{L}_\rho - \{0\}$.

Consider now φ in the interior of $\mathscr{L}_\rho^*$, i.e. $\varphi|\mathscr{L}_\rho - \{0\} > 0$. Since χ_α is also strictly positive on $\mathscr{L}_\rho - \{0\}$ there exist two positive constants c and C such that

$$c\chi_\alpha(\lambda(\rho\gamma)) \leq \varphi(\lambda(\rho\gamma)) \leq C\chi_\alpha(\lambda(\rho\gamma))$$

(recall that $\mathscr{L}_\rho$ is closed by definition) for all $\gamma \in \Gamma$. As mentioned before the exponential growth rate of

$$\#\{[\gamma] \in [\Gamma] : \chi_\alpha(\lambda(\rho\gamma)) \leq s\}$$

is finite. Since the periods of the cocycle $\varphi \circ \beta$ are $\varphi \circ \beta(\gamma, \gamma_+) = \varphi(\lambda(\rho\gamma))$, we obtain that $\varphi \circ \beta$ is of finite and positive exponential growth.

Conversely, if $\varphi \circ \beta$ has finite exponential growth rate then Ledrappier [16] (Lemma 3.7), applied to the cocycle $\varphi \circ \beta$, says that

$$\inf_{[\gamma]\in[\Gamma]} \frac{\varphi(\lambda(\rho\gamma))}{|\gamma|} > 0.$$

The linear functional φ is then strictly positive on the closure of $\{\lambda(\rho\gamma)/|\gamma| : [\gamma] \in [\Gamma]\}$. Since this closure is compact and generates the limit cone (Corollary 7.6) we conclude the proof. □

From now on we shall write $\beta_\varphi : \Gamma \times \partial\Gamma \to \mathbb{R}$ for the Hölder cocycle

$$\beta_\varphi(\gamma, x) := \varphi \circ \beta(\gamma, x)$$

and h_φ for its exponential growth rate

$$h_\varphi = \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s}.$$

We can now deduce the first item of Theorem C:

Theorem 7.8. *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation and consider φ in the interior of $\mathscr{L}_\rho^*$. Then*

$$h_\varphi t e^{-h_\varphi t} \#\{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq t\} \to 1$$

when $t \to \infty$.

Proof. Recall that the periods of the cocycle β_φ are

$$\beta_\varphi(\gamma, \gamma_+) = \varphi(\lambda(\rho\gamma)).$$

The theorem is thus a direct consequence of the fact that the cocycle β_φ has finite and positive exponential growth rate (Lemma 7.7), together with Corollary 4.1. □

Dual cocycle and Gromov product. Consider the Weyl group W of G and denote by w_0 the biggest element of W associated to the choice of $\mathfrak{a}^+$. The *opposition involution* $\mathrm{i}\colon \mathfrak{a} \to \mathfrak{a}$ is $\mathrm{i} := -w_0$; it sends the Weyl chamber $\mathfrak{a}^+$ to itself and $\mathrm{i}(\lambda(g)) = \lambda(g^{-1})$ for every $g \in G$. This property implies that the Hölder cocycle

$$\overline{\beta_\varphi} := \varphi \circ \mathrm{i}\,\beta$$

is a dual cocycle of β_φ.

Consider some simple root $\alpha \in \Pi$ and the equivariant mappings $\xi_\alpha\colon \mathscr{F} \to \mathbb{P}(V_\alpha)$ and $\xi_\alpha^*\colon \mathscr{F} \to \mathbb{P}(V_\alpha^*)$ for Λ_α and Λ_α^*, respectively, where $\Lambda_\alpha\colon G \to \mathrm{PGL}(V_\alpha)$ is the representation given by Tits [30] (see Proposition 7.2) for α.

We define the *Gromov product* $\mathscr{G}_\Pi\colon \partial^2\mathscr{F} \to \mathfrak{a}$ as follows: Since $\{\omega_\alpha : a \in \Pi\}$ is a basis of $\mathfrak{a}^*$ the same occurs for $\{\chi_\alpha : \alpha \in \Pi\}$. The element $\mathscr{G}_\Pi(x, y)$ is thus determined by $\chi_\alpha(\mathscr{G}_\Pi(x, y))$ for every $\alpha \in \Pi$. Consider the Euclidean norm $\| \ \|_\alpha$ on V_α determined by the formula

$$\log \|\Lambda_\alpha(g)\|_\alpha = \chi_\alpha(a(g)).$$

We then define $\chi_\alpha(\mathscr{G}_\Pi(x, y))$ as

$$\chi_\alpha(\mathscr{G}_\Pi(x, y)) := \log \frac{|\theta(v)|}{\|\theta\|_\alpha \|v\|_\alpha}$$

for some $\theta \in \xi_\alpha^*(x)$ and $v \in \xi_\alpha(y)$.

Lemma 7.9. *For every $g \in G$ and $x, y \in \partial^2\mathscr{F}$ one has*

$$\mathscr{G}_\Pi(gx, gy) - \mathscr{G}_\Pi(x, y) = -(\mathrm{i} \circ \sigma(g, x) - \sigma(g, y)).$$

Proof. The lemma follows from the formula

$$\log \frac{|\theta \circ g^{-1}(gv)|}{\|\theta \circ g^{-1}\| \|gv\|} - \log \frac{|\theta(v)|}{\|\theta\| \|v\|} = -\log \frac{\|g\theta\|}{\|\theta\|} + \log \frac{\|gv\|}{\|v\|},$$

for a norm on a vector space V, every $g \in \mathrm{PGL}(V)$ and $(\theta, v) \in \mathbb{P}(V^*) \times \mathbb{P}(V) - \Delta$, together with the definition of $\mathscr{G}_\Pi$. □

Lemma 7.9 directly implies the following:

Lemma 7.10. *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation and consider φ in the interior of the dual cone $\mathscr{L}_\rho^*$, then the function $[\cdot,\cdot]_\varphi\colon \partial^2\Gamma \to \mathbb{R}$ defined as*

$$[x, y]_\varphi = \varphi \circ \mathscr{G}_\Pi(\zeta(x), \zeta(y))$$

is a Gromov product for the pair of dual cocycles $\{\beta_\varphi, \overline{\beta_\varphi}\}$.

Benoist [2] introduced the notion of (r, ε)-proximal on $\mathscr{F}$:

Definition 7.2. We say that $g \in G$ is *(r, ε)-proximal on $\mathscr{F}$* if for every simple root $\alpha \in \Pi$ the transformation $\Lambda_\alpha g$ is (r, ε)-proximal.

The following lemmas are the direct extension to this setting of Lemmas 5.6 and 5.7. Fix some norm $\| \|_{\mathfrak{a}}$ on $\mathfrak{a}$.

Lemma 7.11 (Benoist [2]). *Let r and δ be two positives numbers. Then there exists $\varepsilon > 0$ such that for any g (r, ε)-proximal on $\mathscr{F}$ one has*

$$\|a(g) - \lambda(g) + \mathscr{G}_\Pi(g_-, g_+)\|_{\mathfrak{a}} \leq \delta.$$

Lemma 7.12. *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation and fix some $r \in \mathbb{R}_+$ and $\varepsilon > 0$. Then the set of $\gamma \in \Gamma$ with*

$$\exp(\|\mathscr{G}_\Pi(\zeta(\gamma_-), \zeta(\gamma_+))\|_{\mathfrak{a}}) > r$$

such that $\rho(\gamma)$ is not (r, ε)-proximal on $\mathscr{F}$ is finite.

Lemma 7.12 follows directly from Tits [30] (Proposition 7.2) and from the analogue lemma for strictly convex representations 5.7.

Recall that h_φ is the exponential growth rate of the Hölder cocycle β_φ:

$$h_\varphi := \limsup_{s \to \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s}.$$

We obtain the following result:

Theorem 7.13. *Let $\rho\colon \Gamma \to G$ be a Zariski-dense hyperconvex representation and consider φ in the interior of the dual cone $\mathscr{L}_\rho^*$. Let μ_φ and $\bar{\mu}_\varphi$ be the Patterson–Sullivan probabilities on $\partial\Gamma$ associated to the cocycles β_φ and $\overline{\beta_\varphi}$. Then there exists $c > 0$ such that*

$$c e^{-h_\varphi t} \sum_{\gamma \in \Gamma\,:\, \varphi(a(\rho\gamma)) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to \bar{\mu}_\varphi \otimes \mu_\varphi$$

when $t \to \infty$. In particular one has

$$c e^{-h_\varphi t} \#\{\gamma \in \Gamma : \varphi(a(\rho\gamma)) \leq t\} \to 1.$$

The proof of Theorem 7.13 follows step by step the method of Section 5.

A bound for the orbital counting problem. Let $X = G/K$ be the symmetric space of G and $o = [K] \in X$. Fix some Euclidean norm $\| \|_{\mathfrak{a}}$ on $\mathfrak{a}$ invariant under the Weyl group such that

$$d_X(o, go) = \|a(g)\|_{\mathfrak{a}}$$

for every $g \in G$. For a discrete subgroup Λ of G define h_Λ as the exponential growth rate of an orbit on X:

$$\begin{aligned} h_\Lambda &:= \limsup_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : d_X(o, \rho(g)o) \le s\}}{s} \\ &= \limsup_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : \|a(g)\|_{\mathfrak{a}} \le s\}}{s}. \end{aligned}$$

We need the following theorem of Quint [21]:

Theorem 7.14 (Quint [21]). *Let Λ be a Zariski-dense discrete subgroup of G. Then there exists a linear form Θ_Λ in the interior of the dual cone $\mathscr{L}^*_\Lambda$ such that*

$$\limsup_{s\to\infty} \frac{\log \#\{g \in \Lambda : \Theta_\Lambda(a(g)) \le s\}}{s} = 1$$

and $h_\Lambda = \|\Theta_\Lambda\|_{\mathfrak{a}}$.

This form is called the *growth form* of the group Λ. Applying Corollary 7.13 to the growth form of a hyperconvex representation one obtains a bound for the orbital counting problem:

Corollary 7.15. *Let $\rho\colon \Gamma \to G$ be Zariski-dense hyperconvex representation. Then there exists $C > 0$ such that*

$$e^{-h_{\rho(\Gamma)}t} \#\{\gamma \in \Gamma : d_X(o, \rho(\gamma)o) \le t\} \le C$$

for every t large enough.

Proof. Write Θ for the growth form for $\Theta_{\rho(\Gamma)}$ of $\rho(\Gamma)$ and $h := h_{\rho(\Gamma)}$. One has

$$\Theta(a(\rho\gamma)) \le \|\Theta\| \|a(\rho\gamma)\| = \|\Theta\| d_X(o, \rho(\gamma)o).$$

Thus

$$\#\{\gamma \in \Gamma : d_X(o, \rho(\gamma)o) \le t\} \le \#\{\gamma \in \Gamma : \Theta(a(\rho\gamma)) \le \|\Theta\| t\}.$$

Applying Theorem 7.14 and Theorem 7.13 one has $h = \|\Theta\|$ and

$$\begin{aligned} h_\Theta &:= \limsup_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : \Theta(\lambda(\rho\gamma)) \le s\}}{s} \\ &= \limsup_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : \Theta(a(\rho\gamma)) \le s\}}{s} = 1. \end{aligned}$$

It then follows that

$$\#\{\gamma \in \Gamma : d_X(o, \rho(\gamma)o) \leq t\} \leq \#\{\gamma \in \Gamma : \Theta(a(\rho\gamma)) \leq ht\}$$

which, applying Theorem 7.13, is asymptotic to $ce^{h\Theta ht} = ce^{ht}$. This finishes the proof. □

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