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# On minimal spheres of area $4\pi$ and rigidity

Laurent Mazet and Harold Rosenberg\*

Abstract. Let M be a complete Riemannian 3-manifold with sectional curvatures between 0 and 1. A minimal 2-sphere immersed in M has area at least  $4\pi$ . If an embedded minimal sphere has area  $4\pi$ , then M is isometric to the unit 3-sphere or to a quotient of the product of the unit 2-sphere with  $\mathbb{R}$ , with the product metric. We also obtain a rigidity theorem for the existence of hyperbolic cusps. Let M be a complete Riemannian 3-manifold with sectional curvatures bounded above by -1. Suppose there is a 2-torus T embedded in M with mean curvature one. Then the mean convex component of M bounded by T is a hyperbolic cusp, *i.e.*, it is isometric to  $T \times \mathbb{R}$  with the constant curvature -1 metric:  $e^{-2t}d\sigma_0^2 + dt^2$  with  $d\sigma_0^2$  a flat metric on T.

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### 1. Introduction

Consider a smooth  $(C^{\infty})$  complete metric on the 2-sphere S whose curvature is between 0 and 1. It is well known that a simple closed geodesic in S has length at least  $2\pi$  (see [4] or Klingenberg's theorem in higher dimension [3], [2]). It is less well known that when such an S has a simple closed geodesic of length exactly  $2\pi$ , then S is isometric to the unit 2-sphere  $\mathbb{S}_1^2$ . This result is proved in [1], and the authors attribute the theorem to E. Calabi.

With this in mind, we consider what happens in a complete 3-manifold M with sectional curvatures between 0 and 1 (henceforth we suppose this curvature condition on M, unless stated otherwise).

Let  $\Sigma$  be an embedded minimal 2-sphere in M. Then the Gauss–Bonnet theorem and the Gauss equation tells us that the area of S is at least  $4\pi$ : indeed we have

$$4\pi = \int_{\Sigma} \bar{K}_{\Sigma} = \int \det(A) + K_{T\Sigma} \le \int_{\Sigma} 1 = A(\Sigma)$$
(1)

with det(A) the determinant of the shape operator which is non-positive. We prove in Theorem 1, that when the area of  $\Sigma$  equals  $4\pi$ , then M is isometric to the unit

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3-sphere  $\mathbb{S}_1^3$  or to a quotient of the product of the unit 2-sphere with  $\mathbb{R}$ ,  $\mathbb{S}_1^2 \times \mathbb{R}$ , with the product metric.

We remark that Theorem 1 does not hold for embedded minimal tori. Given  $\varepsilon$  greater than zero, there are Berger spheres with curvatures between 0 and 1, which contain embedded minimal tori of area less than  $\varepsilon$ . But a minimal sphere always has area at least  $4\pi$ .

It would be interesting to know what happens in higher dimensions. In the unit n-sphere  $\mathbb{S}_1^n$ , a compact minimal hyper-surface  $\Sigma$  always has volume at least the volume of the equatorial n-1 sphere  $\mathbb{S}_1^{n-1}$ . Is there a rigidity theorem when one allows metrics on  $\mathbb{S}^n$  (= M) of sectional curvatures between 0 and 1? Two questions arise. First, does an embedded minimal hyper-sphere  $\Sigma$  in M have volume at least the volume of  $\mathbb{S}_1^{n-1}$ . If this is so, and if  $\Sigma$  is an embedded minimal hyper-sphere with volume exactly the volume of  $\mathbb{S}_1^{n-1}$ , is M isometric to  $\mathbb{S}_1^n$  or to  $\mathbb{S}_1^{n-1} \times \mathbb{R}$ ?

In the same spirit as Theorem 1, we prove a rigidity theorem for hyperbolic cusps. We recall that a 3-dimensional hyperbolic cusp is a manifold of the form  $T \times \mathbb{R}$ with T a 2-torus and the hyperbolic metric  $e^{-2t} d\sigma_0^2 + dt^2$  with  $d\sigma_0^2$  a flat metric on T. In Theorem 2, we prove that if M is a complete Riemannian manifold with sectional curvatures bounded above by -1 and T is a constant mean curvature-1 torus embedded in M then the mean convex side of T in M is isometric to a hyperbolic cusp.

### **2.** Minimal spheres of area $4\pi$ and rigidity of 3-manifolds

In this section, we prove a rigidity result for a Riemannian 3-manifold M whose sectional curvatures are between 0 and 1. As explained in the introduction, any minimal sphere in such a manifold has area at least  $4\pi$ .

We denote by  $\mathbb{S}_1^n$  the sphere of dimension *n* with constant sectional curvature 1. We then have the following result.

**Theorem 1.** Let M be a complete Riemannian 3-manifold whose sectional curvatures satisfy  $0 \le K \le 1$ . Assume that there exists an embedded minimal sphere  $\Sigma$  in Mwith area  $4\pi$ . Then the manifold M is isometric either to the sphere  $\mathbb{S}_1^3$  or to a quotient of  $\mathbb{S}_1^2 \times \mathbb{R}$ .

*Proof.* Let  $\Phi$  be the map  $\Sigma \times \mathbb{R} \to M$ ,  $(p,t) \mapsto \exp_p(tN(q))$  where N is a unit normal vector field along  $\Sigma$ . In the following we focus on  $\Sigma \times \mathbb{R}_+$ ; by symmetry of the configuration, the study is similar for  $\Sigma \times \mathbb{R}_-$ .

 $\Sigma$  is compact, so there is an  $\varepsilon$  such that  $\Phi$  is an immersion and even an embedding on  $\Sigma \times [0, \varepsilon)$ . Let us define

 $\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } \Sigma \times [0, \varepsilon)\};$ 

 $\varepsilon_0$  can be equal to  $+\infty$ . Using  $\Phi$ , we pull back the Riemannian metric of M to  $\Sigma \times [0, \varepsilon_0)$ . This metric can be written  $ds^2 = d\sigma_t^2 + dt^2$  where  $d\sigma_t^2$  is a smooth family of metrics on  $\Sigma$ . With this metric,  $\Phi$  becomes a local isometry from  $\Sigma \times [0, \varepsilon_0)$  to M and  $(\Sigma \times [0, \varepsilon_0), ds^2)$  has sectional curvatures between 0 and 1. Moreover,  $\Sigma_0$  is minimal and has area  $4\pi$ . Actually, we will prove the following facts.

**Claim.** The metric  $d\sigma_0^2$  has constant sectional curvature 1 so  $(\Sigma, d\sigma_0^2)$  is isometric to  $\mathbb{S}_1^2$ . Moreover, we have two cases:

- (1)  $\varepsilon_0 = \pi/2$  and  $d\sigma_t^2 = \sin^2 t d\sigma_0^2$ , or
- (2)  $\varepsilon_0 = +\infty$  and  $d\sigma_t^2 = d\sigma_0^2$ .

Let us denote by  $\Sigma_t = \Sigma \times \{t\}$  the equidistant surfaces. We denote by H(p, t) the mean curvature of  $\Sigma_t$  at the point (p, t) with respect to the unit normal vector  $\partial_t$ . We also define  $\lambda(p, t) \ge 0$  such that  $H + \lambda$  and  $H - \lambda$  are the principal curvature of  $\Sigma_t$  at (p, t). We notice that  $\lambda = 0$  if  $\Sigma_t$  is umbilical at (p, t).

The surfaces  $\Sigma_t$  are spheres, so, using the Gauss equation, the Gauss–Bonnet formula implies that

$$4\pi = \int_{\Sigma_t} \bar{K}_{\Sigma_t} = \int_{\Sigma_t} (H+\lambda)(H-\lambda) + K_t = \int_{\Sigma_t} H^2 - \lambda^2 + K_t$$

where  $\overline{K}_{\Sigma_t}$  is the intrinsic curvature of  $\Sigma_t$  and  $K_t$  is the sectional curvature of the ambient manifold of the tangent space to  $\Sigma_t$ . Since  $K_t \leq 1$ , we obtain the following inequality:

$$\int_{\Sigma_t} \lambda^2 = \int_{\Sigma_t} H^2 + K_t - 4\pi \le \int_{\Sigma_t} H^2 + A(\Sigma_t) - 4\pi$$
(2)

where  $A(\Sigma_t)$  is the area of  $\Sigma_t$ . In the following, we denote by F(t) the right-hand side of this inequality.

**Claim 1.** *F* is vanishing on  $[0, \varepsilon_0)$ .

Since  $\Sigma_0$  is minimal and has area  $4\pi$ , we have F(0) = 0. We notice that this implies that  $\lambda(p,0) = 0$ , so  $\Sigma_0$  is umbilical and  $K_{T\Sigma_0} = 1$ . Thus  $(\Sigma_0, d\sigma_0)$  is isometric to  $\mathbb{S}_1^2$ .

We have the usual formulae:

$$\frac{\partial}{\partial t}A(\Sigma_t) = -\int_{\Sigma_t} 2H$$
 and  $\frac{\partial H}{\partial t} = \frac{1}{2}(\operatorname{Ric}(\partial_t) + |A_t|^2)$  (3)

where  $A_t$  is the shape operator of  $\Sigma_t$  and Ric is the Ricci tensor of  $\Sigma \times [0, \varepsilon_0)$ . Since the sectional curvatures of  $M \times [0, \varepsilon_0)$  are non-negative, Ric is non-negative. So the

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second formula above implies that H is non-decreasing and thus  $H \ge 0$  everywhere. Let us now compute and estimate the derivative of F:

$$F'(t) = \int_{\Sigma_t} (2H\frac{\partial H}{\partial t} - 2H^3) - \int_{\Sigma_t} 2H$$
  
= 
$$\int_{\Sigma_t} H(\operatorname{Ric}(\partial_t) + |A_t|^2 - 2H^2 - 2)$$
  
= 
$$\int_{\Sigma_t} H((\operatorname{Ric}(\partial_t) - 2) + ((H + \lambda)^2 + (H - \lambda)^2 - 2H^2))$$
  
= 
$$\int_{\Sigma_t} H((\operatorname{Ric}(\partial_t) - 2) + 2\lambda^2)$$
  
$$\leq 2\int_{\Sigma_t} H\lambda^2$$

where the last inequality comes from  $\operatorname{Ric}(\partial_t) - 2 \leq 0$  because of the hypothesis on the sectional curvatures. If we choose  $\varepsilon < \varepsilon_0$ , there is a constant  $C \geq 0$  such that  $H \leq C$  on  $\Sigma \times [0, \varepsilon]$ . So for  $t \in [0, \varepsilon]$ , using the inequality (2), we get  $F'(t) \leq 2CF(t)$ . Then  $F(t) \leq F(0)e^{2Ct} = 0$  on  $[0, \varepsilon]$ . So  $F \leq 0$  on  $[0, \varepsilon_0)$  and, because of (2), F = 0 on  $[0, \varepsilon_0)$ ; this finishes the proof of Claim 1.

The first consequence of Claim 1 is that all the equidistant surfaces  $\Sigma_t$  are umbilical (see inequality (2)); so  $\lambda \equiv 0$ . In the computation of the derivative of *F*, this implies that

$$\int_{\Sigma_t} H(\operatorname{Ric}(\partial_t) - 2) = 0.$$

Since  $H(\operatorname{Ric}(\partial_t) - 2) \leq 0$  everywhere, we obtain

$$H(\operatorname{Ric}(\partial_t) - 2) = 0 \quad \text{everywhere.}$$
(4)

Moreover the umbilicity and (3) imply that  $\frac{\partial H}{\partial t} = \frac{1}{2}\text{Ric}(\partial_t) + H^2$ . We now prove the following claim.

**Claim 2.** Let  $(p,t) \in \Sigma \times [0, \varepsilon_0)$  (t > 0) be such that H(p,t) > 0 then H(q,t) > 0 for any  $q \in \Sigma$ 

In other words, when the mean curvature is positive at a point of an equidistant, it is positive at any point of this equidistant. We recall that H is increasing in the t variable, so when it becomes positive it stays positive.

So assume that H(p,t) > 0 and consider  $\Omega = \{q \in \Sigma \mid H(q,t) > 0\}$  which is a nonempty open subset of  $\Sigma$ . Let  $q \in \Omega$ . Since H(q,t) > 0,  $\operatorname{Ric}(\partial_t)(q,t) = 2$  by (4). Thus  $\operatorname{Ric}(\partial_t)(r,t) = 2$  for any  $r \in \overline{\Omega}$ . So if  $r \in \overline{\Omega}$ , then, for s < t,  $\operatorname{Ric}(\partial_t)(r,s) > 0$ for s close to t and, by (3), this implies that H(r,t) > 0 and  $r \in \Omega$ . So  $\Omega$  is closed and  $\Omega = \Sigma$ . This finishes the proof of Claim 2. Vol. 89 (2014)

Let us assume that there is an  $\varepsilon_1 > 0$  such that H(p, t) = 0 for  $(p, t) \in \Sigma \times [0, \varepsilon_1]$ and H(p, t) > 0 for any  $(p, t) \in \Sigma \times (\varepsilon_1, \varepsilon_0)$ . Because of the evolution equation of H, this implies that  $\operatorname{Ric}(\partial_t) = 0$  on  $\Sigma \times [0, \varepsilon_1]$ . On  $\Sigma \times (\varepsilon_1, \varepsilon_0)$ , we have  $\operatorname{Ric}(\partial_t) = 2$ because of (4). So by continuity of  $\operatorname{Ric}(\partial_t)$ , we get a contradiction and then we have two possibilities:

(1) H = 0 on  $\Sigma \times [0, \varepsilon_0)$  and  $\operatorname{Ric}(\partial_t) = 0$  on  $\Sigma \times [0, \varepsilon_0)$ ;

(2) H > 0 on  $\Sigma \times (0, \varepsilon_0)$  and  $\operatorname{Ric}(\partial_t) = 2$  on  $\Sigma \times [0, \varepsilon_0)$ .

In the first case, this implies that the sectional curvature of any 2-plane orthogonal to  $\Sigma_t$  is zero. Thus  $d\sigma_t^2 = d\sigma_0^2$ . Since the map  $\Phi$  ceases to be an immersion only if  $d\sigma_t^2$  becomes singular this implies that  $\varepsilon_0 = +\infty$ . Thus  $\Sigma \times \mathbb{R}_+$  with the induced metric is isometric to  $\mathbb{S}_1^2 \times \mathbb{R}_+$  and  $\Phi$  is a local isometry from  $\mathbb{S}_1^2 \times \mathbb{R}_+$  to M.

In the second case, the sectional curvature of any 2-plane orthogonal to  $\Sigma_t$  is equal to 1. The sectional curvature of  $\Sigma_t$  is also 1, since the inequality in (2) is an equality by Claim 1. Thus  $d\sigma_t^2 = \sin^2 t \, d\sigma_0$  and  $\varepsilon_0 = \pi/2$ . This also implies that  $\Phi(p, \pi/2)$  is a point. So  $\Sigma \times [0, \pi/2]$  with the metric  $ds^2$  is isometric to a hemisphere of  $\mathbb{S}_1^3$  and the map  $\Phi$  is a local isometry from that hemisphere to M.

Doing the same study for  $\Sigma \times \mathbb{R}_{-}$ , we get in the first case a local isometry  $\Phi: \mathbb{S}_{1}^{2} \times \mathbb{R} \to M$  and in the second case a local isometry  $\Phi: \mathbb{S}_{1}^{3} \to M$ . Since  $\mathbb{S}_{1}^{2} \times \mathbb{R}$  and  $\mathbb{S}_{1}^{3}$  are simply connected,  $\Phi$  is then the universal cover of M and M is then isometric to a quotient of  $\mathbb{S}_{1}^{2} \times \mathbb{R}$  or  $\mathbb{S}_{1}^{3}$ . Since  $\Phi$  is injective on  $\Sigma$  this implies that in the second case,  $\Phi$  is actually injective and then a global isometry.  $\Box$ 

**Remark 1.** In the proof, since  $\Phi$  is injective on  $\Sigma$ , the possible quotients of  $\mathbb{S}_1^2 \times \mathbb{R}$  are either  $\mathbb{S}_1^2 \times \mathbb{R}$  or its quotient by the subgroup generated by an isometry of the form  $\mathbb{S}_1^2 \times \mathbb{R} \to \mathbb{S}_1^2 \times \mathbb{R}$ ,  $(p, t) \mapsto (\alpha(p), t + t_0)$  with  $\alpha$  an isometry of  $\mathbb{S}_1^2$  and  $t_0 \neq 0$ .

**Remark 2.** Something can be said about constant mean curvature  $H_0$  spheres in a Riemannian 3-manifold with sectional curvatures between 0 and 1. Indeed, the computation (1) implies that the area of  $\Sigma$  is larger than  $\frac{4\pi}{1+H_0^2}$ , which is the area of a geodesic sphere in  $\mathbb{S}_1^3$  of mean curvature  $H_0$ . Moreover, if  $\Sigma$  has area  $\frac{4\pi}{1+H^2}$ , the above proof can be adapted to prove that the mean convex side of  $\Sigma$  is isometric to a spherical cap of  $\mathbb{S}_1^3$  with constant mean curvature  $H_0$  (see Theorem 2 below, for a similar result in the hyperbolic case).

**Remark 3.** Let M be a Riemannian n-manifold whose sectional curvatures are between 0 and 1 and let  $\Sigma$  be a minimal 2-sphere in M. A computation similar to (1) proves also that the area of  $\Sigma$  is larger than  $4\pi$ . It also implies that, if  $\Sigma$  has area  $4\pi$ ,  $\Sigma$  is totally geodesic and isometric to  $\mathbb{S}_1^2$ .

## 3. Existence of hyperbolic cusps

Let  $(\mathbb{T}^2, g)$  be a flat 2 torus, the manifold  $\mathbb{T}^2 \times \mathbb{R}_+$  with the complete Riemannian metric  $e^{-2t}g + dt^2$  is a hyperbolic 3-dimensional cusp.  $\mathbb{T}^2 \times \mathbb{R}$  is actually isometric to the quotient of a horoball of  $\mathbb{H}^3$  by a  $\mathbb{Z}^2$  subgroup of isometries of  $\mathbb{H}^2$  leaving the horoball invariant. Any  $\mathbb{T}^2 \times \{t\}$  has constant mean curvature 1. The following theorem says that, in certain 3-manifolds, a constant mean curvature 1 torus is necessarily the boundary of a hyperbolic cusp.

**Theorem 2.** Let M be a complete Riemannian 3-manifold with its sectional curvatures satisfying  $K \leq -1$ . Assume that there exists a constant mean curvature 1 torus T embedded in M. Then T separates M and its mean convex side is isometric to a hyperbolic cusp.

As a consequence, the existence of this torus implies that M can not be compact. The proof uses the same ideas as in Theorem 1

*Proof.* Let us consider the map  $\Phi: T \times \mathbb{R}_+ \to M$ ,  $(p, t) \mapsto \exp_p(tN(p))$  where N is the unit normal vector field normal to T such that N is the mean curvature vector of T. Let us define

 $\varepsilon_0 = \sup \{ \varepsilon > 0 \mid \Phi \text{ is an immersion on } T \times [0, \varepsilon) \}.$ 

Using  $\Phi$ , we pull back the Riemannian metric of M to  $T \times [0, \varepsilon_0)$ ; it can be written  $ds^2 = dt^2 + d\sigma_t^2$ . We define  $T_t = T \times \{t\}$  the equidistant surfaces to  $T_0$ . We also denote by H(p, t) the mean curvature of the equidistant surfaces at (p, t) with respect to  $\partial_t$ . We finally define  $\lambda(p, t)$  such that  $H + \lambda$  and  $H - \lambda$  are the principal curvatures of  $T_t$  at (p, t).

The surfaces  $T_t$  are tori so, by the Gauss equation and the Gauss–Bonnet formula, we have

$$0 = \int_{T_t} \overline{K}_{T_t} = \int_{T_t} H^2 - \lambda^2 + K_t$$

where  $K_t$  is the sectional curvature of the ambient manifold of the tangent space to  $T_t$ . Since  $K_t \leq -1$ , we obtain the inequality

$$\int_{T_t} \lambda^2 = \int_{T_t} H^2 + K_t \leq \int_{T_t} H^2 - A(T_t).$$

We denote by F(t) the right-hand term of the above inequality. By hypothesis, H(p, 0) = 1 so F(0) = 0 and  $F(t) \ge 0$  for any  $t \ge 0$ . Let us compute the derivative of F:

$$F'(t) = \int_{T_t} \left( 2H \frac{\partial H}{\partial t} - 2H^3 \right) + \int_{T_t} 2H$$
  
= 
$$\int_{T_t} H\left( \operatorname{Ric}(\partial_t) + |A_t|^2 - 2H^2 + 2 \right) = \int_{T_t} H\left( (\operatorname{Ric}(\partial_t) + 2) + 2\lambda^2 \right).$$

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Since H(p,0) = 1, we can consider  $\varepsilon \in (0, \varepsilon_0)$  such that  $0 < H \leq C$  on  $T \times [0, \varepsilon]$ . Since  $\operatorname{Ric}(\partial_t) + 2 \leq 0$  we get

$$F'(t) \leq \int_{T_t} 2H\lambda^2 \leq 2CF(t).$$

Thus  $F(t) \leq F(0)e^{2Ct}$  for  $t \in [0, \varepsilon]$ ; this implies F(t) = 0 on that segment. We then obtain  $\lambda = 0$  on  $T \times [0, \varepsilon]$  (the equidistant surfaces are umbilical) and  $\operatorname{Ric}(\partial_t) = -2$ since H > 0. Thus H satisfies the differential equation  $\frac{\partial H}{\partial t} = -2 + 2H^2$ . This gives that H = 1 on  $T \times [0, \varepsilon]$  since H = 1 on  $T_0$ . Thus we can let  $\varepsilon$  tend to  $\varepsilon_0$  to obtain that F(t) = 0 on  $[0, \varepsilon_0)$  and  $\operatorname{Ric}(\partial_t) = -2$  and H = 1 on  $T \times [0, \varepsilon_0)$ . Since  $0 = \int_{T_t} H^2 + K_t$  and  $K_t \leq -1$ , it follows that  $K_t = -1$  for all t in the interval. We then have proved that the sectional curvature of  $T \times [0, \varepsilon_0)$  with the metric  $ds^2$  is equal to -1 for any 2-plane. Moreover, we get that  $d\sigma_0^2$  is flat and that  $d\sigma_t^2 = e^{-2t} d\sigma_0^2$ . This implies that  $\Phi$  is actually an immersion on  $T \times \mathbb{R}_+$  ( $\varepsilon_0 = +\infty$ ) and  $T \times \mathbb{R}_+$  is isometric to a hyperbolic cusp.  $\Phi$  is then a local isometry from this hyperbolic cusp to M.

To finish the proof, let us prove that  $\Phi$  is in fact injective. If this is not the case, let  $\varepsilon_1 > 0$  be the smallest  $\varepsilon$  such that  $\Phi$  is not injective on  $T \times [0, \varepsilon]$ . This implies that there exist p and q in T such that either

- $\Phi(p,0) = \Phi(q,\varepsilon_1)$ , or
- $\Phi(p,\varepsilon_1) = \Phi(q,\varepsilon_1)$  (with  $p \neq q$  in this case).

Let U and V be respective neighborhoods of (p, 0) (or  $(p, \varepsilon_1)$ ) in  $T_0$  (or  $T_{\varepsilon_1}$ ) and  $(q, \varepsilon_1)$  in  $T_{\varepsilon_1}$  such that  $\Phi$  is injective on them. Since  $\varepsilon_1$  is the smallest one,  $\Phi(U)$  and  $\Phi(V)$  are two constant mean curvature 1 surfaces in M that are tangent at  $\Phi(q, \varepsilon_1)$ . Moreover, in the first case,  $\Phi(U)$  is included in the mean convex side of  $\Phi(V)$  so by the maximum principle  $\Phi(U) = \Phi(V)$ . Thus  $\Phi(T_0)$  would be equal to  $\Phi(T_{\varepsilon_1})$  which is impossible since these two surfaces do not have the same area. In the second case,  $\Phi(U)$  is included in the mean convex side of  $\Phi(V)$  and then  $\Phi$  is not injective on  $T_s$  for s near t, s < t, which is a contradiction.

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