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Positively curved Riemannian metrics with logarithmic symmetry rank bounds

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Abstract. We prove an obstruction at the level of rational cohomology to the existence of positively curved metrics with large symmetry rank. The symmetry rank bound is logarithmic in the dimension of the manifold. As one application, we provide evidence for a generalized conjecture of H. Hopf, which states that no symmetric space of rank at least two admits a metric with positive curvature. Other applications concern product manifolds, connected sums, and manifolds with nontrivial fundamental group.

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A well-known conjecture of Hopf states that $\mathbb{S}^2 \times \mathbb{S}^2$ admits no metric of positive sectional curvature. More generally, one might ask whether any nontrivial product of compact manifolds admits a metric with positive sectional curvature.

Another way to generalize this conjecture is to observe that $\mathbb{S}^2 \times \mathbb{S}^2$ is a compact, rank two symmetric space. While the compact, one-connected rank one symmetric spaces, i.e., $\mathbb{S}^n$, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, and $\mathrm{Ca}\mathbb{P}^2$, admit metrics with positive sectional curvature, it is conjectured that no symmetric space of rank greater than one admits such a metric (see, for example, Ziller [29]).

Since so little was known about these questions, K. Grove proposed a research program in which attention is restricted to metrics with large symmetry. Beginning with Hsiang and Kleiner [17] and continuing with Grove–Searle [14], Rong [22], Fang and Rong [9], and Wilking [26], much has been achieved under the additional assumption of symmetry (see also Wilking [27] and Grove [13] for surveys).

Our first result provides evidence for the generalized conjecture of Hopf under the assumption of symmetry:

Theorem A. *Suppose M^n has the rational cohomology of a one-connected, compact symmetric space N. If M admits a positively curved Riemannian metric with symmetry rank at least $2\log_2(n) + 7$, then N is a product of spheres times either a rank one symmetric space or a rank p Grassmannian $\mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ with $p \in \{2, 3\}$.*

Recall that the symmetry rank is defined as the rank of the isometry group. The assumption that the symmetry rank is at least r is equivalent to the existence of an effective, isometric T^r-action on M.

If we restrict to the case where N is an irreducible symmetric space, then product manifolds are excluded and N has rank at most three. See Theorem 3.3 for a more detailed statement. For example, N cannot be $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^{n-4}$ or $\mathbb{S}^k \times \mathbb{S}^{n-k}$ with $1 < k < 16$.

The obstruction (see Theorem C) we prove in order to obtain Theorem A is at the level of rational cohomology in small degrees. Since taking products with spheres does not affect cohomology in small degrees, and since the Grassmannians $\mathrm{SO}(2+q)/\mathrm{SO}(2) \times \mathrm{SO}(q)$ and $\mathrm{SO}(3+q)/\mathrm{SO}(3) \times \mathrm{SO}(q)$ have the same rational cohomology ring in small degrees as the complex and quaternionic projective spaces, respectively, our methods cannot exclude them.

The second main result is related to the Bott conjecture and a second conjecture of Hopf. Recall that the Bott conjecture states that a nonnegatively curved manifold is rationally elliptic, which, in particular, implies that the Euler characteristic is positive if and only if the odd Betti numbers vanish (see Chapter 32 of Felix, Halperin, and Thomas [10]). The conjecture of Hopf states that the Euler characteristic of an even-dimensional, positively curved manifold is positive. Hence the conjectures together would imply that even-dimensional, positively curved manifolds have vanishing odd Betti numbers. The second part of the following theorem provides some evidence for this statement:

Theorem B. *Let $n \geq c \geq 2$, and let M^n be a connected, closed, positively curved Riemannian manifold with symmetry rank at least $2\log_2(n) + \frac{c}{2} - 1$. The following hold:*

- *The Betti numbers $b_{2i}(M)$ for $2i < c$ agree with those of $\mathbb{S}^n$, $\mathbb{CP}^{\frac{n}{2}}$, or $\mathbb{HP}^{\frac{n}{4}}$.*
- *If $n \equiv 0 \bmod 4$, then $b_{2i+1}(M) = 0$ for $2i + 1 < c$.*

In order to explain our main topological result, which is the crucial step in proving Theorems A and B, we make the following definition:

Definition. For a closed, one-connected manifold M, we say that $H^*(M;\mathbb{Q})$ is 4-periodic up to degree c if there exists $x \in H^4(M;\mathbb{Q})$ such that the map $H^i(M;\mathbb{Q}) \to H^{i+4}(M;\mathbb{Q})$ given by $y \mapsto xy$ is a surjection for $0 \leq i < c-4$ and an injection for $0 < i \leq c-4$. If $H^*(M;\mathbb{Q})$ is 4-periodic up to degree $\dim(M)$, we simply say that $H^*(M;\mathbb{Q})$ is 4-periodic.

In particular, if $x \neq 0$ in the definition, then $H^{4s}(M;\mathbb{Q}) \cong \mathbb{Q}$ for $0 \leq s < \frac{c}{4}$. However we abuse notation slightly by allowing $x = 0$, hence we say that a rationally $(c-1)$-connected space has 4-periodic rational cohomology up to degree c.

Examples of n-manifolds with 4-periodic rational cohomology are $\mathbb{S}^n$, $\mathbb{CP}^{\frac{n}{2}}$, $\mathbb{HP}^{\frac{n}{4}}$, $\mathbb{S}^2 \times \mathbb{HP}^{\frac{n-2}{4}}$, and $\mathbb{S}^3 \times \mathbb{HP}^{\frac{n-3}{4}}$. By taking a product of one of these spaces with any rationally $(c-1)$-connected space, we obtain examples of spaces with 4-periodic rational cohomology up to degree c.

We can now state the main cohomological obstruction to the existence of positively curved metrics with large symmetry rank.

Theorem C. *Let $n \geq c \geq 2$. If M^n is a closed, one-connected Riemannian manifold with positive sectional curvature and symmetry rank at least $2\log_2(n) + \frac{c}{2} - 1$, then $H^*(M;\mathbb{Q})$ is 4-periodic up to degree c.*

We remark that, if $c > \frac{n}{2}$, the conclusion together with Poincaré duality implies that M has 4-periodic rational cohomology. On the other hand, $c > \frac{n}{2}$ already implies that M is homotopy equivalent to $\mathbb{S}^n$, $\mathbb{CP}^{\frac{n}{2}}$, or $\mathbb{HP}^{\frac{n}{4}}$ (see Theorem 2 in Wilking [26]). Similarly, $c > \frac{n}{3}$ and $n \geq 6000$ already implies that $H^*(M;k)$ is 4-periodic for any coefficient field k (see Theorem 5 in [26]). In our applications, we will think of c as a fixed constant, which is small relative to n.

For example, taking $c = 16$ and restricting to the situation where M has the rational cohomology of a compact symmetric space, we obtain Theorem A by comparing this obstruction with the classification of symmetric spaces. See Section 3 for details.

Another, more immediate consequence of Theorem C follows by taking $c = 6$ and concluding that the fourth Betti number of M is at most 1:

Corollary. *No nontrivial connected sum with summands $\mathbb{CP}^n$ and $\mathbb{HP}^{\frac{n}{2}}$ admits a positively curved metric with symmetry rank at least $2\log_2(4n)$.*

On the other hand, the manifolds $\mathbb{CP}^n \# \mathbb{CP}^n$, $\mathbb{CP}^n \# \mathbb{HP}^{\frac{n}{2}}$, and $\mathbb{HP}^{\frac{n}{2}} \# \mathbb{HP}^{\frac{n}{2}}$ admit metrics, called Cheeger metrics, with nonnegative curvature (see Cheeger [6]).

A final corollary, which we prove in the discussion following Theorem 2.2, relates to a conjecture of Chern. The conjecture is that, for a positively curved manifold, every abelian subgroup of the fundamental group is cyclic. While this holds for spherical space forms (see Wolf [28]) and even-dimensional manifolds by a classical theorem of Synge, there are counterexamples in general (see Shankar [24], Bazaikin [2], and Grove–Shankar [15]). However modified versions of the Chern conjecture have been verified under the additional assumption that the symmetry rank is at least a linear function of the dimension (see, for example, Wilking [26], Frank–Rong–Wang [11], Wang [25], and Rong–Wang [23]).

Corollary. *If M^{4n+1} is a connected, closed manifold with positive curvature and symmetry rank at least $2\log_2(4n+1)$, then $\pi_1(M)$ acts freely and isometrically on some positively curved rational homology $(4k+1)$-sphere. As a consequence, $\pi_1(M) \cong \pi' \times \pi''$ where π' is cyclic with order a power of two and π'' has odd order.*

In the proofs of these results, we use many ideas from [26], [19], including Wilking's connectedness theorem in [26] and the periodicity theorem in [19]. These theorems place restrictions on the cohomology of a closed, positively curved manifold in the presence of totally geodesic submanifolds of small codimension (see Section 1). Since fixed-point sets of isometries are totally geodesic, these become powerful tools in the presence of symmetry. The connection made in [26] to the theory of error-correcting codes also plays a role. Here we will use the Griesmer bound, which is well suited to the logarithmic symmetry rank bound with which we are working.

We actually obtain a stronger version of Theorem C, namely Theorem 2.2. It states that, given the assumptions of Theorem C, there exists a c-connected inclusion $P \subseteq M$ of a compact submanifold P such that $H^*(P;\mathbb{Q})$ is 4-periodic. Moreover, one can ensure that $\dim P \equiv \dim M \bmod 4$ and $\dim P \geq c+4 \geq 6$. The advantage of this statement is that one can apply Poincaré duality to conclude the following about P:

- the subring of $H^*(P;\mathbb{Q})$ made up of elements of even degree is isomorphic to that of $\mathbb{S}^n$, $\mathbb{CP}^{\frac{n}{2}}$, $\mathbb{HP}^{\frac{n}{4}}$, or $\mathbb{S}^2 \times \mathbb{HP}^{\frac{n-2}{4}}$, and
- if $\dim M \equiv 0 \bmod 4$, then P has vanishing odd-dimensional cohomology.

For $i < c$, the map $H^i(M;\mathbb{Q}) \to H^i(P;\mathbb{Q})$ induced by inclusion is an isomorphism, so one can use these observations to conclude Theorem B.

This paper is organized as follows. In Section 1, we quote preliminary results and prove a lemma using the Griesmer bound. In Section 2, we prove Theorems 2.2 and C. In Section 3, we study the topological obstructions imposed by Theorem C and prove Theorem A.

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1. Preliminaries and the Griesmer bound

An important result for this work is Wilking's connectedness theorem:

Theorem 1.1 (Connectedness Theorem, [26]). *Suppose M^n is a closed Riemannian manifold with positive sectional curvature.*

(1) *If N^{n-k} is a closed, embedded, totally geodesic submanifold of M, then $N \hookrightarrow M$ is $(n-2k+1)$-connected.*

(2) *If $N_1^{n-k_1}$ and $N_2^{n-k_2}$ are closed, embedded, totally geodesic submanifolds of M with $k_1 \leq k_2$, then $N_1 \cap N_2 \hookrightarrow N_2$ is $(n-k_1-k_2)$-connected.*

Recall an inclusion $N \hookrightarrow M$ is called c-connected if $\pi_i(M, N) = 0$ for all $i \leq c$. It follows from the relative Hurewicz theorem that the induced map $H_i(N; \mathbb{Z}) \to H_i(M; \mathbb{Z})$ is an isomorphism for $i < c$ and a surjection for $i = c$. The following is a topological consequence of highly connected inclusions of closed, orientable manifolds:

Theorem 1.2 ([26]). *Let M^n and N^{n-k} be connected, closed, orientable manifolds. If $N \hookrightarrow M$ is $(n-k-l)$-connected with $n-k-2l > 0$, then there exists $e \in H^k(M; \mathbb{Z})$ such that the maps $H^i(M; \mathbb{Z}) \to H^{i+k}(M; \mathbb{Z})$ given by $x \mapsto ex$ are surjective for $l \leq i < n-k-l$ and injective for $l < i \leq n-k-l$.*

In particular, in the case where $l = 0$, the integral cohomology of M is k-periodic according to the following definition:

Definition 1.3. For a space M, a coefficient ring R, and a positive integer c, we say that $H^*(M; R)$ is k-periodic up to degree c if M is connected and there exists $x \in H^k(M; R)$ such that the map $H^i(M; R) \to H^{i+k}(M; R)$ given by $y \mapsto xy$ is a surjection for $0 \leq i < c-k$ and an injection for $0 < i \leq c-k$.

If, in addition, M is a c-dimensional, closed, R-orientable manifold, we say that $H^*(M; R)$ is k-periodic.

In [19], the action of the Steenrod algebra was exploited to prove the following:

Theorem 1.4 (Periodicity Theorem, [19]). *Let M^n be a closed, one-connected Riemannian manifold with positive sectional curvature. Let $N_1^{n-k_1}$ and $N_2^{n-k_2}$ be connected, closed, embedded, totally geodesic submanifolds that intersect transversely.*

(1) *If $2k_1 + 2k_2 \leq n$, the rational cohomology rings of M, N_1, N_2, and $N_1 \cap N_2$ are 4-periodic.*

(2) *If $3k_1 + k_2 \leq n$ and N_2 is simply connected, the rational cohomology rings of N_2 and $N_1 \cap N_2$ are 4-periodic.*

Next we record two additional results concerning torus actions on positively curved manifolds:

Theorem 1.5 (Berger, [3], [14]). *Suppose T is a torus acting by isometries on a closed, positively curved manifold M^n. If n is even, then the fixed-point set M^T is nonempty, and if n is odd, then a codimension one subtorus has nonempty fixed-point set.*

Theorem 1.6 (Maximal symmetry rank, [14]). *If T^r is a torus acting effectively by isometries on a closed, positively curved manifold M^n, then $r \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. Moreover, if equality holds and M is one-connected, M is diffeomorphic to $\mathbb{S}^n$ or $\mathbb{CP}^{\frac{n}{2}}$.*

A corollary of Theorem 1.6 is the following, which we will use in the proof of Theorem 2.2:

Corollary 1.7. *Let M^n be closed, positively curved manifold with symmetry rank r such that*

$$r \geq 2\log_2(n) + \frac{c}{2} - 1 - \delta(n)$$

where $\delta(n)$ is 0 if n is even and 1 if n is odd. If $n \geq c \geq 2$, then, in fact, $n \geq c + 8$.

Proof of Corollary 1.7. The assumption $n \geq c \geq 2$ implies $r \geq 2$. By Theorem 1.6, we have $n \geq 3$. But now the bound on r implies $r \geq 3$, so Theorem 1.6 implies $n \geq 5$. Repeating this argument twice more, we conclude $n \geq 9$, which implies

$$\frac{n+1}{2} \geq r \geq 2\log_2(9) + \frac{c}{2} - 2 > \frac{c+8}{2}.$$

It follows that $n \geq c + 8$, as claimed. □

Finally, we use the Griesmer bound from the theory of error-correcting codes to prove the following proposition. The estimates are specifically catered to our application. The proof indicates the general bounds required.

Lemma 1.8. *Let $n \geq c \geq 2$. Assume T is a torus that acts effectively by isometries on a positively curved manifold M^n with fixed point x. Let $\delta(n) = 0$ if n is even and $\delta(n) = 1$ if n is odd.*

(1) *If*

$$\dim T \geq 2\log_2 n + \frac{c}{2} - 1 - \delta(n),$$

there exists an involution $\sigma \in T$ such that the component M_x^σ of the fixed-point set of σ that contains x satisfies $\mathrm{cod}(M_x^\sigma) \equiv 0 \bmod 4$ *and* $0 < \mathrm{cod}(M_x^\sigma) \leq \frac{n-c}{2}$.

(2) *Let σ be an involution as above such that M_x^σ has minimal codimension. If*

$$\dim T \geq \log_2 n + \frac{c}{2} + 1 + \log_2(3) - \delta(n),$$

there exists an involution $\tau \in T$ satisfying $M_x^\tau \not\subseteq M_x^\sigma$, $\mathrm{cod}(M_x^\tau) \equiv 0 \bmod 4$, $\mathrm{cod}(M_x^\sigma \cap M_x^\tau) \equiv 0 \bmod 4$, *and* $0 < \mathrm{cod}(M_x^\tau) \leq \frac{n-c}{2}$.

By the connectedness theorem, the inclusions $M_x^\sigma \hookrightarrow M$, $M_x^\sigma \cap M_x^\tau \hookrightarrow M_x^\tau$, and $M_x^\tau \hookrightarrow M$ are c-connected. Since $c \geq 2$, this implies that all three submanifolds are one-connected if M is. In particular, $M_x^\sigma \cap M_x^\tau = M_x^{\langle\sigma,\tau\rangle}$ where $\langle\sigma,\tau\rangle$ is the subgroup generated by σ and τ.

The only part in the proof where we use positive curvature is to conclude that $n \geq 10$ in the first statement by Corollary 1.7. Given this, the bound on $\dim T$ implies

$$\dim T > \log_2 \left\lceil \frac{n-c+1}{2} \right\rceil + \left\lceil \frac{c}{2} \right\rceil + 1.$$

Similarly, the bound in the second statement together with a proof like that of Corollary 1.7 implies

$$\dim T > \log_2 \left\lceil \frac{n-c+1}{2} \right\rceil + \left\lceil \frac{c}{2} \right\rceil + 2.$$

We proceed to the proof. The first step is to establish the following inequality:

Lemma 1.9. *If $n \geq c \geq 2$ and $r > \lceil \frac{c}{2} \rceil + \log_2 \lceil \frac{n-c+1}{2} \rceil$, then*

$$\left\lfloor \frac{n}{2} \right\rfloor < \sum_{i=0}^{r-1} \left\lceil 2^{-i-1} \left\lceil \frac{n-c+1}{2} \right\rceil \right\rceil.$$

Proof of Lemma 1.9. We proceed by contradiction. Suppose the opposite inequality holds. The bounds on n, c, and r imply that $r \geq \lceil \frac{c}{2} \rceil + 1$ and that $\lceil \frac{c}{2} \rceil \geq 1$, hence we may split the sum into two pieces and estimate as follows:

$$\left\lfloor \frac{n}{2} \right\rfloor \geq \sum_{i=0}^{r-\lceil \frac{c}{2} \rceil - 1} 2^{-i-1} \left\lceil \frac{n-c+1}{2} \right\rceil + \sum_{i=r-\lceil \frac{c}{2} \rceil}^{r-1} 1.$$

Calculating the geometric sum and rearranging, we obtain

$$\left\lceil \frac{n-c+1}{2} \right\rceil \geq 2^{r-\lceil \frac{c}{2} \rceil} \left(\left\lceil \frac{n-c+1}{2} \right\rceil + \left\lceil \frac{c}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor \right).$$

Observing that the integers n, c, and $n-c+1$ cannot all be even, we conclude that the term in parentheses is at least 1, hence taking logarithms yields a contradiction to the assumed bound on r. $\square$

We proceed to the proof of Proposition 1.8.

Proof. Set $s = \dim T$. Choose a basis of $T_x M$ such that the image of $\mathbb{Z}_2^s \subseteq T$ under the isotropy representation $T \to \mathrm{SO}(T_x M)$ lies in a copy of $\mathbb{Z}_2^m \subseteq T^m \subseteq \mathrm{SO}(T_x M)$ where $m = \lfloor \frac{n}{2} \rfloor$. Observe that we are identifying the 2×2 matrix blocks $\pm I_2$ with $\pm 1 \in \mathbb{Z}_2$. Denote the map $\mathbb{Z}_2^s \to \mathbb{Z}_2^m$ by ι, and observe that ι is injective since the action of T is effective.

Consider the first statement. The bound on s and the assumption $n \geq c \geq 2$ imply $s \geq 2$. Consider the map $\mathbb{Z}_2^s \to \mathbb{Z}_2$ that sends $\sigma \in \mathbb{Z}_2^s$ to the Hamming weight of $\iota(\sigma)$, reduced modulo 2. The Hamming weight of $\iota(\sigma)$ is the number of nontrivial entries of $\iota(\sigma) \in \mathbb{Z}_2^m$, hence the Hamming weight of $\iota(\sigma)$ is equal to half of the codimension of M_x^σ. Since the map $\mathbb{Z}_2^s \to \mathbb{Z}_2$ is a homomorphism, we conclude that there exists $\mathbb{Z}_2^{s-1} \subseteq \mathbb{Z}_2^s$ such that every $\sigma \in \mathbb{Z}_2^{s-1}$ has the property that $\iota(\sigma)$ has even Hamming weight, which is to say that $\mathrm{cod}(M_x^\sigma) \equiv 0 \bmod 4$.

It therefore suffices to prove the existence of $\sigma \in \mathbb{Z}_2^{s-1} \setminus \{\mathrm{id}\}$ with $\mathrm{cod}(M_x^\sigma) \leq \frac{n-c}{2}$. Suppose that no such σ exists. Then every $\sigma \in \mathbb{Z}_2^{s-1} \setminus \{\mathrm{id}\}$ has $\mathrm{cod}(M_x^\sigma) \geq \left\lceil \frac{n-c+1}{2} \right\rceil$. Equivalently, the Hamming weight of the image of every $\sigma \in \mathbb{Z}_2^{s-1} \setminus \{\mathrm{id}\}$ is at least $\frac{1}{2} \left\lceil \frac{n-c+1}{2} \right\rceil$. We now apply the Griesmer bound from the theory of error-correcting codes:

Theorem (Griesmer bound, [12]). *If $\mathbb{Z}_2^r \to \mathbb{Z}_2^m$ is a homomorphism such that every nontrivial element in the image has Hamming weight at least w, then*

$$m \geq \sum_{i=0}^{r-1} \left\lceil \frac{w}{2^i} \right\rceil.$$

This bound implies

$$\left\lfloor \frac{n}{2} \right\rfloor \geq \sum_{i=0}^{(s-1)-1} \left\lceil 2^{-i-1} \left\lceil \frac{n-c+1}{2} \right\rceil \right\rceil.$$

By the comments following the statement of Proposition 1.8, we have

$$s - 1 > \log_2 \left\lceil \frac{n-c+1}{2} \right\rceil + \left\lceil \frac{c}{2} \right\rceil,$$

hence we have a contradiction to Lemma 1.9, as desired.

We now prove the second statement of Proposition 1.8. First, observe that the lower bound on s implies $s \geq 4$. Let $\sigma \in \mathbb{Z}_2^s$ be as in the statement. By reordering the basis of $T_x M$, if necessary, we may assume that all of the nontrivial entries of $\iota(\sigma)$ come before the trivial entries. Let w be the Hamming weight of $\iota(\sigma)$, so that $\iota(\sigma) \in \mathbb{Z}_2^m$ takes the form

$$\iota(\sigma) = (-1, -1, \dots, -1, 1, 1, \dots, 1)$$

where w is the number of (-1)s.

We define three linear maps $\mathbb{Z}_2^s \to \mathbb{Z}_2$. For the first, assign $\tau \in \mathbb{Z}_2^s$ to the Hamming weight of $\iota(\tau)$, reduced modulo 2. Equivalently, the first map assigns τ to the product of the entries in $\iota(\tau)$. For the second, assign τ to the product of the last $(m - w)$ entries of $\iota(\tau)$. For the third map, assign $\tau \in \mathbb{Z}_2^s$ to the first component of $\iota(\tau)$.

The intersection of the kernels of these three maps contains a $\mathbb{Z}_2^{s-3}$. Let $\tau \in \mathbb{Z}_2^{s-3}$. By the definition of the first two maps, $\mathrm{cod}(M_x^\tau) \equiv 0 \bmod 4$ and $\mathrm{cod}(M_x^\sigma \cap M_x^\tau) \equiv$

0 mod 4. By the definition of the third map, $M_x^\tau \not\subseteq M_x^\sigma$. It therefore suffices to prove that some $\tau \in \mathbb{Z}_2^{s-3} \setminus \{\mathrm{id}\}$ has $\mathrm{cod}(M_x^\tau) \leq \frac{n-c}{2}$.

Consider the composition

$$\mathbb{Z}_2^{s-3} \subseteq \mathbb{Z}_2^s \to \mathbb{Z}_2^m \to \mathbb{Z}_2^{m-1}$$

where the last map is the projection onto the last $(m-1)$ components. By our choice of $\mathbb{Z}_2^{s-3}$, this composition is injective and the Hamming weights of the images of $\tau \in \mathbb{Z}_2^{s-3}$ in $\mathbb{Z}_2^m$ and $\mathbb{Z}_2^{m-1}$ are the same. Hence if every $\tau \in \mathbb{Z}_2^{s-3} \setminus \{\mathrm{id}\}$ has $\mathrm{cod}(M_x^\tau) \geq \frac{n-c+1}{2}$, then there exists a homomorphism $\mathbb{Z}_2^{s-3} \to \mathbb{Z}_2^{m-1}$ such that the image of every nontrivial $\tau \in \mathbb{Z}_2^{s-3}$ has Hamming weight at least $\frac{1}{2}\left\lceil \frac{n-c+1}{2} \right\rceil$. Applying the Griesmer bound, we conclude

$$\left\lfloor \frac{n}{2} \right\rfloor - 1 \geq \sum_{i=0}^{(s-3)-1} \left\lceil 2^{-i-1} \left\lceil \frac{n-c+1}{2} \right\rceil \right\rceil.$$

Now the bound on s implies that the $i = s-3$ term in the sum would be 1, hence we may add one to both sides of this inequality to conclude

$$\left\lfloor \frac{n}{2} \right\rfloor \geq \sum_{i=0}^{(s-2)-1} \left\lceil 2^{-i-1} \left\lceil \frac{n-c+1}{2} \right\rceil \right\rceil.$$

As established after the statement of Proposition 1.8, the bound on s implies

$$s-2 > \log_2 \left\lceil \frac{n-c+1}{2} \right\rceil + \left\lceil \frac{c}{2} \right\rceil,$$

so we have another contradiction to Lemma 1.9. This concludes the proof of Proposition 1.8. □

2. Proof of Theorem C

In this section, we use the following notation:

Definition 2.1. For integers n, let $\delta(n)$ be 0 if n is even and 1 if n is odd, and for $n \geq c \geq 2$, let

$$f_c(n) = 2\log_2 n + \frac{c}{2} - 1 - \delta(n).$$

Given an isometric action of an r-torus on a closed, positively curved n-manifold with $r \geq 2\log_2 n + \frac{c}{2} - 1$, Theorem 1.5 implies that a subtorus of dimension $r - \delta(n) \geq f_c(n)$ has a fixed point. Using this, one can conclude Theorem C from the following:

Theorem 2.2. *Let $n \geq c \geq 2$. Assume M^n is a closed, one-connected, positively curved manifold, and assume a torus T acts effectively by isometries with* $\dim T \geq f_c(n)$. *For all $x \in M^T$, there exists $H \subseteq T$ such that $H^*(M_x^H;\mathbb{Q})$ is 4-periodic and the inclusion $M_x^H \hookrightarrow M$ is c-connected.*

Moreover, H may be chosen to satisfy $\dim(M_x^H) \equiv n \bmod 4$, $\dim(M_x^H) \geq c+4$, *and the property that every free group action $\pi \times M \to M$ commuting with the action of T restricts to a π-action on M_x^H.*

Here, and throughout this section, we use the notation M^H to denote the fixed-point set of $H \subseteq T$, and we write M_x^H for the component of M^H containing x.

In the case where M is not simply connected, we consider the universal cover $\widetilde{M}$ of M. The torus action on M induces an action by a torus $\widetilde{T}$ of the same dimension on $\widetilde{M}$. Moreover, the fundamental group $\pi = \pi_1(M)$ acts freely on $\widetilde{M}$ and its action on $\widetilde{M}$ commutes with the action of $\widetilde{T}$. Theorem 2.2 with $c = 2$ implies the existence of a 2-connected inclusion $N \hookrightarrow \widetilde{M}$ of a compact submanifold N such that π acts freely on N, N has 4-periodic rational cohomology, $\dim N \equiv \dim M \bmod 4$, and $\dim N \geq 6$.

Adding the assumption that $\dim M \equiv 1 \bmod 4$, we conclude that $\dim N \geq 9$. By 4-periodicity and Poincaré duality, it follows that N is a simply connected rational homology sphere. This proves the first part of the corollary stated in the introduction. The second statement follows directly from Theorem D in Davis [8]. Indeed, one only has to check that $\pi_1(M)$ acts by orientation-preserving isometries, which follows from the classical theorem of Weinstein.

We note that, in the case where $\dim M \equiv 3 \bmod 4$, the corresponding conclusion is that $\pi_1(M)$ acts freely on a simply connected rational homology $(4k+3)$-sphere or a simply connected rational $\mathbb{S}^3 \times \mathbb{HP}^k$. However this does not appear to be very restrictive. It is known, for example, that every finite group acts freely on some rational homology 3-sphere (see Cooper and Long [7]). Moreover by taking the $(k+1)$-fold join of this action, one immediately obtains a free action of this group on a simply connected rational homology $(4k+3)$-sphere (see also Browder and Hsiang [5] for an earlier proof of this latter fact). Hence this argument does not immediately yield an analogous obstruction if $\dim M \equiv 3 \bmod 4$.

We spend the rest of this section on the proof of Theorem 2.2. First observe that the assumption in the theorem implies $n \geq c + 8 \geq 10$ by Corollary 1.7, so the theorem holds vacuously in dimensions less than 10. We may therefore proceed with the induction step. For this purpose, we assume the following:

- $c \geq 2$,
- M is a closed, one-connected, positively curved n-manifold with $n \geq c$,
- T is a torus acting effectively by isometries on M with $\dim T \geq f_c(n)$, and
- x is a fixed point in M^T.

To simplify the statement we wish to prove, we make the following definition:

Definition 2.3. Let c, M, T, and x be as above. Denote by $\mathcal{C}$ the set of M_x^H where $H \subseteq T$ ranges over subgroups such that

- the inclusion $M_x^H \hookrightarrow M$ is c-connected,
- $\dim(M_x^H) \equiv n \bmod 4$,
- $\dim(M_x^H) \geq c + 4$, and
- every free group action $\pi \times M \to M$ commuting with the action of T restricts to a π-action on M_x^H.

Observe that our goal is to prove the following:

Claim. *There exists $M_x^H \in \mathcal{C}$ with 4-periodic rational cohomology.*

Our first step is to draw a conclusion from our induction hypothesis. To state the lemma, we require one more definition:

Definition 2.4. For a submanifold $N \subseteq M$ on which T acts, let $\ker(T|_N) \subseteq T$ denote the kernel of the induced T-action on N. Also let $\operatorname{dk} N = \dim \ker(T|_N)$, that is, the dimension of the kernel of the induced T-action on N.

Since T is fixed, the quantity $\operatorname{dk} N$ is well defined. We now put our induction hypothesis to use:

Lemma 2.5. *Some $M_x^H \in \mathcal{C}$ has 4-periodic rational cohomology, or the following holds: For all $Q, N \in \mathcal{C}$ with $Q \subseteq N \subseteq M$ and $\dim Q < n$,*

(1) $\dim Q > n/2^{(\operatorname{dk} Q)/2}$ *and*

(2) *if $k < n/(3 \cdot 2^{\operatorname{dk} N})$, then*

$$\dim Q > \begin{cases} k & \text{if } 2\operatorname{dk} N - \operatorname{dk} Q \geq -3, \\ 2k & \text{if } 2\operatorname{dk} N - \operatorname{dk} Q \geq -1, \\ 3k & \text{if } 2\operatorname{dk} N - \operatorname{dk} Q \geq 0. \end{cases}$$

Proof. Suppose for a moment that there exists $Q \in \mathcal{C}$ such that $\dim Q < n$ and $\dim Q \leq n/2^{(\operatorname{dk} Q)/2}$. Then $T/\ker(T|_Q)$ is a torus acting effectively on Q with dimension

$$\dim T - \operatorname{dk} Q \geq f_c(n) - 2\log_2(n) + 2\log_2(\dim Q) = f_c(\dim Q).$$

Since $\dim Q < n$, the induction hypothesis implies the existence of a subgroup $H' \subseteq T/\ker(T|_Q)$ such that

- $Q_x^{H'}$ has 4-periodic rational cohomology,
- $Q_x^{H'} \hookrightarrow Q$ is c-connected,
- $\dim\left(Q_x^{H'}\right) \equiv \dim Q \bmod 4$,
- $\dim \dim\left(Q_x^{H'} \dim\right) \geq c + 4$, and
- every free group action $\pi \times Q \to Q$ that commutes with the action of $T/\ker(T|_Q)$ restricts to a π-action on $Q_x^{H'}$.

Letting H be the inverse image of H' under the quotient map $T \to T/\ker(T|_Q)$, we conclude that $Q_x^{H'} = M_x^H$. Moreover, since $Q \in \mathcal{C}$, we have $M_x^H \in \mathcal{C}$. Hence $M_x^H \in \mathcal{C}$ and has 4-periodic rational cohomology.

We may assume therefore that no such Q exists. Letting Q and N be as in the assumption of the lemma, we immediately obtain the estimate $\dim Q > n/2^{(\mathrm{dk}\, Q)/2}$. The second estimate on $\dim Q$ follows directly from the first together with the estimate on k. $\square$

Since our goal is to prove that some $M_x^H \in \mathcal{C}$ has 4-periodic rational cohomology, we assume from now on the second statement of Lemma 2.5.

Next we begin the study of fixed-point sets of involutions. Using Proposition 1.8 and the periodicity theorem, we prove the following:

Lemma 2.6. *Some $M_x^H \in \mathcal{C}$ has 4-periodic rational cohomology, or there exists an involution $\sigma \in T$ such that $M_x^\sigma \in \mathcal{C}$ and*

$$0 < \operatorname{cod}(M_x^\sigma) \leq \min\left(\frac{n-c}{2}, \frac{n}{3}\right).$$

Proof. Recall that $x \in M$ has been fixed. Also recall that $\dim T \geq f_c(n)$. By the first part of Proposition 1.8, there exists an involution $\sigma \in T$ satisfying $\operatorname{cod}(M_x^\sigma) \equiv 0 \bmod 4$ and $0 < \operatorname{cod}(M_x^\sigma) \leq \frac{n-c}{2}$.

By choosing σ among all such involutions so that $\operatorname{cod}(M_x^\sigma)$ is minimal, we ensure that $\mathrm{dk}(M_x^\sigma) \leq 2$. Indeed, if $\mathrm{dk}(M_x^\sigma) \geq 3$, then a $\mathbb{Z}_2^3$ would fix M_x^σ and we could choose $\sigma' \in \mathbb{Z}_2^3 \backslash \langle\sigma\rangle$ with $\operatorname{cod}(M_x^{\sigma'}) \equiv 0 \bmod 4$. Because the action of T is effective, we would have $0 < \operatorname{cod}(M_x^{\sigma'}) < \operatorname{cod}(M_x^\sigma)$, a contradiction to the minimality of $\operatorname{cod}(M_x^\sigma)$.

Suppose for a moment that $\mathrm{dk}(M_x^\sigma) = 2$. There exists a $\mathbb{Z}_2^2$ in T that fixes M_x^σ, so we can choose $\sigma' \in \mathbb{Z}_2^2 \setminus \langle\sigma\rangle$. It follows that $M_x^\sigma \subseteq M_x^{\sigma'} \subseteq M$ with both inclusions strict. Since σ' and σ are involutions in T, M_x^σ is the transverse intersection of $M_x^{\sigma'}$ and $M_x^{\sigma\sigma'}$. Moreover, Lemma 2.5 implies $\operatorname{cod}(M_x^\sigma) < n/2$ since $\mathrm{dk}(M_x^\sigma) = 2$, hence

$$2\operatorname{cod}(M_x^{\sigma'}) + 2\operatorname{cod}(M_x^{\sigma\sigma'}) = 2\operatorname{cod}(M_x^\sigma) < n.$$

The periodicity theorem implies that $H^*(M;\mathbb{Q})$ is 4-periodic. Since $M = M_x^{\langle \mathrm{id}\rangle} \in \mathcal{C}$, the proof is complete in this case.

Finally, suppose that $\mathrm{dk}(M_x^\sigma) \leq 1$. By Lemma 2.5,

$$\mathrm{cod}(M_x^\sigma) < \left(n - \frac{n}{\sqrt{2}}\right) < \frac{n}{3},$$

hence we just need to show that $M_x^\sigma = M_x^{\langle\sigma\rangle} \in \mathcal{C}$.

First, the bound on $\mathrm{cod}(M_x^\sigma)$ and the connectedness theorem imply that $M_x^\sigma \hookrightarrow M$ is c-connected. Second, our choice of σ implies $\dim(M_x^\sigma) \equiv n \bmod 4$. Third, the bounds $\mathrm{cod}(M_x^\sigma) \leq \frac{n-c}{2}$ and $n \geq c + 8$, the latter coming from Corollary 1.7, imply that $\dim(M_x^\sigma) \geq \frac{n+c}{2} \geq c + 4$. Finally, the assumption that $c \geq 2$ implies that $\dim(M_x^\sigma) \geq \frac{n}{2}$. By the connectedness theorem, M_x^σ is the unique component of M^σ with dimension at least $n/2$, which implies that every π-action on M that commutes with T preserves M_x^σ and hence restricts to a π-action on M_x^σ. These conclusions imply $M_x^\sigma \in \mathcal{C}$. □

Since our goal is to prove that some $M_x^H \in \mathcal{C}$ has 4-periodic rational cohomology, we may assume the existence of an involution $\sigma \in T$ as in this lemma. In other words, we may assume that the pair (M, σ) satisfies Property $(*)$ according to the following definition:

Definition 2.7. We say that (N, σ) satisfies Property $(*)$ if $N \in \mathcal{C}$ and σ is an involution in $T/\ker(T|_N)$ such that $N_x^\sigma \in \mathcal{C}$ and

$$0 < \mathrm{cod}_N(N_x^\sigma) \leq \min\left(\frac{\dim N - c}{2}, \frac{n}{3 \cdot 2^{\mathrm{dk}\, N}}\right).$$

Here and throughout the rest of the proof, $\mathrm{cod}_R\, Q = \mathrm{cod}\, Q - \mathrm{cod}\, R$ denotes the codimension of $Q \subseteq R$. As established before the definition, there exists at least one pair satisfying Property $(*)$. We focus our attention on a particular minimal pair. Specifically, among pairs (N, σ) satisfying Property $(*)$ with minimal $\dim N$, we choose one with minimal $\mathrm{cod}_N(N_x^\sigma)$.

With N fixed, we will denote by $\bar{T}$ the quotient of T by the kernel $\ker(T|_N)$ of the induced T-action on N. Observe that $\bar{T}$ acts effectively on N and has dimension $\dim T - \mathrm{dk}\, N$. Moreover, we wish to emphasize that the involution σ lies in $\bar{T}$.

The strategy for the rest of the proof is to choose a second involution in $\bar{T}$ in a certain minimal way, analyze the consequences of our minimal choices to prove Lemma 2.9 below, then to conclude the proof of Theorem 2.2.

To begin, we prove the following:

Lemma 2.8. *There exists an involution* $\tau \in \bar{T}$ *such that* $\mathrm{cod}_N(N_x^\tau) \equiv 0 \bmod 4$, $\mathrm{cod}_N(N_x^{\langle\sigma,\tau\rangle}) \equiv 0 \bmod 4$, *and*

$$0 < \mathrm{cod}_N(N_x^\tau) \leq \frac{\dim N - c}{2}.$$

Moreover, for any such τ,

- *$N_x^\tau \in \mathcal{C}$, and*
- *$N_x^{\sigma\tau} \in \mathcal{C}$ and $N_x^{\langle\sigma,\tau\rangle} \in \mathcal{C}$ if $N_x^\sigma \cap N_x^\tau$ is not transverse.*

Proof. The existence of such a τ follows from Proposition 1.8 once we establish that

$$\dim \overline{T} \geq \log_2(\dim N) + \frac{c}{2} + 1 + \log_2(3) - \delta(\dim N).$$

Moreover, we see that this is the case by combining the following facts:

- $\dim \overline{T} = \dim T - \operatorname{dk} N$ by definition of $\operatorname{dk} N$,
- $\dim T \geq f_c(n)$ by assumption,
- $4 \leq \operatorname{cod}_N(N_x^\sigma) \leq n/(3 \cdot 2^{\operatorname{dk} N})$ because (N, σ) satisfies Property $(*)$, and
- $\delta(\dim N) = \delta(n)$ because $\dim N \equiv n \bmod 4$.

For the second claim, let τ be any involution satisfying these properties. First let $H \subseteq T$ be such that $N = M_x^H$. If $p\colon T \to \overline{T}$ is the projection map, then $N_x^\tau = M_x^{\langle H, p^{-1}(\tau)\rangle}$, where $\langle H, p^{-1}(\tau)\rangle$ is the subgroup of T generated by H and $p^{-1}(\tau)$. Similarly, $N_x^{\sigma\tau} = M_x^{\langle H, p^{-1}(\sigma\tau)\rangle}$ and $N_x^{\langle\sigma,\tau\rangle} = M_x^{\langle H, p^{-1}(\langle\sigma,\tau\rangle)\rangle}$.

Second, the inclusions $N_x^\sigma \cap N_x^\tau \hookrightarrow N_x^\tau \hookrightarrow N$ are c-connected by the connectedness theorem together with the upper bounds on $\operatorname{cod}_N(N_x^\sigma)$ and $\operatorname{cod}_N(N_x^\tau)$. In particular, $N_x^\sigma \cap N_x^\tau$ is connected since $c \geq 2$, so we have $N_x^{\langle\sigma,\tau\rangle} = N_x^\sigma \cap N_x^\tau$. Also by the connectedness theorem, the inclusion $N_x^{\langle\sigma,\tau\rangle} \hookrightarrow N_x^{\sigma\tau}$ is $(c+1)$-connected. Since $N \in \mathcal{C}$, this proves that the inclusions of N_x^τ, $N_x^{\sigma\tau}$, and $N_x^{\langle\sigma,\tau\rangle}$ into M are c-connected.

Third, the dimensions of N_x^τ, $N_x^{\sigma\tau}$, and $N_x^{\langle\sigma,\tau\rangle}$ are congruent to n modulo 4 since $\dim N \equiv n \bmod 4$, $\operatorname{cod}_N(N_x^\sigma) \equiv 0 \bmod 4$, $\operatorname{cod}_N(N_x^\tau) \equiv 0 \bmod 4$, $\operatorname{cod}_N(N_x^{\langle\sigma,\tau\rangle}) \equiv 0 \bmod 4$, and

$$\operatorname{cod}_N(N_x^{\sigma\tau}) \equiv \operatorname{cod}_N(N_x^\sigma) + \operatorname{cod}_N(N_x^\tau) \equiv 0 \bmod 4.$$

Fourth, the previous paragraph together with the minimality of $\operatorname{cod}_N(N_x^\tau)$ implies $\dim(N_x^\tau) \geq 4 + \dim(N_x^{\langle\sigma,\tau\rangle})$, hence

$$\dim(N_x^\tau) \geq 4 + \big(\dim N - \operatorname{cod}_N(N_x^\sigma) - \operatorname{cod}_N(N_x^\tau)\big) \geq c + 4.$$

If $N_x^\sigma \cap N_x^\tau$ is not transverse, then the codimension a of $N_x^{\langle\sigma,\tau\rangle} \subseteq N_x^{\sigma\tau}$ is positive. By the previous paragraph, $a \equiv 0 \bmod 4$ and hence $a \geq 4$. Hence

$$\dim(N_x^{\sigma\tau}) > \dim(N_x^{\langle\sigma,\tau\rangle}) = \dim N - \operatorname{cod}_N(N_x^\sigma) - \operatorname{cod}_N(N_x^\tau) + a \geq c + 4.$$

Finally, let $\pi \times M \to M$ be a free group action commuting with the T-action on M. We wish to show that the π-action restricts to π-actions on N_x^τ, $N_x^{\langle\sigma,\tau\rangle}$, and $N_x^{\sigma\tau}$. This follows from the following observations:

- By assumption, $N \in \mathcal{C}$, so the π-action restricts to a π-action on N.
- By assumption, the dimensions of N_x^σ and N_x^τ in N are at least $\frac{1}{2}\dim N$, hence the connectedness theorem implies that N_x^σ and N_x^τ are the unique such components of N^σ and N^τ, respectively. It follows that the π-action preserves N_x^σ and N_x^τ.
- Since π preserves N_x^σ and N_x^τ, it also preserves $N_x^\sigma \cap N_x^\tau = N_x^{\langle\sigma,\tau\rangle}$.
- Since π preserves $N_x^{\langle\sigma,\tau\rangle}$, the fact that $N_x^{\langle\sigma,\tau\rangle} \subseteq N_x^{\sigma\tau}$ implies that π also preserves $N_x^{\sigma\tau}$.

This concludes the proof that $N_x^\tau \in \mathcal{C}$ and that $N_x^{\sigma\tau}, N_x^{\langle\sigma,\tau\rangle} \in \mathcal{C}$ if $N_x^\sigma \cap N_x^\tau$ is not transverse. □

We now choose $\tau \in \bar{T}$ such that

$$\mathrm{cod}_N(N_x^\tau) \equiv \mathrm{cod}_N(N_x^{\langle\sigma,\tau\rangle}) \equiv 0 \bmod 4$$

and

$$0 < \mathrm{cod}_N(N_x^\tau) \leq \frac{\dim N - c}{2}$$

and such that $\mathrm{cod}_N(N_x^\tau)$ is minimal among all such choices.

Having chosen τ, we use the minimality of $\dim N$ and $\mathrm{cod}_N(N_x^\tau)$ to obtain the following:

Lemma 2.9. *Both of the following hold:*

(1) $\mathrm{dk}(N_x^{\sigma\tau}) - \mathrm{dk}\, N \geq 2$ *or the intersection* $N_x^\sigma \cap N_x^\tau$ *is transverse in* N.

(2) $\mathrm{dk}(N_x^\tau) - \mathrm{dk}\, N \leq 3$ *with equality only if* $N_x^\sigma \cap N_x^\tau$ *is not transverse in* N.

Proof. We prove the first statement by contradiction. We assume therefore that $\mathrm{dk}(N_x^{\sigma\tau}) \leq 1 + \mathrm{dk}\, N$ and that $N_x^\sigma \cap N_x^\tau$ is not transverse. The first assumption implies that $\bar{T}/\ker(\bar{T}|_{N_x^{\sigma\tau}}) = T/\ker(T|_{N_x^{\sigma\tau}})$ has dimension at least $\dim T - \mathrm{dk}\, N - 1$. Let $\bar{\sigma}$ denote the image of σ under the projection $\bar{T} \to \bar{T}/\ker(\bar{T}|_{N_x^{\sigma\tau}})$, and observe that $(N_x^{\sigma\tau})_x^{\bar{\sigma}} = N_x^{\langle\sigma,\tau\rangle}$. The second assumption implies that the inclusion $N_x^{\langle\sigma,\tau\rangle} \subseteq N_x^{\sigma\tau}$ has positive codimension. Moreover, this codimension is at most $\frac{1}{2}\mathrm{cod}_N(N_x^\sigma)$ by the minimality of $\mathrm{cod}_N(N_x^\tau)$. Putting these facts together, we see that $(N_x^{\sigma\tau}, \bar{\sigma})$ satisfies Property $(*)$. Since $\dim(N_x^{\sigma\tau}) < \dim N$, this is a contradiction to the minimality of $\dim N$.

Proceeding to the second statement, suppose for a moment that $\mathrm{dk}(N_x^\tau) \geq 4 + \mathrm{dk}\, N$. Then there exists a 4-torus inside $\bar{T}$ that fixes N_x^τ. It follows that we may choose a nontrivial involution $\iota \neq \tau$ inside this 4-torus such that

$$\mathrm{cod}_N(N_x^\iota) \equiv \mathrm{cod}_N(N_x^{\langle\sigma,\iota\rangle}) \equiv 0 \bmod 4.$$

Because the action of $\bar{T}$ is effective, $\iota \notin \langle\tau\rangle$ implies that $N_x^\tau \subseteq N_x^\iota \subseteq N$ with both inclusions strict. Moreover, since $N_x^\tau \not\subseteq N_x^\sigma$, it follows for free that $N_x^\iota \not\subseteq N_x^\sigma$. Hence we have a contradiction to the minimality of $\mathrm{cod}_N(N_x^\tau)$, and we may conclude that $\mathrm{dk}(N_x^\tau) \leq 3 + \mathrm{dk}\, N$.

For the equality case, suppose that $\mathrm{dk}(N_x^\tau) = 3 + \mathrm{dk}\, N$ and that $N_x^\sigma \cap N_x^\tau$ is transverse in N. Since a 3-torus inside $\bar{T}$ fixes N_x^τ, we may choose an involution $\iota \in \bar{T} \setminus \langle\tau\rangle$ such that $\mathrm{cod}_N(N_x^\iota) \equiv 0 \bmod 4$ and $N_x^\tau \subseteq N_x^\iota \subseteq N$ with all inclusions strict. Since $N_x^\sigma \cap N_x^\tau$ is transverse, it follows that $\mathrm{cod}_N(N_x^{\langle\sigma,\iota\rangle}) \equiv 0 \bmod 4$ as well, hence we have another contradiction to the minimality of $\mathrm{cod}_N(N_x^\tau)$. □

We are ready to conclude the proof of Theorem 2.2. We do this by breaking the proof into cases and showing in each case that N_x^τ has 4-periodic rational cohomology or that $N_x^\sigma \cap N_x^\tau$ is not transverse and $N_x^{\langle\sigma,\tau\rangle}$ has 4-periodic rational cohomology. Since we have already established that $N_x^\tau \in \mathcal{C}$ and that $N_x^{\langle\sigma,\tau\rangle} \in \mathcal{C}$ if $N_x^\sigma \cap N_x^\tau$ is not transverse, this would conclude the proof of Theorem 2.2. The three cases are as follows:

Case 1: $2\,\mathrm{dk}\, N - \mathrm{dk}(N_x^\tau) \leq -2$.

Case 2: $2\,\mathrm{dk}\, N - \mathrm{dk}(N_x^\tau) \geq -1$ and $N_x^\sigma \cap N_x^\tau$ is not transverse.

Case 3: $2\,\mathrm{dk}\, N - \mathrm{dk}(N_x^\tau) \geq -1$ and $N_x^\sigma \cap N_x^\tau$ is transverse.

Clearly one of these cases occurs, so our task will be complete once we show, in each case, that N_x^τ has 4-periodic rational cohomology or that $N_x^\sigma \cap N_x^\tau$ is not transverse and $N_x^{\langle\sigma,\tau\rangle}$ has 4-periodic rational cohomology. We assign each case its own lemma.

Lemma 2.10 (Case 1). *If* $2\,\mathrm{dk}\, N - \mathrm{dk}(N_x^\tau) \leq -2$, *one of the following holds:*

(1) N_x^τ *has* 4*-periodic rational cohomology, or*

(2) $N_x^{\langle\sigma,\tau\rangle}$ *has* 4*-periodic rational cohomology and* $N_x^\sigma \cap N_x^\tau$ *is not transverse.*

Proof. First observe that $\mathrm{dk}(N_x^\tau) \geq 2 + \mathrm{dk}\, N$ since, by definition, $\mathrm{dk}\, N \geq 0$. We may therefore choose an involution $\iota \in \bar{T}$ such that $N_x^\tau \subseteq N_x^\iota \subseteq N$ with both inclusions strict. In addition, we may assume $\mathrm{cod}_N(N_x^\iota) \equiv 0 \bmod 4$ in the case where $\mathrm{dk}(N_x^\tau) \geq 3 + \mathrm{dk}\, N$. Choose a basis for the tangent space $T_x N$ so that the images of σ, τ, and ι under the isotropy representation $\phi\colon \bar{T} \to \mathrm{SO}(T_x N)$ have the following block representations:

$$\begin{aligned}
\phi(\sigma) &= \mathrm{diag}\begin{pmatrix} -I & -I & -I & I & I & I \end{pmatrix},\\
\phi(\tau) &= \mathrm{diag}\begin{pmatrix} -I & -I & I & -I & -I & I \end{pmatrix},\\
\phi(\iota) &= \mathrm{diag}\begin{pmatrix} -I & I & I & -I & I & I \end{pmatrix},
\end{aligned}$$

where the blocks have size b, $a-b$, $k-a$, $m-b$, $(l-a)-(m-b)$, and $\dim(N_x^{\langle\sigma,\tau\rangle})$, where $k = \mathrm{cod}_N(N_x^\sigma)$, $l = \mathrm{cod}_N(N_x^\tau)$, and $m = \mathrm{cod}_N(N_x^\iota)$, and where $a = \mathrm{cod}\left(N_x^{\langle\sigma,\tau\rangle} \subseteq N_x^{\sigma\tau}\right)$ and $b = \mathrm{cod}\left(N_x^{\langle\sigma,\iota\rangle} \subseteq N_x^{\sigma\iota}\right)$.

Suppose for a moment that $b = 0$ or $b = a$. Our choice of τ implies that $\mathrm{cod}(N_x^{\langle\sigma,\tau\rangle}) \equiv 0 \bmod 4$ and hence $a \equiv 0$. Since $b = 0$ or $b = a$, this means $b \equiv 0 \bmod 4$ and hence

$$\mathrm{cod}(N_x^\iota) = m \equiv k + m - b = \mathrm{cod}(N_x^{\langle\sigma,\iota\rangle}) \bmod 4.$$

By our choice of τ, we must have that $m \equiv 2 \bmod 4$. By our choice of ι, we must have $\mathrm{dk}(N_x^\tau) \leq 2 + \mathrm{dk}\, N$. Combining this with the assumption in this case, we conclude $\mathrm{dk}\, N = 0$ and $\mathrm{dk}(N_x^\tau) = 2 + \mathrm{dk}\, N$. The first of these equalities implies $N = M$ by Lemma 2.5. Using Lemma 2.5 again, we conclude $\dim(N_x^\tau) \geq \frac{1}{2}\dim N$. Since N_x^τ is fixed by a 2-torus, there exists an involution $\tau' \in \bar{T} \setminus \langle\tau\rangle$ such that $N_x^\tau \subseteq N_x^{\tau'} \subseteq N$ with both inclusions strict. Hence N_x^τ is the transverse intersection in N of $N_x^{\tau'}$ and $N_x^{\tau\tau'}$, and since the codimensions of these submanifolds satisfies

$$2\,\mathrm{cod}_N(N_x^{\tau'}) + 2\,\mathrm{cod}_N(N_x^{\tau\tau'}) = 2\,\mathrm{cod}_N(N_x^\tau) < n,$$

the periodicity theorem implies that N_x^τ has 4-periodic rational cohomology.

Now suppose that $0 < b < a$. First observe that $a > 0$ implies that $N_x^\sigma \cap N_x^\tau$ is not transverse. Second, observe that $(N_x^{\sigma\tau})_x^\iota$ and $(N_x^{\sigma\tau})_x^{\tau\iota}$ intersect transversely in $N_x^{\sigma\tau}$, have codimensions b and $a-b$, respectively, and have intersection $N_x^{\langle\sigma,\tau\rangle}$. The codimensions b and $a-b$ are positive, and they satisfy

$$2b + 2(a-b) = 2a \leq \dim(N_x^\tau) - k + 2a = \dim(N_x^{\sigma\tau})$$

by Lemmas 2.5 and 2.9. It follows from the periodicity theorem that $N_x^{\langle\sigma,\tau\rangle}$ has 4-periodic rational cohomology. This concludes the proof in Case 1. □

Lemma 2.11 (Case 2). *If $2\,\mathrm{dk}\, N - \mathrm{dk}(N_x^\tau) \geq -1$ and $N_x^\sigma \cap N_x^\tau$ is not transverse, then $N_x^{\langle\sigma,\tau\rangle}$ has 4-periodic rational cohomology.*

Proof. First observe that $\mathrm{dk}(N_x^{\sigma\tau}) \geq 2 + \mathrm{dk}\, N$ by Lemma 2.9, hence we may choose an involution $\iota \in \bar{T}$ such that $N_x^{\sigma\tau} \subseteq N_x^\iota \subseteq N$ with both inclusions strict. Choose a basis for the tangent space $T_x N$ so that the images of σ, τ, and ι under the isotropy representation $\phi\colon \bar{T} \to \mathrm{SO}(T_x N)$ have the following block representations:

$$\begin{aligned}
\phi(\sigma) &= \mathrm{diag}\begin{pmatrix}-I & -I & -I & I & I & I\end{pmatrix},\\
\phi(\tau) &= \mathrm{diag}\begin{pmatrix}-I & I & I & -I & -I & I\end{pmatrix},\\
\phi(\iota) &= \mathrm{diag}\begin{pmatrix}I & -I & I & -I & I & I\end{pmatrix},
\end{aligned}$$

where the blocks have size $a, b, k-a-b, m-b, (l-a)-(m-b)$, and $\dim(N_x^{\langle\sigma,\tau\rangle})$. Here k, l, m, a, and b have the same geometric meaning as in Case 1. The difference is in the order of the blocks, which indicate that $N_x^{\sigma\tau} \subseteq N_x^{\iota}$ in this case while $N_x^{\tau} \subseteq N_x^{\iota}$ in Case 1.

Observe that $a > 0$ because $N_x^{\sigma} \cap N_x^{\tau}$ is not transverse. In addition, observe that the assumption in this case implies $\dim(N_x^{\tau}) \geq 2k$ by Lemma 2.5. Finally, by replacing ι by $\iota\sigma\tau$ if necessary, we may assume that $b \leq \frac{k-a}{2}$.

First suppose $b > 0$. Then $(N_x^{\tau})_x^{\iota}$ and $(N_x^{\tau})_x^{\sigma\iota}$ intersect transversely in N_x^{τ} with intersection $N_x^{\langle\sigma,\tau\rangle}$. Since the codimensions, b and $k-a-b$, are positive and satisfy

$$2b + 2(k-a-b) \leq 2k \leq \dim(N_x^{\tau}),$$

the periodicity theorem implies that $N_x^{\langle\sigma,\tau\rangle} = (N_x^{\tau})_x^{\iota} \cap (N_x^{\tau})_x^{\sigma\iota}$ has 4-periodic rational cohomology.

Now suppose $b = 0$. Then $(N_x^{\sigma\tau\iota})_x^{\sigma}$ and $(N_x^{\sigma\tau\iota})_x^{\sigma\tau}$ intersect transversely inside $N_x^{\sigma\tau\iota}$ with positive codimensions a and m. Using the estimates $a \leq k$ and $\dim(N_x^{\tau}) \geq 2k$, it follows that

$$3a + m \leq \dim(N_x^{\tau}) - k + 2a + m = \dim(N_x^{\sigma\tau\iota}).$$

Moreover, $(N_x^{\sigma\tau\iota})_x^{\sigma\tau} = N_x^{\sigma\tau} \in \mathcal{C}$ by Lemma 2.8, so $(N_x^{\sigma\tau\iota})_x^{\sigma\tau}$ is one-connected. By the periodicity theorem, $N_x^{\langle\sigma,\tau\rangle} = (N_x^{\sigma\tau\iota})_x^{\sigma} \cap (N_x^{\sigma\tau\iota})_x^{\sigma\tau}$ has 4-periodic rational cohomology. This concludes the proof in Case 2. $\square$

Lemma 2.12 (Case 3). *If* $2\,\mathrm{dk}\,N - \mathrm{dk}(N_x^{\tau}) \geq -1$ *and* $N_x^{\sigma} \cap N_x^{\tau}$ *is transverse, then* N_x^{τ} *has* 4*-periodic rational cohomology.*

Proof. Let $k = \mathrm{cod}_N(N_x^{\sigma})$ and $l = \mathrm{cod}_N(N_x^{\tau})$. As in the proof of Case 2, we have $\dim(N_x^{\tau}) \geq 2k$.

First we consider the case where $\mathrm{dk}(N_x^{\tau}) \geq 2 + \mathrm{dk}\,N$. This implies the existence of an involution $\iota \in \overline{T}$ such that $N_x^{\tau} \subseteq N_x^{\iota} \subseteq N$ with both inclusions strict. By replacing ι by $\tau\iota$ if necessary, we may assume that its codimension m satisfies $m \leq \frac{l}{2}$. Since $N_x^{\sigma} \cap N_x^{\tau}$ is transverse, $N_x^{\sigma} \cap N_x^{\iota}$ is as well. Since the codimensions of this transverse intersection satisfy

$$2k + 2m \leq 2k + l \leq n,$$

the periodicity theorem implies that N_x^{τ} has 4-periodic rational cohomology.

Second we consider the case where

$$\mathrm{dk}(N_x^{\langle\sigma,\tau\rangle}) \geq 2 + \mathrm{dk}(N_x^{\tau}).$$

This implies the existence of an $\iota \in \overline{T}$ such that $\iota^2|_{N_x^{\tau}} = \mathrm{id}$ and such that

$$(N_x^{\tau})_x^{\sigma} = N_x^{\langle\sigma,\tau\rangle} \subseteq (N_x^{\tau})_x^{\iota} \subseteq N_x^{\tau}$$

with both inclusions strict. It follows that $(N_x^\tau)_x^\iota$ and $(N_x^\iota)_x^{\sigma\iota}$ intersect transversely in N_x^τ and have codimensions, say, b and $k-b$. Since

$$2b + 2(k-b) = 2k \le \dim(N_x^\tau),$$

the periodicity theorem implies that N_x^τ has 4-periodic rational cohomology.

Third, we consider the case where $2\,\mathrm{dk}\,N - \mathrm{dk}(N_x^\tau) \ge 0$. Lemma 2.5 implies $\dim(N_x^\tau) \ge 3k$, hence $3k + l \le n$. Since N_x^σ and N_x^τ intersect transversely, the periodicity theorem implies N_x^τ has 4-periodic rational cohomology.

Finally, if none of these three possibilities occurs, the assumption in this case implies that $\mathrm{dk}\,N = 0$, $\mathrm{dk}(N_x^\tau) = 1 + \mathrm{dk}\,N$, and $\mathrm{dk}(N_x^{\langle\sigma,\tau\rangle}) \le 2 + \mathrm{dk}\,N$. Using Lemma 2.5, we can further conclude $N = M$ and $\dim(N_x^{\langle\sigma,\tau\rangle}) \ge \frac{1}{2}\dim N$. Hence

$$2k + 2l = 2\,\mathrm{cod}(N_x^{\langle\sigma,\tau\rangle}) \le n,$$

so the periodicity theorem applied to the transverse intersection of N_x^σ and N_x^τ implies that N_x^τ has 4-periodic rational cohomology. This concludes the proof of Case 3, and hence concludes the proof of Theorem 2.2. □

3. From Theorem C to Theorem A

The proof of Theorem A contains three steps. The first step classifies one-connected, compact, irreducible symmetric spaces that have 4-periodic rational cohomology up to degree 16. We will also need to prove the basic fact that an n-dimensional, one-connected, compact, irreducible symmetric space with $H^i(M;\mathbb{Q}) = 0$ for all $3 < i < 16$ is $\mathbb{S}^2$, $\mathbb{S}^3$, or $\mathbb{S}^n$. The second step is a lemma about product manifolds whose rational cohomology is 4-periodic up to degree 16. The final step combines these lemmas to classify one-connected, compact symmetric spaces whose rational cohomology is 4-periodic up to degree 16. From Theorem C, these results immediately imply Theorem A.

The first lemma concerns one-connected, compact, irreducible symmetric spaces:

Lemma 3.1. *Let M^n be a one-connected, compact, irreducible symmetric space.*

(1) *If $H^4(M;\mathbb{Q}) \cong \mathbb{Q}$ and $H^*(M;\mathbb{Q})$ is 4-periodic up to degree 16, then M is*
  - *$\mathbb{CP}^q$ or $\mathrm{SO}(2+q)/\mathrm{SO}(2)\times\mathrm{SO}(q)$ with $q = \frac{n}{2}$ or*
  - *$\mathbb{HP}^q$ or $\mathrm{SO}(3+q)/\mathrm{SO}(3)\times\mathrm{SO}(q)$ with $q = \frac{n}{4}$ or $q = \frac{n}{3}$, respectively.*

(2) *If $H^i(M;\mathbb{Q}) = 0$ for all $3 < i < 16$, then M is $\mathbb{S}^2$, $\mathbb{S}^3$, or $\mathbb{S}^n$.*

Observe that periodicity up to degree $c \ge 16$ implies either that $H^4(M;\mathbb{Q}) \cong \mathbb{Q}$ or that M is rationally $(c-1)$-connected. In the former case, $\dim H^{16}(M;\mathbb{Q}) \ge 1$

by periodicity up to degree 16, so we may as well assume that $n \geq 16$ in the first statement of the lemma.

The only facts about 4-periodicity up to degree 16 that we will use in the proof are the following: $b_i = b_{i+4}$ for $0 < i < 12$, $b_4 \leq 1$, and $b_4 = 0$ only if $b_i = 0$ for all $0 < i < 16$. Here and throughout the section, b_i denotes the i-th Betti number of M.

Proof. We use Cartan's classification of simply connected, irreducible compact symmetric spaces. We also keep Cartan's notation. See Helgason [16] for a reference.

One possibility is that M is a simple Lie group. The rational cohomology of M is therefore that of a product of spheres $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_s}$ for some $s \geq 1$ where the n_i are odd. In fact, the dimensions of these sphere are known and are listed in Table 1 (see Mimura and Toda [20] for a reference). Since M is simply connected, we may assume

$$3 = n_1 \leq n_2 \leq \cdots \leq n_s.$$

By the Künneth theorem, $H^4(M;\mathbb{Q}) = 0$, so we must be in the case where $H^i(M;\mathbb{Q})$ is zero for all $3 < i < 16$. But the data in Table 1 imply that $s = 1$ and hence that $M = \mathbb{S}^3$. This completes the proof in the case that M is a simple Lie group.

Table 1. Dimensions of spheres.

G	$n_1, n_2, \ldots, n_s$
$\mathrm{Sp}(n)$	$3, 7, \ldots, 4n-1$
$\mathrm{Spin}(2n+1)$	$3, 7, \ldots, 4n-1$
$\mathrm{Spin}(2n)$	$3, 7, \ldots, 4n-5, 2n-1$
$\mathrm{U}(n)$	$1, 3, \ldots, 2n-1$
$\mathrm{SU}(n)$	$3, 5, \ldots, 2n-1$
G_2	$3, 11$
F_4	$3, 11, 15, 23$
E_6	$3, 9, 11, 15, 17, 23$
E_7	$3, 11, 15, 19, 23, 27, 35$
E_8	$3, 15, 23, 27, 35, 39, 47, 59$

Now we consider the irreducible spaces which are not Lie groups. We have that $M = G/H$ for some compact Lie groups G and H where G is simple. The possible pairs (G, H) fall into one of seven classical families or are one of 12 exceptional examples. First, it will be clear in each case that $M \neq \mathbb{S}^n$ and $\dim M \geq 4$ implies $H^i(M;\mathbb{Q}) \neq 0$ for some $3 < i < 16$, hence the second part of the lemma follows. To prove the first part of the lemma, we calculate the first 15 Betti numbers in each of the $7 + 12 = 19$ cases, then we compare the results to the requirement that they be 4-periodic as described above. We summarize the results in Tables 2 and 3.

Table 2. Classical one-connected, compact, irreducible symmetric spaces of dimension at least 16 that are not listed in the conclusion of Lemma 3.1. The pairs (p, q) satisfy $4 \leq p \leq q$ for the real Grassmannians and $2 \leq p \leq q$ for the complex and quaternionic Grassmannians.

G/H	$P_{G/H}(t) - 1$ if $\mathrm{rk}(G) = \mathrm{rk}(H)$	Reference if not	Obstruction
$\mathrm{SU}(n)/\mathrm{SO}(n), \quad n \geq 6$	–	[4]	$b_5 > 0$
$\mathrm{SU}(2n)/\mathrm{Sp}(n), \quad n \geq 4$	–	[4]	$b_5 > 0$
$\mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$	–	[21]	$1 < b_4$
$\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$	$2t^4 + \cdots$	–	$1 < b_4$
$\mathrm{Sp}(n)/\mathrm{U}(n), \quad n \geq 4$	$t^2 + t^4 + 2t^6 + \cdots$	–	$b_2 < b_6$
$\mathrm{Sp}(p+q)/\mathrm{Sp}(p) \times \mathrm{Sp}(q)$	$t^4 + 2t^8 + \cdots$	–	$b_4 < b_8$
$\mathrm{SO}(2n)/\mathrm{U}(n), \quad n \geq 5$	$t^2 + t^4 + 2t^6 + \cdots$	–	$b_2 < b_6$

Table 3. Exceptional one-connected, compact irreducible symmetric spaces.

G/H	$P_{G/H}(t) - 1$ if $\mathrm{rk}(G) = \mathrm{rk}(H)$	Reference if not	Obstruction
$\mathrm{E}_6/\mathrm{Sp}(4)$	–	[18]	$b_9 > 0$
$\mathrm{E}_6/\mathrm{F}_4$	–	[1]	$b_9 > 0$
$\mathrm{E}_6/\mathrm{SU}(6) \times \mathrm{SU}(2)$	$t^4 + t^6 + 2t^8$	–	$b_4 < b_8$
$\mathrm{E}_6/\mathrm{SO}(10) \times \mathrm{SO}(2)$	$t^4 + 2t^8 + \cdots$	–	$b_4 < b_8$
$\mathrm{E}_7/\mathrm{SU}(8)$	$t^8 + \cdots$	–	$b_4 < b_8$
$\mathrm{E}_7/\mathrm{SO}(12) \times \mathrm{SU}(2)$	$t^4 + 2t^8 + \cdots$	–	$b_4 < b_8$
$\mathrm{E}_7/\mathrm{E}_6 \times \mathrm{SO}(2)$	$t^4 + t^8 + 2t^{12} + \cdots$	–	$b_8 < b_{12}$
$\mathrm{E}_8/\mathrm{SO}(16)$	$t^8 + \ldots$	–	$b_4 < b_8$
$\mathrm{E}_8/\mathrm{E}_7 \times \mathrm{SU}(2)$	$t^4 + t^8 + 2t^{12} + \cdots$	–	$b_8 < b_{12}$
$\mathrm{F}_4/\mathrm{Sp}(3) \times \mathrm{SU}(2)$	$t^4 + 2t^8 + \cdots$	–	$b_4 < b_8$
$\mathrm{F}_4/\mathrm{Spin}(9)$	$t^8 + \cdots$	–	$b_4 < b_8$
$\mathrm{G}_2/\mathrm{SO}(4)$	$t^4 + t^8$	–	$b_8 > b_{12}$

To explain our calculations, we first consider the case $M = G/H$ where G and H have equal rank. Let $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_s}$ and $\mathbb{S}^{m_1} \times \cdots \times \mathbb{S}^{m_s}$ denote the rational homotopy types of G and H, respectively. Then one has the following formula for the Poincaré polynomial of M (see Borel [4]):

$$P_M(t) = \sum_{i \geq 0} b_i(M)t^i = \frac{(1 - t^{n_1+1}) \cdots (1 - t^{n_s+1})}{(1 - t^{m_1+1}) \cdots (1 - t^{m_s+1})}.$$

For each simple Lie group G, the dimensions of the spheres are listed in Table 1. When $\mathrm{rank}(G) = \mathrm{rank}(H)$, we compute the Poincaré polynomial of M and list the relevant terms in Tables 2 and 3. In the case where $\mathrm{rank}(G) \neq \mathrm{rank}(H)$, we simply cite a source where the cohomology is calculated.

The tables give the pair (G, H) realizing the space, the first few terms of the Poincaré polynomial if $\mathrm{rank}(G) = \mathrm{rank}(H)$, and the relevant Betti number inequalities that show M is not rationally 4-periodic up to degree 16.

We remark that, in Table 2, we exclude spaces with dimension less than 16, as such spaces M have $H^{16}(M;\mathbb{Q}) = 0$ and therefore $H^4(M;\mathbb{Q}) = 0$ by periodicity. We also exclude the rank one Grassmannians and the rank two and rank three real Grassmannians, as these are the spaces that appear in the conclusion of the lemma. □

With the first step complete, we prove the following lemma about general products $M = M' \times M''$ whose rational cohomology is 4-periodic up to degree c. The lemma roughly states that, if a product has 4-periodic rational cohomology, then one of the factors has 4-periodic rational cohomology. Moreover, most of the cohomology is concentrated in that factor. In the proof, we use the full strength of periodicity, not simply the corollary that the Betti numbers are 4-periodic. For example, while $\mathbb{S}^4 \times \mathbb{S}^8 \times \mathbb{S}^{16} \times \cdots \times \mathbb{S}^{2^k}$ has 4-periodic Betti numbers, its cohomology is not 4-periodic.

Lemma 3.2. *Assume that $H^1(M;\mathbb{Q}) = 0$, that $H^*(M;\mathbb{Q})$ is 4-periodic up to degree c with $c \geq 9$, and that $M = M' \times M''$ with $\dim H^4(M';\mathbb{Q}) \geq \dim H^4(M'';\mathbb{Q})$. If M is not rationally $(c-1)$-connected, then $H^4(M';\mathbb{Q}) \cong \mathbb{Q}$ and the following hold:*

(1) *$H^*(M';\mathbb{Q})$ is 4-periodic up to degree c,*

(2) *$H^i(M'';\mathbb{Q}) = 0$ for $3 < i < c$, and*

(3) *if $H^2(M';\mathbb{Q}) \neq 0$ or $H^3(M';\mathbb{Q}) \neq 0$, then $H^2(M'';\mathbb{Q}) = H^3(M'';\mathbb{Q}) = 0$.*

Proof of lemma. For simplicity we denote the Betti numbers of M, M', and M'' by b_i, b_i', and b_i'', respectively. Observe that we must show that $H^*(M';\mathbb{Q})$ is 4-periodic, that $b_i'' = 0$ for $3 < i < c$, and that $b_2'' = b_3'' = 0$ if $b_2' > 0$ or $b_3' > 0$.

Let $x \in H^4(M;\mathbb{Q})$ be an element inducing periodicity. If $x = 0$, then $c \geq 8$ implies M is rationally $(c-1)$-connected. Assume therefore that $b_4(M) = 1$ (i.e., that $x \neq 0$).

We first claim that $b'_4 = 1$. Suppose instead that $0 = b'_4 = b''_4$. The Künneth theorem implies $1 = b_4 = b'_2 b''_2$, and hence $b'_2 = b''_2 = 1$. Using periodicity and the Künneth theorem again, we have

$$0 = b_1 = b_5 = b'_5 + b''_5 + b'_3 + b''_3,$$

and hence that all four terms on the right-hand side are zero. Similarly, we have

$$2 = b'_2 + b''_2 = b_2 = b_6 = b'_6 + b''_6.$$

Finally, we obtain

$$1 = b_4 = b_8 \geq b'_2 b''_6 + b'_6 b''_2 = b'_6 + b''_6 = 2,$$

a contradiction. Assume therefore that $b'_4 = 1$ and hence that $b''_4 = b'_2 b''_2 = 0$.

Let $p\colon M \to M'$ be the projection map. It follows from $b'_4 = b_4 = 1$ and the Künneth theorem that the composition

$$H^4(M') \cong H^4(M') \otimes H^0(M'') \hookrightarrow \bigoplus_{j=0}^{4} H^{4-j}(M') \otimes H^j(M'') \xrightarrow{\times} H^4(M)$$

is an isomorphism. Choose $\bar{x} \in H^4(M'; \mathbb{Q})$ with $p^*(\bar{x}) = x$. We claim that $\bar{x}$ induces periodicity in $H^*(M')$ up to degree c.

First, $b'_4 = 1$ implies that multiplication by $\bar{x}$ induces a surjection $H^0(M') \to H^4(M')$. Second, consider the commutative diagram

$$\begin{tikzcd}
H^i(M') \arrow[r, hook] \arrow[d] & \bigoplus_j H^{i-j}(M') \otimes H^j(M'') \arrow[r, "\times"] & H^i(M) \arrow[d] \\
H^{i+4}(M') \arrow[r, hook] & \bigoplus_j H^{i+4-j}(M') \otimes H^j(M'') \arrow[r, "\times"] & H^{i+4}(M),
\end{tikzcd}$$

where the vertical arrows from left to right are given by multiplication by $\bar{x}$, $\bar{x} \otimes 1$, and x, respectively. Because multiplication by x is injective for $0 < i \leq c - 4$, it follows that multiplication by $\bar{x}$ is injective in these degrees as well. It therefore suffices to check that multiplication by $\bar{x}$ is surjective for $0 < i < c - 4$. We accomplish this by a dimension counting argument. Specifically, we claim $b'_i = b'_{i+4}$ for $0 < i < c - 4$. Indeed, for all $0 \leq i < c - 4$, we have from periodicity, the Künneth theorem, and injectivity of multiplication by $\bar{x}$ the following estimate:

$$\sum_{j=0}^{i} b'_{i-j} b''_j = b_i = b_{i+4} \geq b''_{i+4} + \sum_{j=0}^{i} b'_{i+4-j} b''_j \geq b''_{i+4} + \sum_{j=0}^{i} b'_{i-j} b''_j.$$

Equality must hold everywhere, proving $b'_i = b'_{i+4}$ and $b''_{i+4} = 0$ for all $0 \leq i < c - 4$. This completes the proof of the first part, as well as the second part, of the lemma.

Finally, suppose that $b_2' > 0$ or $b_3' > 0$. Then

$$b_4' + (b_2' + b_3')b_2'' \leq b_4 + b_5 = 1 + b_1 = b_4'$$

implies $b_2'' = 0$, and

$$b_6' + (b_2' + b_3')b_3'' \leq b_5 + b_6 = b_2 = b_2' = b_6'$$

implies $b_3'' = 0$. □

We are ready to prove Theorem A. In fact, we prove the following stronger theorem:

Theorem 3.3. *Suppose M^n has the rational cohomology of a one-connected, compact symmetric space. Let $c \geq 16$, and assume M admits a metric with positive curvature and symmetry rank at least $2\log_2 n + \frac{c}{2} - 1$. There exists a (possibly trivial) product S of spheres, each of dimension at least c, such M has the rational cohomology of*

(1) *S,*

(2) *$S \times R$ with $R \in \{\mathbb{CP}^q, \mathrm{SO}(2+q)/\mathrm{SO}(2) \times \mathrm{SO}(q)\}$, or*

(3) *$S \times R \times Q$ with $R \in \{\mathbb{HP}^q, \mathrm{SO}(3+q)/\mathrm{SO}(3) \times \mathrm{SO}(q)\}$ and $Q \in \{*, \mathbb{S}^2, \mathbb{S}^3\}$.*

Consider the special case where M is a product of spheres. This theorem implies that each sphere has dimension at least c, which is at least 16, so N cannot be $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^{n-4}$ or $\mathbb{S}^k \times \mathbb{S}^{n-k}$ with $1 < k < 16$, as claimed in the introduction.

Observe that the Lie group E_8 has the rational cohomology of a product of spheres in dimensions $3, 15, \ldots$. It follows that the rational cohomology of $\mathrm{E}_8 \times \mathbb{HP}^3$ is 4-periodic up to degree 15, so we must take $c \geq 16$ in the statement of this theorem.

Proof. Let N^n be a one-connected, compact symmetric space such that $H^*(N;\mathbb{Q}) \cong H^*(M;\mathbb{Q})$. Assuming without loss of generality that $n > 0$, Lemma 1.6 implies $n \geq 16$. Write $N = N_1 \times \cdots \times N_t$ where the N_i are irreducible symmetric spaces and $b_4(N_1) \geq b_4(N_i)$ for all i.

Theorem C implies that $H^*(N;\mathbb{Q})$ is 4-periodic up to degree c. If $H^4(N;\mathbb{Q}) = 0$, then N and hence each N_i is rationally $(c-1)$-connected. Since $c \geq 16$, Lemma 3.1 implies that N is a product of spheres of dimension at least c.

Suppose therefore that $H^4(N_1;\mathbb{Q}) \cong H^4(N;\mathbb{Q}) \cong \mathbb{Q}$. By Lemma 3.2, N_1 is 4-periodic up to degree c and $H^j(N_i;\mathbb{Q}) = 0$ for $3 < j < 16$ and $i > 1$. If N_1 is $\mathbb{CP}^q$ or $\mathrm{SO}(2+q)/\mathrm{SO}(2) \times \mathrm{SO}(q)$, then taking $M' = N_1$ and $M'' = N_2 \times \cdots \times N_t$ in Lemma 3.2 and applying Lemma 3.1, we conclude that every N_i with $i > 1$ is a sphere of dimension at least c. This concludes the proof in this case.

If N_1 is not $\mathbb{CP}^q$ or $\mathrm{SO}(2+q)/\mathrm{SO}(2) \times \mathrm{SO}(q)$, Lemma 3.1 implies N_1 is $\mathbb{HP}^q$ or $\mathrm{SO}(3+q)/\mathrm{SO}(3) \times \mathrm{SO}(q)$. If $b_2(N_i) = b_3(N_i) = 0$ for all $i > 0$, then once again we have that each N_i with $i > 1$ is a sphere of dimension at least c. Otherwise, we may reorder the N_i so that $b_2(N_2) > 0$ or $b_3(N_2) > 0$. By Lemma 3.1, N_2 is $\mathbb{S}^2$ or $\mathbb{S}^3$, and by taking $M' = N_1 \times N_2$ and $M'' = N_3 \times \cdots \times N_t$ in Lemma 3.2, we conclude that N_i is a sphere of dimension at least c for all $i > 2$. This concludes the proof. $\square$

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